ON NORMABILITY OF A SPACE OF MEASURABLE REAL FUNCTIONS

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Abstract. Let (X, S, μ) be a σ-finite measure space. Denote by $\mathcal{M}$ the class of all S-measurable functions that are finite almost everywhere on X. Gribanov in [G] considers the topology of convergence in measure on sets of finite measure on $\mathcal{M}$. This topological space is normable if and only if X is a union of finite many atoms of finite measure.

Introduction

The metric space (s, ρ) of all real sequences endowed with the Fréchet metric

$$\rho(a, b) = \sum_i 2^{-i} \frac{|a_i - b_i|}{1 + |a_i - b_i|}, \text{ where } a = \{a_i\}_i, b = \{b_i\}_i \in s$$

has been thoroughly investigated by several authors for it offers a convenient background for studying diverse properties of real sequences (cf.[KŠ],[EŠ],[N],[TZs]). The space (s, ρ) has hoverwer also an unfavourable property, namely it is non-normable (see [ ]).

Gribanov considers in [G] the following generalization of (s, ρ): Let (X, S, μ) be a σ-finite measure space such that $\mu(X) > 0$. Then $X = \cup_i X_i$ for some sequence $\{X_i\}_i$ of pairwise disjoint S-measurable sets of positive finite measure. Denote by $\mathcal{M}$ the class of all S-measurable functions that are finite almost everywhere on X. We will identify members of $\mathcal{M}$ if they equal a.e. on X. Put

$$d(f, g) = \sum_i \frac{1}{2^i \mu(X_i)} \int_{X_i} \frac{|f - g|}{1 + |f - g|} d\mu$$

for all $f, g \in \mathcal{M}$.

Observe that this construction yields a generalization of (s, ρ) indeed, since if we take for X the set of all natural numbers $\mathbb{N}$, for S the potential set of $\mathbb{N}$, for μ the counting measure on $\mathbb{N}$ (i.e. $\mu(A)$=card(A) for $A \subset \mathbb{N}$ finite and $\mu(A) = +\infty$ for A infinite) and put $X_i = \{i\}$ for all $i \in \mathbb{N}$, then $\mathcal{M}$ reduces to s and d to ρ, respectively.

It can be shown that several properties of (s, ρ) hold for $(\mathcal{M}, d)$ as well, e.g. $(\mathcal{M}, d)$ is a complete metric space ([G],Theorem 2). It is the purpose of this paper to characterize σ-finite measure spaces (X, S, μ) so as $(\mathcal{M}, d)$ be normable.

Main Results

Throughout this section (X, S, μ) will be a σ-finite (possibly finite) measure space. The symbol χ_A will stand for the characteristic function of the set $A \subset X$.

First we establish how d-convergence (denoted $f_n \xrightarrow{d} f$) and convergence in measure (denoted $f_n \xrightarrow{\mu} f$) of measurable functions interact and when they coincide:

Proposition 1. *Let $f, f_n \in \mathcal{M}$ $(n \in \mathbb{N})$. The following are equivalent:*

(i) $f_n \xrightarrow{d} f$*;*
(ii) $f_n \xrightarrow{\mu} f$ *on every S-measurable set of finite measure;*
(iii) $f_n \xrightarrow{\mu} f$ *on X_i for all $i \in \mathbb{N}$.*

Proposition 2. *We have*

(i) *given $f, f_n \in \mathcal{M}(n \in \mathbb{N})$, $f_n \xrightarrow{\mu} f$ implies $f_n \xrightarrow{d} f$;*
(ii) *given $f, f_n \in \mathcal{M}(n \in \mathbb{N})$, $f_n \xrightarrow{d} f$ implies $f_n \xrightarrow{\mu} f$ if and only if μ is finite.*

Now we turn to investigating the normability of $(\mathcal{M}, d)$.

Lemma. *Suppose there exists a norm $\|\cdot\|$ on $\mathcal{M}$ such that given $f, f_n \in \mathcal{M}$ $(n \in \mathbb{N})$, $f_n \xrightarrow{d} f$ implies $f_n \xrightarrow{\|\cdot\|} f$.*

Then there exists a $\gamma > 0$ and an $n \in \mathbb{N}$ such that for all $f \in \mathcal{M}$ with unit norm

$$(*) \qquad \int_{\bigcup_1^n X_i} |f| d\mu \geq \gamma.$$

Proof. Assume the contrary. Then for each $n \in \mathbb{N}$ there is an $f_n \in \mathcal{M}$ of unit norm such that

$$\frac{1}{n} > \int_{\bigcup_1^n} |f_n| d\mu \geq \int_{X_i} |f_n| d\mu \text{ for all } 1 \leq i \leq n,$$

so $\{f_n\}_n$ converges in mean to $f_0 = 0$ on X_i for all $i \in \mathbb{N}$, which implies its convergence in measure to f_0 on every X_i ([H]), hence by Proposition 1(iii) $f_n \xrightarrow{d} f_0$.

On the other hand $f_n \overset{\|\cdot\|}{\nrightarrow} f_0$, since $\|f_n\| = 1$ for all $n \in \mathbb{N}$ and $\|f_0\| = 0$. □

Theorem. *The following are equivalent:*

(i) *$(\mathcal{M}, d)$ is normable;*
(ii) *X is a union of finite many atoms of finite measure.*

Proof. Suppose (ii). Then the functions from $\mathcal{M}$ have finite range since measurable functions are constant on atoms. It means that $\mathcal{M}$ coincides with the integrable functions on X, which are normable by the norm

$$\|f\| = \int_X |f| d\mu.$$

This norm however generates the topology of $(\mathcal{M}, d)$, since by [T], Theorem 3 (i)⇒(ii), in our case convergence in mean is equivalent to convergence in measure which is in turn equivalent to d-convergence by Proposition 2.

Conversely, suppose there exists a norm $\|\cdot\|$ on $\mathcal{M}$ which generates the topology of $(\mathcal{M}, d)$ and (ii) fails to hold. Then X is decomposable into a denumerable sequence $\{X_n\}_{n=1}^{\infty}$ of sets of finite positive measure. Further given $f, f_n \in \mathcal{M}$ ($n \in \mathbb{N}$), $f_n \xrightarrow{d} f$ implies $f_n \xrightarrow{\|\cdot\|} f$.

Then in view of the Lemma there exists a $\gamma > 0$ and an $n \in \mathbb{N}$ such that (*) holds for every $f \in \mathcal{M}$ of unit norm. The function $g = \chi_{X \setminus \bigcup_{i=1}^{n} X_i}$ is nonvanishing, consequently $f = \frac{g}{\|g\|} \in \mathcal{M}$ and clearly $\|f\| = 1$. It means by (*) that $0 = \int_{\bigcup_{i=1}^{n} X_i} |g| d\mu \geq \gamma \|g\|$, hence $g \equiv 0$ which is a contradiction. $\square$

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