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On generalized metric properties of the Fell hyperspace

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Abstract It is shown that if X contains a closed uncountable discrete subspace, then the Tychonoff plank embeds in the hyperspace $CL(X)$ of the non-empty closed subsets of X with the Fell topology τ_F as a closed subspace. As a consequence, a plethora of properties is proved to be equivalent to normality and metrizability, respectively, of $(CL(X), \tau_F)$. Countable paracompactness, pseudonormality and other weak normality properties of the Fell topology are also characterized.

Keywords Fell topology · Vietoris topology · Tychonoff plank · Extent · Spread · Metrizable · Lindelöf · (hereditary) normal · Monotonically normal · δ-Normal · Pseudonormal · Countably paracompact

Mathematics Subject Classification (2010) 54B20 · 54C25 · 54D15 · 54D20

1 Introduction

The *Fell topology* τ_F on the space of (non-empty) closed subsets of a topological space X is a fundamental construct due to its usefulness in various areas of mathematics and applications [3,4,19]. Since the Fell topology is a hit-and-miss type topology (i.e., a typical base element for τ_F consists of closed sets that hit finitely many open subsets of X and miss a compact subset of X), it has frequently been compared to another classical well-studied hit-and-miss

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topology, the *Vietoris topology* τ_V, for motivation to obtain results and ideas about τ_F. This is far from an automatic transfer of properties and results (see e.g., characterizations of normality of these topologies [14,17,27]); on the other side, for various properties, one can find common characterizations of even general hit-and-miss topologies (e.g., for separation, countability axioms, metrizability, completeness [3,28]).

In this paper, we continue studying the Fell topology and obtain new results about various generalized metric properties of τ_F. The motivation came from analogous properties of the Vietoris topology (results of several authors spread in the literature); however, instead of separately proving these results for the Fell topology, we present a technique that produces the proofs simultaneously. The technique relies on some embedding results presented in Sect. 2, which allow us, in Sect. 3, to prove the equivalence of a large number of properties of τ_F to its normality, and we also look at properties that are strictly weaker than normality, but we show they still coincide with normality for τ_F, fixing a result of [20] in the process. Finally, in Sect. 4, we characterize more generalized metric properties of τ_F showing them equivalent to its metrizability through a method that ties the analogous (known) results of the Vietoris topology to those of the Fell topology.

First, we introduce some notation and terminology: throughout the paper, X is a Hausdorff space. Denote by 2^X (resp. $CL(X)$) the (non-empty) closed subsets of X. For $S \subseteq X$, put

$$S^- = \{A \in 2^X : A \cap S \neq \emptyset\},\ S^+ = \{A \in 2^X : A \subseteq S\}.$$

For a finite collection $\mathscr{S}$ of subsets of X denote

$$\mathscr{S}^- = \bigcap_{S \in \mathscr{S}} S^-.$$

The *Vietoris topology* [22] τ_V on 2^X has as a subbase elements of the form U^-, U^+, where U is open in X. The *Fell topology* [11] τ_F on 2^X has as a subbase the collection

$$\{U^- : U \text{ open in } X\} \cup \{(X \setminus K)^+ : K \text{ compact in } X\};$$

so a typical base element for τ_F is of the form $(X \setminus K)^+ \cap \mathscr{N}^-$, where $K \subseteq X$ is compact, and $\mathscr{N}$ is a finite collection of X-open sets. We will denote by $cl_F(\mathscr{S})$ the τ_F-closure of $\mathscr{S} \subseteq CL(X)$. All subspaces of 2^X will carry the relative Fell topology.

The Fell topology on 2^X is always compact, and, if X is locally compact, it is also Hausdorff [11]. This in turn implies that $(CL(X), \tau_F)$ is Hausdorff; in fact, $(CL(X), \tau_F)$ is Hausdorff (regular, Tychonoff, respectively) iff X is locally compact [3, Proposition 5.1.2]. Moreover, $(CL(X), \tau_F)$ is normal (paracompact, Lindelöf, respectively) iff X is locally compact, Lindelöf [14, Theorem 1] , and $(CL(X), \tau_F)$ is metrizable iff X is locally compact, second countable [3, Theorem 5.1.5]. We will also use that the Fell and the Vietoris topology are *admissible*, i.e., X embeds as a closed subspace in $(CL(X), \tau_F)$ and $(CL(X), \tau_V)$, respectively. Moreover, if $A \in CL(X)$, then $(CL(A), \tau_F)$ is a closed subspace of $(CL(X), \tau_F)$. The Vietoris and Fell topologies coincide on $CL(X)$ iff X is compact [3].

2 Embedding into the Fell hyperspace

Embedding techniques had been successfully used to obtain normality-type results for hyperspaces (see [17] for the Vietoris topology and [7] for the Wijsman topology). In this section, we explore embeddability of various spaces into the Fell hyperspace.

Recall the definition of the *spread* and *extent*, respectively, of a topological space X:

$$s(X) = \omega \cdot \sup\{|D| : D \text{ is a discrete subspace of } X\},$$
$$e(X) = \omega \cdot \sup\{|D| : D \text{ is a closed discrete subspace of } X\}.$$

Proposition 1 *If $s(X) > \omega$, then $(\omega_1 + 1) \times (\omega + 1)$ embeds into $(2^X, \tau_F)$.*

Proof Let D be a discrete subspace of X of size ω_1, $D = \{x_\nu : \nu < \omega_1\}$. For each $x \in D$, fix an open neighborhood $U(x)$ such that $U(x) \cap D = \{x\}$ and denote by $\preccurlyeq$ the product order on $T_0 = (\omega_1 + 1) \times (\omega + 1)$ (as usual, $a \prec b$ means that $a \preccurlyeq b$ and $a \neq b$). Let $\Lambda = \{\lambda_\alpha : 1 \leq \alpha \leq \omega_1\}$ be the infinite limit ordinals in $\omega_1 + 1$ and put $\lambda_0 = 0$. If α is a successor, denote by α' its predecessor.

For convenience, put $\varphi(0', \beta) = X$ for each $\beta \leq \omega$ and define the function $\varphi : T_0 \to 2^X$ as follows:

$$\varphi(\alpha, \beta) = \begin{cases} \varphi(\alpha', \beta) \setminus \bigcup_{\nu \in [\lambda_\alpha, \lambda_\alpha + \beta]} U(x_\nu) & \text{if } (\alpha, \beta) \in (\omega_1 \setminus \Lambda) \times \omega, \\ \bigcap\{\varphi(\bar{\alpha}, \bar{\beta}) : (\bar{\alpha}, \bar{\beta}) \prec (\alpha, \beta)\}, & \text{if } \alpha \in \Lambda \text{ or } \beta = \omega. \end{cases}$$

If we take distinct $A, B \in \varphi(T_0)$, then there is $x \in D$ with $U(x)$ missing one of A, B and hitting the other, so $U(x)^- \cap \varphi(T_0)$, and $(X \setminus \{x\})^+ \cap \varphi(T_0)$ are disjoint $\varphi(T_0)$-neighborhoods of A, B. Consequently, $(\varphi(T_0), \tau_F)$ is Hausdorff, so to show that $\varphi : T_0 \to (\varphi(T_0), \tau_F)$ is a homeomorphism it suffices to show that φ is continuous, since T_0 is compact and φ is one-to-one. If $(\alpha, \beta) \in (\omega_1 \setminus \Lambda) \times \omega$, then (α, β) is isolated in T_0, so we can assume that one of α, β is a limit ordinal. Then, $\varphi(\alpha, \beta) = \bigcap\{\varphi(\bar{\alpha}, \bar{\beta}) : (\bar{\alpha}, \bar{\beta}) \prec (\alpha, \beta)\}$, so we just need to consider the τ_F-neighborhood $(X \setminus K)^+ \cap \varphi(T_0)$ of $\varphi(\alpha, \beta)$ for some compact $K \subseteq X$. Then, there exists $(\bar{\alpha}, \bar{\beta}) \prec (\alpha, \beta)$ with $\varphi(\bar{\alpha}, \bar{\beta}) \cap K = \emptyset$, which implies that $\varphi((\bar{\alpha}, \alpha] \times (\bar{\beta}, \beta]) \subset (X \setminus K)^+ \cap \varphi(T_0)$. □

The *Tychonoff plank* is defined as $T = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$, where $\omega_1 + 1$ and $\omega + 1$ are both endowed with the order topology. It is well known that T is not normal [18].

Corollary 1 *1. If $s(X) > \omega$, the Tychonoff plank embeds into $(CL(X), \tau_F)$.*
2. If $e(X) > \omega$, the Tychonoff plank embeds into $(CL(X), \tau_F)$ as a closed subspace.

Proof (1) follows from Proposition 1.

(2) If D is an uncountable closed discrete subset of X, we can choose $U(x_0) = X \setminus (D \setminus \{x_0\})$ in the proof of Proposition 1, so $\varphi((\omega_1, \omega)) = \emptyset$. Observe that $A \in CL(X) \setminus \varphi(T)$ iff there exists $(\bar{\alpha}, \bar{\beta}) \prec (\alpha, \beta)$ so that for $i_1 = \lambda_{\bar{\alpha}} + \bar{\beta}$ and $i_2 = \lambda_\alpha + \beta$ we have $x_{i_1} \in A$, $x_{i_2} \notin A$. It follows that $U(x_{i_1})^- \cap (X \setminus \{x_{i_2}\})^+$ is a τ_F-neighborhood of A disjoint to $\varphi(T)$, so $\varphi(T)$ is a closed subspace of $(CL(X), \tau_F)$.

□

Proposition 2 *Let κ be an ordinal and X have a closed discrete set of size $|\kappa|$. Then, κ embeds as a closed subspace of $(CL(X), \tau_F)$.*

Proof Let $D = \{x_\alpha : \alpha < \kappa\}$ be a closed discrete subspace of X, and (without loss of generality) assume $D \neq X$, fix some $x_{-1} \in X \setminus D$. For each $0 \leq \alpha < \kappa$, choose an open set $U(x_\alpha)$ with $\{x_\alpha\} = D \cap U(x_\alpha)$, put $D_\alpha = \{x_\beta : \beta \geq \alpha\}$ and define the function $\varphi : \kappa \to CL(X)$ via $\varphi(\alpha) = D_\alpha$. Then, φ is clearly injective; moreover,

- φ *is continuous*; indeed, if $\alpha < \kappa$ is a limit ordinal, then $D_\alpha = \bigcap_{\beta < \alpha} D_\beta$, so it suffices to consider the τ_F-neighborhood $(X \setminus K)^+$ of $\varphi(\alpha) = D_\alpha$ for some compact $K \subseteq X$. Then, $D_\alpha \cap K = \emptyset$, so $D_\beta \cap K = \emptyset$ for some $\beta < \alpha$, i.e., $(\beta, \alpha] \subset \varphi^{-1}((X \setminus K)^+)$.

– φ *is open*: consider the open set $(\beta, \alpha]$ for some $-1 \leq \beta < \alpha < \kappa$. Then,

$$D_\alpha \in U(x_\alpha)^- \cap (X \setminus \{x_\beta\})^+ \cap \varphi(\kappa) \subseteq \varphi((\beta, \alpha]).$$

– $\varphi(\kappa)$ *is a closed subspace*: if $A \in CL(X) \setminus \varphi(\kappa)$, then either $A \setminus D \neq \emptyset$ and then $A \in (X \setminus D)^- \subseteq CL(X) \setminus \varphi(\kappa)$ or $A \subset D$ and there exist $0 \leq \beta < \alpha < \kappa$ with $x_\beta \in A$, $x_\alpha \notin A$, which implies $A \in U(x_\beta)^- \cap (X \setminus \{x_\alpha\})^+ \subseteq CL(X) \setminus \varphi(\kappa)$.

□

Corollary 2 *If $s(X) > \omega$ (resp. $e(X) > \omega$), then ω_1 and $\omega_1 + 1$ embed in $(CL(X), \tau_F)$ as (closed) subspaces.*

Proof See Corollary 1 and Proposition 2. □

Corollary 3 *1. Let $\mathscr{P}$ be a (closed) hereditary topological property that the Tychonoff plank does not have. If $(CL(X), \tau_F)$ has property $\mathscr{P}$, then $s(X) = \omega$ (resp. $e(X) = \omega$).*

2. *Let $\mathscr{P}$ be a (closed) hereditary topological property that ω_1 does not have. If $(CL(X), \tau_F)$ has property $\mathscr{P}$, then $s(X) = \omega$ (resp. $e(X) = \omega$).*

Proof Follows from Corollary 1 and Corollary 2. □

3 Normality-related properties of the Fell topology

Theorem 1 *Let $\mathscr{P}$ be a closed hereditary property such that having countable extent with property $\mathscr{P}$ implies Lindelöfness. If $(CL(X), \tau_F)$ has property $\mathscr{P}$, then X is Lindelöf.*

Proof The Tychonoff plank (or ω_1) does not have property $\mathscr{P}$, because it has countable extent and is not Lindelöf. It follows from Corollary 3 that $e(X) = \omega$ and since, by admissibility, X has property $\mathscr{P}$, it is Lindelöf. □

Corollary 4 *Let $\mathscr{P}$ be a closed hereditary property such that having countable extent with property $\mathscr{P}$ is equivalent to Lindelöfness. Then, the following are equivalent:*

1. *$(CL(X), \tau_F)$ is T_2 with property $\mathscr{P}$,*
2. *$(CL(X), \tau_F)$ is Lindelöf,*
3. *X is locally compact and Lindelöf.*

Proof (1)⇒(3) follows from Theorem 1 and [3, Proposition 5.1.2], (3)⇒(2) is known [14, Theorem 1], and (2)⇒(1) is clear. □

The previous results immediately imply characterizations of various properties for the Fell topology, which turn out to be all equivalent to normality of $(CL(X), \tau_F)$ by [14, Theorem 1]. Most of these properties are well known [6]. Recall that X is a *D-space* [2] if for every open neighborhood assignment N, one can find a closed discrete $D \subset X$ such that $\{N(x) : x \in D\}$ covers X; moreover, X is a *(weakly) aD-space* [2] if for each closed $F \subset X$ (for $F = X$) and each open cover $\mathscr{U}$ of X, there is a locally finite $A \subset F$ and $\phi : A \to \mathscr{U}$ with $a \in \phi(a)$ and $F \subset \cup\phi(A)$.

Corollary 5 *The following are equivalent:*

1. *$(CL(X), \tau_F)$ is Lindelöf,*
2. *$(CL(X), \tau_F)$ is paracompact,*

3. $(CL(X), \tau_F)$ *is subparacompact,*
4. $(CL(X), \tau_F)$ *is metacompact,*
5. $(CL(X), \tau_F)$ *is submetacompact,*
6. $(CL(X), \tau_F)$ *is* σ*-metacompact,*
7. $(CL(X), \tau_F)$ *is screenable,*
8. $(CL(X), \tau_F)$ *is paralindelöf,*
9. $(CL(X), \tau_F)$ *is* T_2 *metalindelöf,*
10. $(CL(X), \tau_F)$ *is* T_2 *submetalindelöf,*
11. $(CL(X), \tau_F)$ *is a* T_2 D*-space,*
12. $(CL(X), \tau_F)$ *is a* T_2 aD*-space,*
13. $(CL(X), \tau_F)$ *is a* T_2 *weakly* aD*-space,*
14. X *is locally compact and Lindelöf.*

Proof Each of the properties (1)–(9) implies (10) (see [6]); (10) implies (13) by [2, Theorem 1.16]. Furthermore, (11)⇒(12)⇒(13), and (13) implies (14) by Theorem 1 and [2, Proposition 1.10]. Finally, assuming (14), we get that $(CL(X), \tau_F)$ is σ-compact by [14, Theorem 1], which yields (1), (11). □

Note that all the properties in the previous corollary are not weaker than normality; however, it is known that normality of the Fell topology is equivalent to all of them [14]. We will show that there are properties weaker than normality, which still imply normality for the Fell topology. Recall that a space X is *pseudonormal* (resp. δ*-normal*) iff any pair of disjoint closed sets, one of which is countable (resp. a regular G_δ), can be separated by disjoint open sets.

Theorem 2 *The following are equivalent:*

1. $(CL(X), \tau_F)$ *is a* T_2 *countably paracompact space,*
2. $(CL(X), \tau_F)$ *is a* T_2 δ*-normal space,*
3. $(CL(X), \tau_F)$ *is a* T_2 *pseudonormal space,*
4. X *is locally compact and either countably compact or Lindelöf.*

Proof For (1)⇒(2), see [21, Theorem 3], and for (2)⇒(3), see [12, Proposition 5.1] using that a Hausdorff Fell hyperspace is Tychonoff as well.

(3)⇒(4) X is locally compact since τ_F is T_2. It suffices to prove that if X is not countably compact, then X is Lindelöf; indeed, let $D = \{x_k : k < \omega\}$ be a closed discrete subset of X and define the closed sets $D_n = \{x_k : k \geq n\}$ for each $n < \omega$. Then $\mathscr{A} = \{D_n : n < \omega\}$ is a countable τ_F-closed set disjoint to the τ_F-closed set $\mathscr{B} = \{\{x\} : x \in X\}$. Since $(CL(X), \tau_F)$ is pseudonormal, we can find disjoint τ_F-open sets $\mathscr{U}, \mathscr{V}$ so that

$$\mathscr{A} \subseteq \mathscr{U} \quad \text{and} \quad \mathscr{B} \subseteq \mathscr{V}.$$

Also, for each $n < \omega$, there exist a finite collection $\mathscr{U}_n$ of open sets and a compact K_n such that

$$D_n \in (X \setminus K_n)^+ \cap \mathscr{U}_n^- \subseteq \mathscr{U}.$$

We will be done if we show that $X = \bigcup_{n<\omega} K_n$: if there were an $x \in X \setminus \bigcup_{n<\omega} K_n$, we could find an open neighborhood V of x and a compact K with

$$\{x\} \in (X \setminus K)^+ \cap V^- \subseteq \mathscr{V}.$$

By compactness of K, and since $\bigcap_{n<\omega} D_n = \emptyset$, this would imply $D_n \cap K = \emptyset$ for some $n < \omega$, and hence $D_n \cup \{x\} \in \mathscr{U} \cap \mathscr{V}$, a contradiction.

(4)⇒(1) If X is countably compact, so is $(CL(X), \tau_F)$ by [13, Proposition 4.1]. If X is locally compact Lindelöf, then $(CL(X), \tau_F)$ is paracompact by [14, Theorem 1]. □

A space X is *weakly normal* [1] iff for any pair of disjoint closed sets $A, B \subset X$, there exists a continuous $f : X \to \mathbb{R}^\omega$ so that $f(A), f(B)$ are disjoint. Normality implies weak normality as well as pseudonormality; however, the space constructed in [26, Example 1] is non-normal, weakly normal (since it contains a coarser separable metric space) and pseudonormal. We will show that this distinction is not present for the Fell topology:

Theorem 3 *The following are equivalent:*

1. $(CL(X), \tau_F)$ *is a* T_2 *weakly normal, countably paracompact space,*
2. $(CL(X), \tau_F)$ *is a* T_2 *weakly normal,* δ*-normal space,*
3. $(CL(X), \tau_F)$ *is a* T_2 *weakly normal, pseudonormal space,*
4. $(CL(X), \tau_F)$ *is normal,*
5. X *is locally compact and Lindelöf.*

Proof Only (3)⇒(5) needs explanation, the rest follows by Theorem 2 and [14]: if X is countably compact, so is $(CL(X), \tau_F)$ by [13, Proposition 4.1]; moreover, a T_2 countably compact weakly normal space is normal [1], so X is Lindelöf by [14]. On the other hand, if X is not countably compact, then it must be Lindelöf by Theorem 2. □

The following theorem is the main result of [20]; however, the proof repeatedly uses the incorrect claim that if A, B are disjoint closed subsets of X, then $CL(A) \times CL(B)$ embeds as a closed subspace in $CL(A \cup B)$, when the hyperspaces are endowed with the Fell topology (indeed, if $f(C, D) = C \cup D$ is the embedding, then $B \notin f(CL(A) \times CL(B))$, but every Fell neighborhood of B intersects $f(CL(A) \times CL(B))$, unless A is compact). We can fix the argument, however, if we work directly inside the Fell hyperspace:

Theorem 4 *The following are equivalent:*

1. $(CL(X), \tau_F)$ *is* T_2 *and all of its* F_σ*-subsets are* δ*-normal,*
2. $(CL(X), \tau_F)$ *is normal,*
3. X *is locally compact and Lindelöf.*

Proof (1)⇒(3) By Theorem 2, we just need to eliminate the possibility that X is countably compact and not Lindelöf; otherwise, X has a countable set C with a limit point $x \notin C$. Let $U \neq X$ be an open neighborhood of x with a compact closure and denote $A = X \setminus U$. We will also assume, without loss of generality, that $\overline{\{x\} \cup C} \subset U$.

Claim If $\mathscr{A} \subseteq CL(A)$ is τ_F-closed, then $\mathscr{A}_D = \{D \cup B : B \in \mathscr{A}\}$ is τ_F-closed for any $D \in U^+$.

[Indeed, let $E \in CL(X) \setminus \mathscr{A}_D$. Then

- either $E \setminus (D \cup A) \neq \emptyset$, and so $E \in (X \setminus (D \cup A))^- \subseteq CL(X) \setminus \mathscr{A}_D$,
- or $E \subseteq D \cup A$, then
  - either there is $d \in D \setminus E$, and so $E \in (X \setminus \{d\})^+ \subseteq CL(X) \setminus \mathscr{A}_D$,
  - or $D \subset E$, and $B = E \cap A \notin \mathscr{A}$. If $(X \setminus K)^+ \cap \mathscr{N}^-$ is a τ_F-basic neighborhood of B missing $\mathscr{A}$ such that $N \subseteq X \setminus D$ for each $N \in \mathscr{N}$, then so is $(X \setminus K \cap A)^+ \cap \mathscr{N}^-$ which, in turn, is a τ_F-neighborhood of E missing $\mathscr{A}_D$.]

Let $\mathscr{A} \subsetneq CL(A)$ be τ_F-closed and $\mathscr{W}$ be $CL(A)$-open so that $\mathscr{A} \subseteq \mathscr{W} \subsetneq CL(A)$. Denote $\mathscr{Z} = \mathscr{A}_{\{x\}} \cup \bigcup\{CL(A)_{\{c\}} : c \in C\}$. Then,

- $\mathscr{B} = \bigcup\{(CL(A) \setminus \mathscr{W})_{\{c\}} : c \in C\}$ *is closed in* $\mathscr{Z}$: if $E \in \mathscr{Z} \setminus \mathscr{B}$, then $E = \{e\} \cup B$ for some $e \in \{x\} \cup C$ and $B \in \mathscr{W}$. Let $(A \setminus K)^+ \cap \bigcap_{i \leq n}(W_i \cap A)^-$ be a $CL(A)$-basic neighborhood of B contained in $\mathscr{W}$, where $W_i \subseteq X \setminus \overline{\{x\} \cup C}$ for all $i \leq n$ ($n < \omega$). Then, $(X \setminus K \cap A)^+ \cap \bigcap_{i \leq n} W_i^- \cap \mathscr{Z}$ is a $\mathscr{Z}$-neighborhood of E missing $\mathscr{B}$.
- $\mathscr{A}_{\{x\}} \cap \mathscr{B} = \emptyset$: clear.
- $\mathscr{A}_{\{x\}}$ *is a regular* G_δ *in* $\mathscr{Z}$: for each $c \in C$, let V_c be an open neighborhood of c with compact closure such that $\overline{V_c}$ misses $A \cup \{x\}$ and denote $\mathscr{O}_c = (X \setminus \overline{V_c})^+$. Note that $cl_F(\mathscr{O}_c) \subseteq (X \setminus V_c)^+$, so

$$\mathscr{A}_{\{x\}} = \bigcap_{c \in C} \mathscr{Z} \cap \mathscr{O}_c \subseteq \bigcap_{c \in C} \mathscr{Z} \cap cl_F(\mathscr{O}_c) \subseteq \bigcap_{c \in C} \mathscr{Z} \cap (X \setminus V_c)^+ = \mathscr{A}_{\{x\}}.$$

It follows, by Claim 3, that $\mathscr{Z}$ is an F_σ-subset of $(CL(X), \tau_F)$, and so it is δ-normal; thus, there exist disjoint $\mathscr{Z}$-open sets $\mathscr{U}$, $\mathscr{V}$ so that $\mathscr{A}_{\{x\}} \subset \mathscr{U}$, and $\mathscr{B} \subset \mathscr{V}$. For each $c \in C$, define the set

$$\mathscr{U}_c = \{B \in CL(A) : \{c\} \cup B \in \mathscr{U}\}.$$

Then,

- $cl_{CL(A)}(\mathscr{U}_c) \subset \mathscr{W}$; otherwise, if $B \in cl_{CL(A)}(\mathscr{U}_c) \setminus \mathscr{W}$, then $\{c\} \cup B \in \mathscr{B} \subset \mathscr{V} \subseteq \mathscr{Z} \setminus cl_{\mathscr{Z}}(\mathscr{U})$, so there is a $\mathscr{Z}$-neighborhood $(X \setminus K)^+ \cap \mathscr{N}^-$ of $\{c\} \cup B$ that misses $\mathscr{U}$. Let $\mathscr{N}_0 = \{N \in \mathscr{N} : B \in N^-\}$. Then, $(A \setminus K)^+ \cap \mathscr{N}_0^-$ is a $CL(A)$-neighborhood of B that misses $\mathscr{U}_c$, a contradiction.
- $\mathscr{A} \subseteq \bigcup_{c \in C} \mathscr{U}_c$: fix $B \in \mathscr{A}$. Then, there exists a τ_F-basic open neighborhood $(X \setminus K)^+ \cap \mathscr{N}^-$ of $\{x\} \cup B$ so that $\mathscr{Z} \cap (X \setminus K)^+ \cap \mathscr{N}^- \subseteq \mathscr{U}$. Denote $\mathscr{N}_1 = \{N \in \mathscr{N} : x \in N\}$. Since x is a cluster point of C, we can find a $c \in (C \setminus K) \cap \bigcap \mathscr{N}_1$. Then $\{c\} \cup B \in \mathscr{Z} \cap (X \setminus K)^+ \cap \mathscr{N}^- \subseteq \mathscr{U}$, so $B \in \mathscr{U}_c$.

It follows by [9, Lemma 1.5.14], that $CL(A)$ is normal; thus, $A = X \setminus U$ is Lindelöf by [14, Theorem 1], and so is $X = (X \setminus U) \cup \overline{U}$, a contradiction. □

4 Metrizability-related properties of the Fell topology

The point of the following result is to show how various properties of the Vietoris topology provide characterizations of the relevant properties for the Fell topology:

Proposition 3 *Let X be a locally compact paracompact space. Let* $\mathscr{P}$ *be a closed hereditary property such that* τ_V *has property* $\mathscr{P}$ *iff* τ_V *is metrizable. If* $(CL(X), \tau_F)$ *has property* $\mathscr{P}$*, then X is metrizable.*

Proof Let K be a compact neighborhood of a given $x \in X$. Since the Fell and Vietoris topologies coincide on compacts, and $CL(K)$ is a closed subspace of $(CL(X), \tau_F)$, then $(CL(K), \tau_V)$ has property $\mathscr{P}$, so K is metrizable. It follows that X is locally metrizable so, by the Smirnov metrization theorem [9, 5.4.A], X is metrizable. □

Using the previous result, our last theorem provides characterization of various properties for τ_F that have been separately established for the Vietoris topology [5,10,16,25].

Theorem 5 *The following are equivalent:*

1. $(CL(X), \tau_F)$ *is* T_2 *and a countable union of metrizable subspaces,*
2. $(CL(X), \tau_F)$ *is* T_2 *and symmetrizable,*
3. $(CL(X), \tau_F)$ *is perfectly normal,*
4. $(CL(X), \tau_F)$ *is monotonically normal,*
5. $(CL(X), \tau_F)$ *is hereditarily normal,*
6. $(CL(X), \tau_F)$ *is metrizable,*
7. X *is locally compact and 2nd countable.*

Proof (7)⇔(6) follows by [3, Theorem 5.1.5]. Since (6) implies (1)–(5), both (3) and (4) imply (5), we just need to prove that (1), (2), (5), respectively, implies (7):

– (1) implies that X and $(CL(X), \tau_F)$ are T_2 and locally compact; moreover, $(CL(X), \tau_F)$ is a sequential space by [24, Theorem 1]. This in turn yields that X is hereditarily Lindelöf by [8, Proposition 2.12]. Finally, being a countable union of metrizable subspaces is a property that satisfies Proposition 3 by [16, Corollary 27], so X is metrizable, and (7) follows.
– (2) implies that $(CL(X), \tau_F)$ is T_2, locally compact and symmetrizable, so it is a Moore space by [6, Theorem 9.13], which is equivalent to (7) by [15, Theorem 7] (an alternative argument could use that X is Lindelöf by [23, Theorem 2], and symmetrizability is a property that satisfies Proposition 3 by [25, Theorem 3]).
– (5) implies that X is locally compact and Lindelöf by [14, Theorem 1], and hereditary normality is another property that satisfies Proposition 3 by [10, Theorem 1], so X is metrizable, and (7) follows.

□

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