# STRONG $\alpha$-FAVORABILITY OF THE (GENERALIZED) COMPACT-OPEN TOPOLOGY

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Abstract. Strong $\alpha$-favorability of the compact-open topology on the space of continuous functions, as well as of the generalized compact-open topology on continuous partial functions with closed domains is studied.

## 1. Introduction

Spaces of partial maps have been studied for various applications throughout the century ([Ku1-2], [AB], [BB], [Ba], [DN1-2], [Fi], [KS], [La], [Se], [St], [Wh], [Za]). In particular, the so-called *generalized compact-open topology* on the space of continuous partial functions with closed domains proved to be a useful tool in mathematical economics ([Ba]), in convergence of dynamic programming models ([La], [Wh]) or more recently in the theory of differential equations ([BC]). This topology was also scrutinized from purely topological point of view e.g. in [BCH], [Ho], [HZ1-2], where among others, separation axioms and some completeness properties (such as Baireness, weak $\alpha$-favorability, Čech-completeness, complete metrizability) of the generalized compact-open topology have been investigated.

Our paper continues in this research by looking at strong $\alpha$-favorability in this setting. Section 3 contains our results on strong $\alpha$-favorability of $\tau_C$ as well as a short proof of a recent theorem of *Holá* on complete metrizability of $\tau_C$.

We will rely on the close connection that exists between the generalized compact-open topology, the ordinary compact-open topology $\tau_{CO}$ ([MN1]) and the Fell topology $\tau_F$ on the hyperspace of nonempty closed subsets of a topological space ([Be], [KT]). This connection and some other auxiliary material is described at the end of Section 1, while in Section 2 we list results about strong $\alpha$-favorability of $\tau_{CO}$ and $\tau_F$, respectively, needed for proving our main results; a generalization of a theorem of *Ma* on weak $\alpha$-favorability of the compact-open topology is also given.

Let $X$ and $Y$ be Hausdorff spaces. Denote by $CL(X)$ the family of nonempty closed subsets of $X$ and by $K(X)$ the nonempty compact subsets of $X$. For any $B \in CL(X)$ and a topological space $Y$, $C(B,Y)$ will stand for the space of all continuous functions from $B$ to $Y$. A partial map is a pair $(B,f)$ such that $B \in CL(X)$ and $f \in C(B,Y)$. Denote by $\mathcal{P} = \mathcal{P}(X,Y)$ the family of all partial maps.

1991 *Mathematics Subject Classification.* 54C35; 54B20, 90D44.

*Key words and phrases.* Generalized compact-open topology, Fell topology, compact-open topology, strong $\alpha$-favorability, weak $\alpha$-favorability.

Typeset by $\mathcal{A}_\mathcal{M}\mathcal{S}$-TEX

Define the so-called *generalized compact-open topology* $\tau_C$ on $\mathcal{P}$ as the topology having subbase elements of the form

$$[U] = \{(B, f) \in \mathcal{P} : B \cap U \neq \emptyset\},$$
$$[K : I] = \{(B, f) \in \mathcal{P} : f(K \cap B) \subset I\},$$

where $U$ is open in $X$, $K \subset X$ is compact and $I$ is an open (possibly empty) subset of $Y$. We can assume that the $I$'s are members of some fixed open base for $Y$.

The compact-open topology $\tau_{CO}$ on $C(X, Y)$ has subbase elements of the form

$$[K, I] = C(X, Y) \cap [K : I] = \{f \in C(X, Y) : f(K) \subset I\},$$

where $K \subset X$ is compact and $I \subset Y$ is open; $C_k(X)$ (see [MN1]) stands for $(C(X, Y), \tau_{CO})$ with $Y = \mathbb{R}$ (the reals). Note, that $C_k(X)$ is a topological group, so a typical basic open neighborhood of $f \in C_k(X)$ is of the form $f + [K, I] = \{f + f' \in C_k(X) : f' \in [K, I]\}$, where $K \in K(X)$ and $I$ is a bounded open neighborhood of zero. We will also use that, if $X$ is a Tychonoff space, then $f + [K, I] \subset f' + [K', I']$ implies $K \supset K'$.

Denote by $\tau_F$ the so-called Fell topology on $CL(X)$ having subbase elements of the form $\{A \in CL(X) : A \cap V \neq \emptyset\}$ with $V$ open in $X$, plus sets of the form $\{A \in CL(X) : A \subset V\}$ with $V$ co-compact in $X$. For notions not defined in the paper see [En].

In the *strong Choquet game* (cf. [Ch] or [Ke]) two players, $\alpha$ and $\beta$, take turns in choosing objects in the topological space $X$ with an open base $\mathcal{B}$: $\beta$ starts by picking $(x_0, V_0)$ from $\mathcal{E}(X) = \mathcal{E}(X, \mathcal{B}) = \{(x, V) \in X \times \mathcal{B} : \ x \in V\}$ and $\alpha$ responds by $U_0 \in \mathcal{B}$ with $x_0 \in U_0 \subset V_0$. The next choice of $\beta$ is some couple $(x_1, V_1) \in \mathcal{E}(X, \mathcal{B})$ with $V_1 \subset U_0$ and again $\alpha$ picks $U_1$ with $x_1 \in U_1 \subset V_1$ etc. Player $\alpha$ wins the run $(x_0, V_0), U_0, \ldots, (x_n, V_n), U_n, \ldots$ provided $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$, otherwise $\beta$ wins. A *winning tactic* for $\alpha$ (cf. [Ch]) is a function $\sigma : \mathcal{E}(X, \mathcal{B}) \to \mathcal{B}$ such that $\alpha$ wins every run of the game compatible with $\sigma$, i.e. such that $U_n = \sigma(x_n, V_n)$ for all $n$. The strong Choquet game is $\alpha$-favorable if $\alpha$ possesses a winning tactic; in this case $X$ is called *strongly $\alpha$-favorable* (or a strong Choquet space - cf. [Ke]). We will need the following facts about the strong Choquet game:

**Proposition 1.1.**

(i) *Let $X$ be metrizable. Then $X$ is completely metrizable if and only if $X$ is strongly $\alpha$-favorable.*
(ii) *If $X$ is locally compact, then $X$ is strongly $\alpha$-favorable.*
(iii) *Let $f : X \to Y$ be continuous, open and onto. If $X$ is strongly $\alpha$-favorable, so is $Y$.*
(iv) *The product of any collection of strongly $\alpha$-favorable spaces is strongly $\alpha$-favorable.*

*Proof.* It is not hard to show that (ii)-(iv) holds (cf. [Ke], Exercise 8.16); as for (i), see [Ch], Theorem 8.7 or [Ke], Theorem 8.17. □

The *Banach-Mazur game* (see [HM] or the Choquet game in [Ke]) is played as the strong Choquet game except that $\beta$'s choice is just a nonempty open set contained in the previous move of $\alpha$. A space $X$ is called *weakly $\alpha$-favorable* if $\alpha$ possesses a *winning strategy* in the Banach-Mazur game (i.e. a function defined on nests of nonempty open sets of odd length picking for $\alpha$ the set that wins the Banach-Mazur game for $\alpha$ no matter what $\beta$ chooses). Note that $\beta$ has no winning strategy in the Banach-Mazur game if and only if $X$ is a Baire space (i.e. countable intersections of dense open sets are dense - cf. [Ke] or [HM]), consequently, weakly $\alpha$-favorable spaces are Baire spaces.

The *restriction mapping*

$$\eta : (CL(X), \tau_F) \times (C(X,Y), \tau_{CO}) \to (\mathcal{P}, \tau_C)$$

is defined as $\eta((B,f)) = (B, f\restriction_B)$.

Clearly, $\eta$ is onto provided continuous partial functions with closed domain are continuously extendable over $X$. We can say more about $\eta$ if we assume that $X, Y$ have *property (P)*, i.e. if $X, Y$ are such that partial continuous functions with closed domains are continuously extendable over $X$ and there exists an open base $\mathcal{V}$ for $Y$ closed under finite intersections such that for each nonempty $K \in \mathcal{K}(X)$ and $V \in \mathcal{V}$, every function $f \in C(K,V)$ is extendable to some $f^* \in C(X,V)$. A fundamental result about $\eta$ is as follows (see [HZ1], Section 3):

**Proposition 1.2.**
(i) *If $X, Y$ have property (P), then $\eta$ is open, continuous and onto.*
(ii) *If $X$ is paracompact and $Y$ is locally convex completely metrizable or if $X$ is $T_4$ and $Y \subset \mathbb{R}$ is an interval, then $X, Y$ have property (P). In particular, $\eta$ is open, continuous and onto in this case.*

## 2. Strong $\alpha$-favorability of $\tau_{CO}$ and $\tau_F$

As for strong $\alpha$-favorability of the Fell topology, we have:

**Theorem 2.1.**
(i) *If $X$ is locally compact, then $(CL(X), \tau_F)$ is locally compact (and hence strongly $\alpha$-favorable).*
(ii) *If $X$ is a strongly $\alpha$-favorable space such that the countable subsets of $X$ are closed, then $(CL(X), \tau_F)$ is strongly $\alpha$-favorable.*

*Proof.* (i) See [Be], Corollary 5.1.4.

(ii) In our case $K(X)$ is a weakly Urysohn family, i.e. if $S \in K(X)$ and $A \subset S^c$, then there exists $T \in K(X)$ with $A \subset T^c \subset S^c$ such that $\overline{E} \subset S^c$ for all countable $E \subset T^c$ (we can choose $T = S$). Consequently, Theorem 5.1 of [Zs] yields the desired result. □

Recall that a Hausdorff space $X$ is *hemicompact* ([En], Excercise 3.4.E) provided in the family of all compact subspaces of $X$ ordered by inclusion there exists a countable cofinal subfamily.

**Theorem 2.2.** *If $X$ is locally compact paracompact and $Y$ is completely metrizable, then $(C(X,Y), \tau_{CO})$ is strongly $\alpha$-favorable.*

*Proof.* The proof of Theorem 5.3.1 in [MN1] can be modified to get the result: write $X = \bigoplus_{t \in T} X_t$, where each $X_t$ is locally compact and hemicompact ([En], Theorem 5.1.27). Then $(C(X_t, Y), \tau_{CO})$ is completely metrizable for all $t \in T$ ([MN1], Exercise 5.8.1(a)). Therefore, $(C(X,Y), \tau_{CO})$ is homeomorphic to the product $\prod_{t\in T}(C(X_t, Y), \tau_{CO})$ ([MN1], Corollary 2.4.7) of completely metrizable spaces, hence, in view of Proposition 1.1 (i) and (iii), $(C(X,Y), \tau_{CO})$ is strongly $\alpha$-favorable. □

A space $X$ is a *q-space* if for each $x \in X$ there is a sequence $\{G_n\}_{n\in\omega}$ of open neighborhoods of $x$ such that whenever $x_n \in G_n$ for all $n$, the set $\{x_n\}_{n\in\omega}$ has a cluster point. Notice that 1st countable or locally compact (even Čech-complete) spaces are q-spaces. The next result generalizes Theorem 1.2 of [Ma] about weak $\alpha$-favorability of the compact-open topology (see also [MN2]):

**Theorem 2.3.** *Let $X$ be a q-space. Then the following are equivalent:*

(i) *$C_k(X)$ is strongly $\alpha$-favorable;*
(ii) *$C_k(X)$ is weakly $\alpha$-favorable;*
(iii) *$X$ is locally compact and paracompact.*

*Proof.* (i)⇒(ii) Clear.

(ii)⇒(iii) $X$ is locally compact by Theorem 4.4 of [MN2], since weakly $\alpha$-favorable spaces are Baire spaces. Paracompactness of $X$ follows from Theorem 1.2 of [Ma].

(iii)⇒(i) See Theorem 2.2 □

**Proposition 2.4.** *Let $X$ be a $T_4$ space with the countable subsets closed and discrete. Then $C_k(X)$ is strongly $\alpha$-favorable.*

*Proof.* Let $(f, U) \in \mathcal{E}(C_k(X))$ with $U = f + [K, I]$ and $\text{diam}(I) < \infty$ (the diameter of $I$). Define $\sigma(f, U) = f + [K, J]$, where $J$ is an open neighbourhood of zero such that $\text{diam}(J) = \frac{1}{2}\text{diam}(I)$. Then $\sigma$ is a winning strategy for $\alpha$: let $(f_0, U_0), V_0, \ldots, (f_n, U_n), V_n, \ldots$ be a run of the strong Choquet game in $C_k(X)$, where

$$U_n = f_n + [K_n, I_n],\ V_n = \sigma(f_n, U_n),$$

$K_n \in K(X)$ and $I_n$ is an open neighborhood of zero $(n \in \omega)$. Then $U_{n+1} \subset V_n \subset U_n$ for each $n \in \omega$, so $K_{n+1} \supset K_n$ and $\text{diam}(I_{n+1}) \leq \frac{1}{2}\text{diam}(I_n)$; consequently, for each $x \in K = \bigcup_{n\in\omega} K_n$, the sequence $\{f_n(x)\}_{n\in\omega}$ converges to some $f(x) \in \mathbb{R}$. Observe that in our case the $K_n$'s are finite and hence $K$ is closed and discrete, so the function $f : K \to \mathbb{R}$ defined above is continuous. If we extend $f$ to some $f^* \in C(X, \mathbb{R})$, we have $f^* \in \bigcap_{n\in\omega} U_n$ and $\alpha$ wins the run. □

## 3. Strong $\alpha$-favorability of $\tau_C$

**Theorem 3.1.** *Assume that $X, Y$ have property (P). If both $(C(X,Y), \tau_{CO})$ and $(CL(X), \tau_F)$ are strongly $\alpha$-favorable, so is $(\mathcal{P}, \tau_C)$.*

*Proof.* The restriction mapping is continuous, open and onto by Proposition 1.2(i), so Proposition 1.1(iv) and (iii) applies. □

The next theorem generalizes Corollary 4.4(i) of [HZ1]:

**Theorem 3.2.** *Let $X$ be a locally compact, paracompact space and $Y$ a locally convex completely metrizable space. Then $(\mathcal{P}, \tau_C)$ is strongly $\alpha$-favorable.*

*Proof.* $(CL(X), \tau_F)$ and $(C(X,Y), \tau_{CO})$ are strongly $\alpha$-favorable by Theorem 2.1(i) and Theorem 2.2, so Proposition 1.2 and Theorem 3.1 yields the desired result. □

As a corollary of Theorem 3.2 we get the following theorem of Holá ([Ho], Theorem 3.3):

**Theorem 3.3.** *Let $X$ be a Tychonoff space and $Y$ a locally convex completely metrizable space. Then the following are equivalent:*

(i) $(\mathcal{P}, \tau_C)$ *is completely metrizable;*
(ii) $X$ *is a locally compact 2nd countable space.*

*Proof.* In view of [Ho] (Theorem 2.4), $(\mathcal{P}, \tau_C)$ is metrizable if and only if $X$ is locally compact and 2nd countable, so the implication (i)$\Rightarrow$(ii) immediately follows. As for (ii)$\Rightarrow$(i), use Theorem 3.2 and Proposition 1.1(i). □

We will now study strong $\alpha$-favorability of $\tau_C$ for $Y = \mathbb{R}$ to show that Theorem 3.2 is not reversible, i.e. that local compactness plus paracompactness is not necessary for strong $\alpha$-favorability of the generalized compact-open topology.

**Theorem 3.4.** *Let $Y = \mathbb{R}$ and $X$ be a $T_4$ strongly $\alpha$-favorable space with the countable subsets closed discrete. Then $(\mathcal{P}, \tau_C)$ is strongly $\alpha$-favorable.*

*Proof.* $(CL(X), \tau_F)$ and $C_k(X)$ are strongly $\alpha$-favorable by Theorem 2.1(ii) and Theorem 2.4, respectively, hence Proposition 1.2(ii) and Theorem 3.1 applies. □

To demonstrate that Theorem 3.2 is not reversible we need (by Theorem 3.4) the following:

**Example 3.5.** *There exists a $T_4$ non-paracompact, strongly $\alpha$-favorable space with the countable subsets closed discrete.*

*Proof.* The space with the required properties is $X = \{x \in \omega_2 : \operatorname{cf} x > \omega\}$, which is a stationary subset of $\omega_2$ and hence $X$ is $T_4$ and, by the Pressing Down Lemma, not paracompact. Further, by the definition of $X$, no countable subset of $X$ clusters, thus, countable subsets of $X$ are closed and discrete. To show that $X$ is strongly $\alpha$-favorable, put $\sigma(x, U) = U$ for every $(x, U) \in \mathcal{E}(X)$ with $U = (a, x] \cap X$.

Then $\sigma$ is a winning tactic for $\alpha$, since if $(x_0, U_0), U_0, \ldots, (x_n, U_n), U_n, \ldots$ is a run of the strong Choquet game compatible with $\sigma$, then there exists some $n_0 \in \omega$ with $x = x_n = x_m$ for all $m, n \geq n_0$ (otherwise $\{x_n\}_n$ would have a subsequence of order type $\omega^*$), whence $x \in \bigcap_{n \in \omega} U_n$. □

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