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ON COMPLETE METRIZABILITY OF THE HAUSDORFF METRIC TOPOLOGY

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ABSTRACT. There exists a completely metrizable bounded metrizable space X with compatible metrics d, d' so that the hyperspace $CL(X)$ of nonempty closed subsets of X endowed with the Hausdorff metric H_d, $H_{d'}$, resp. is α-favorable, β-favorable, resp. in the strong Choquet game. In particular, there exists a completely metrizable bounded metric space (X, d) such that $(CL(X), H_d)$ is not completely metrizable.

1. INTRODUCTION

The *Hausdorff metric topology* τ_{H_d} on the hyperspace $CL(X)$ of nonempty closed subsets of a given metric space (X, d) is one of the oldest and best-studied hypertopologies due to its applicability to various areas of mathematics [1, 2, 4, 20]. The main reason for this interest is the following well known fact [4, §3.2.]: if (X, d) is a bounded complete metric space, then $(CL(X), H_d)$ is a complete metric space, where H_d is the *Hausdorff metric* on $CL(X)$ defined as

$$H_d(A_0, A_1) = \sup\{|d(x, A_0) - d(x, A_1)| : x \in X\}, \text{ for } A_0, A_1 \in CL(X), \tag{1}$$

and $d(x, A) = \inf\{d(x, a) : a \in A\}$ is the distance from $x \in X$ to $A \in CL(X)$. If d is not bounded, H_d is only an infinite-valued distance, which generates the topology τ_{H_d} on $CL(X)$; moreover, since $d' = \min\{1, d\}$ is an equivalent to d bounded metric on X and $\tau_{H_{d'}} = \tau_{H_d}$, we get

Theorem 1.1.
If (X, d) is complete, then $(CL(X), \tau_{H_d})$ is completely metrizable.

Various completeness-type properties of the Hausdorff metric topology are stock theorems in topology, e.g. $(CL(X), \tau_{H_d})$ is compact (resp. totally bounded) iff X is [4, 17]; more recently, local compactness [12], and cofinal completeness [6] have been characterized for $(CL(X), \tau_{H_d})$; however, despite the above considerations and other partial results (see below), a characterization of complete metrizability of $(CL(X), \tau_{H_d})$ is unknown. Observe that the Hausdorff distance is sensitive to its generating metric, more precisely, $\tau_{H_d} = \tau_{H_{d'}}$ iff d, d' are uniformly equivalent metrics on X [4, Theorem 3.3.2.],

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thus, it is not automatic to argue that *complete metrizability* of (X,d) is sufficient for complete metrizability of $(CL(X),\tau_{H_d})$ even though it is clearly necessary, since (X,d) embeds as a closed subspace of $(CL(X),\tau_{H_d})$. It is the purpose of this note to demonstrate that complete metrizability of (X,d), in fact, is not sufficient for complete metrizability of $(CL(X),\tau_{H_d})$, contrary to some claims in the literature [3].

To put this question in perspective, briefly review the known results related to complete metrizability of the Hausdorff metric topology: it was *Effros* [16, Lemma] who showed that for $(CL(X),\tau_{H_d})$ to be Polish (i.e. completely metrizable and separable), it is sufficient that (X,d) is completely metrizable and totally bounded, which is in turn also necessary, since separability of $(CL(X),\tau_{H_d})$ is equivalent to total boundedness of X [4, Theorem 3.2.3.], and X sits in $(CL(X),\tau_{H_d})$ as a closed subspace. It is possible to improve on this results using the work of *Costantini* [10] about another related hyperspace topology, the so-called *Wijsman topology* τ_{W_d} [4]: to explain this, it is useful to view $CL(X)$ as sitting in the space $C(X)$ of real-valued continuos functions defined on X via the identification $A \leftrightarrow d(\cdot,A)$, since, by (1), $(CL(X),\tau_{H_d})$ is then a subspace of $C(X)$ with the uniform topology, while $(CL(X),\tau_{W_d})$ is a subspace of $C(X)$ with the topology of pointwise convergence. This immediately implies that $\tau_{W_d} \subseteq \tau_{H_d}$, in particular, G_δ-subsets of $(CL(X),\tau_{W_d})$ are G_δ-subsets of $(CL(X),\tau_{H_d})$ as well, which helps us to prove

Theorem 1.2.
If (X,d) is Polish, then $(CL(X),\tau_{H_d})$ is completely metrizable.

Proof. It follows from [10] that $CL(X)$ is a G_δ-set of $(CL(\widetilde{X}),\tau_{W_{\widetilde{d}}})$, where $(\widetilde{X},\widetilde{d})$ is the completion of (X,d). Thus, $CL(X)$ is also G_δ in $(CL(\widetilde{X}),\tau_{H_{\widetilde{d}}})$, therefore, by Theorem 1.1, $(CL(X),\tau_{H_d})$ is completely metrizable, since mapping $A \in CL(X)$ onto the $\widetilde{X}$-closure of A is an isometric embedding of $(CL(X),H_d)$ into $(CL(\widetilde{X}),H_{\widetilde{d}})$ [16]. □

Knowing that $\tau_{W_d} = \tau_{H_d}$ on $CL(X)$ iff (X,d) is totally bounded [4, Theorem 3.2.3.], it is not surprising that in the above results of Effros and Costantini the Hausdorff metric and Wijsman topologies interact in studying complete metrizability of the hyperspaces, however, when a totally bounded metric is not available on X, i.e. when X is a non-separable metric space, the two topologies have no effect on each other. Therefore the wealth of completeness results on the Wijsman topology [5, 11, 27, 8, 13] is not applicable in our case, which demonstrates a fundamental difference between these topologies.

Since complete metrizability of a metrizable space is equivalent to its Čech-completeness (i.e. being G_δ in a compactification [17]), the recent characterization of local compactness of $(CL(X),\tau_{H_d})$ by *Costantini, Levi, Pelant* in [12, Corollary 15], as well as of the intermediary property of cofinal completeness of $(CL(X), \tau_{H_d})$ by *Beer, Di Maio* in [6, Theorem 3.9.] must be mentioned here, as they both imply complete metrizability of $(CL(X), \tau_{H_d})$.

The main results of this paper, proved in Section 3, use topological games, namely the so-called strong Choquet game and the Banach-Mazur game, which are reviewed in Section 2, along with some relevant results about them. As mentioned in the abstract and introduction, our results will demonstrate that complete metrizability of (X, d) does not guarantee the same for the Hausdorff metric topology, more specifically, $(CL(X), \tau_{H_d})$ may not have any closed-hereditary completeness property, since it contains a closed copy of the rationals; however, we will show this hyperspace still contains a dense completely metrizable subspace, and thus, is a Baire space.

2. Preliminaries

Given a metric space (X, d), $A \in C(X)$ and $\varepsilon > 0$, denote by

$$B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$$

the open ε-hull of A, and use $B_d(x, \varepsilon)$ instead of $B_d(\{x\}, \varepsilon)$ for the open ε-ball about x. In addition to (1), there is an equivalent definition for the Hausdorff distance H_d:

$$H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\},$$

whenever $A_0, A_1 \in CL(X)$ [4, 17].

In the *strong Choquet game* $Ch(Z)$ (cf. [9, 19]) players α and β take turns in choosing objects in the topological space Z with an open base $\mathscr{B}$: β starts by picking (z_0, V_0) from $\mathscr{E} = \{(z, V) \in Z \times \mathscr{B} : \ z \in V\}$ and α responds by $U_0 \in \mathscr{B}$ with $z_0 \in U_0 \subseteq V_0$. The next choice of β is $(z_1, V_1) \in \mathscr{E}$ with $V_1 \subseteq U_0$ and again α picks U_1 with $z_1 \in U_1 \subseteq V_1$ etc. Player α wins the run $(z_0, V_0), U_0, \ldots, (z_n, V_n), U_n, \ldots$ provided $\bigcap_n U_n = \bigcap_n V_n \neq \emptyset$; otherwise, β wins. A *strategy* in $Ch(Z)$ for α (resp. β) is a function $\sigma : \mathscr{E}^{<\omega} \to \mathscr{B}$ (resp. $\sigma : \mathscr{B}^{<\omega} \to \mathscr{E}$) such that

$$z_n \in \sigma((z_0, V_0), \ldots, (z_n, V_n)) \subseteq V_n \quad \text{for all } ((z_0, V_0), \ldots, (z_n, V_n)) \in \mathscr{E}^{<\omega}$$
$$(\text{resp. } \sigma(\emptyset) = (z_0, V_0) \text{ and } V_n \subseteq U_{n-1}, \text{where } \sigma(U_0, \ldots, U_{n-1}) = (z_n, V_n)$$
$$\text{for all } (U_0, \ldots, U_{n-1}) \in \mathscr{B}^n, \ n \geq 1).$$

A strategy σ for α (resp. β) is a *winning strategy*, if α (resp. β) wins every run of $Ch(Z)$ compatible with σ, i.e. such that $\sigma(z_0, V_0), \ldots, (z_n, V_n)) = U_n$ for all $n < \omega$ (resp. $\sigma(\emptyset) = (z_0, V_0)$ and $\sigma(U_0, \ldots, U_{n-1}) = (z_n, V_n)$ for all $n \geq 1$). The strong Choquet game $Ch(Z)$ *is α-, β-favorable*, respectively, provided α, resp. β has a winning strategy in $Ch(Z)$. This game has been studied in general topological spaces [22, 7, 15, 14, 28], however, the two fundamental results about it concern metrizable ones:

- **Choquet** [9, 19] A metrizable space X is completely metrizable if and only if $Ch(X)$ is α-favorable.

- **Debs-Porada-Telgársky** [14, 24, 25] A metrizable space X contains a closed copy of the rationals if and only if $Ch(X)$ is β-favorable.

The *Banach-Mazur game* $BM(Z)$ (see [18], also referred to as the Choquet game [19]) is played as the strong Choquet game, except β's choice is only a nonempty open set contained in the previous choice of α. The notions of α-, β-favorability of $BM(Z)$ are defined analogously to those of $Ch(Z)$. Two key results about the Banach-Mazur game are as follows:

- **Oxtoby** [23, 26] A metrizable space X contains a dense completely metrizable subspace if and only if $BM(X)$ is α-favorable.
- **Oxtoby-Krom** [23, 18, 19] A topological space X is a Baire space (i.e. countable intersctions of dense open subsets of X are dense) if and only if $BM(X)$ is not β-favorable.

3. Main results

Our main result is as follows:

Theorem 3.1. *There exists a bounded metric space (X, d) such that*

(1) *X is completely metrizable,*
(2) *$(CL(X), H_d)$ contains a closed copy of the rationals; in particular, $(CL(X), H_d)$ is not completely metrizable,*
(3) *$(CL(X), H_d)$ is α-favorable in the Banach-Mazur game; in particular, $(CL(X), H_d)$ is a Baire space.*

Proof. (1) Consider the product space $\mathbb{R}^\omega$, where $\mathbb{R}$ has the discrete topology. This topology is metrizable by the Baire metric

$$d(f, g) = \frac{1}{\min\{n+1 : \ f(n) \neq g(n)\}}$$

for $f, g \in \mathbb{R}^\omega$. Denote $F = \{x \in \mathbb{R}^\omega : x(0) \neq 0 \text{ and } x(k) = 0 \text{ for all } k > 0\}$, and put $X = \mathbb{R}^\omega \setminus F$. It is clear that F is closed in $\mathbb{R}^\omega$, so X is an open subspace of the complete space $(\mathbb{R}^\omega, d)$, and hence, (X, d) is completely metrizable.

(2) By the Debs-Porada-Telgársky Theorem, we need to show that $(CL(X), H_d)$ is β-favorable in the strong Choquet game: let $\{I_n^0 \subset \mathbb{R} \setminus \{0\} : n < \omega\}$ be a sequence of pairwise disjoint closed bounded intervals, and denote by I_0 their union. For each $t \in I_0$ define $x_t^0 \in X$ via

$$x_t^0(k) = \begin{cases} t, & \text{if } t \in I_n^0, \ k = 0 \text{ or } k > n+1, \\ 0, & \text{if } t \in I_n^0, \ 1 \leq k \leq n+1. \end{cases}$$

Define $A_0 = \{x_t^0 : t \in I_0\} \in CL(X)$, $\mathbf{V}_0 = B_{H_d}(A_0, 1)$, and let $(A_0, \mathbf{V}_0)$ be β's initial step in $Ch(CL(X), H_d)$. Let $\mathbf{U}_0 = B_{H_d}(A_0, \frac{1}{n_0})$ be α's response, where $1 \leq n_0 < \omega$. Proceeding inductively, assume we have defined a partial run $(A_0, \mathbf{V}_0), \mathbf{U}_0, \ldots, (A_m, \mathbf{V}_m), \mathbf{U}_m$ of the strong Choquet game in $(CL(X), H_d)$, where

$$\mathbf{U}_i = B_{H_d}\left(A_i, \frac{1}{\sum\limits_{j \le i} n_j}\right)$$

for some $1 \le n_i < \omega$ whenever $i \le m$. Moreover, for each $1 \le i \le m$ a sequence $\{I_n^i \subset I_{n_{i-1}+1}^{i-1} : n < \omega\}$ of pairwise disjoint closed bounded intervals with union I_i be chosen, as well as $x_t^i \in X$ for each $t \in I_0$ so that $x_t^i = x_t^{i-1}$ whenever $t \in I_0 \setminus I_i$, and for $t \in I_i$

$$x_t^i(k) = \begin{cases} x_t^{i-1}(k), & \text{if } t \in I_n^i,\ k \le \sum\limits_{j<i} n_j, \\ 0, & \text{if } t \in I_n^i,\ \sum\limits_{j<i} n_j < k \le 1 + n + \sum\limits_{j<i} n_j, \\ t, & \text{if } t \in I_n^i,\ k > 1 + n + \sum\limits_{j<i} n_j. \end{cases}$$

Then let $A_i = \{x_t^i : t \in I_0\}$ and $\mathbf{V}_i = B_{H_d}(A_i, \frac{1}{1+\sum_{j<i} n_j})$. Choose a sequence of pairwise disjoint closed bounded intervals $\{I_n^{m+1} \subset I_{n_m+1}^m : n < \omega\}$ with union I_{m+1}, and define $x_t^{m+1} = x_t^m$ for each $t \in I_0 \setminus I_{m+1}$, and for $t \in I_{m+1}$ put

$$x_t^{m+1}(k) = \begin{cases} x_t^m(k), & \text{if } t \in I_n^{m+1},\ k \le \sum\limits_{i \le m} n_i, \\ 0, & \text{if } t \in I_n^{m+1},\ \sum\limits_{i \le m} n_i < k \le 1 + n + \sum\limits_{i \le m} n_i, \\ t, & \text{if } t \in I_n^{m+1},\ k > 1 + n + \sum\limits_{i \le m} n_i. \end{cases}$$

Define $A_{m+1} = \{x_t^{m+1} : t \in I_0\}$ and $\mathbf{V}_{m+1} = B_{H_d}(A_{m+1}, \frac{1}{1+\sum_{i \le m} n_i})$.

Claim 3.1.1. $\mathbf{V}_{m+1} \subseteq \mathbf{U}_m$.

Indeed, if $A \in \mathbf{V}_{m+1}$, then $A \subseteq B_d(A_{m+1}, \frac{1}{1+\sum_{i \le m} n_i})$, so for all $a \in A$ there is some $x_t^{m+1} \in A_{m+1}$ with $d(a, x_t^{m+1}) < \frac{1}{1+\sum_{i \le m} n_i}$, which implies

$$a(k) = x_t^{m+1}(k) \text{ for all } k \le \sum_{i \le m} n_i. \tag{2}$$

If $t \in I_0 \setminus I_{m+1}$, then

$$d(a, x_t^m) = d(a, x_t^{m+1}) < \frac{1}{1 + \sum\limits_{i \le m} n_i} < \frac{1}{\sum\limits_{i \le m} n_i};$$

if $t \in I_{m+1}$, then $t \in I_n^{m+1}$ for some $n < \omega$. It follows from the definition of x_t^{m+1}, and (2) that

$$d(a, x_t^m) \le \frac{1}{1 + \sum\limits_{i \le m} n_i} < \frac{1}{\sum\limits_{i \le m} n_i},$$

so we have $A \subseteq B_d(A_m, \frac{1}{\sum_{i \le m} n_i})$. A similar argument shows that

$$A_{m+1} \subseteq B_d\left(A, \frac{1}{1 + \sum\limits_{i \le m} n_i}\right) \text{ implies } A_m \subseteq B_d\left(A, \frac{1}{\sum\limits_{i \le m} n_i}\right),$$

thus, $A \in \mathbf{U}_m$. As a consequence of Claim 3.1.1, we have that putting $\sigma_{Ch}(\emptyset) = (A_0, \mathbf{V}_0)$, and $\sigma_{Ch}(\mathbf{U}_0, \ldots, \mathbf{U}_m) = (A_{m+1}, \mathbf{V}_{m+1})$ whenever $m < \omega$, defines a strategy for player β in the strong Choquet game on $(CL(X), H_d)$. We will be done if we prove

Claim 3.1.2. *σ_{Ch} is a winning strategy for β in $Ch(CL(X), H_d)$.*

To show this, consider a run

$$(A_0, \mathbf{V}_0), \mathbf{U}_0, \ldots, (A_m, \mathbf{V}_m), \mathbf{U}_m, \ldots$$

of $Ch(CL(X), H_d)$ compatible with σ_{Ch}, and assume $A \in \bigcap_{m<\omega} \mathbf{V}_m$. If we choose some $t \in \bigcap_{m<\omega} I_{n_m+1}^m$, note that for every $m < \omega$,

$$x_t^m(k) = 0 \text{ for all } 0 < k \le 1 + \sum_{i \le m} n_i. \tag{3}$$

Since $A \in \mathbf{V}_0$, there is some $a \in A$ with $d(x_t^0, a) < 1$, thus,

$$a(0) = x_t^0(0) = t. \tag{4}$$

Since $a \in X$, there exists $0 < k$ so that

$$a(k) \ne 0. \tag{5}$$

Choose $m < \omega$ so that $k \le 1 + \sum_{i \le m} n_i$. Since $A \in \mathbf{V}_m$, there exists an $x_{t'}^m \in A_m$ with $d(a, x_{t'}^m) < \frac{1}{1 + \sum_{i \le m} n_i}$, which implies that

$$x_{t'}^m(0) = a(0), \text{ and} \tag{6}$$

$$x_{t'}^m(k) = a(k). \tag{7}$$

Using (4),(6) we get

$$t' = x_{t'}^m(0) = a(0) = x_t^0(0) = t,$$

so $t' = t$. This would yield, by (7),(3), that

$$a(k) = x_{t'}^m(k) = x_t^m(k) = 0,$$

which contradicts (5). In conclusion, we got that $\bigcap_{m<\omega} \mathbf{V}_m = \emptyset$, and so β wins in $Ch(CL(X), H_d)$.

(3) Let $\mathbf{V}_0$ be β's initial step in $BM(CL(X), H_d)$, where $\mathbf{V}_0 = B_{H_d}(A_0, \frac{1}{n_0})$ for some $A_0 \in CL(X)$ and $n_0 \geq 1$. For each $a_0 \in A_0$ define $x_{a_0} \in X$ via

$$x_{a_0}(k) = \begin{cases} a_0(k), & \text{if } k < 2n_0 - 1, \\ n_0 + 1, & \text{if } k \geq 2n_0 - 1, \end{cases}$$

put $C_0 = \{x_{a_0} : a_0 \in A_0\}$. Then $C_0 \in CL(X)$, and $H_d(A_0, C_0) \leq \frac{1}{2n_0}$. Define $\mathbf{U}_0 = B_{H_d}(C_0, \frac{1}{2n_0})$. Then $\mathbf{U}_0 \subseteq \mathbf{V}_0$ (since if $A \in \mathbf{U}_0$, then $H_d(A, C_0) < \frac{1}{2n_0}$, so $H_d(A_0, A) \leq H_d(A, C_0) + H_d(C_0, A_0) < \frac{1}{n_0}$), so we can take $\mathbf{U}_0$ as α's first step in $BM(CL(X), H_d)$.

Assume we have defined a partial run $\mathbf{V}_0, \mathbf{U}_0, \ldots, \mathbf{V}_m, \mathbf{U}_m$ of the Banach-Mazur game in $(CL(X), H_d)$, where

$$\mathbf{V}_i = B_{H_d}\left(A_i, \frac{1}{n_i}\right) \text{ and } \mathbf{U}_i = B_{H_d}\left(C_i, \frac{1}{2n_i}\right)$$

for some $2n_{i-1} \leq n_i < \omega$ whenever $i \leq m$ (for convenience, define $n_{-1} = \frac{1}{2}$). Moreover, for each $i \leq m$ let $C_i = \{x_{a_i} : a_i \in A_i\}$, where

$$x_{a_i}(k) = \begin{cases} a_i(k), & \text{if } k < 2n_i - 1, \\ 1 + \sum\limits_{j \leq i} n_j, & \text{if } k \geq 2n_i - 1. \end{cases} \tag{8}$$

Take $\mathbf{V}_{m+1} = B_{H_d}(A_{m+1}, \frac{1}{n_{m+1}}) \subseteq \mathbf{U}_m$. For any $a_{m+1} \in A_{m+1}$ define

$$y_{a_{m+1}}(k) = \begin{cases} a_{m+1}(k), & \text{if } k < n_{m+1}, \\ 2 + \sum\limits_{i \leq m} n_i, & \text{if } k \geq n_{m+1}. \end{cases}$$

Then $\{y_{a_{m+1}} : a_{m+1} \in A_{m+1}\} \in \mathbf{V}_{m+1} \subseteq \mathbf{U}_m$, so there exists $x_{a_m} \in C_m$ for some $a_m \in A_m$ so that $d(y_{a_{m+1}}, x_{a_m}) < \frac{1}{2n_m}$. If $n_{m+1} < 2n_m$, then

$$y_{a_{m+1}}(2n_m - 1) = 2 + \sum_{i \leq m} n_i \text{ and}$$
$$x_{a_m}(2n_m - 1) = 1 + \sum_{i \leq m} n_i,$$

so $d(y_{a_{m+1}}, x_{a_m}) \geq \frac{1}{2n_m}$, which is impossible, thus, $n_{m+1} \geq 2n_m$. It also follows from $\mathbf{V}_{m+1} \subseteq \mathbf{U}_m$ that $H_d(A_{m+1}, C_m) < \frac{1}{2n_m}$. Hence, for each $a_m \in A_m$ there exists $a_{m+1} \in A_{m+1}$ with $d(x_{a_m}, a_{m+1}) < \frac{1}{2n_m}$, so

$$a_{m+1}(k) = \begin{cases} a_m(k), & \text{if } k < 2n_m - 1, \\ 1 + \sum\limits_{i \leq m} n_i, & \text{if } k = 2n_m - 1. \end{cases} \tag{9}$$

Define $C_{m+1} = \{x_{a_{m+1}} : a_{m+1} \in A_{m+1}\}$, where

$$x_{a_{m+1}}(k) = \begin{cases} a_{m+1}(k), & \text{if } k < 2n_{m+1} - 1, \\ 1 + \sum\limits_{i \leq m+1} n_i, & \text{if } k \geq 2n_{m+1} - 1, \end{cases} \tag{10}$$

and put $\mathbf{U}_{m+1} = B_{H_d}(C_{m+1}, \frac{1}{2n_{m+1}})$. Note that $H_d(A_{m+1}, C_{m+1}) \le \frac{1}{2n_{m+1}}$, so $\mathbf{U}_{m+1} \subseteq \mathbf{V}_{m+1}$, since if $A \in \mathbf{U}_{m+1}$, then $H_d(C_{m+1}, A) < \frac{1}{2n_{m+1}}$, thus, $H_d(A_{m+1}, A) \le H_d(A_{m+1}, C_{m+1}) + H_d(C_{m+1}, A) < \frac{1}{n_{m+1}}$. This means that putting $\sigma_{BM}(\mathbf{V}_0, \dots, \mathbf{V}_m) = \mathbf{U}_m$ for all $m < \omega$ defines a strategy for α in $BM(CL(X), H_d)$.

Claim 3.1.3. *σ_{BM} is a winning strategy for α in $BM(CL(X), H_d)$.*

To show this, consider a run $\mathbf{V}_0, \mathbf{U}_0, \dots, \mathbf{V}_m, \mathbf{U}_m, \dots$ of the Banach-Mazur game in $(CL(X), H_d)$ compatible with σ_{BM}. For any $m < \omega$ and $a_m \in A_m$ we get an $a_{m+1} \in A_{m+1}$ satisfying (9). Then for any $a_0 \in A_0$ we can define the nonempty

$$A_1[a_0] = \{a_1 \in A_1 :\ a_1(k) = a_0(k) \text{ for all } k < 2n_0 - 1\}.$$

Assume, by induction, that we have defined $A_m[a_{m-1}] \neq \emptyset$ for some $a_{m-1} \in A_{m-1}$ and $m \ge 1$. For every $a_m \in A_m[a_{m-1}]$ put

$$A_{m+1}[a_m] = \{a_{m+1} \in A_{m+1} :\ a_{m+1}(k) = a_m(k) \text{ for all } k < 2n_m - 1\},$$

which is nonempty by (9); for convenience, also define $A_0[a_{-1}] = A_0$. Denote

$$P = \{(a_m)_{m\ge 0} : a_m \in A_m[a_{m-1}] \text{ for all } m \ge 0\},$$

and for any $p = (a_m)_{m\ge 0} \in P$ define s_p as follows:

$$s_p(k) = \begin{cases} a_0(k), & \text{if } k < 2n_0 - 1, \\ a_m(k), & \text{if } 2n_{m-1} - 1 \le k < 2n_m - 1,\ m \ge 1. \end{cases} \tag{11}$$

Note, by (9), that $s_p(2n_m - 1) = 1 + \sum_{i\le m} n_i$ for every $m < \omega$, so $s_p \in X$ for each $p \in P$. Denote by S the X-closure of the set $\{s_p : p \in P\}$.

Given any $s_p \in S$, we have a sequence $p = (a_m)_{m\ge 0} \in P$ such that $a_i(k) = a_{i-1}(k)$ for all $1 \le i \le m$ and $k < 2n_i - 1$, which implies by (11) that $a_m(k) = s_p(k)$ for all $k < 2n_m - 1$.

It follows that $d(s_p, A_m) \le d(s_p, a_m) \le \frac{1}{2n_m}$, so $d(s, A_m) \le \frac{1}{2n_m} < \frac{1}{n_m}$ for each $s \in S$, thus,

$$S \subseteq B_{H_d}\left(A_m, \frac{1}{n_m}\right). \tag{12}$$

Furthermore, for each $1 \le i \le m$, $A_i \in \mathbf{V}_i \subseteq \mathbf{U}_{i-1}$, so for any $a_i \in A_i$ there exists $a_{i-1} \in A_{i-1}$ with $d(x_{a_{i-1}}, a_i) < \frac{1}{2n_{i-1}}$, which means that $a_i(k) = x_{a_{i-1}}(k)$ for each $k \le 2n_{i-1} - 1$, so by (8),

$$a_i(k) = a_{i-1}(k) \text{ for each } k < 2n_{i-1} - 1; \tag{13}$$

moreover, if $i > m$ we can choose by (9), $a_i \in A_i$ so that (13) is satisfied. It follows that $a_i \in A_i[a_{i-1}]$ for all $1 \le i$, thus, $p = (a_i)_{i\ge 0} \in P$ and

$s_p(k) = a_m(k)$ for all $k < 2n_m - 1$. This implies that $d(a_m, S) \leq d(a_m, s_p) \leq \frac{1}{2n_m} < \frac{1}{n_m}$, so

$$A_m \subseteq B_{H_d}\left(S, \frac{1}{n_m}\right). \tag{14}$$

In conclusion, by (12), (14) we have that $H_d(A_m, S) < \frac{1}{n_m}$, thus, $S \in \mathbf{V}_m$, which implies that $S \in \bigcap_{m<\omega} \mathbf{V}_m$, and so α wins. $\square$

Corollary 3.2. *There exists a completely metrizable bounded metric space X with compatible metrics d, d' so that $Ch(CL(X), H_d)$ is α-favorable and $Ch(CL(X), H_{d'})$ is β-favorable.*

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