TURTLE BEATS CARL LEWIS!
(INFINITIES - STUFF THAT MAKES PEOPLE GO NUTS)

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Mathematics is often erroneously referred to as the science of common sense.
Edward Kasner and James R. Newman

Mathematics may be defined as the subject in which we never know what we are talking about nor whether what we are saying is true.
Bertrand Russel

... *this was sometime a paradox, but now time gives it proof.*
William Shakespeare

I hope that the above wise men convinced everybody that the following presentation is going to be about very serious mathematics. What I try to achieve in this paper is to introduce a new world, where nearly everything defies our experience, a world in which amusing and amazing things happen.

First however a word of caution: it took about 2000 years for the most clever to find out the rules of this world, and learn how to properly behave in it. In the process several of them went nuts (literally)!

Aristotle and Zeno

The great Greek philosopher Aristotle set the rules of logic, the way of reasoning, more than 2000 years ago. Soon, however, it turned out that something is wrong with the picture of the world Aristotle created. It became evident after Zeno of Elea had produced his famous *paradoxes*, i.e. results of reasoning (using of logic) which conflict with experience in the real world. It was a pretty big scandal, as Aristotle was a highly honored scholar of ancient Greece.

One of the best known paradoxes of Zeno is about Achilles and the turtle (if you substitute Carl Lewis for Achilles you get the headline from the title):

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Problem 1

Since Achilles was noted for his swiftness and the turtle for its slowness, the turtle is given a head start when they race each other. Zeno argues that Achilles first must reach the point where the turtle was initially. By then, the turtle will have moved beyond that point. Now the situation is the same as it was at the start of the race. The turtle has a head start on Achilles. Achilles must again reach the point where the turtle was (when Achilles reached the point where the turtle got his first head start). But by the time he arrived, the turtle has moved on. Now the situation is the same as it was at the start of the race. The turtle has a head start on Achilles. And so on. Achilles can never catch, let alone pass, the turtle, so the **turtle wins the race** !!!!!

Incredible! But it seems to be O.K. as far as the reasoning is concerned. The Greeks went nuts (case # 1).

The thing to blame for this tragedy is called *infinity.* Indeed, when we say "and so on", we mean that one can do the same thing over and over again infinitely many times! But then (it makes sense, or what!) it will never end!

This is not true.

We can explain this using a notion from Calculus, so-called *geometric series:*

Let q be a positive number less than 1. Then the infinite sum

$$1 + q + q^2 + q^3 + q^4 + \cdots + q^{n-1} + \ldots$$

is meaningful, and we can calculate its value using the formula

$$1 + q + q^2 + q^3 + \cdots + q^{n-1} + \cdots = \frac{1}{1-q}.$$

The solution of the above paradox is maybe better understandable with specific distances and speeds given. Let's say, that the original head start for the turtle is 100 yards, and that Achilles can run 10 times as fast as the turtle, thus

$$v_{Achilles} = 10 v_{turtle}.$$

In this case, by the time Achilles reaches the turtle's starting point, the turtle has moved $10 = \frac{100}{10} = 100 \times 0.1$ yards from that point. By the time Achilles reaches the 110-yard point of the race, the turtle will be at $111 = 100 + 100 \times 0.1 + 100 \times (0.1)^2$ yards. When Achilles is at 111 yards, the turtle is at $111.1 = 100 + 100 \times 0.1 + 100 \times (0.1)^2 + 100 \times (0.1)^3$ yards, and so on. As the race progresses, the turtle is heading for the point at *exactly*

$$\begin{gathered} 100 + 100 \times 0.1 + 100 \times (0.1)^2 + 100 \times (0.1)^3 + \cdots + 100 \times (0.1)^{n-1} + \cdots = \\ = 100 \times (1 + 0.1 + (0.1)^2 + (0.1)^3 + \cdots + (0.1)^{n-1} + \ldots) = \\ = 100 \frac{1}{1-0.1} = \frac{1000}{9} = 111.111\ldots \text{ yards.} \end{gathered}$$

When they both get to that point, Achilles will catch the turtle at $111.111\ldots$ yards, pass it in the next instant, and go on to win the race. Hence, **there is no paradox.**

Georg Cantor

Let's move further in time, to Germany in the late nineteenth century, where a mathematician, named *Georg Cantor* (1845-1918), decided to go against intuition. He is the founder of modern Set Theory and was fairly successful in dealing with infinities, **however** eventually also went nuts (case # 2) while trying to solve his own paradoxes. Nevertheless, he still managed to discover the following remarkable creature, the *Cantor set.*

Problem 2

Consider the closed interval $C_0 = [0, 1]$. Cut out the middle third of it, the open interval $(\frac{1}{3}, \frac{2}{3})$. Denote by C_1 what is left, i.e. the union of $[0, \frac{1}{3}]$ and $[\frac{2}{3}, 1]$. Proceed in the same way with these remaining intervals, cut out their middle third and look at the remaining portion, the union of intervals $[0, \frac{1}{9}], [\frac{2}{9}, \frac{1}{3}], [\frac{2}{3}, \frac{7}{9}], [\frac{8}{9}, 1]$ and denote it by C_2. Now continue with these 4 remaining intervals taking out their middle third part, you'll get 8 new intervals (2 from each in C_2). **And so on!**

Thus, we can construct a set C_n for every natural number n cutting out the middle thirds of the intervals which constitute the previous set C_{n-1}. The Cantor set C is then composed of those points which lie in every one of the C_n's. Be careful, we are in the world of infinities again!

Question: Have we really created something?

Yes! The cutting points are firm (e.g. $\frac{1}{3}, \frac{1}{9}, \frac{2}{3}, 1$), they will be in every one of the C_n's, and therefore also in C.

Take a different view on C now, measure the overall length of the intervals removed from $[0, 1]$ in the process of construction of C:

- from C_0 we left out $(\frac{1}{3}, \frac{2}{3})$ of length $\frac{1}{3}$
- from C_1 we left out $(\frac{1}{9}, \frac{2}{9}), (\frac{7}{9}, \frac{8}{9})$ of total ength $2 \times \frac{1}{9} = 2(\frac{1}{3})^2$
- from C_2 we left out $(\frac{1}{27}, \frac{2}{27}), (\frac{7}{27}, \frac{8}{27}), (\frac{19}{27}, \frac{20}{27}), (\frac{25}{27}, \frac{26}{27})$ of total length

$$4 \times \frac{1}{27} = 2^2(\frac{1}{3})^3$$

...

- from C_n we left out $(\frac{1}{3^n}, \frac{2}{3^n}), \dots, (\frac{3^n-2}{3^n}, \frac{3^n-1}{3^n})$ of total length $2^{n-1}(\frac{1}{3})^n$.

...

Consequently, we removed non-overlapping intervals of total length

$$\frac{1}{3} + 2(\frac{1}{3})^2 + 2^2(\frac{1}{3})^3 + \cdots + 2^{n-1}(\frac{1}{3})^n + \cdots =$$
$$= \frac{1}{3}(1 + \frac{2}{3} + (\frac{2}{3})^2 + \cdots + (\frac{2}{3})^{n-1} + \dots) =$$
$$\text{(geometric series with } q = 2/3)$$
$$= \frac{1}{3}(\frac{1}{1-2/3}) = \frac{1}{3} \times 3 = 1.$$

All in all, we deleted intervals of length equal the length of the initial interval $[0, 1]$.

We have left out everything!(?)

Incredible for our senses, normal in the world of infinities.

This is however not the end of the story. Cantor was truely an exceptional mathematician, he was curious about unusual things. Even for the great mathematicians of the nineteenth century his work was not really understandable, his questions seemed to be rather meaningless. Here is a sample:

Problem 3

There is no doubt, there are infinite objects. For example, if we wanted to list all the natural numbers (thinking there is only finite many of them) we would necessarily fail, since no matter how many of them is already listed, there is at least one more, namely the next natural number (we can always go on = "and so on").

Another, perhaps more visual, example is the collection of all the points on the interval $[0, 1]$. Again, an infinite object, since between any two points there is a distinct third, namely the middle point (we can always go on = "and so on").

Question: Is the "number" of natural numbers the same as the "number" of points on the interval $[0, 1]$?

What a nonsense to ask something like this; both are infinite and done! Cantor pondered a bit longer about the meaningfulness of this question, and found the answer using a nice transparent argument (what we now call the *Cantor diagonalization*).

And what an answer!

There are **MORE** points on the interval $[0, 1]$, than there are natural numbers.

I could barely breath, when I first learnt about it.

The first thing Cantor had to resolve was the problem of *comparing* two infinite objects. In order to better understand what he did, think for a moment about the following:

Problem 4

Suppose we have a bunch of horses and cowboys. What is the fastest way of determining, if there is the same number of horses as cowboys?

It is surely not by counting (we are not interested in the exact amounts). Simply command: mount the horse?

If every cowboy is on a horse and on every horse there is a cowboy, their number is the same. This is an **assignment**: to every cowboy a horse and conversely, to every horse a cowboy is assigned. There is a **one-to-one correspondence** between the cowboys and horses.

This is the idea which helps to compare the "number" of elements of infinite objects (or their "size"):

If we can assign to every natural number a distinct point on the interval $[0, 1]$ **AND** *to every point on* $[0, 1]$ *a distinct natural number, then these two infinite objects have the same "number" of elements.*

Cantor's result shows that there is **no way** of setting up a one-to-one correspondence between the natural numbers and the points on $[0, 1]$, there's going to be lot of points on $[0, 1]$ (horses) which will not correspond to any natural number (cowboys).

It turned out that Cantor's question was far from childish, he actually proved that there are **infinitely many distinct infinities.**

Welcome to Infiniteland

After Cantor's discoveries, mathematicians started to compare and classify infinities around us. One of my favorites is that there is *exactly the same number of points in the Cantor set, than in the initial interval* $C_0 = [0, 1]$. If we compare this with the fact that after measuring the Cantor set, we found it to be of measure *zero* (we left out from $[0, 1]$ intervals of total length 1), then we are entitled to call C *the big nothing.*

More examples:

Example 1.
Any two closed intervals (segments) contain the **same** *number of points.*

Example 2. *There is* **exactly** *as many even numbers as natural numbers!*

Example 3. *There are* **exactly** *as many rational numbers (the fractions) as natural numbers, consequently there are* **MORE** *points on* $[0, 1]$ *even than the rationals.*

(It is clearly not possible to list the rationals as the naturals, since there is no such thing as the "next rational number", but it is possible to list them in some order as a sequence)

Example 4. *(pretty tough) There are* **exactly** *as many points on the closed interval* $[0, 1]$ *as on the half-open interval* $(0, 1]$.

(Select the numbers $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots, \frac{1}{n}, \ldots$ from $[0, 1]$. A 1-1 correspondence between $[0, 1]$ and $(0, 1]$ can be defined as follows:

$$0 \to 1, 1 \to \frac{1}{2}, \frac{1}{2} \to \frac{1}{3}, \ldots, \frac{1}{n} \to \frac{1}{n+1}, \ldots,$$

and $x \to x$ for all the remaining x's from [0,1])

Example 5. *The real number line has* **exactly** *as many points as the open interval* $(0, 1)$

(Think "inverse tangent")

Example 6. *The plane has* **exactly** *as many points as the real line.*