ON SUBSPACES OF MEASURABLE REAL FUNCTIONS

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Abstract. Let (X, S, μ) be a measure space. Let $\Phi : \mathbb{R} \to \mathbb{R}$ be a continuous function. Topological properties of the space of all measurable real functions f such that $\Phi \circ f$ is Lebesgue-integrable are investigated in the space of measurable real functions endowed with the topology of convergence in measure.

Introduction

Let (X, S, μ) be a measure space. Denote by $\mathcal{M}$ the space of all measurable real functions on X. As usual the symbol $L_p(\mu)$ stands for the set of all functions $f \in \mathcal{M}$ for which $\int_X |f|^p d\mu < +\infty$ $(p \geq 1)$.

It is shown in [4] that the Riemann-integrable functions on the interval $[a, b]$ $(a, b \in \mathbb{R})$ constitute a meager set in the space of all Lebesgue-integrable functions on $[a, b]$ furnished with the topology of mean convergence. Then a natural question arises to establish the largeness of Lebesgue-integrable functions, or more generally of L_p spaces in the space $\mathcal{M}$ with an appropriate topology.

Making allowance for this we could pursue the analogy further by examining the class $A(\Phi)$ of all mesurable real functions f such that $\Phi \circ f$ is Lebesque-integrable, where $\Phi : \mathbb{R} \to \mathbb{R}$ is an arbitrary but fixed continuous function.

In favour of this we need a proper topology on $\mathcal{M}$. Let $E(f, g; r) = \{x \in X; |f(x) - g(x)| > r\}$, where $f, g \in \mathcal{M}, r > 0$. Define the pseudo-metric ϱ on $\mathcal{M}$ as follows ([1]):

$$\varrho(f, g) = \inf\{r > 0; \mu(E(f, g; r)) \leq r\} \quad (f, g \in \mathcal{M}).$$

Given $f_n, f \in \mathcal{M}$ $(n \in \mathbb{N})$ we say that f_n converges in measure to f if, for each $r > 0$ $\lim\limits_{n\to\infty} \mu(E(f_n, f; r)) = 0$.

It is known that the ϱ-convergence is equivalent to the convergence in measure, further $(\mathcal{M}, \varrho)$ is a complete pseudo-metric space ([1],p.80).

Define the following sets:

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$$A_\alpha(\Phi) = \{f \in \mathcal{M}; \int_X |\Phi \circ f| d\mu \leq \alpha\} \ (\alpha \geq 0),$$
$$A(\Phi) = \{f \in \mathcal{M}; \int_X |\Phi \circ f| d\mu < +\infty\},$$
where $\Phi : \mathbb{R} \to \mathbb{R}$ is an arbitrary but fixed continuous function.

The symbol χ_A stands for the characteristic function of $A \subset X$.

Main Results

First we point out to which Borel class $A_\alpha(\Phi)$ and $A(\Phi)$, respectively belong $(\alpha \geq 0)$. We have

Theorem 1. *The set $A_\alpha(\Phi)$ is closed in $(\mathcal{M}, \varrho)$ for all $\alpha \geq 0$.*

Proof. Let $f \in \mathcal{M}, f_n \in A_\alpha(\Phi)$ and $\varrho(f_n, f) \to 0$ $(n \to \infty)$. Then by a well-known theorem of Riesz there exists a subsequence $\{f_{n_k}\}_{k=1}^\infty$ of $\{f_n\}_{n=1}^\infty$ converging a.e. on X to f. Consequently $|\Phi \circ f_{n_k}| \to |\Phi \circ f|$ a.e. on X, thus in view of the Fatou Lemma
$$\int_X |\Phi \circ f| d\mu = \int_X (\lim_{k\to\infty} |\Phi \circ f_{n_k}|) d\mu \leq \liminf_{k\to\infty} \int_X |\Phi \circ f_{n_k}| d\mu \leq \alpha,$$
so $f \in A_\alpha(\Phi)$. $\square$

Corollary 1. *The set $A(\Phi)$ is an F_σ-subset of $(\mathcal{M}, \varrho)$.*

Proof. It follows from Theorem 1, since $A(\Phi) = \cup_{n=1}^\infty A_n(\Phi)$. $\square$

Remark 1. In the sequel we will use the fact that $A(\Phi)$ *is meager in* $(\mathcal{M}, \varrho)$ *if and only if* $\mathcal{M} \setminus A_\alpha(\Phi)$ *is dense in* $\mathcal{M}$ *for all* $\alpha > 0$. Indeed, the sufficiency follows from Theorem 1 (resp. Corollary 1). Conversely, $(\mathcal{M}, \varrho)$ is a complete pseudo-metric space and therefore a Baire space as well (cf.[3],p.19), i.e. every nonempty open subset of $\mathcal{M}$ is nonmeager in $(\mathcal{M}, \varrho)$. $\square$

Now we are prepared to determine the category of $A(\Phi)$ in $\mathcal{M}$.

Theorem 2. *Suppose that*

(1) *for each $\varepsilon > 0$ there exists $E \in S$ such that $0 < \mu(E) < \varepsilon$.*

Let Φ be unbounded. Then $A(\Phi)$ is meager in $(\mathcal{M}, \varrho)$.

Proof. Let $f \in A_\alpha(\Phi)$ (where $\alpha > 0$), $\varepsilon > 0$, further $0 < \mu(E) < \varepsilon$ for some $E \in S$. Choose $t_0 \in \mathbb{R}$ such that
$$|\Phi(t_0)| > \frac{1}{\mu(E)}(\alpha - \int_{X\setminus E} |\Phi \circ f| d\mu).$$

Then for $g = f \cdot \chi_{X\setminus E} + t_0 \cdot \chi_E \in \mathcal{M}$ we have
$$\int_X |\Phi \circ g| d\mu = \int_{X\setminus E} |\Phi \circ f| d\mu + |\Phi(t_0)|\mu(E) > \alpha, \text{ thus } g \in \mathcal{M} \setminus A_\alpha(\Phi).$$

On the other hand $E(f, g; \varepsilon) \subset E$, so $\varrho(f, g) < \varepsilon$ (see Remark 1). $\square$

Theorem 3. *Let (X, S, μ) be a non-σ-finite measure space. Suppose that either Φ is bounded or (1) does not hold.*

Then $A(\Phi)$ is meager in $(\mathcal{M}, \varrho)$ if and only if $|\Phi|^{-1}(0, +\infty) = \{t \in \mathbb{R}; |\Phi(t)| > 0\}$ is dense in $\mathbb{R}$.

Proof. Suppose that $|\Phi|^{-1}(0, +\infty)$ is dense in $\mathbb{R}$. Let $\alpha > 0$ and $f \in A_\alpha(\Phi)$. Then f can be considered as a uniform limit of a sequence of elementary measurable functions ([2], p.86). Hence we can find an elementary measurable function $g = \sum_{n=1}^{\infty} a_n \chi_{E_n}$ (with $X = \bigcup_{n=1}^{\infty} E_n$) in every ε-neighbourhood of f in $(\mathcal{M}, \varrho)$ $(\varepsilon > 0)$ such that $\Phi(a_n) \neq 0$ for all $n \in \mathbb{N}$.

Since (X, S, μ) is not σ-finite we can find $m \in \mathbb{N}$ for which $\mu(E_m) = +\infty$. It follows that

$$\int_X |\Phi \circ g| d\mu \geq \int_{E_m} |\Phi \circ g| d\mu = |\Phi(a_m)| \mu(E_m) = +\infty,$$

hence $g \in \mathcal{M} \setminus A_\alpha(\Phi)$. Further see Remark 1.

Conversely, suppose that there exist $\delta > 0, t \in \mathbb{R}$ such that $\Phi(t') \equiv 0$, for every $t' \in I = (t - \delta, t + \delta)$. Define $f(x) \equiv t$, which is evidently in $A(\Phi)$. Choose an arbitrary $g \in \mathcal{M}$ from the δ-neighbourhood of f. Then we can find $0 < r_0 < \delta$ such that $E = E(f, g; r_0)$ is of measure less than δ. Then $t - r_0 \leq g(x) \leq t + r_0$, consequently $g(x) \in I$, thus

$$\int_X |\Phi \circ g| d\mu = \int_{X \setminus E} |\Phi \circ g| d\mu + \int_E |\Phi \circ g| d\mu = \int_E |\Phi \circ g| d\mu = a. \tag{2}$$

If (1) does not hold then $a = 0$ for a suitably small δ, further if Φ is bounded then $a \leq K\mu(E) \leq K r_0 < +\infty$ for some $K > 0$. It is now clear from (2) that under our assumptions $\int_X |\Phi \circ g| d\mu < +\infty$, so $g \in A(\Phi)$. Accordingly $A(\Phi)$ contains a nonempty open ball. □

Before we state the appropriate theorem for σ-finite spaces define the function

$$\phi(c, \varepsilon) = \max_{t \in [c - \varepsilon, c + \varepsilon]} |\Phi(t)|, \text{ where } c \in \mathbb{R}, \varepsilon > 0.$$

Theorem 4. *Let (X, S, μ) be a σ-finite measure space and $\{X_n\}_{n=1}^{\infty}$ be a measurable decomposition of X with $\mu(X_n) < +\infty$. Suppose that either Φ is bounded or (1) does not hold. Then $A(\Phi)$ is meager in $(\mathcal{M}, \varrho)$ if and only if*

$$\forall \varepsilon > 0 \ \forall c_n \in \mathbb{R} \ (n \in \mathbb{N}) : \ \sum_{n=1}^{\infty} \mu(X_n) \cdot \phi(c_n, \varepsilon) = +\infty. \tag{3}$$

Proof. First suppose that (3) holds. Choose arbitrary $\alpha \geq 0, \varepsilon > 0$ and $f \in A_\alpha(\Phi)$.

Examine f on the finite measure space $(X_n, S|_{X_n}, \mu|_{X_n})$ $(n \in \mathbb{N})$. There exists a sequence of simple measurable functions which converges a.e. to f on X_n, further the convergence a.e. implies convergence in measure on finite measure spaces ([1],p.78). It means that for every $n \in \mathbb{N}$ there exists a simple measurable function $g_n = \sum_{i=1}^{k(n)} c_{n,i} \chi_{X_{n,i}}$ (where $k(n) \in \mathbb{N}, c_{n,i} \in \mathbb{R}, X_{n,i} \in S|_{X_n}$) such that $\mu(X_n \cap E(f, g_n; \frac{\varepsilon}{2})) \leq \frac{\varepsilon}{2^{n+1}}$.

Define the function $g = \sum_{n=1}^{\infty} g_n \in \mathcal{M}$. We have

$$\mu(E(f,g;\frac{\varepsilon}{2})) = \sum_{n=1}^{\infty} \mu(X_n \cap E(f,g_n;\frac{\varepsilon}{2})) \leq$$
$$\leq \sum_{n=1}^{\infty} \frac{\varepsilon}{2^{n+1}} = \frac{\varepsilon}{2}, \text{ so } \varrho(f,g) \leq \frac{\varepsilon}{2}.$$

For every $n \in \mathbb{N}$ let c_n be that of the numbers $c_{n,1}, \ldots, c_{n,k(n)}$ for which $\phi(c_{n,i}, \frac{\varepsilon}{2})$ is the least $(1 \leq i \leq k(n))$. Choose $d_{n,i} \in [c_{n,i} - \frac{\varepsilon}{2}, c_{n,i} + \frac{\varepsilon}{2}]$ such that $|\Phi(d_{n,i})| = \phi(c_{n,i}, \frac{\varepsilon}{2})$ and put $h = \sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} d_{n,i} \chi_{X_{n,i}} \in \mathcal{M}$. Then $\varrho(h,g) \leq \frac{\varepsilon}{2}$, thereby $\varrho(f,h) \leq \varrho(f,g) + \varrho(g,h) \leq \varepsilon$.

On the other hand from (3) we have

$$\int_X |\Phi \circ h| d\mu = \sum_{n=1}^{\infty} \int_{X_n} |\Phi \circ h| d\mu = \sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} \phi(c_{n,i}, \frac{\varepsilon}{2}) \cdot \mu(X_{n,i}) \geq$$
$$\geq \sum_{n=1}^{\infty} \sum_{i=1}^{k(n)} \phi(c_n, \frac{\varepsilon}{2}) \cdot \mu(X_{n,i}) = \sum_{n=1}^{\infty} \phi(c_n, \frac{\varepsilon}{2}) \cdot (\sum_{i=1}^{k(n)} \mu(X_{n,i})) =$$
$$= \sum_{n=1}^{\infty} \phi(c_n, \frac{\varepsilon}{2}) \cdot \mu(X_n) = +\infty.$$

It means that $h \in \mathcal{M} \setminus A_\alpha(\Phi)$ (see Remark 1).

Conversely, if contrary to (3) $\sum_{n=1}^{\infty} \phi(c_n, \varepsilon_0) \leq \alpha$ for some $\alpha, \varepsilon_0 > 0$ and $c_n \in \mathbb{R}$ $(n \in \mathbb{N})$, then $f = \sum_{n=1}^{\infty} c_n \cdot \chi_{X_n} \in A_\alpha(\Phi)$. Choose $g \in \mathcal{M}$ such that $\varrho(f,g) < \delta$ $(0 < \delta < \varepsilon_0)$. One can find an $0 < r_0 < \delta$, for which the measure of $E = E(f,g;r_0)$ is less than δ.

We have

$$\int_X |\Phi \circ g| d\mu = (\sum_{n=1}^{\infty} \int_{X_n \setminus E} |\Phi \circ g| d\mu) + \int_E |\Phi \circ g| d\mu \leq$$
$$\leq (\sum_{n=1}^{\infty} \int_{X_n \setminus E} \phi(c_n, \delta) d\mu) + \int_E |\Phi \circ g| d\mu \leq (\sum_{n=1}^{\infty} \phi(c_n, \varepsilon_0) \cdot \mu(X_n)) +$$
$$+ \int_E |\Phi \circ g| d\mu \leq \alpha + \int_E |\Phi \circ g| d\mu.$$

Reasoning analoguous to that of at the end of the proof of Theorem 3 works. $\square$

Remark 2. Observe that Theorems 2-4 determine the category of $A(\Phi)$ in $(\mathcal{M}, \varrho)$ for every continuous Φ and measure space (X, S, μ), respectively. However some of these theorems overlap, e.g. in one direction Theorem 3 holds for σ-finite measure spaces as well (the necessity of the density of $|\Phi|^{-1}(0, +\infty)$ for $A(\Phi)$ being meager), but in reverse it is false.

Indeed, let (X, S, μ) be an arbitrary σ-finite measure space. Let $\{X_n\}_{n=1}^{\infty}$ be a measurable decomposition of X such that $\mu(X_n) < +\infty$ for all $n \in \mathbb{N}$. Define the sequence $r_0 = 1, r_n = \frac{1}{2} \min\{r_{n-1}, \frac{1}{2^n \cdot \mu(X_n)}\}$ if $\mu(X_n) > 0$ and $r_n = \frac{1}{2} r_{n-1}$ if $\mu(X_n) = 0$ $(n \in \mathbb{N})$. Let

$$\Phi(t) = \begin{cases} 1, & \text{for } t \le 0 \\ r_n, & \text{for } t = n \ (n \in \mathbb{N}) \\ \text{linear}, & \text{elsewhere.} \end{cases}$$

Then Φ is a nonincreasing, positive, bounded continuous function.

On the other hand setting $c_n = \frac{2n+1}{2}$ $(n \in \mathbb{N})$ we get $\phi(c_n, \frac{1}{2}) = r_n$, thus $\sum_{n=1}^{\infty} \phi(c_n, \frac{1}{2}) \cdot \mu(X_n) \le \sum_{n=1}^{\infty} \frac{1}{2^n} = 2$. Consequently by Theorem 4 $A(\Phi)$ is nonmeager in $(\mathcal{M}, \varrho)$. $\square$

Corollary 2. *Let $p \ge 1$. Then $L_p(\mu)$ is nonmeager in $(\mathcal{M}, \varrho)$ if and only if μ is finite and bounded away from zero.*

Proof. Suppose that μ is not bounded away from zero (i.e. (1) holds). Since the function $\Phi(t) = |t|^p$ $(p > 0)$ is continuous and unbounded Theorem 2 yields the desired result at once.

Assume now the converse of (1) and consider a non-σ-finite measure space (X, S, μ). Then $L_p(\mu)$ is meager in $(\mathcal{M}, \varrho)$ by Theorem 3.

Suppose further that (X, S, μ) is σ-finite. Let $\{X_n\}_{n=1}^{\infty}$ be a measurable decomposition of X with $\mu(X_n) < +\infty$ $(n \in \mathbb{N})$. It is easy to check that $\phi_(c, \varepsilon) \ge \varepsilon^p$ for all $\varepsilon > 0$ and $c \in \mathbb{R}$

Consequently we get for every $c_n \in \mathbb{R}$ $(n \in \mathbb{N})$ that

$$\sum_{n=1}^{\infty} \mu(X_n) \cdot \phi(c_n, \varepsilon) \ge \sum_{n=1}^{\infty} \mu(X_n) \cdot \varepsilon^p = \varepsilon^p \cdot \mu(X) = +\infty,$$

provided $\mu(X) = +\infty$. Then in virtue of Theorem 4 $L_p(\mu)$ is meager in $\mathcal{M}$.

Finally if (X, S, μ) is a finite measure space then putting $c_n = 0$ for all $n \in \mathbb{N}$ and $\varepsilon = 1$ we can see that (3) is not fulfilled, thus Theorem 4 completes the proof. $\square$

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