HOW WE UNDERSTAND NUMBERS

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ABSTRACT

There is evidence to suggest that all numerical formats are not processed by the same internal analog representation. The multiple analog representation models of numerical processing propose that there are individual analog representations per numerical format. Our research has expanded on these models in assessing whether there are multiple analog representations within a single numerical format. The data suggest that relative frequencies in fact do involve multiple internal quantity representations.

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INTRODUCTION

The cognitive literature on numbers and numerical representations supports an innate ability to determine and discriminate between numerical quantities. This ability is demonstrated in animal (Brannon & Terrace, 1998; Cantlon & Brannon, 2006; Davis & Perusse, 1988; Matsuzawa, 1985; Meck & Church, 1983), pre-verbal infant (Kobayashi, Hiraki, & Hasegawa, 2005; Mack, 2006; Starkey, Spelke, & Gelman, 1983; Starkey & Cooper, 1980), and adult research (Buckley & Gillman, 1974; Dehaene, 1995; Dehaene, 1990; Hinrichs, Yurko, & Hu, 1981; Moyer & Landauer, 1967; Zhang & Wang, 2005). PET scans provide support that specific brain areas are designated to process numerical stimuli (Burbaud, et. al, 1995; Burbaud, et. al, 1999; Dehaene, et. al, 1996; Kiefer & Dehaene, 1997; Rueckert, et. al, 1996). Finally, there is considerable functional evidence that individual's perceptual errors when viewing numbers are similar to errors when viewing actual quantities (Buckley, 1974; Dehaene & Akhavein, 1995; Dehaene, 1990; Hinrichs, 1981; Moyer, 1967; Moyer & Landauer, 1973; Welford,1960; Zhang, 2005). This supports the supposition that numbers are processed in ways similar to those of physical quantities.

There are a number of theories that attempt to model the relation between numerical symbols and the quantities that they represent. The single representation model suggests that all numerical stimuli are converted into a single analog representation (McCloskey, Sokol, & Goodman, 1986). The triple code model proposes that there are three representations for numerical stimuli: a verbal, a visual, and an analog code (Dehaene, 1992). The multiple representation model suggests that there is an individual representation for each numerical format (Gonzalez & Koler, 1982). In this paper I will examine the response times of individuals comparing two relative frequencies. The comparisons will involve both relative frequencies on the same scale and relative frequencies on different scales. This data will provide insight into whether all relative frequencies are processed within a single analog representation or whether there are multiple analog representations – one for each scale of the relative frequency.

Non-verbal Understanding of Numerosity

Multiple animal studies, involving a variety of numerical stimuli, have demonstrated an ability to discriminate between numerosities and rank numerosities as larger or smaller (Brannon, 1998; Cantlon, 2006; Davis, 1988; Matsuzawa, 1985; Meck, 1983). These results provide evidence that indicate an innate numerical competence for the ability to distinguish between different quantities. These results also provide evidence that internal numerical representation existed before the language of numerical symbols was developed.

Researchers have demonstrated that primates are able to rank numerosities as larger or smaller. Brannon and Terrace (1998) trained two monkeys to discriminate between numerosities 1-4 and to place them in increasing order. To train the monkeys, a touch screen monitor simultaneously displayed four numerosity sets and it was the monkeys' task to order the numerosities in ascending order (eg. 1 arrow right 2 arrow right 3 arrow right 4). The monkeys were then tested with numerosity sets of 5-9 to examine whether they were able to properly order the novel stimuli. In addition, the novel stimuli were varied in terms of size, shape, and color to control for non-numerical cues. The authors found that the monkeys were above chance in responding to the new stimuli 5-9. In a similar study, Cantlon (2006) trained two monkeys to order the numerical values 1-9. The authors then had the monkeys determine whether the numbers 10, 15, 20, and 30 were larger or smaller than the numbers 1-9. The monkeys responded above chance to comparing the new stimuli to the learned stimuli in the larger/smaller comparison task. The authors argue that such spontaneous behavior of ordering new stimuli demonstrates an innate understanding for numerosities.

The ability to discriminate between numerosities has also been found in pre-verbal infants. Pre-verbal infants have not developed the ability to understand language nor the symbols that have been developed to represent numbers. However, in the past 30 years, a number of experiments have demonstrated that pre-verbal infants can determine whether two numerosity sets have the same quantity value or different value (Kobayashi, 2005; Mack, 2006; Starkey, 1980; Starkey, 1983). Pre-verbal infants have not been introduced to the concepts of numbers and thus their ability to discriminate between numerosities suggests that this ability is innate. Starkey (1980) conducted an experiment with 6- to 7-month-old infants using a habituation-recovery task. Infants were repeatedly shown a slide with a certain number of dots on it. When the infants decreased their looking time, an indication of habituation to the slide, the experimenters presented a slide with a different number of dots on it. They found that the children would look at the new slide longer which indicates that the infants were able to discriminate between the two different quantities. In a similar study, Wynn (1992) presented infants with either 1, 2, or 3 toys and then the toys were placed behind a screen. The researcher would then drop the screen, revealing either 1, 2, or 3 toys, and the infants' looking times were recorded. They found that if the number of toys presented did not correspond with the number of toys revealed when the screen was dropped, that the infants looked longer at the incorrect result. This demonstrates that the infants were able to recognize a difference in the number of toys presented and thus an ability to decipher between two different numerosities.

There is also evidence that pre-verbal infants are able to transform a quantity that is presented as a number of auditory sounds into a visual quantity representation. Starkey et al. (1983) conducted an experiment in which 6- to 8-month-old infants were presented with an auditory stimulus of 2 or 3 drumbeats. The infants were simultaneously presented two slides, one with 2 objects and one with 3 objects. After the auditory stimulus was presented, the slides remained in front of the infants for another 10 seconds. The infants' looking times were recorded. The authors found that the infants looked longer at the slide of objects that corresponded to the number of drum beats presented. This experiment demonstrates that the infants were able to relate the number of drum beats heard to the number of objects presented.

Further evidence that the representations of numerosities are non-language based comes from research with individuals that have language disorders. Henschen (1920) ran a number of case studies with humans, in which language and calculation abilities were shown to be independent of each other. An excellent example of the discrimination between language and numerical comprehension is a patient named IH. IH's language comprehension was poor and his production of language was limited to repeated phrases. However, on single and multi-digit calculation tasks, he was found to be completely competent, often scoring at or near the maximum score for each test (Cappelletti, Kopelman, & Butterworth, 2002). This further supports the independence of numerical and lexical based language processing.

The literature involving animals, pre-verbal infants, and individuals with language disorders demonstrates that the ability to represent numerosities is innate. This innate ability in non-language producing animals and infants strongly suggests that numerical representations and language are distinct.

Neuroanatomy of Numerosity

To examine the biological basis for representing numerosities, a number of experiments have examined the neural circuitry involved in number processing. There is evidence to suggest that there are specific brain areas in which quantities are processed, and these areas are distinct from those devoted to processing language.

As with most cognitive processes, there are specific brain areas that are involved in processing and representing numbers. Dehaene, Piazza, Pinel, and Cohen (2003) analyzed research involving functional magnetic resonance imaging (fMRI) to examine the activation of the parietal lobe and numerical representations. Based on multiple research studies (Burband, 1999; Chochon, Cohen, van de Moortele, & Dehaene, 1999; Pesenti, Thioux, Seron, & De Volder, 2000), the horizontal segment of the intraparietal sulcus (HIPS) has been found to be consistently activated when quantity processing is involved. The HIPS has been shown to be activated during mental arithmetic and numerical comparisons (Chochon, 1999). It has also been shown to be activated for numerical words but not for general language comprehension (Dehaene, 1995). In addition, activation does not depend on the numerical format that is presented (Dehaene, 2003).

PET scan results also support the conclusion that the inferior parietal lobe is involved in number processing. It has been found that different areas of the parietal lobe are activated, depending on the numerical manipulation. The frontal and inferior parietal area is activated bilaterally during subtraction problems (Burbaud, 1995; Rueckert, 1996). Intraparietal activation was discovered during multiplication of two digit numbers (Dehaene, 1996) and brain activity was found to be left-lateralized for multiplication problems, right-lateralized during comparison tasks, and bilateral during subtraction problems (Dehaene, 1996; Kiefer, 1997).

Research with primates suggests that neurons are activated for specific numerosities. Nieder, Freedman, and Miller (2002) examined macaque monkeys' neuron firing for different numerosities. The researchers presented the monkeys with two different computer displays of dots and trained them to determine whether the two displays had the same number of dots or not. The monkeys were trained on numerosities 1-5 and the dots varied in location and physical size on the computer screen. They were trained until they achieved a better than chance success rate. The researchers then inserted electrodes into the monkeys' lateral prefrontal cortex. They found that specific neurons fired for specific numerosities. For example, a neuron might fire maximally for the three dot display and slightly less for the two or four dot display. To ensure that the monkeys' neuronal firing was not simply due to a physical characteristic of the display, the authors presented the monkeys with novel displays in which the physical appearance of the dots varied compared to the originally trained set. The firing of the neurons was found to be dependent solely on the numerosity presented, not on any visual cue. The monkeys' ability to apply what they had been trained on to new stimuli indicates an expansion of an ability that is innate, which the training alone could not provide.

Analog Representation of Numerosity

The preponderance of evidence suggests that numerosities, presented by a numerical symbol, are converted into an internal analog representation (Dehaene, 1992). An analog representation is a representation of quantity that exists on a continuous dimension, just like distance, time, and length (Carey, 2001). The perception of numerical stimuli by individuals is similar to how physical stimuli are perceived on a continuous dimension.

There is a mathematical equation that relates how physical stimuli are perceived by individuals: the Weber-Fechner law (Fechner, 1948). The Weber-Fechner law mathematically demonstrates that there is a discriminability between two stimuli that is a function of their ratio. The differential equation is

$$dP = k(dS/S)$$

where dP is the change in perception, k is a constant that is determined experimentally, dS is the change in the stimulus, and S is the stimulus at that instant. Using this equation, it becomes apparent that it is easier to discriminate between the numbers 4 and 5 than it is 9 and 10. Mathematically, the difference between 4 and 5 is perceived as 1/5 and the difference between 9 and 10 is perceived as 1/10. Thus the perception is that there is a smaller difference between 9 and 10 than the numbers 4 and 5, even though mathematical difference is the same. If the equation is rearranged:

$$p = k \ln(S/S_0)$$

where p is perception, k is a constant, S is the stimulus presented, and S_0 is the base threshold of the stimulus (Fechner, 1948), the equation demonstrates that discriminating between two stimuli is based on a logarithmic function. In order to change the perception of a continuous function, the stimulus must be increased by a multiplicative factor. For example, if an original stimulus is given the value 3, in order to perceive that a new stimulus is double that of the original, it must have the value of 3^2 or 9. This is due to the logarithmic relationship between stimuli and the actual perception of the stimuli.

Due to the inflexibility of Weber-Fechner's law, Steven's (1956) revamped the original equation. Steven's power law

$$\log R = a \log (S-S_0) + \log k$$

includes an additional parameter and thus makes the equation applicable to more types of stimuli (ex. pain, pressure, sound, light) than the Weber-Fechner law. This equation supersedes the Weber-Fechner equation because it describes a broader range of stimuli. For our purposes, this equation is the newest mathematical expression of the perception of numerical stimuli.

Research has consistently demonstrated that numerical stimuli are also subject to Steven's power law (Buckley, 1974; Dehaene, 1990; Dehaene, 1992; Dehaene, 1995; Hinrichs, 1981; Moyer, 1967; Moyer, 1973). That is, numerical stimuli produce the same pattern of data as other continuous stimuli (Carey, 2001; Fechner, 1948; Stevens, 1956; Stevens & Harris, 1962; Stevens & Guirao, 1963; Stevens & Mack, 1959; Walker, 2002). Most notably, research with adults, who have acquired the language of numbers, has demonstrated that numerical symbols are transformed into an internal analog representation and the results are consistent with Steven's equation (Dehaene, 1992). Support for an internal analog representation is most effectively demonstrated with experiments involving the distance effect.

In a classic experiment, Moyer and Landauer (1967) discovered what is now known as the distance effect. The authors presented the participants with the numbers 1-

9 and asked them to determine which of two numbers was larger or smaller. The authors found that the participants' reaction times were an inverse function of the distance between the two numbers presented. More explicitly, the *distance effect* is when two numbers that are close together take a longer time to discriminate between than two numbers that are farther apart. Moyer furthered this research when he completed an experiment where he examined the relationship between reaction time and memory retrieval (Moyer, 1973). Participants were presented with a pair of animal names and were asked to determine which animal was larger. They again found that the reaction times of the participants were inversely related to the difference in animal size. Moyer and Landauer determined that the distance effect can be described by the Welford (1960) function:

$$RT = a + k*\log[L/(L-S)]$$

where RT is the response time, a and k are constants, L is the larger number being compared, and S is the smaller number being compared. Parkman (1971) scrutinized Moyer and Landauer's (1967) conclusion and claimed that they overestimated the importance of (L-S) and underestimated the importance of simply S. Parkman came to this conclusion because S was more highly correlated with RT than (L-S). Moyer and Landauer (1973) reanalyzed their data, as well as Parkman's (1971) data and found that log(L/L-S) was more highly correlated with RT than S. This defense demonstrated that both numbers, the larger and smaller numbers being compared, are important when running a comparison task. In addition, the research shows that the Welford model is a reliable representation of the distance effect.

The distance effect is a robust effect that has been repeatedly found with adult humans. These result have not only been found when comparing numerosities, but the use of dots (Buckley & Gillman, 1974), with single Arabic digits (Buckley & Gillman, 1974; Dehaene, 1995; Moyer, 1967) and with verbal notation of numbers (Dehaene, 1995). In addition, the effect has been found when multi-digit comparison tasks were performed (Dehaene, 1990; Hinrichs, 1981; Zhang, 2005).

The major criticism against the analog representation is not the representation itself, but in regards to the automaticity of the representation. Pansky and Algom (2002) found that automatic processing of quantity information can be eliminated if the stimuli are manipulated. Participants were shown two 3X3 matrices. Each matrix contained the number 8, the number 2, or an asterisk. Participants were asked to judge either the numerical magnitude or the numerosity in the matrix. To judge the numerical magnitude, participants had to select the matrix that had the numbers with the higher magnitudes in it. To judge the numerosity, participants had to select the matrix with the higher quantity of numbers in it, regardless of the magnitude of those numbers. The researchers found that the automatic processing of the numbers could be eliminated by changing the format of the numbers and asterisks. When the asterisks were enlarged, numerical magnitude was judged faster than numerosity. Thus the automatic processing of numerical magnitude was easier than discerning the numerosity of numbers in the matrix because of the large size of the asterisks. When the authors reduced the size of the asterisks, numerosity was judged faster than numerical magnitude. The small size of the asterisks made it easier to count the numbers and the automatic processing of the numerical magnitude was reduced. The authors concluded that automatic processing of numbers can be altered by way of presentation. In another experiment, Dehaene (1995) found that he could eliminate automatic processing by having participants compare word numbers (ex. ONE) and Arabic numbers (ex. 1). When the participants were asked to compare two stimuli and respond to whether the stimuli were in the same format, both written as words or both written as Arabic numbers, automatic processing was eliminated. These experiments do not counter the theory of an analog representation; rather, they demonstrate that quantities are not always automatically processed.

A secondary criticism against the analog representation is the possibility of intermediate representation. An intermediate representation could exist between processing the numerical stimuli and converting the stimuli into an analog representation. Noel and Seron (1997) found that a single numerical format can have different numerical biases due to differing intermediate representations. The experiments demonstrated that the presentation of a number has an impact on how a number is processed. In one of the experiments the participants were shown a pair of numbers. There were two different structures in which numbers could be presented. The two structures were either a tens-hundreds word (ex. 1200=12 hundred) or a thousands-hundred word (1200=1 thousand 2 hundred). The pairs of numbers were either the same structure or one of each structure and the participants were asked to respond to which number was larger. They found that it took the participants less time if both of the numbers had the same structure and less time for the tens-hundred structure. The authors concluded that the different structures of these numbers, that represent the exact same quantity, produce two different intermediate representations. In addition, this suggests that numerical stimuli are not automatically converted into an analog representation but rather, are first converted into an intermediate representation. Alternatively, these results could demonstrate that there are multiple analog representations, which will be discussed later in the paper.

Models of Numerical Representation

As demonstrated, there is a general consensus that numerical stimuli are converted into analog representations. However, there is a current debate on whether there is a single analog representation that all numerical formats are converted into or multiple analog representations in which each format has its own representation. There are a number of models that have been proposed that attempt to explain the current data involving numerical representations.

Single Representation

McCloskey developed a model that proposes that all numerical inputs are transformed into a single analog representation. The numerical symbol is converted into an analog representation and the analog representation consists of the quantity plus the power of ten associated with the number. For example, the Arabic number 4031 would be represented as {4}10EXP3, {3}10EXP1, {1}10EXP 0. The numbers in braces would be represented as quantities and the EXP numbers represent the power of ten associated with each quantity. The analog representation is processed or manipulated the same way, regardless of its original format. From this manipulation, an output is produced (McCloskey, 1986).

McCloskey proposes that numbers are categorized into lexical stacks in both the analog representation stage and the number production stage. The lexical stacks consist of ones, teens and tens. The ones stack consists of the numbers 1-9, the teens stack consists of numbers 10-19, and the tens stack consists of 20, 30, 40, 50, 60, 70, 80, and

90. In addition, there are multiplier words (e.g. hundred) that are used to expand the number of tens places of the number (McCloskey, 1986).

Research on numerical representation involving dots and digits has found that there is a single underlying representation for dots and digits. Buckley and Gillman (1974) found that four different formats presented to the participants had a single underlying abstract representation. The participants were shown either Arabic numerals, a regular dot pattern, an irregular dot pattern, or a random dot pattern. The regular dot pattern was similar to what appears on a pair of dice. The irregular dot pattern consisted of the dice pattern, with one of the dots “misplaced”. The random dot pattern was a random pattern. Buckley had the participants perform a larger/smaller comparison task involving each of the formats. The results demonstrated that the distance effect occurred for each of the formats. In addition, the authors statistically scaled the data on multidimensions. When the data was represented on a two dimensional graph, it revealed that the four different formats were processed similarly. These results support that, for at least dots and digits, there is a single underlying analog representation.

McCloskey’s research with brain damaged patients supports his single representation model and the existence of lexical stack representations (McCloskey, 1992; McCloskey & Macaruso, 1995; McCloskey, Sokol, Caramazza, & Goodman-Schulma, 1990). In a frequently referenced case study, McCloskey (1986) examined a patient named HY who suffered from brain damage. HY was given an array of mathematical tests that demonstrated that he was completely competent in comprehending and manipulating numbers. However, when asked to verbally produce the correct answer, HY’s responses became significantly more erroneous. The authors found that his errors occurred in the same lexical stack. For example, if the correct number was a ones number, HY would produce an incorrect ones digit. These findings support that the patient was transforming all of his numbers into analog representations, he just failed in his production stage to be able to produce the correct words (phonological representation) to represent those analogs. The erroneous answers he did produce were not random, but rather were in the same lexical stack as the correct answer.

As demonstrated, some of the research on numerical cognition supports the single representation model. The model is concrete and makes specific predictions. However, it is limited in its ability to explain all the results found in numerical cognition.

Triple Code Model

Research in the field of cognitive arithmetic has revealed some of the underlying processes involved in numerical representation. There are currently three major models that illustrate arithmetic processing: Ashcrafts' (1987) network retrieval model, Siegler and Shrager's (1984) distribution of associations model, and Campbell's (1987a; 1985) network interference model. A common factor in each of the three models is that mathematical facts, like addition and multiplication problems, are stored in memory in an organized fashion (Ashcraft, 1992). This research suggests that an analog representation of quantity is not always activated when numerical stimuli are presented. Instead, there is an inter network of information, that is recalled when an individual is presented with a mathematical problem. A quantity representation would not be necessary, thus not activated, when a mathematical fact is needed.

Based on this cognitive arithmetic research and evidence, Dehaene (1992) rejected McCloskey's (1986) model and created a new model termed the Triple Code

Model. The triple code consists of a verbal, visual, and analog code and each of these codes activates the parietal area of the brain. The verbal code is used to manipulate verbally spoken numbers, as well as numbers written as words. This code is located in the left angular gyrus. The visual code is used for visual number forms, such as Arabic digits. This takes place in the posterior superior parietal system. The analog code is used to represent numerical quantities. The horizontal intraparietal system (HIPS) is activated in both hemispheres for analog representations (Dehaene, 2003).

The triple code model proposes a network that consists of an analog, visual, and verbal code, however, one of the codes can be activated, without the activation of the other (Dehaene, 1992). The analog code, which is a quantity representation, is activated when estimations are required. The visual and verbal codes store memorized facts (ex. multiplication tables) and thus would be activated for numerical fact retrieval (Dehaene, 1992; Gonzalez & Kolers, 1987). Dehaene (1992) proposes that quantity retrieval is not necessary in order to recall memorized facts, which is further supported by Ashcraft's (1992) research. Thus the analog, visual, and verbal codes can be activated independently of each other. These propositions have also been supported with research involving PET scans (Dehaene, 2002).

Dehaene (2002) has found that the triple-code model predicts the results of various mental disorders. In split-brain patients, the model predicts that the left hemisphere can complete calculations because the three codes exist in the left hemisphere. The right hemisphere of the brain, which consists of only the analog representation (HIPS), can recognize and understand the quantity amount of a number, but is not able to verbally communicate the results. The model accurately predicts these results.

Dehaene (2002) argues that the triple-code model predicts the neurological route of number processing in the brain. For multiplication tables and rudimentary addition problems, the numbers are first transformed into verbal representations which recall the word sequence from memory. This process takes place in the left-cortico-subcortical loop involving the basal ganglia and the thalamus. This first route is used for memorized mathematical facts. The second route is termed an *indirect route* and is used for quantity retrieval. The numbers are encoded as representations in the left and right inferior parietal areas. Calculations on the representations produce a quantity that is transmitted from the left inferior parietal cortex to the left-hemispheric perisylvian language network for verbal production. This indirect route is used when the calculation is not memory based.

Dehaene's (Dehaene, 1990; Dehaene, 1992; Dehaene, 1995; Dehaene, 2002; Dehaene, Piazza, Pinel, & Cohen, 2003; Dehaene et al.,1996) research in the field of numerical cognition is extensive. His contributions range from a theory of memorized mathematical facts to the neuro-circuitry of numerical processing. However, his current triple code model fails to address recent research on multiple formats and the subsequent biases associated with each format.

Multiple Quantity Representations

There is extensive research that rejects both McCloskey's (1986) claim of a single abstract representation and Dehaene's (1992) triple code model. The overriding premise of the research demonstrates that different formats of numbers result in different response times, error types, and biases and thus the possibility of more than one analog representation (Cohen, Ferrell, & Johnson, 2002; Gonzalez & Kolers, 1982; Takahashi & Green, 1983).

Examining the theory of multiple representations, Takahashi (1983) found that numbers written as words are not processed the same as numerical symbols. The author examined two Japanese scripts, Kanji (number symbols) and Kana (verbally written numbers). In addition to the type of stimuli, the authors also analyzed whether the physical size of the stimuli would impact a larger/smaller comparison task. The congruent condition consisted of a number that was physically larger, as well as numerically larger. The incongruent condition consisted of a number that was physically larger, however its numerical value was smaller. The stimuli presented were the numbers 1-9 written in Kanji and Kana, in two separate sessions. The participants were asked to respond to which of the two numbers presented was numerically larger. The authors found that the participants reacted faster to the Kanji stimuli than the Kana. For the Kanji numbers, the participants responded quickest to the congruent condition, then the same condition and responded slowest for the incongruent condition. They found that the distance effect occurred throughout the conditions for the Kanji script. For the Kana condition, there was a main effect for physical size of the stimuli. Participants responded the same for the congruent and the same condition, but responded significantly slower for the incongruent condition. The most interesting finding is that the distance effect produced different results for the two scripts. The data suggests that there are different biases involved in the two formats and thus this difference suggests that the two scripts have separate analog representations.

Gonzalez and Koler (1982) also refute the single analog representation model in support of the multiple representation model. In addition, they propose that the individual representations per format may be inaccessible to each other. The researchers examined whether Roman numerals and Arabic numbers are processed differently. In experiment one, the participants were shown 45 sums that were either in Roman numeral form, Arabic form, or a combination of the two. The participants job was to determine whether the equation, p+q=n, was correct or incorrect. The overall finding was that as the number of Roman numerals increased in the equation, the response time also increased. Graphing the data, as the number of Roman numerals increased, the intercept also increased. This could be solely due to inexperienced with Roman numerals or that Roman numerals simply take longer to be translated into an analog representation. If this was the case, than as the number of Roman numerals increased, the intercept would also increase but the slopes of the lines would be equivalent. However, the slopes of the response times for Arabic numerals also differed from the slopes involving Roman numerals. The varying slopes represent multiple biases in converting of the Roman and Arabic stimuli into analog representations. Thus this finding supports a multiple analog representation in which Roman and Arabic numerical formats have different analog representations. In the same study, the researchers analyzed whether lack of familiarity with Roman numerals had impacted the results found in the previous experiment. The participants were given practice trials with Roman numerals (I through X) until they responded to the value of the number at a rate close to 10% of their ability on Arabic trials. The participants were then presented with the same equations from the previous experiment and the findings were replicated. Thus, the authors concluded that participants' familiarity with Roman numerals had no impact on the differential biases found for Arabic and Roman numerals.

Gonzalez and Koler's (1982) theory that numerical formats each have an individual analog representation has only recently been analyzed with numerical quantities less than one. Cohen (2002) completed a number of experiments to analyze production and judgment tasks in order to access the perception and understanding of proportions. In the first experiment, participants were asked to estimate the proportion of black dots to white dots in either decimal (ex. .01) or relative frequency (ex. 1 in 100) format that appeared on the screen. There were 50,000 black dots on a screen and the number of white dots formed proportions ranging from .0001 to .01, however the participants were uninformed of this range. In the second experiment, participants were either in the decimal, relative frequency, or display group. They were either presented with a number (decimal/relative frequency) and then were asked to represent it on the dot screen by adding white dots to the screen filled with 50,000 black dots or they were shown a display screen and were asked to replicate it on the response screen. The third experiment was a replication of the first experiment, however the number of dots on the screen varied between 500 and 50,000 and only three different proportions were presented (.2, .02, .002). The fourth experiment was similar to the second experiment however the total number of dots varied between 500 and 50,000, only three proportions were presented (.2, .02, .002), and there were only two conditions (decimal and relative frequency group). In the fifth experiment, participants were asked to convert between decimals and relative frequencies and between relative frequencies and decimals. The results of the fifth experiment revealed that individuals were unable to systematically convert decimals into relative frequencies or vice versa. In fact, the participants did not even create the correct ordinal relationship between the two formats. Instead, the participants produced large relative frequencies for small decimal values and vice versa. The inability to easily convert between decimals and relative frequencies suggests that the two numerical formats do not share an analog representation.

At this point, the current numerical models agree that numerosities and numerical symbols are represented as analogs. However, the theories disagree on how many analog representations exist. The most recent research suggests that some formats may share an internal analog representation, while other formats are individualized. It is plausible that innate formats like dots could share an internal representation with integers due to the development of the integer notation being based on externally represented numerosities. With this logic, it is also possible that, different symbolic notations, such as Roman numerals and Arabic numerals, have individual analog representations.

Theoretical Summary

The field of numerical cognition has revealed that numerical representations are an innate cognitive process that is independent of the development of language. Numerical representations have been shown to exist as analog representations and research involving brain scans and electrodes has shown that specific brain areas and neurons are specified to process quantity information. The primary discrepancy in the field of numerical cognition is the models. The single representation model claims that all numerical formats are transformed into the same analog representation. The triple code model claims that the analog representation is not always accessed and this activation depends on the method of delivering the quantity representation, not necessarily on the format. The multiple representation model claims that there are individual analog representations per format of the quantity. My research has examined whether comparing two relative frequencies involves a single or multiple analog representations.

Relative Frequencies

The current research will examine relative frequencies and their cognitive representations. The experiments involve comparing relative frequencies with various denominators. Relative frequencies are a unique numerical format. Within the single numerical format, there are multiple scales in which a quantity can be presented. These scales are based on the relative frequency's denominator. For example, the same quantity could be presented as 2 in 4 or 1 in 2. However, the first example is presented as 2 parts on a scale that is broken into 4 sections and the second example is presented as 1 part on a scale that is broken into 2 sections. Analyzing this numerical format will allow for an examination of whether analog representations are based on the numerical format as a totality and thus the example above would produce equivalent response times on a larger/smaller comparison task or whether there can be multiple analog representations within a single numerical format and thus the example above would not produce equivalent response times on a larger/smaller comparison task. In addition to the models, a secondary reason for examining relative frequencies is the current discrepancy on whether relative frequencies are naturally understood. Research has consistently demonstrated that elementary children have difficulty understand quantities less than one (Mack, 1995; Ni & Zhou, 2005; Sophian, Garyantes, & Chang, 1997). However, Gigerenzer and Hoffrage (1995) claim that relative frequencies are naturally understood.

The research for both claims is persuasive. The current research will provide compelling evidence to resolve these two current debates in numerical cognition.

At the most basic level, children need to learn that whole numbers are infinitely divisible. Research has demonstrated that 6-9 year old children have difficulty understanding both the proportion notation of fractions and the quantities they represent (Hartnett & Gelman, 1998; Mack, 1995). By the third grade, children have demonstrated that they have learned that whole numbers are divisible, however, they still do not understand the fractional notation (Mack, 1995). In elementary school, specifically during the third to sixth grades, children are taught about numerical symbols that represent quantities between 0 and 1 (Smith, Solomon, & Carey, 2005). It is at this time that children learn that 2/6 does not mean 2 objects and 6 objects, or 2 whole parts with 6 pieces per whole part, but rather 2 of 6 parts of 1 whole. Research has demonstrated that 5-7 year olds cannot interpret the symbols and notations used to represent fractions (Sophian, Garyantes, & Chang, 1997). Elementary school children often make errors with fractions, due to intrusion of knowledge about integers (Ni & Zhou, 2005). The application of knowledge about integers onto fractional notation has been termed the "whole number bias." The whole number bias causes preschool children to deny quantities between 0 and 1 and middle school children to quantify fractions with a higher denominator as larger than a fraction with a smaller denominator (Ni & Zhou, 2005). Smith, Solomon, and Carey (2005) have demonstrated that even sixth graders still classify fractions with a high denominator as larger than a fraction with a smaller denominator. For example, 1 in 4 would be reported as a larger quantity than 1 in 3 because 4 is larger than 3. These errors are understandable, in a sense, because increasing the value of the denominator actually means that there is less per part; this increase in denominator represents a quantity decrease and this is counterintuitive (Sophian, Garyantes, and Chang,1997). Smith et al. (2005) asked third, fourth, fifth, and sixth grade children why fractions consist of two numbers. The authors found that 43% of the children could not explain the relationship between the two numbers, 31% gave ambiguous answers, and 24% could provide a clear explanation. This research demonstrates that children in elementary school clearly do not understand the symbolic notations of fractions.

Unlike fractions, Gigerenzer and Hoffrage (1995) claim that humans have a natural understanding for frequencies. Natural frequencies are the likelihood of event occurring in or during an organism's life. Through experience, humans become natural statisticians in their ability to determine whether an event is likely or unlikely to occur (Gigerenzer & Hoffrage, 1995). For example, research has shown that humans are sensitive to frequency patterns in language. This consists of the frequency in which specific letters or words are used and how words tend to be paired together (Hertwig & Gigerenzer, 1998; Hintzman, 1976). Natural frequencies are easily understood by adults because they consist of a series of whole numbers. Though natural frequencies are easily understood by adults, adults, like children, have difficulty understanding the association between proportion notation and the quantities the notation represents (Gigerenzer & Hoffrage, 1999).

Gigerenzer and Hoffrages's application of natural frequencies has involved Bayesian reasoning problems. Bayesian problems involve multiple probabilities in which a single probability needs to be inferred. A number of studies have found that individuals solve a higher percentage of Bayesian problems correctly when they are presented as relative frequencies rather than as percentages (Brase, 2008; Gigerenzer & Hoffrage, 1995). The relative frequencies are presented as X out of Y and it is theorized by the authors that this format is more accurate because the use of whole numbers makes the computations easier. Thus, the numerical format of relative frequencies is more naturally understood by adults than any other numerical format representing proportions.

Relative frequencies, like integers, represent quantities and thus as those quantities become quantitatively closer in value, it should become more difficult for the participants to decipher between the values. Thus, the expected result of comparing two relative frequencies would be the Welford function or distance effect. In addition, as the comparisons between relative frequencies become more challenging, it is expected that the slope of the distance effect will also increase. This expectation is based on research that Sekuler and Mierkiewicz completed with kindergartner, first-, fourth-, and seventh grade children (Sekuler & Mierkiewicz, 1977). The authors had the children complete a larger/smaller task with Arabic numerals. They found that the younger children's distance effect had a steeper slope than the older children due to the perceived difficulty of the task.

Analog Representation(s) of Relative Frequencies

As mentioned earlier, extensive research has established that integers are converted into quantity representations that are termed analog representations. This thesis will examine whether relative frequencies are also converted into analog representations or if they simply consist of mathematical facts. If relative frequencies are mentally represented as quantities, the data will produce the distance effect. It is likely that relative frequencies are represented as analogs because they cognitively represent quantities and there is no research to the contrary.

In order to compare two relative frequencies, the symbolic representations of the two relative frequencies must first be transformed into their analog representations. This thesis proposes three possible processes that could occur as the two frequencies are being compared.

The first theory is that the comparison is made directly within the single analog representation of relative frequencies and thus the comparison is quick. If relative frequencies, like integers, consist only of a single analog representation, then the comparisons should produce quick reaction times. The data should represent shallow slopes and have a small intercept value. There would be a shallow slope because the comparison of values would be easy across values and thus a shallow distance effect would be produced. The data would also have a small intercept value because there is only one conversion step, from a symbolic representation into an analog representation, before the quantities would be compared. This theory is based off of Gigerenzer and Hoffrage's research (1995, 1999) that natural frequencies are easily understood by adults.

The second theory is that a mathematical conversion must occur in order to transform the relative frequencies into a comparable format. The denominator of a relative frequency is not always the same value and thus the quantities are not always represented on the same scale. For example, a person could be asked to compare 3 in 20 and 14 in 100. In order to accurately state which quantity is larger, the person may convert the 3 in 20 to 15 in 100 to then compare the relative frequencies on the same scale (out of 100). If this theory is correct, the intercept value is expected to be large because the reaction time is consumed by the mathematical conversion. The comparison of the two quantities, on the same scale, would then be relatively quick across all the values compared and thus the data would produce a shallow slope.

The third theory is that the two analog representations are simply difficult to compare. This result would be demonstrated by a large slope value when relative frequencies of varying denominators are being compared. This could occur due to multiple factors. First, if relative frequencies, with different denominators, have different analog representations, the connection between these two representations could be weak. This weak association between analog representations could be caused by the rarity in which individuals are exposed to or use relative frequencies. This rarity in comparing relative frequencies, even those that have a large mathematical difference between them, causes the task to be difficult and thus large response times are the result. As the values of the frequencies become closer in value, the response times become substantially effected due to both the weak association between the analog representations and the comparison of two numbers close in value. The overall result is a steep slope for conditions in which multiple denominators are being compared.

EXPERIMENT 1

In Experiment 1, relative frequencies with varying denominators were examined to determine whether varying the denominator had an effect on individuals' reaction times. The relative frequencies were presented as X in Y. There were five different conditions, in which the second number, Y, of the relative frequency was manipulated. The participants were presented one condition at a time and completed 110 trials per condition. A single trial consisted of two relative frequencies presented on a computer screen, one number above the other, and the participants were asked to respond whether the bottom relative frequency was larger or smaller than the top.

Conditions one and two examined whether individuals were able to compare similar formats of relative frequencies in a larger/smaller comparison task. In condition one, the participants compared relative frequencies presented as X in 1000 to relative frequencies presented as X in 1000. In this condition, the participants could potentially only attend to the first number in the relative frequency and still respond correctly. In condition two, the participants compared relative frequencies presented as X in 736 and X in 736. The participants could, again, only attend to the first number and ignore the denominator and still respond correctly. However, the purpose of this condition was to examine whether the denominator has any influence on response time, compared to condition one.

Conditions three and four examined whether individuals were able to compare different formats of relative frequencies in a larger/smaller comparison task. In condition three, the standard was presented as X in 736 and the probe was presented as X in 1000. In condition four, the standard was presented as X in 1000 and the probe was presented as X in 736. These conditions reveal whether individuals were able to consistently convert between two different numerical representations of relative frequencies. Conditions three and four also examined whether presentation order of the numerical stimulus had an effect on response time.

Condition five examined whether individuals were able to compare relative frequencies that are presented with varying denominators. The relative frequencies presented were reduced to their lowest common denominator. The purpose of condition five was to expand the range of denominators presented to the participant and thus to challenge the participants' ability to convert between numerical representations.

METHOD

Participants

The experiment consisted of 45 participants, 25 females and 20 males. The participants were all over the age of 18 and were recruited from the University of North Carolina at Wilmington subject pool. The participants received course credit for their participation. The participants were unaware of the purpose of the experiment.

Materials

The experiment was performed on a MS Windows based computers and the stimuli were presented on 40cm computer monitor. The project instructor programmed the experiments using C++ and Java.

Stimuli

The conditions consisted of relative frequencies with the second number of the frequency varying, per condition. The relative frequencies were presented on the same screen, one above the other. The relative frequency presented on top was termed the standard and the bottom relative frequency, the probe. The participants sat 50cm from the screen. The individual integers presented on the computer screen were 4cm wide and 5 cm tall.

Condition 1: X in 1000 compared to X in 1000

All of the relative frequencies, presented in this condition, were presented as X in 1000. The standard and the probe relative frequencies presented to the participants were randomly selected from the range of 250 in 1000 to 750 in 1000.

Condition 2: X in 736 compared to X in 736

All of the relative frequencies, presented in this condition, were presented as X in 736. The standard and the probe relative frequencies presented to the participants were randomly selected from the range of 184 in 736 to 552 in 736. This range is mathematically proportionate to the range used in condition one.

Condition 3: X in 736 and X in 1000

The numerical format of the standard was X in 736. The probe was presented as X in 1000. The range of the standard was between 184 in 736 and 552 in 736. The range of the probe was between 250 in 1000 and 750 in 1000.

Condition 4: X in 1000 and X in 736

The format of the standard was X in 1000. The probe was presented as X in 736. The range of the standard was between 250 in 1000 and 750 in 1000. The range of the probe was between 184 in 736 and 552 in 736.

Condition 5: Lowest Common Denominator

All of the relative frequencies presented were presented with the lowest common denominator and thus the denominator varied between the standard and the probe between trials. The standard and the probe were randomly selected from the range of 250 in 1000 and 750 in 1000.

Procedure

When the participants arrived to the cognition lab, they were greeted and asked to fill out a consent form and a demographic survey. The participants were then brought into a small room that contained a single computer. The participants were then instructed that they would be completing five sets of trials. They were then asked to read through the instructions and advise the experimenter if they had any questions. The following instructions were presented on the computer screen:

In this experiment, you will be presented two relative frequencies (for example "1 in 10" and "5 in 10") representing different quantities. These two relative frequencies will be presented one above the other:

1 in 10

5 in 10

Your job is to judge whether the bottom relative frequency is larger or smaller than the top relative frequency. You will respond by pressing one of the two keys on the keyboard. At the beginning of the experiment, the computer will tell you which key to press if the bottom relative frequency is larger than the top relative frequency and which key to press if the bottom relative frequency is smaller than the top relative frequency. Please respond as quickly and as accurately as you can. Remember, speed is important, but accuracy is essential. Do you have any questions?

Each trial consisted of the standard quantity and probe quantity being presented, one above the other, on a single screen. The participant then determined whether the bottom quantity was larger or smaller than the top quantity, and pressed the L button, for a response of larger or S button, for a response of smaller. The screen remained visible until a response was made. There was a blank screen between each trial that lasted 2000ms.

Each condition consisted of 10 practice trials and 100 experimental trials. The order of the five conditions were randomized per participant. The participants' reaction times were recorded in milliseconds. Each participant participated in all five conditions during a 90 minute session.

RESULTS

Of the fifty-two participants that completed the experiment, the data of forty-five participants were examined. One participant was immediately excluded from the experiment, before beginning the task, due to being underage. The other six participants were removed because their error rate in the two easiest conditions, 10001000 and 736736, was higher than 10% in at least one of the conditions (Mean Error Rate = 74.0%). The high error rates demonstrated that the participants either did not understand the purpose of the experimental task or simply were not performing the task very accurately and thus were not included in the statistical analysis.

To allow the participants a chance to practice using the larger and smaller response buttons on the keyboard and to practice responding as accurately and quickly as they could, the participants completed 10 practice trials before starting the experiment. The initial data gathered consisted of response times for 100 trials, per participant, for each of the 5 conditions: 10001000 (M=1190ms, SD=564ms, ME=4.69%), 736736 (M=1339ms, SD=743ms, ME=4.73%), 7361000 (M=3305ms, SD=2763ms, ME=23.11%), 1000736 (M=3290ms, SD=2814ms, ME=20.82%), and lcd (M=4598ms, SD=3366ms, ME=20.18%).

Due to the nature of response time data, low and high response time cut-off values were determined per condition. The high response time cut-off threshold was determined by past research in which removing no more than 2% of the data is the recommended limit (Ratcliff, 1993). The top 2% of the response time data, per condition, was excluded from our analysis (Ratcliff, 1993). The cut-off values consisted of any response times greater than 3872ms for the 10001000 condition, greater than 5084ms for the 736736 condition, greater than 17303ms for the 7361000 condition, greater than 18439ms for the

1000736 condition, and greater than 21008ms for the lcd condition. The low response time cut-off threshold was also determined by condition. As a group, the participants' response times were examined in 50ms increments, per condition. Per 50ms increment, the percentage of correct responses was determined. The low cut-off threshold was determined to be the value in which that response time value and all lesser response time values the participants responded with an accuracy of 51% correct or higher. Responding with the correct answer more than 50% of the time demonstrated that the participants were completing the task as requested (Ratcliff, 1993).

The initial examination of the distribution of the data revealed that the data was positively skewed. In order to normalize the distribution, a log (base 10) function was applied to the response times. This transformation allowed the data to be examined with normal statistical assumptions. All further analysis were completed on the log transformed data.

To analyze whether the distance effect was present in each of the five conditions, a series of linear regressions on both a symmetrical log function (log |stimulus-probe|) and the Welford function (log max/(max-min)) of the log of the response times were completed. The analysis revealed that the symmetrical log function (log |stimulus-probe|) consistently produced a higher r^2 value on the log (base 10) response time data, than the Welford function (log max/(max-min) (See Table 1).

Linear regressions were completed on the symmetrical log function for each of the participant's log (base 10) response time data, for each of the five conditions. These analyses produced each of the participant's slope, intercept, and r^2 values per condition. These values were subsequently analyzed through a series of mixed model ANOVAs.

Table 1. The r^2 values for the symmetrical log function compared to the Welford function for each condition.

	10001000	736736	7361000	1000736	lcd
Log|s-p|	.051	.042	.054	.038	.019
Welford	.050	.039	.048	.035	.019

A mixed model ANOVA with subject as the random effect and condition as the within subject variable revealed a significant main effect of condition on slope, $F(4,220) = 9.125$, $p<.001$, MSE=.008. Tukey's post-hoc test revealed that the 7361000 condition (M=-.1626, SD=.1184) and the 1000736 condition (M=-.1492, SD=.1232) produced significantly higher slopes than the 10001000 condition (M=-.0902, SD=.0410), 736736 condition (M=-.0866, SD=.0404), and the lcd condition (M=-.0748, SD=.0851) (Figure 1). The analysis also revealed that the 7361000 and 1000736 conditions were not significantly different from each other and the 10001000 condition, 736736 condition, and lcd condition were not significantly different from each other. In addition, a one-sample t-test ($t(134)=-16.4482$, $p<.01$) demonstrated that the three most shallow slope conditions (10001000, 736736, and lcd) were significantly different from zero.

A mixed model ANOVA with subject as the random effect and condition as the within subject variable revealed a significant main effect of condition on intercept, $F(4,220) = 67.313$, $p<.001$, MSE=.034. Tukey's post-hoc test revealed that the 10001000 condition (M=2.9534, SD=.1190) and the 736736 condition (M=2.9966, SD=.1337) were not significantly different from each other, however both conditions had a significantly smaller intercept than the 7361000 condition (M=3.2545, SD=.2164) and the 1000736 condition (M=3.2694, SD=.2369). The 7361000 condition and 1000736 condition were found not to be significantly different from each other. In addition, the lcd condition had a significantly larger intercept than the other four conditions (M=3.5040, SD=.1844) (Figure 2).

A mixed model ANOVA with subject as the random effect and condition as the within subject variable revealed a significant main effect of condition on r^2, $F(4,220) =$

Figure 1. This graph demonstrates the mean slope values for each condition, for Experiment 1.

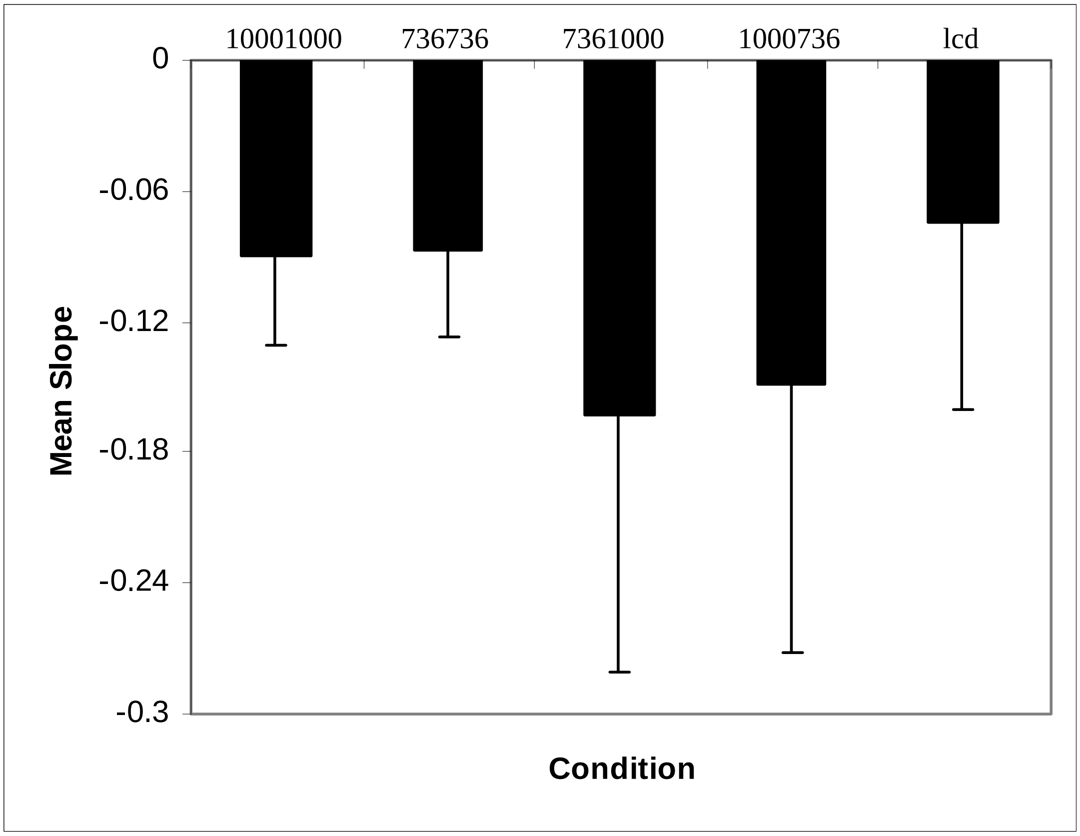
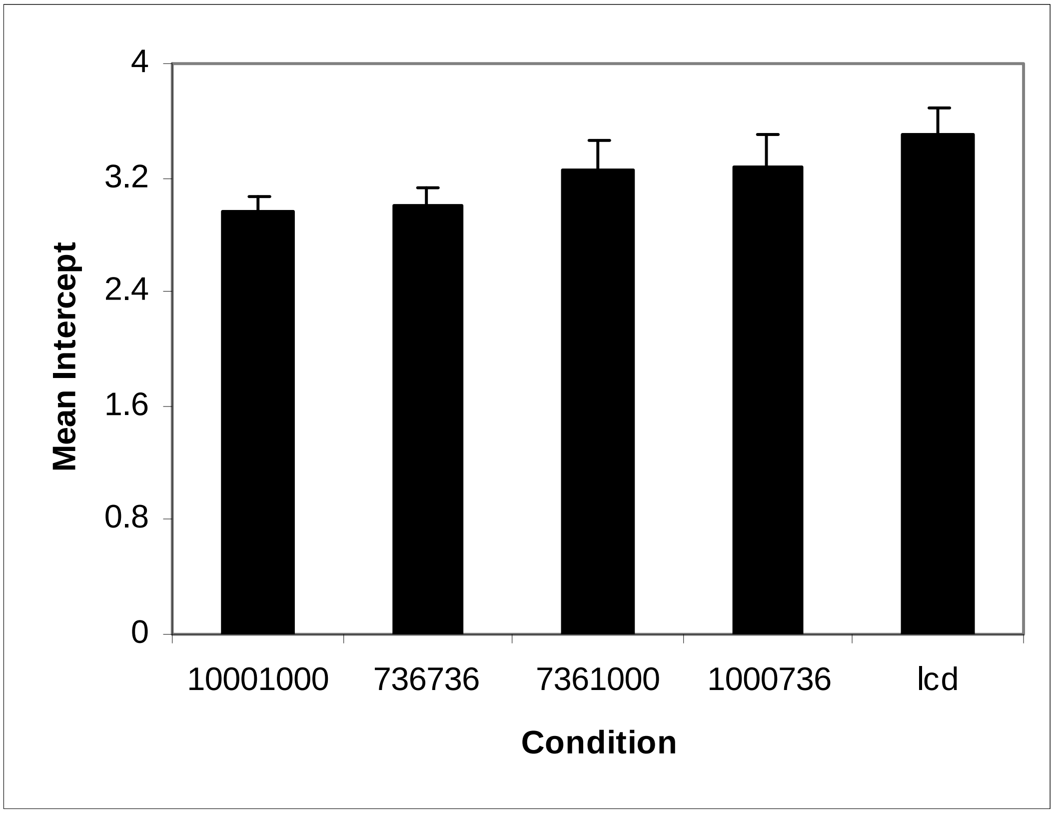
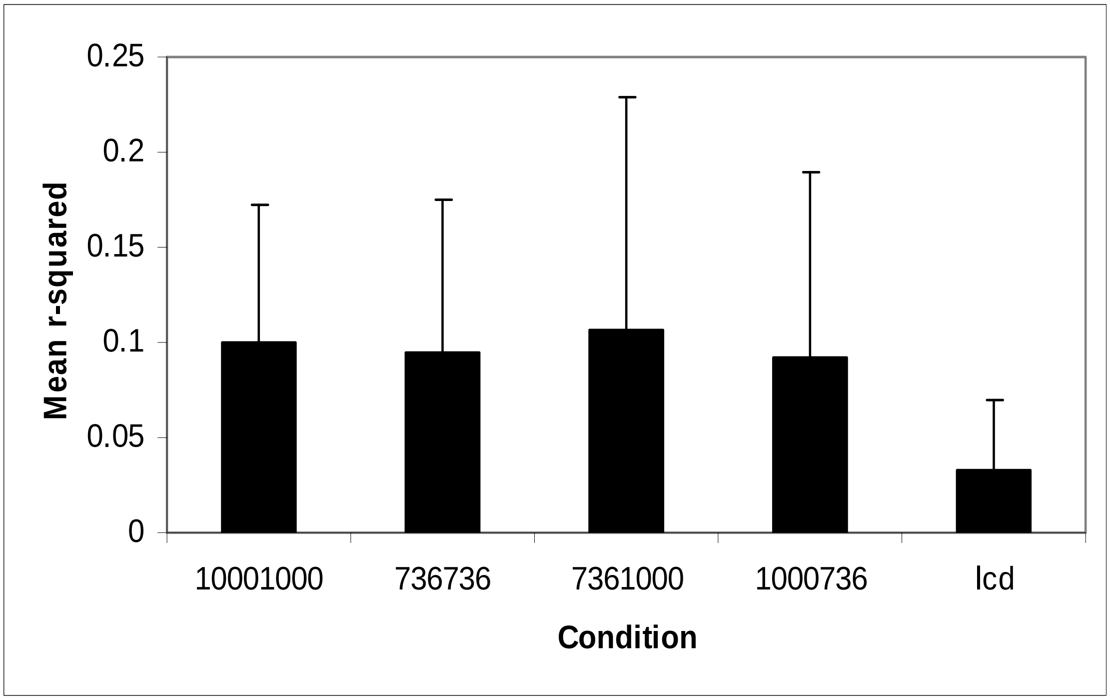


Figure 2. This graph demonstrates the mean intercept values for each condition, for Experiment 1.



5.492, $p<.001$, MSE=.007. Tukey's post-hoc test revealed that the lcd condition (M=.0324, SD=.0369) accounted for less of the variance than the 10001000 (M=.1003, SD=.0723), 736736 (M=.0952, SD=.0797), 7361000 (M=.1070, SD=.1216), and 1000736 (M=.0925, SD=.0976) conditions (Figure 3), which were not significantly different from each other.

Figure 3. This graph demonstrates the mean r^2 values for each condition, for Experiment 1.



DISCUSSION

The appearance of the distance effect, within each of the five conditions, suggests that relative frequencies (1) have an analog quantity representation and (2) the analog representation is not of the components of the relative frequency, but rather of the proportion the numerical frequency represents. First, the pattern of data revealed in Experiment 1 is consistent with the distance effect (Moyer & Landauer, 1967) produced with other analog tasks (Buckley & Gillman, 1974; Carey, 2001). This has traditionally been interpreted as supporting the assumption of analog representation (Dehaene, 1990; Dehaene, 1995; Hinrichs, 1981; Zhang, 2005). The same conclusion is appropriate here. Second, the slopes of the distance effect in the same denominator conditions suggest that these analog representations are of the proportion rather than the numerator of the relative frequency. If the participants based their judgments by comparing the two numerators and ignored the denominators, the distance effect would be based on the mathematical difference between those two integers. Instead, the participants were attending to the quantity represented by the relative frequency as demonstrated by the distance effect being based off of the mathematical difference between the proportions that the relative frequencies represented. If the numerator values were being compared, the slope values for the 10001000 and 736736 conditions would have been significantly different from each other. However, the slope and intercept values were not significantly different from each other in the 10001000 and 736736 conditions. Thus the pattern of data for the two conditions were essentially equivalent and this could have only occurred if the proportional values were used in both conditions.

The slopes for the two conditions in which the denominators of the probe and the standard are the same (i.e., 10001000 and 736736 conditions) were (1) the same and (2) shallow. The shallow slopes of the 10001000 and 736736 conditions indicate an efficient comparison between the quantities being compared. Theoretically, the most efficient comparisons should be made within a single analog representation. Given that one would logically suspect that relative frequencies with the same denominator would share a quantity representation and the data demonstrate a very efficient comparison process, we conclude that relative frequencies with a common denominator share a quantity representation. If all numerical representations existed within a single analog representation, we would expect to find that comparing quantities should be equally easy for all the conditions.

The lcd condition, in which there were many different denominators, produced an equivalent slope value to the same denominator conditions. The shallow slope demonstrates an efficient comparison between two quantities. Thus, when relative frequencies with varying denominators were compared, we conclude that the comparison occurred within a single analog representation.

The slopes for the two-denominator conditions (i.e., 7361000 and 1000736 conditions) were the same (i.e., there was no effect of presentation order) and quite steep. The data reveals that the quantity comparisons in the two-denominator conditions were slow and imprecise. It was hypothesized that with only two denominators being compared that these conditions would produce faster overall response times than the lcd condition, in which the denominator consistently varied. However, the slope for the 7361000 and 1000736 conditions were steeper than the lcd condition, suggesting that the task was even more difficult than the lcd condition. The pattern of data suggests that the comparisons are not being made within the same analog representation, rather, that the comparisons are made between two different analog representations.

The intercept values explain some of the inconsistency in the slope results. The lcd condition produced a very high intercept value. The production of a high intercept value by the participants appears to unveil the difficulty of the task and how the participants were able to simplify the consistently changing denominators. The high intercept value together with the shallow slope suggests that the participants applied a mathematical strategy to convert the numerical stimuli into a common denominator or numerical format. This conversion explains how two relative frequencies with different denominators could be compared within a single analog representation. Thus, it appears that the lcd condition caused participants to convert the numerical stimuli into a common denominator or perhaps a more common format (i.e. decimals) before the comparison was made within a single analog representation.

In the 7361000 and 1000736 conditions, the intercept value was significantly higher than the same denominator conditions, but significantly less than the lcd condition. It is not clear what the extra intercept time involves in the two-denominator condition, however, it is apparent that it does not involve a mathematical conversion. The application of a mathematical strategy would result in an easier comparison between the two quantities and thus result in a shallow slope for these conditions. Instead, the slope for the 7361000 and 1000736 conditions is greater than 1.5 times steeper than the other conditions.

In Experiment 2, we examined the effect of reaction time when participants were asked to compare a relative frequency and a decimal in a larger/smaller comparison task. The decimal format was examined for two primary reasons. First, the advantage of examining the decimal format is the ability to present equivalent values as either a relative frequency (X in 1000) or as a decimal and have them be visually similar. The use of decimals allowed us to infer whether the decimal format and the relative frequency format have separate or overlapping analog representations. Secondly, decimals and relative frequencies are completely separate numerical formats. Examining the pattern of data produced from comparing two different numerical formats, and thus two analog representations, expanded the research completed in Experiment 1 and thus aided in a more comprehensive explanation and interpretation of all the results. Thus, Experiment 2 expanded, as well as confirmed the data from Experiment 1.

EXPERIMENT 2

Experiment 2 was quite similar to Experiment 1. The participants were asked to compare two quantities in a larger/smaller comparison task. One of the quantities was presented as a decimal and the other as a relative frequency. The participants' response times were recorded.

METHOD

Participants

Participants consisted of 41 college students, 23 females and 18 males. The participants were all over the age of 18 and were recruited from the University of North Carolina at Wilmington. The participants did not consist of individuals who completed

Experiment 1. The participants received course credit for their participation. The participants were unaware of the purpose of the experiment.

Materials

The apparatus used in Experiment 2 was the same as Experiment 1.

Stimuli

In Experiment 2, two numerical stimuli were compared in a larger/smaller comparison task. The comparisons consisted of one quantity being presented as a relative frequency and one quantity as a decimal. Six conditions were examined.

Condition 1: X in 1000 is to a decimal

In this condition, the quantity presented at the top of the screen was a relative frequency displayed as X in 1000. The relative frequencies ranged in value from 250 in 1000 to 750 in 1000. The quantity presented at the bottom of the screen was a decimal. The decimal quantities had the same proportional range as the relative frequencies, from 0.250 to 0.750.

Condition 2: A decimal is to X in 1000

Condition two consisted of the same numerical stimuli being presented as condition one. The only difference was that the decimal was presented at the top of the screen and the relative frequency at the bottom of the screen.

Condition 3: X in 736 is to a decimal

In this condition, the quantity presented at the top of the screen was a relative frequency displayed as X in 736. The relative frequencies ranged in value from 184 in 736 to 552 in 736. The quantity presented at the bottom of the screen was a decimal.

The decimal quantities had the same proportional range as the relative frequencies, from 0.250 to 0.750.

Condition 4: A decimal is to X in 736

Condition four consisted of the same numerical stimuli being presented as condition three. The only difference was that the decimal was presented at the top of the screen and the relative frequency at the bottom of the screen.

Condition 5: X in lowest common denominator is to a decimal

In this condition, the quantity presented at the top of the screen was a relative frequency displayed as X in lowest common denominator. The relative frequencies in this condition were divided through by the numerator and denominator's greatest common factor. The proportional range for the relative frequencies was 0.250 to 0.750. The quantity presented at the bottom of the screen was a decimal. The decimal quantities had the same proportional range as the relative frequencies, from 0.250 to 0.750.

Condition 6: A decimal is to X in lowest common denominator

Condition six consisted of the same numerical stimuli being presented as condition five. The only difference was that the decimal was presented at the top of the screen and the relative frequency at the bottom of the screen.

Procedure

When the participants arrived to the cognition lab, they were greeted and asked to fill out a consent form and a demographic survey. The participants were then brought into a small room that contained a single computer. The participants were then instructed that they would be completing six sets of trials. They were then asked to read through the instructions and advise the experimenter if they had any questions. The following instructions were presented on the computer screen:

In this experiment, you will be presented one relative frequency (for example "1 in 10") and one decimal, representing different quantities. These two quantities will be presented one above the other:

1 in 10

.5

Your job is to judge whether the bottom quantity is larger or smaller than the top quantity. You will respond by pressing one of the two keys on the keyboard. At the beginning of the experiment, the computer will tell you which key to press if the bottom quantity is larger than the top quantity and which key to press if the bottom quantity is smaller than the top quantity. Please respond as quickly and as accurately as you can. Remember, speed is important, but accuracy is essential. Do you have any questions?

Each trial consisted of the stimulus and probe quantities being presented, one above the other, on a single screen. The participants' task was to determine whether the bottom quantity was larger or smaller than the top quantity, and pressed the L button, for a response of larger or S button, for a response of smaller. The screen remained visible until a response was made. There was a blank screen between each trial that lasted 2000ms.

Each condition consisted of 10 practice trials and 100 experimental trials. The order of the six conditions was randomized per participant. The participants' reaction times were recorded in milliseconds. Each participant completed all six conditions during a 90 minute session.

RESULTS

Of the fifty-four participants that completed the experiment, the data of forty-one participants were examined. One participant was removed for not completing all six conditions. Twelve participants were removed from the data set because their error rates in the two easiest conditions, 1000deci and deci1000, were higher than 10% in either condition (Mean Error Rate = 51.9%). The high error rates demonstrated that the participants either did not understand the purpose of the experimental task or simply were not performing the task very accurately and thus were not included in the statistical analysis.

Similar to Experiment 1, the first ten trials of each condition were considered practice trials and they were completed before the experimental trials began, so that the participant could get acquainted with the task. The practice trials were not statistically examined. The initial data gathered consisted of response times for 100 trials, per participant, for each of the 6 conditions: 1000deci (M=1487ms, SD=890ms, ME=2.46%), deci1000 (M=1530ms, SD=832ms, ME=2.37%), 736deci (M=3405ms, SD=2601ms, ME=20.63%), deci736 (M=3296ms, SD=2452ms, ME=19.61%), and lcddeci (M=3425ms, SD=2530ms, ME=13.63%), decilcd (M=3481ms, SD=2588ms, ME=13.46%).

The slowest and quickest response times were removed from the data set by the same criteria used in Experiment 1. The slowest 2% of the response time data were removed per condition (Ratcliff, 1993). The cut-off values consisted of any response times greater than 6378ms for the 1000deci condition, greater than 6467ms for the deci1000 condition, greater than 16168ms for the 736deci condition, greater than 15254ms for the deci736 condition, greater than 16978ms for the lcddeci condition, and greater than 15868ms for the decilcd condition. Like Experiment 1, the faster response times were eliminated until the responses consistently demonstrated a 51% or better accuracy threshold, per condition (Ratcliff, 1993). In the 1000deci and deci1000 conditions, the participants always responded above 51% correct and thus all of the data were included.

Like Experiment 1, the initial distribution of the data was positively skewed. To normalize the distribution of the data, the response time data was again transformed by using the log (base 10) of the response time.

To examine whether the symmetrical log function fit the data better than the Welford function, a linear regression was completed on the data for each of the two functions, per condition. The analysis revealed that the symmetrical log function (log |stimulus-probe|) consistently had a higher r^2 value on the transformed log (base 10) response time data, than the Welford function (log max/(max-min), with each of the six conditions (See Table 2).

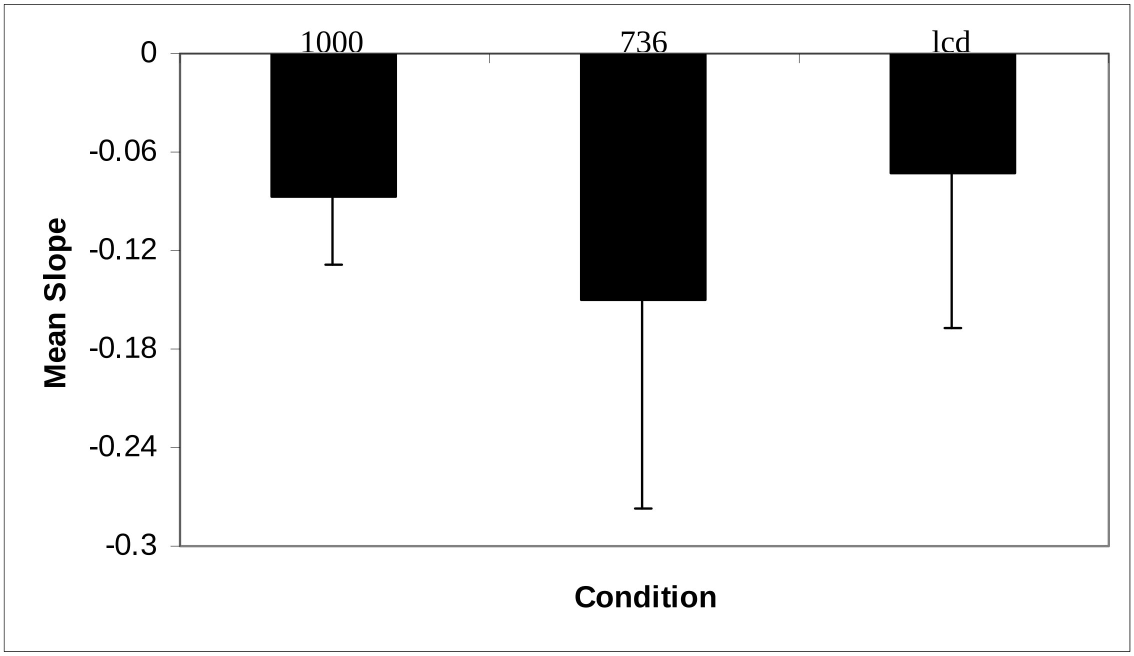
The dependent variables under examination were the participants' slope, intercept, and r^2 values for each of the three conditions To obtain these values, linear regressions were completed on the symmetrical log function for each of the participant's log (base 10) response time data, for each of the three conditions. A series of 2X3 mixed model ANOVAs were completed to examine the dependent variables.

A 2 (deci first/deci second) X 3 (1000/736/lcd) mixed model ANOVA with subject as the random effect and condition and order as the within subject variables uncovered a significant main effect of condition on slope, $F(2,240) = 14.856$, $p<.001$, MSE=.009. Tukey's post-hoc test revealed that the 736 condition (M=-.1493, SD=.1272) had a significantly higher slope than both the 1000 condition (M=-.0872, SD=.0407) and the lcd condition (M=-.0724, SD=.0954) (Figure 4). There was no significant difference

Table 2. The r^2 values for the symmetrical log function compared to the Welford function for each condition.

	1000deci	deci1000	736deci	deci736	lcddeci	decilcd
log|s-p|	.041	.039	.032	.026	.015	.005
Welford	.040	.037	.028	.024	.014	.004

Figure 4. This graph demonstrates the mean slope values for each condition, for Experiment 2.



between the 1000 and lcd conditions. The analysis also revealed that the decimal order was not significant and there was not a significant interaction between condition and order. In addition, a one-sample t-test ($t(163)=-13.4877$, $p<.01$) demonstrated that the two most shallow slope conditions (1000, lcd) were significantly different from zero.

A 2 (deci first/deci second) X3 (1000/736/lcd) mixed model ANOVA with subject as the random effect and condition and order as the within subject variables revealed that there was a significant main effect of condition on intercept, $F(2,240) = 66.563$, $p<.001$, $MSE=.030$. Tukey's post-hoc test demonstrated that all three conditions were significantly different from each other. The 1000 condition ($M=3.0489$, $SD=.1347$) had the smallest intercept, then the 736 condition ($M=3.2689$, $SD=.1866$), and the lcd condition ($M=3.3531$, $SD=.1908$) produced the largest value (Figure 5). There was no significant effect of order and there was no significant interaction between condition and order.

A 2X3 mixed model ANOVA with subject as the random effect and condition as the within subject variable demonstrated that there was a significant main effect of condition on r^2, $F(2,240) = 16.271$, $p<.001$, $MSE=.007$. Tukey's post-hoc test revealed that the lcd condition ($M=.0407$, $SD=.0485$) was found to account for significantly less of the variance than the 736 ($M=.1141$, $SD=.1120$) and 1000 conditions ($M=.0884$, $SD=.0723$) (Figure 6). The 736 and 1000 conditions were found to not be significantly different from each other. The analysis also revealed no significant effect of order and no significant interaction between condition and order.

Figure 5. This graph demonstrates the mean intercept values for each condition, for Experiment 2.

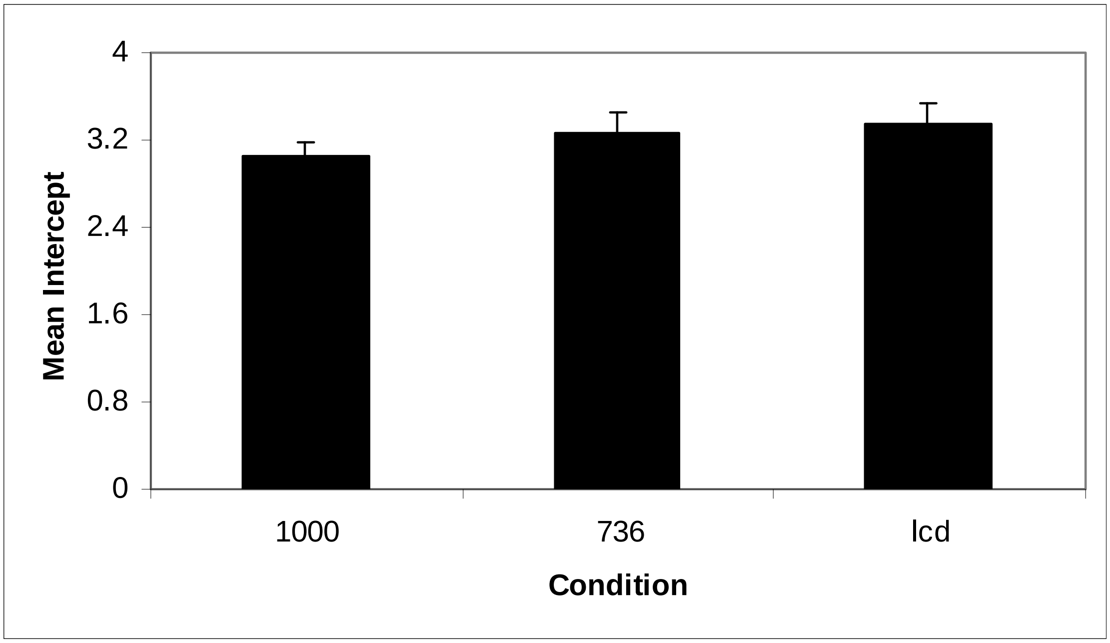
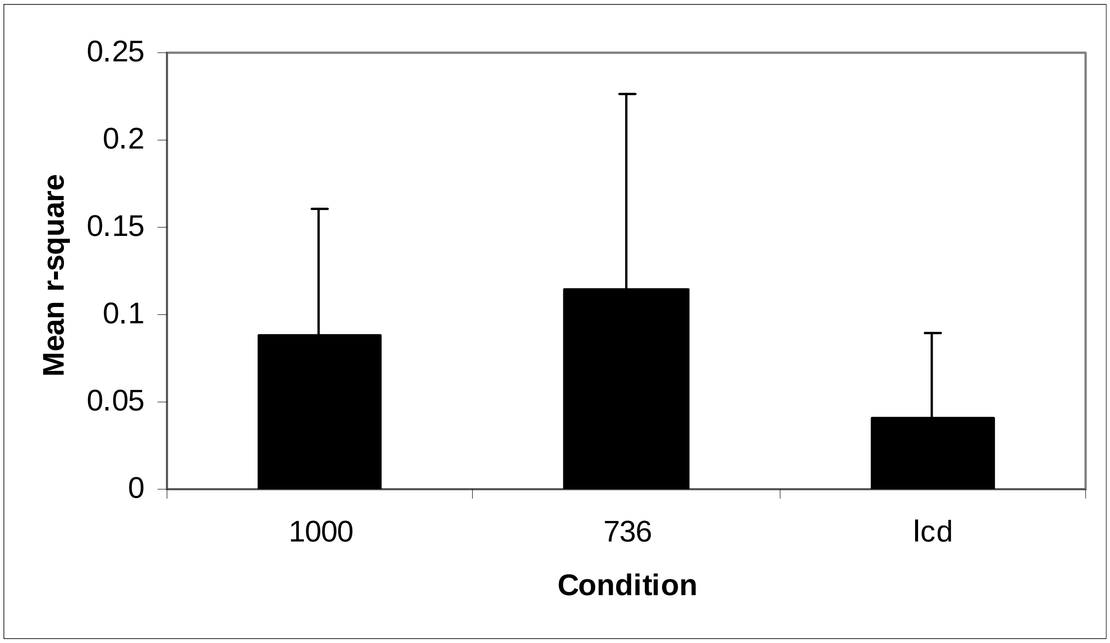


Figure 6. This graph demonstrates the mean r^2 values for each condition, for Experiment 2.



DISCUSSION

Like Experiment 1, the distance effect was found in each of the conditions of Experiment 2 and thus the numerical comparisons were again completed between analog representations. The data demonstrate that the participants did not simply compare the components of the numerical stimuli, but rather accessed the quantity representations of both formats. The results of Experiment 2 were consistent of that of Experiment 1 with the 1000 condition replicating the findings of the 10001000 and 736736 conditions, the 736 condition replicating the findings of the 7361000 and 1000736 conditions, and the lcd condition of Experiment 2 replicating the results of the lcd condition of Experiment 1.

The 1000 condition replicated the findings for the 10001000 condition and 736736 condition in Experiment 1. For the 1000 condition in Experiment 2, the slope of the data was shallow and the intercept value was small. The difference between the two experiments is that Experiment 1 compared a relative frequency to a relative frequency where Experiment 2 compared a relative frequency to a decimal. The parallel results from the two experiments, leads to an interesting conclusion. Theoretically, the data suggest that relative frequencies with 1000 as the denominator share the same analog representation as decimals. In the 10001000 condition of Experiment 1, the data reveal an efficient comparison between the two quantities, within a single analog representation, as is evident by a small slope and intercept value. The pattern of data for the 1000 condition of Experiment 2, is equivalent to that found in the 10001000 condition of Experiment 1. Thus the data suggest that comparing a relative frequency out of 1000 and a decimal occurs within a single analog representation.

The lcd condition from Experiment 2 replicated the findings for Experiment 1, with a high intercept value and a shallow slope. The shallow slope, like that produced in the lcd condition of Experiment 1, suggests that the relative frequency and the decimal are converted into a single analog representation. The high intercept value would suggest, like Experiment 1, that a mathematical strategy is applied to one or both of the numerical stimuli before the comparison is completed. In Experiment 2, the numerical comparison was made between one relative frequency and one decimal. It is likely that the relative frequency was either converted into a relative frequency out of 1000 or a decimal before the larger/smaller comparison was made.

The data from the 736 condition in Experiment 2 revealed a high slope and relatively high intercept value, replicating the findings of Experiment 1. The high slope demonstrates that the comparison between the two quantities became increasingly difficult as the values became closer in value. Again, the relatively high intercept value suggests that to compare a relative frequency out of 736 and a decimal that some sort of preparatory act needs to take place. However, similar to the 7361000 and 1000736 conditions from Experiment 1, the large slope value suggests that no mathematical strategy was used in this condition. Like Experiment 1, the data suggests that numerical comparisons between a decimal and a relative frequency out of 736 are not being processed within a single analog representation. Rather, the combined results in this condition suggest that the comparisons are between two separate analog representations and these comparisons are laborious.

GENERAL DISCUSSION

The results are consistent in both experiments. The conditions that consisted of comparing two relative frequencies with the same denominator produced quick reaction times, with small intercept and slope values. The numerical comparisons in these conditions were processed quickly. The combination of these results suggests that relative frequencies with the same denominator are processed within the same analog representation. The conditions in which the denominator consistently changed caused the participants to perform some sort of preparatory act before the comparison was completed. It's likely that the participants used a mathematical rule to convert one or both of the numerical stimuli so that both quantities had the same denominator. The task is made easier by converting the stimuli because then the stimuli can be compared within a single analog representation. Lastly, the conditions in which only two denominators were compared produced significantly higher response times than the other three conditions. The pattern of data from these conditions suggests that the numerical comparisons are not occurring within a single analog representation, rather via multiple representations.

Analog Representation

The cognitive literature has consistently demonstrated that the ability to discriminate between quantity representations is present in animals, pre-verbal infants, and adults. A considerable amount of the research has focused on integers and the transformation between numerical symbol into an analog representation. The current research is the first to analyze whether relative frequencies, a numerical format that represents quantities less than one, are also translated into an analog representation. Our findings suggest that the adult human brain is not simply structured to process whole numbers, but also proportions.

The same denominator conditions from Experiment 1 both produced the distance effect and thus demonstrated that an analog representation was accessed. In both of these conditions, the participants could have simply compared the numerators of the two quantities being compared and they would have responded correctly. Analyzing the conditions individually does not answer the question of whether the analog representation is simply of the numerator of each relative frequency being compared or if it is the analog representation of the proportion of each of the relative frequencies being compared. However, comparing the pattern of response times for both conditions based on the values of the numerators in the relative frequencies versus the proportion values of the relative frequencies can answer the question. For example, the average response times of comparing the relative frequency of 300 in 1000 to 400 in 1000 could be compared to the average response times of comparing 300 in 736 to 400 in 736. If the participants were solely attending to the numerators, then the response times for both conditions would be equivalent because the comparison is the same (300 to 400). If the participants were comparing the proportional value of each of the relative frequencies than the response time for the 736736 condition would be faster than the 10001000 condition. The faster response time would be due to the larger mathematical difference between the proportion 300 in 736 and 400 in 736 (.136) than 300 in 1000 and 400 in 1000 (.100). Examining the pattern of data for the 10001000 and 736736 conditions concurrently suggests that the analog representation being accessed is that of the proportional value of the relative frequencies. These results suggests that even when the denominator of the relative frequency did not play a role in determining the correct response, the proportion that the relative frequency represented was being translated into an analog representation.

To analyze the presence of the distance effect, in both experiments, a series of regressions were completed on the individual conditions. The results demonstrated the presence of the distance effect in both conditions comparing relative frequencies with the same denominator, as conditions comparing relative frequencies with different denominators. As mentioned earlier, the quick and efficient comparisons of relative frequencies with the same denominator suggest that the relative frequencies are being compared within the same analog representation. In contrast, the extended response times of comparing relative frequencies with different denominators, suggest that these comparisons involve multiple analog representations. The combined results suggest that the distance effect can be produced within a single analog representation, as well as when two separate analog representations are being compared. Within a single analog representation, the distance effect is produced because the two quantities fall on the same continuous scale and the closer the quantities are to each other, the more difficult it becomes to discriminate between them. It appears that when two quantities, that have separate analog representations, are compared that the two continuous scales that the quantities fall upon are also compared. Analog representations for different numerical formats (ex. Arabic and Roman numerals) have been shown to produce different patterns of data (Gonzalez & Koler, 1982). These patterns of data are based on the numerical biases that exist for each of the formats. A possible cause of numerical biases could be due to how the quantities are aligned on the analog scale for that specific format. Thus, when two quantities that have separate analog representations are compared, it is not solely the comparison of two quantities but rather the totality of comparing two quantities and where the two quantities are represented on two different analog scales. The combination of comparing the quantities and scales of each numerical stimuli would explain the difficulty in comparing relative frequencies that do not have the same denominator and thus do not share the same analog representation.

Numerical Models and Relative Frequencies

The results of both experiments suggest that relative frequencies with different denominators have separate analog representations. The 10001000, 736736, and lcd conditions from Experiment 1 and the 1000deci, deci1000, lcddeci, and decilcd conditions from Experiment 2 provided evidence that relative frequencies with the same denominator have the same analog representation. The numerical comparisons in these conditions were quick and produced shallow slopes. However, the 7361000 and 1000736 conditions from Experiment 1 and the 736deci and deci736 conditions from Experiment 2 produced longer response times with steeper slopes which suggests the involvement of multiple analog representations. Comparing our results to the current numerical models, our results counter McClosky's (1986) single analog representation model and Dehaene's (1992) triple code model, and provide support for Gonzalez and Koler's multiple representations model. McCloskey's (1986) model would predict that relative frequencies would be translated into the same analog representation as every other numerical format. If this were the case, comparisons between two relative frequencies or a relative frequency and a decimal would result in a consistent shallow slope. The slope value would be consistent and small for every condition because any comparison between two numbers would be taking place within a single analog representation and comparisons within a single analog representation are efficient. McClosky's model would have been supported if the same pattern of data that was produced in the 10001000, 736736, 1000deci, and deci1000 condition was produced by each of the other conditions. This was not the case. Instead, the 7361000, 1000736, 736deci, and deci736 conditions produced much steeper slopes than the other conditions.

Dehaene's (1992) triple code model assumes that different numerical presentations are processed in different modules of his model. The model consists of an analog, visual, and verbal code. The analog code is a quantity representation and thus this is the code that relative frequencies would be translated into. The visual and verbal codes involve memorized mathematical facts (ex. multiplication tables) and thus do not apply to this research. Our presentation of numerical stimuli were consistently presented as Arabic numerals and thus Dehane's model would predict that the stimuli would all be converted into a single analog representation and thus have the same predictions as McCloskey's (1986) model. However, again, varying the denominator of the relative frequencies was found to effect the slope values for the different conditions and thus comparing relative frequencies with varying denominators is not equally efficient. Thus, the triple code model does not adequately explain the results of our experiments.

Our data associate well with Gonzalez and Koler's (1982) multiple analog representation model. The multiple analog model proposes that there is an analog representation per numerical format and it is unknown whether these individual analog representations can interact without the use of a mathematical strategy. Our data expands on this model by suggesting that there are multiple analog representations within a single numerical format, relative frequencies. The data also appear to suggest that each analog representation is based on the relative frequency's denominator. In both Experiments 1 and 2 we found that when a relative frequency out of 736 was compared to a relative frequency out of 1000 or compared to a decimal that these comparisons had a significantly different slope value than when the denominators were the same or in the lcd condition.

The extended response times for the 736 conditions suggest that comparing relative frequencies with different denominators is quite difficult. Comparing two separate analog representations appears to be like comparing two quantities that are measured with separate scales. For example, it would be like comparing a measurement on a meter stick versus a yard stick. It is not solely the comparison of two values, rather a comparison of where the values fall on two scales. Without using a mathematical conversion, this comparison is arduous and thus comparing a measurement on one stick versus the other takes time and becomes significantly more difficult as the comparisons become closer in value. The 736 conditions, with the resulting high slope value and extended response times, suggest that comparisons between relative frequencies with different denominators cause separate analog representation to interact.

Relative Frequencies Are Not Natural

Research in numerical cognition consistently provided evidence for the theory that humans innately understand integers. The research has not adequately demonstrated whether or not quantities less than one, like relative frequencies, have an innate cognitive representation. Gigerenzer and Hoffrage (1995) argue that relative frequencies are naturally understood. The authors have established that the relative frequency format increases individuals' accuracy on Bayesian Reasoning problems. Gigerenzer and

Hoffrage (1995) make no conclusions about the internal representation of relative frequencies.

Our response time data provides insight into whether relative frequencies are truly naturally understood. If relative frequencies are innately understood, individuals should be able to compare two quantities accurately and relatively quickly. The 10001000 and 736736 conditions from Experiment 1 provide such evidence. The shallow slope in both conditions demonstrates that the task was completed with ease and the values of the numerical stimuli had minimal effect on the participants' response times. In addition, participants consistently completed the conditions in less than two seconds suggesting that the cognitive processing of the quantities is efficient. Overall, relative frequencies that have the same denominator provide evidence of the naturalness of relative frequencies.

The 7361000, 1000736, and the lcd conditions from Experiment 1 suggest that relative frequencies are not innately understood or if they are understood, relative frequencies with different denominators are difficult to compare. The high slope values from the 7361000 and 1000736 conditions demonstrate that comparing quantities with different denominators dramatically affects the participants' reaction time. This data was found to be significantly different from the 10001000 and 736736 conditions from Experiment 1 and thus suggest that all relative frequencies are not processed the same. In the lcd condition, in order to be able to compare two relative frequencies, participants had to first convert one of the quantities. If relative frequencies are naturally processed, a mathematical strategy should not be necessary to complete a comparison. It appears that when comparisons are completed with different denominators, and thus different analog representations, relative frequencies do not produce a pattern of data that suggests they are naturally understood.

It is reasonable to suggest that relative frequencies with the same denominator share an analog representation and that relative frequencies with different denominators do not share an analog representation. Relative frequencies with the same denominator are being compared on the same scale. The scale is determined by the value of the denominator. The denominator is critical to the analog scale because it determines how many parts per whole. Comparing two relative frequencies, with different denominators, involves comparing two scales that are broken down into a different number of parts. Thus it is understandable that each scale or each relative frequency with a different denominator value has an individual analog representation.

Future Research

There is an anomaly in our data that needs further exploration. In the 7361000 and 1000736 conditions, the participants were consistently presented with two relative frequencies, one as X in 736 and one as Y in 1000. In this condition, it appears that a direct comparison was completed, with no mathematical conversions. In the lcd condition, the participants were presented with two relative frequencies and the denominators of these relative frequencies consistently varied. In this condition, the participants converted one relative frequency to have the same denominator as the other relative frequency. The question arises, why was a mathematical strategy used in one condition but not the other?

There is limited research with quantities less than one. Further research is needed with different numerical representations of proportions and how their internal representations interact. For example, does the presentation of quantities as fractions (3/4 and 4/5) produce the same pattern of response time data as the same quantities in relative frequency format (3 in 4 and 4 in 5). The study of internal representations and quantities less than one is a novel area of numerical cognition and thus has extensive avenues to explore.

Conclusion

Our research has demonstrated that relative frequencies are understood as the proportions they represent and are transformed into analog representations. The data suggest that there are multiple representations and these representations are based on the value of the relative frequency's denominator. Quantities that are represented with the same denominator provide evidence that relative frequencies are naturally understood. However, the internal representations of relative frequencies with different denominators do not appear to interact efficiently. This suggests that relative frequencies are not innately understood.

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