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ON A TYPICAL PROPERTY OF FUNCTIONS

JÁNOS T. TÓTH — LÁSZLÓ ZSILINSZKY

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ABSTRACT. Let $s$ be the space of all real sequences endowed with the Fréchet metric $\rho$. Consider the space $\mathcal{F}$ of all functions $f : \mathbb{R} \to \mathbb{R}$ with the uniform topology. Denote by $\mathcal{U}$ the class of all functions $f \in \mathcal{F}$ for which the set $\{\{a_i\}_i \in s ; \sum_i f(a_i) \text{ converges} \}$ is $\sigma$-superporous in $(s, \rho)$. Then $\mathcal{U}$ is residual in $\mathcal{F}$, both $\mathcal{U}$ and $\mathcal{F} \setminus \mathcal{U}$ are dense-in-itself and $\mathcal{U}$ is a Baire space in the relative topology.

Introduction

Let $(s, \rho)$ be the metric space of all real sequences with the Fréchet metric

$$\rho(a, b) = \sum_{i=1}^{\infty} 2^{-i} \frac{|a_i - b_i|}{1 + |a_i - b_i|},$$

where $a = \{a_i\}_i, b = \{b_i\}_i \in s$.

Denote by $B(a, r)$ the open ball centred at $a \in s$ with radius $r > 0$ in $(s, \rho)$. Let $E \subset s$, $a \in s$ and $r > 0$. Define

$$\gamma(a, r, E) = \sup \{r' > 0 ; \exists a' \in s \ B(a', r') \subset B(a, r) \setminus E \}.$$ 

We say that $E$ is porous at $a$ if

$$\limsup_{r \to 0^+} \frac{\gamma(a, r, E)}{r} > 0.$$ 

Further, the set $E \subset s$ is said to be superporous at $a \in s$ (see [7], [8]), if $E \cup F$ is porous at $a$ whenever $F \subset s$ is porous at $a$. We say that $E$ is superporous if it is superporous at each of its points, further $E$ is $\sigma$-superporous if it is a countable union of superporous sets.

Denote by $\mathbb{Q}$ the set of all rational numbers, by $\chi_M$ the characteristic function of $M \subset \mathbb{R}$, and by $\mathbb{R}$ the set $\mathbb{R} \cup \{\pm \infty\}$.

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It is known that the set of all real sequences \( \{a_i\}_{i} \) such that \( \sum a_i \) converges constitutes a meager set in \((s, g)\) ([2], [6]). It is not hard to generalize this result realizing that the set

\[
A(f) = \left\{ \{a_i\}_{i} \in s; \sum_{i} f(a_i) \text{ converges} \right\}
\]

is meager in \((s, g)\) for every nonvanishing continuous function \( f: \mathbb{R} \to \mathbb{R} \). In fact, these sets are even “poorer” since, as we will show, \( A(f) \) is \( \sigma \)-superporous for a broad class of functions \( f \). More precisely, if \( \mathcal{U} \) stands for the class of all functions \( f: \mathbb{R} \to \mathbb{R} \) (not necessarily continuous) for which \( A(f) \) is \( \sigma \)-superporous in \( s \), then \( \mathcal{U} \) constitutes a residual set in the space \((\mathcal{F}, d)\) of all real functions of one real variable with the sup-metric \( d(f, g) = \min\left\{ 1, \sup_{x \in \mathbb{R}} |f(x) - g(x)| \right\} \), where \( f, g \in \mathcal{F} \). Besides, we will investigate various topological properties of \( A(f) \) in \((s, g)\) and of \( \mathcal{U} \) in \((\mathcal{F}, d)\).

**Properties of \( A(f) \)**

First we examine the density of \( A(f) \).

**THEOREM 1.** The set \( A(f) \) is either empty or dense in \((s, g)\).

**Proof.** Suppose \( A(f) \neq \emptyset \) and \( \{b_i\}_i \in A(f) \). Let \( a = \{a_i\}_i \in s \) and \( \varepsilon > 0 \). Choose \( j \in \mathbb{N} \) such that \( 2^{-j} < \varepsilon \). Put \( c_i = a_i \) for \( i \leq j \), and \( c_i = b_i \) for \( i > j \). Then evidently \( c = \{c_i\}_i \in A(f) \) and \( g(a, c) < \varepsilon \). \( \square \)

Define the following sets for \( f \in \mathcal{F} \) and \( p, q \in \mathbb{N} \):

\[
A_{pq}(f) = \left\{ \{a_i\}_i \in s; \forall m, n > q, m < n \ |f(a_{m+1}) + \cdots + f(a_n)| \leq \frac{1}{p} \right\}.
\]

**LEMMA 1.** Suppose \( \alpha > 0 \) and \( x_0 \in \mathbb{R} \). Let \( f_0(x) = \max\{0, 2 - \frac{1}{\alpha} |x - x_0|\} \), \( x \in \mathbb{R} \). Then \( A_{pq}(f_0) \) is superporous for every \( p, q \in \mathbb{N} \).

**Proof.** Let \( a \in A_{pq}(f_0) \). Suppose \( F \subset s \) is an arbitrary set porous at \( a \). Then we have a number \( \beta > 0 \) such that for all \( n \geq q \) there exist \( r_n, r'_n \) such that \( \beta r_n < r'_n < r_n < 2^{-n} \) and \( a' \in s \) for which

\[
B(a', r'_n) \subset B(a, r_n) \setminus F.
\]

(1)

Denote \( m_n = \min\{k \in \mathbb{N}; 2^{-k} < r'_n\} \) and \( \varepsilon_n = 2^{-m_n} \). Then we have

\[
m_n > q \quad \text{and} \quad r'_n > \varepsilon_n \geq \frac{r'_n}{2}.
\]

(2)
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Define \( b \in s \) as follows:

\[
\begin{aligned}
  b_i &= a_i' \quad \text{if } i \neq m_n + 1, \\
  b_{m_n + 1} &= \begin{cases} 
    x_0 & \text{if } a_{m_n + 1}' \notin (x_0 - \frac{\alpha}{4}, x_0 + \frac{\alpha}{4}), \\
    x_0 + \frac{\alpha}{2} & \text{if } a_{m_n + 1}' \in (x_0 - \frac{\alpha}{4}, x_0 + \frac{\alpha}{4}).
  \end{cases}
\end{aligned}
\]

Then we get

\[
\frac{\varepsilon_n}{2} > g(a', b) = 2^{-m_n - 1} \cdot \frac{|a_{m_n + 1}' - b_{m_n + 1}|}{1 + |a_{m_n + 1}' - b_{m_n + 1}|} \geq 2^{-m_n - 1} \cdot \frac{\alpha}{4} = \frac{\alpha}{4 + \alpha} \cdot \frac{\varepsilon_n}{2},
\]

and, by (2), we have

\[
\frac{\varepsilon_n}{2} > g(a', b) \geq \frac{\alpha}{4 + \alpha} \cdot \frac{\varepsilon_n}{2} \geq \frac{\alpha}{4(4 + \alpha)} \cdot r'_n. \tag{3}
\]

Put \( \delta = \frac{\alpha}{4 + \alpha} \cdot g(a', b) \) and choose an arbitrary \( c \in B(b, \delta) \). Then we get

\[
\frac{\varepsilon_n}{2} \cdot \frac{|c_{m_n + 1} - b_{m_n + 1}|}{1 + |c_{m_n + 1} - b_{m_n + 1}|} \leq g(c, b) < \delta,
\]

thus in view of (3)

\[
|c_{m_n + 1} - b_{m_n + 1}| < \frac{2\delta}{\varepsilon_n} < \frac{2}{\varepsilon_n} \cdot \frac{\alpha}{2} \cdot \frac{\varepsilon_n}{4 + \alpha} = \frac{\alpha}{4},
\]

consequently, \( c_{m_n + 1} \in (x_0 - \frac{3\alpha}{4}, x_0 + \frac{3\alpha}{4}) \) (see the definition of \( b_{m_n + 1} \)).

Observe now that \( |f_0(c_{m_n + 1})| > 1 \geq \frac{1}{p} \), so

\[
c \in s \setminus A_{pq}(f_0). \tag{4}
\]

Using (3), we have \( \varepsilon_n - g(a', b) > \frac{\varepsilon_n}{2} > \frac{\alpha}{4 + \alpha} \cdot \frac{\varepsilon_n}{2} > \delta \), therefore \( B(b, \delta) \subset B(a', \varepsilon_n) \subset B(a', r'_n) \). In virtue of (4) and (1), there holds

\[
B(b, \delta) \subset B(a', r'_n) \setminus A_{pq}(f_0) \subset B(a, r_n) \setminus (F \cup A_{pq}(f_0)).
\]

It means that \( \gamma(a, r_n, F \cup A_{pq}(f_0)) \geq \delta \geq \left( \frac{\alpha}{4 + \alpha} \right)^2 \cdot \frac{r'_n}{4} > \left( \frac{\alpha}{4 + \alpha} \right)^2 \cdot \frac{\beta}{4} \cdot r_n, \)

thus

\[
\limsup_{r \to 0^+} \frac{\gamma(a, r, F \cup A_{pq}(f_0))}{r} \geq \left( \frac{\alpha}{4 + \alpha} \right)^2 \cdot \frac{\beta}{4} > 0.
\]

Therefore \( F \cup A_{pq}(f_0) \) is porous at \( a \). \( \square \)
THEOREM 2. Let $f : \mathbb{R} \to \mathbb{R}$ be a function for which there exists $x_0 \in \mathbb{R}$ such that
\[
\liminf_{x \to x_0} |f(x)| > 0.
\]

Then $A(f)$ is $\sigma$-superporous in $(s, g)$.

Proof. First consider $x_0 \in \mathbb{R}$. Then by (5), there exist $h > 0$ and $\alpha > 0$ such that
\[
|f(x)| \geq h
\]
for all $x \in (x_0 - 2\alpha, x_0 + 2\alpha)$. Let $a \in A(f)$. By (6), the interval $(x_0 - 2\alpha, x_0 + 2\alpha)$ contains only a finite number of terms of $a$. Thereby $a \in A(f_0)$, where $f_0$ is defined in Lemma 1. Hence $A(f) \subset A(f_0)$. It suffices to observe that $A(f_0) = \bigcap_{p, q} A_{pq}(f_0)$ and use Lemma 1.

If $x_0 = \pm \infty$, then, by (5), one can easily find $x'_0 \in \mathbb{R}$ and $\alpha > 0$ such that (6) is fulfilled for every $x \in (x'_0 - 2\alpha, x'_0 + 2\alpha)$, which converts this case to the previous one.

Remark 1. It is worth noticing which classes of functions fulfil (5). Some examples follow:

(i) Functions that are lower (upper) semicontinuous at an $x_0 \in \mathbb{R}$ such that $f(x_0) > 0$ ($f(x_0) < 0$). This can be inferred from the definition of semicontinuous functions and Theorem 2.

(ii) Nonvanishing functions with closed graph (in the product topology – cf. [3], [4]). To show this, recall that each function $f \in \mathcal{F}$ having closed graph is a Baire 1 function (cf. [3; Theorem 1']). Thus the set of its continuity points $C_f$ is dense in $\mathbb{R}$ ([5; p. 235]). It means that every $x \in \mathbb{R}$ is a limit of a sequence $x_i \in C_f$ ($i \in \mathbb{N}$). Thus, by [4; Theorem 1], $f(x_i) \to f(x)$ as $i \to \infty$. Consequently, $|f(x_0)| > 0$ for some $x_0 \in C_f$, since otherwise $f \equiv 0$. Hence, we have $\liminf_{x \to x_0} |f(x)| = |f(x_0)| > 0$.

(iii) Nonvanishing, monotone functions. That is clear from Theorem 2 since, if $f$ is nonvanishing and increasing (decreasing), then (5) holds for $x_0 = +\infty$ ($x_0 = -\infty$).

Properties of $\mathcal{U}$

Introduce an auxiliary set
\[
\mathcal{U}_0 = \{f \in \mathcal{F} ; \ f \text{ satisfies (5) for some } x_0 \in \mathbb{R}\}.
\]

We have
LEMMa 2. The set $U_0$ is dense and open in $(F,d)$, thus $F \setminus U_0$ is nowhere dense in $(F,d)$.

Proof. Choose $f \in U_0$. Then for some $x_0 \in \mathbb{R}$ there exists $h > 0$ and a neighbourhood $I$ of $x_0$ such that (6) holds for each $x \in I$. Put $\varepsilon_0 = \frac{h}{2}$. For every $g \in B(f,\varepsilon_0)$ we get that $\|g(x)\| \geq \|f(x)\| - \|f(x) - g(x)\| \geq h - \varepsilon_0 = \varepsilon_0 > 0$ for each $x \in I$. Consequently $g \in U_0$, thus $U_0$ is open in $F$.

To show the density of $U_0$ in $F$, choose $f \in F$ and $\varepsilon > 0$. Put $I = (0,1)$. Define $M = \{x \in \mathbb{R}; \text{ either } x \in X \setminus I, \text{ or } x \in I \text{ and } |f(x)| \geq \frac{\varepsilon}{4}\}$ and $M' = \mathbb{R} \setminus M$. Define a function $g = f \cdot \chi_M + \frac{\varepsilon}{4} \cdot \chi_{M'}$. Then $|f(x) - g(x)| = \left|f(x) - \frac{\varepsilon}{4}\right| \cdot \chi_{M'}(x) \leq \left(|f(x)| + \frac{\varepsilon}{4}\right) \cdot \chi_{M'}(x) \leq \frac{\varepsilon}{2}$ for all $x \in \mathbb{R}$. Further for $x \in I$ we have $|g(x)| = |f(x)| \cdot \chi_M(x) + \frac{\varepsilon}{4} \cdot \chi_{M'}(x) \geq \frac{\varepsilon}{4} > 0$. Accordingly $g \in U_0 \cap B(f,\varepsilon)$.

Since $(F,d)$ is a complete metric space, the following theorem is meaningful:

Theorem 3. The set $U$ is residual in $(F,d)$.

Proof. It is an easy consequence of Lemma 2 and the fact that $U_0 \subset U$ (see Theorem 2).

Remark 2. In connection with the inclusion $U_0 \subset U$ notice that $U_0 \neq U$. Indeed, we will show that $\chi_{\mathbb{R} \setminus \mathbb{Q}} \in U \setminus U_0$.

In favour of this, introduce the set $A_k(x) = \\{a_i \in s; a_k = x\}$ for every $k \in \mathbb{N}$, $x \in \mathbb{R}$. Choose $a \in A_k(x)$ ($k \in \mathbb{N}$, $x \in \mathbb{R}$) and a set $F \subset s$ which is porous at $a$. Then there exist $\beta > 0$, sequences $r_n, r'_n > 0$ and $a' \in s$ such that $r_n \searrow 0$, $\beta r_n < r'_n < r_n < 2^{-k+1}$ and

$$B(a',r'_n) \subset B(a,r_n) \setminus F.$$  \hfill(7)

Define the sequence $b = \{b_i\}_{i \in s}$ as follows:

$$b_i = a'_i \quad \text{if} \quad i \neq k,$$

$$b_k = \begin{cases} a'_k - \frac{2^{k-1}r'_n}{1 - 2^{k-1}r'_n} & \text{if } a'_k < x, \\ a'_k + \frac{2^{k-1}r'_n}{1 - 2^{k-1}r'_n} & \text{if } a'_k \geq x. \end{cases}$$

Put $\delta = \frac{r'_n}{2}$. Then $g(b,a') = \delta$, thus

$$B(b,\delta) \subset B(a',r'_n).$$  \hfill(8)
Furthermore, if \( c \in B(b, \delta) \), then \( r_n' > \rho(b, c) \geq 2^{-k} \cdot \frac{|b_k - c_k|}{1 + |b_k - c_k|} \), so

\[
|b_k - c_k| < \frac{2^{k-1}r_n'}{1 - 2^{k-1}r_n'}. 
\]

Therefore \( c_k \neq x \) since, according to the definition of \( b \), we have \( |b_k - x| \geq \frac{2^{k-1}r_n'}{1 - 2^{k-1}r_n'} \). In view of (7), (8), it means that

\[
B(b, \delta) \subset B(a', r_n') \setminus A_k(x) \subset B(a, r_n) \setminus (F \cup A_k(x)) .
\]

Consequently, we get

\[
\gamma(a, r_n, F \cup A_k(x)) \geq \delta > \frac{\beta}{2} \cdot r_n. 
\]

Hence

\[
\limsup_{r \to 0^+} \frac{\gamma(a, r, F \cup A_k(x))}{r} \geq \frac{\beta}{2} > 0. 
\]

So we have proved that \( A_k(x) \) is superporous at \( a \). It is now sufficient to observe that

\[
A(\chi_{\mathbb{R} \setminus Q}) \subset \bigcup_{k \in \mathbb{N}} A_k(p_n),
\]

where \( Q = \{p_1, \ldots, p_n, \ldots\} \).

In virtue of Lemma 2, the set \( U \cup U_0 \) is dense in \( F \) and, evidently, \( U \neq F \). Consequently, \( U \) is not closed in \( F \), hence neither is a complete subspace of \( (F, d) \). Nevertheless, we have:

**Theorem 4.** The space \((U, d)\) is a Baire space.

**Proof.** By Lemma 2, \( U_0 \) is open in the complete metric space \((F, d)\), thus \((U_0, d)\) is a Baire space ([1; Proposition 1.14]). Furthermore, \( U_0 \) is dense in \( U \) (see Lemma 2), hence \((U, d)\) is a Baire space as well ([1; Proposition 1.15]).

**Theorem 5.** Each point of \( U \) (\( F \setminus U \)) is a point of condensation of \( U \) (\( F \setminus U \)).

**Proof.** Let \( 0 < \varepsilon < 1 \). One can find a nonvanishing function \( f \in U \) (\( f \in F \setminus U \)). Then \( f(x_0) \neq 0 \) for some \( x_0 \in \mathbb{R} \). Define \( f_c = f + cf(x_0) \cdot \chi_{\{x_0\}} \) for each \( c > 0 \). We have \( A(f) = A(f_c) \) (\( c > 0 \)). Now, it is easy to check that \( f_c \in B(f, \varepsilon) \cap U \) (\( f_c \in B(f, \varepsilon) \cap (F \setminus U) \)), \( f_c \neq f \) for every \( 0 < c < \frac{\varepsilon}{|f(x_0)|} \).

**Remark 3.** In the light of Theorems 3–5, the set \( U_0 = \bigcup_{f \in U_0} A(f) \) would be worth studying. What we know is that \( U_0 = \bigcup_{f \in U_0} A(f) \) is \( \sigma \)-superporous in \( s \). To show this enumerate intervals with rational endpoints as \( I_1, I_2, \ldots \), further denote the midpoint of \( I_n \) by \( q_n \) (\( n \in \mathbb{N} \)). Define the functions \( f_n(x) = (1 - |q_n - x|) \cdot \chi_{I_n}(x) \) for \( x \in \mathbb{R} \). Now it suffices to notice that \( f_n \in U_0 \) (\( n \in \mathbb{N} \)) and \( U_0 = \bigcup_{n=1}^{\infty} A(f_n) \).
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