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On hereditary Baireness of the Vietoris topology

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Abstract

It is shown that a metrizable space X, with completely metrizable separable closed subspaces, has a hereditarily Baire hyperspace $K(X)$ of nonempty compact subsets of X endowed with the Vietoris topology τ^v. In particular, making use of a construction of Saint Raymond, we show in ZFC that there exists a non-completely metrizable, metrizable space X with hereditarily Baire hyperspace $(K(X), \tau^v)$; thus settling a problem of Bouziad. Hereditary Baireness of $(K(X), \tau^v)$ for a Moore space X is also characterized in terms of an auxiliary product space and the strong Choquet game. Finally, using a result of Kunen, a non-consonant metrizable space having completely metrizable separable closed subspaces is constructed under CH. © 2001 Elsevier Science B.V. All rights reserved.

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0. Introduction

A topological space (X, τ) is said to be *consonant* if the upper Kuratowski topology on the hyperspace of closed subsets of X coincides with the cocompact topology. Consonance was introduced by Dolecki, Greco and Lechicki in [10,11] and has been subsequently studied by several authors (see, e.g., [1,3–5,24]). It has been established that Čech-complete spaces, in particular, completely metrizable spaces, are consonant [11]. On

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the other hand, a completely regular 1st countable consonant space is a Prohorov space (see [5]) and hence is hereditarily Baire.

An interesting problem in this respect, posed by Nogura and Shakhmatov [24, Problem 11.4], is to find a non-completely metrizable, metrizable consonant space. It is not possible in the realm of separable co-analytic spaces [5], and the answer is independent within the analytic spaces [4].

It is also known (see [3]) that the hyperspace $K(X)$ of all nonempty compact subsets of a metrizable consonant space X endowed with the Vietoris topology τ^v, is hereditarily Baire. If we compare this result with the above mentioned problem of Nogura and Shakhmatov, it is natural to consider the following question of Bouziad: *does there exist a ZFC example of a non-completely metrizable, metrizable space X such that the hyperspace $(K(X), \tau^v)$ is hereditarily Baire*?

It is one of the purposes of this paper to affirmatively answer this question (see Theorem 4.8), making use of a ZFC construction of Saint Raymond [26] of a non-completely metrizable, metrizable space, each separable closed subspace of which is completely metrizable. In fact, we show that all metrizable spaces having completely metrizable separable closed subspaces have hereditarily Baire hyperspaces (cf. Corollary 4.7).

In the light of these results another natural question arises: *does there exist a non-consonant metrizable space with completely metrizable separable closed subspaces*? The answer is yes under CH, as it is demonstrated in Theorem 5.2 using a result of K. Kunen. A natural candidate for a ZFC solution of this problem would be the above mentioned space of Saint Raymond; on the other hand, this space is a non-separable hereditarily Baire space, which is neither analytic nor co-analytic (see Remark 4.9), hence it is also a good candidate for a ZFC solution of the Nogura–Shakhmatov problem mentioned above.

Baireness of $(K(X), \tau^v)$ was first studied in [20] using the Banach–Mazur game (see [25] or [15]). This method was then generalized in [29,30] to get results concerning Baireness of various hypertopologies. Another topological game, the so-called strong Choquet game (see [7] or [15]), was then employed in [31,32] to characterize complete metrizability of hypertopologies. Note that complete metrizability of $(K(X), \tau^v)$ is equivalent to complete metrizability of X (since, for metrizable X, the Vietoris topology on $K(X)$ coincides with the Hausdorff metric topology on $K(X)$, see [17]); however, hereditary Baireness of X is only necessary, not sufficient for hereditary Baireness of $K(X)$ (see Remark 4.2). Results of Section 4 shed more light on hereditary Baireness of $(K(X), \tau^v)$ through results of Debs [8] (see also Telgársky's paper [27]) concerning a characterization of hereditary Baireness using the strong Choquet game.

1. Preliminaries

Throughout the paper (X, τ) is a Hausdorff space and $K(X)$ the set of nonempty compact subsets of X. Denote by ω the nonnegative integers. The Vietoris topology τ^v on $K(X)$ (cf. [21]) has as a base sets of the form

$$\langle U_0,\ldots,U_n\rangle = \left\{A\in K(X)\colon\ A\subset \bigcup_{k\leqslant n} U_k \text{ and } A\cap U_k\neq\emptyset \text{ for all } k\leqslant n\right\},$$

where $U_0,\ldots,U_n\in\tau$ and $n\in\omega$; denote by $\mathcal{B}^v$ this canonical base. Given $\boldsymbol{x}\in X^\omega$, denote by $\overline{\boldsymbol{x}}$ the τ-closure of the range of $\boldsymbol{x}$ in X. In general, $\overline{A}$ will stand for the τ-closure of $A\subset X$.

Put $X^\star=\{\boldsymbol{x}\in X^\omega\colon\ \overline{\boldsymbol{x}}\in K(X)\}$, and for $U_0,\ldots,U_n\subset X$ write

$$\langle U_0,\ldots,U_n\rangle^\star = \left\{\boldsymbol{x}\in X^\star\colon\ \overline{\boldsymbol{x}}\subset \bigcup_{k\leqslant n} U_k \text{ and } \boldsymbol{x}(k)\in U_k \text{ for all } k\leqslant n\right\}.$$

Then the family $\mathcal{B}^\star$ of the sets $\langle U_0,\ldots,U_n\rangle^\star$ with $U_0,\ldots,U_n\in\tau$, forms a base for a topology $\tau^\star$ on $X^\star$. Observe, that $\tau^\star$ is finer than the relative Tychonoff product topology on $X^\star$; thus, $(X^\star,\tau^\star)$ is a Hausdorff space.

Recall, that (X,τ) is a *developable* space, provided it has a countable development, i.e., a sequence $\{\mathcal{G}_n\}_n$ of open covers of X such that for each $x\in X$ and $U\in\tau, x\in U$, $\mathrm{St}(x,\mathcal{G}_n)=\bigcup\{G\in\mathcal{G}_n\colon\ x\in G\}\subset U$ for some $n\in\omega$. A regular, developable space is a *Moore space*.

Lemma 1.1. *If (X,τ) is metrizable, then $(X^\star,\tau^\star)$ is a Moore space.*

Proof. Let d be a compatible metric on X, and H the corresponding Hausdorff metric on $K(X)$ (which is compatible with the Vietoris topology on $K(X)$ [21]). The symbol $B_d(x,n)$ (respectively $B_H(K,n)$) will stand for the open d-ball about $x\in X$ (respectively open H-ball about $K\in K(X)$) of radius $1/n$. For every $\boldsymbol{x}\in X^\star$ and $n>0$, put $t=t(\boldsymbol{x},n)=\min\{k\geqslant n\colon\ \overline{\boldsymbol{x}}\subset\bigcup_{i\leqslant k}B_d(\boldsymbol{x}(i),3n)\}$ and define

$$B^\star(\boldsymbol{x},n)=\bigl\{\boldsymbol{y}\in X^\star\colon\ \forall i\leqslant t(\boldsymbol{x},n),\ d\bigl(\boldsymbol{y}(i),\boldsymbol{x}(i)\bigr)<1/n \text{ and } H(\overline{\boldsymbol{x}},\overline{\boldsymbol{y}})<1/n\bigr\}.$$

Then $B^\star(\boldsymbol{x},n)$ is $\tau^\star$-open, since $\boldsymbol{x}\in\langle B_d(\boldsymbol{x}(0),3n),\ldots,B_d(\boldsymbol{x}(t),3n)\rangle^\star\subset B^\star(\boldsymbol{x},n)$. For all $n>0$, denote $\mathcal{G}_n=\{B^\star(\boldsymbol{x},n)\colon\ \boldsymbol{x}\in X^\star\}$ and fix some $\boldsymbol{x}\in X^\star$. Consider a $\tau^\star$-neighborhood $\boldsymbol{U}^\star=\langle U_0,\ldots,U_m\rangle^\star$ of $\boldsymbol{x}$ and find an $n_0>0$ so that for each $i\leqslant m$, $B_d(\boldsymbol{x}(i),n_0)\subset U_i$. Then $\overline{\boldsymbol{x}}\in\langle U_0,\ldots,U_m\rangle=\boldsymbol{U}$ and $\mathrm{St}(\overline{\boldsymbol{x}},\mathcal{H}_n)\subset\boldsymbol{U}$ for some $n\geqslant n_0$, where $\mathcal{H}_n=\{B_H(K,n)\colon\ K\in K(X)\}$.

We will show that $\mathrm{St}(\boldsymbol{x},\mathcal{G}_{3(n+m)})\subset\boldsymbol{U}^\star$: if $\boldsymbol{y}\in\mathrm{St}(\boldsymbol{x},\mathcal{G}_{3(n+m)})$ then there exists some $\boldsymbol{x}'\in X^\star$ such that $\boldsymbol{x},\boldsymbol{y}\in B^\star(\boldsymbol{x}',3(n+m))$. It means that $\boldsymbol{x}(i),\boldsymbol{y}(i)\in B_d(\boldsymbol{x}'(i),3(n+m))$ for each $i\leqslant t(\boldsymbol{x}',3(n+m))$ and $\overline{\boldsymbol{x}},\overline{\boldsymbol{y}}\in B_H(\overline{\boldsymbol{x}'},3(n+m))\subset B_H(\overline{\boldsymbol{x}'},n)$; thus, $d(\boldsymbol{x}(i),\boldsymbol{y}(i))<2/(3(n+m))<1/n$ for all $i\leqslant t(\boldsymbol{x}',3(n+m))$, so $\boldsymbol{y}(i)\in B_d(\boldsymbol{x}(i),n)\subset U_i$ for each $i\leqslant m$ (since $m\leqslant t(\boldsymbol{x}',3(n+m))$). Furthermore, $\overline{\boldsymbol{y}}\in\mathrm{St}(\overline{\boldsymbol{x}},\mathcal{H}_n)\subset\boldsymbol{U}$, so $\overline{\boldsymbol{y}}\subset\bigcup_{i\leqslant m}U_i$ and hence $\boldsymbol{y}\in\boldsymbol{U}^\star$. This proves that $(X^\star,\tau^\star)$ is developable.

As for regularity of $(X^\star,\tau^\star)$, observe that $\langle\overline{U_0},\ldots,\overline{U_n}\rangle^\star$ is the $\tau^\star$-closure of $\langle U_0,\ldots,U_n\rangle^\star\in\mathcal{B}^\star$. $\square$

2. Games

For details of the following exposition of games we refer the reader to [9], where the authors consider a so-called transitive game G played by two players α and β on the *domain* $D = \mathrm{Dom}(G)$ equipped with two transitive relations $<_\alpha$ and $<_\beta$. These relations determine the *rule* of the game as follows: player β picks some $u_0 \in \mathrm{Dom}(G)$ first, then at the nth move, with $n > 0$, player α chooses $u_n <_\alpha u_{n-1}$, if n is odd and player β chooses $u_n <_\beta u_{n-1}$, if n is even. The sequence $\{u_n\colon n \in \omega\}$ is then a *run* of the game G. A *strategy* is a function $\sigma : \bigcup_{n\in\omega} D^n \to D$. If we specify a *win condition* for player α (respectively β), we can define a *winning strategy* for player α (respectively β) as a strategy σ, such that α (respectively β) wins every run $\{u_n\colon n \in \omega\}$ of G compatible with σ, i.e., such that $u_n = \sigma(u_0, \ldots, u_{n-1})$ for all odd n (respectively, for all even n). The game G is called *γ-favorable* (for $\gamma \in \{\alpha, \beta\}$), provided γ possesses a winning strategy. Two games G and H are *equivalent*, provided for any $\gamma \in \{\alpha, \beta\}$, G is γ-favorable if and only if H is γ-favorable.

Let G and H be two transitive games for which we denote by the same symbols $<_\alpha$ and $<_\beta$ the relations defining their respective rules. If γ is one of the players, then $\overline{\gamma}$ will denote the opponent player. A *game morphism* from G onto H is a mapping $\varphi : \mathrm{Dom}(G) \to \mathrm{Dom}(H)$ satisfying for any $\gamma \in \{\alpha, \beta\}$, $u, u' \in \mathrm{Dom}(G)$ and $v \in \mathrm{Dom}(H)$ the following conditions:

(M1) $u' <_\gamma u \Longrightarrow \varphi(u') <_\gamma \varphi(u)$;

(M2) $v <_\gamma \varphi(u) \Longrightarrow \exists u'\colon u' <_\gamma u$ and $\varphi(u') <_{\bar{\gamma}} v$;

(M2′) $\forall v\, \exists u'\colon \varphi(u') <_\alpha v$;

(M3) γ wins the run $\{u_n\colon n \in \omega\}$ in $G \Leftrightarrow \gamma$ wins the run $\{\varphi(u_n)\colon n \in \omega\}$ in H.

Note that (M2′) is a consequence of (M2) if the following condition is fulfilled:

(M4) $\exists a \in \mathrm{Dom}(G)\colon \forall v \in \mathrm{Dom}(H), v <_\beta \varphi(a)$.

Theorem DSR. *If there exists a game morphism from a transitive game G onto another transitive game H, then G and H are equivalent.*

Proof. See [9], Theorem 4.5. □

Given a topological space X with an open base $\mathcal{B}$ define

$$\mathcal{E}(X, \mathcal{B}) = \big\{(x, U) \in X \times \mathcal{B}\colon x \in U\big\}.$$

The so-called *strong Choquet game* $\Gamma(X, \mathcal{B})$ with $\mathcal{E}(X, \mathcal{B})$ as domain, is played in accordance with the rule defined by the following relations $<_\alpha$, $<_\beta$ on $\mathcal{E}(X, \mathcal{B})$:

$$(x', U') <_\alpha (x, U) \iff U' \subset U \text{ and } x' = x,$$

$$(x', U') <_\beta (x, U) \iff U' \subset U.$$

Player α wins the run $\{(x_n, U_n)\colon n \in \omega\}$ in $\Gamma(X, \mathcal{B})$, provided $\bigcap_{n\in\omega} U_n \neq \emptyset$; otherwise β wins.

If $\Gamma(X, \mathcal{B})$ is γ-favorable for some open base $\mathcal{B}$ for X ($\gamma \in \{\alpha, \beta\}$), then $\Gamma(X, \mathcal{B}')$ is γ-favorable for each open base $\mathcal{B}'$ for X. Indeed, if we consider a mapping $h : \mathcal{E}(X, \mathcal{B}) \to \mathcal{B}$ such that $x \in h(x, U) \subset U$ for each $(x, U) \in \mathcal{E}(X, \mathcal{B})$ and define the relation $<_\alpha^h$ on $\mathcal{E}(X, \mathcal{B})$ as $(x', U') <_\alpha^h (x, U) \iff U' \subset h(x, U)$ and $x' = x$, then

Proposition 2.1. *The game $\Gamma_h(X, \mathcal{B})$ governed by the relations $<_\alpha^h$ and $<_\beta$ is equivalent to $\Gamma(X, \mathcal{B})$.*

We may therefore use the symbol $\Gamma(X)$ for the strong Choquet game on X without specifying the base $\mathcal{B}$.

Consider two collections $\{(X_s, \mathcal{B}_s)\colon s \in S\}$ and $\{(Y_s, \mathcal{D}_s)\colon s \in S\}$ of topological spaces. Let $G_s = \Gamma(X_s, \mathcal{B}_s)$ and $H_s = \Gamma(Y_s, \mathcal{D}_s)$; further, assume that there exist strong Choquet game morphisms $\varphi_s : \mathcal{E}(X_s, \mathcal{B}_s) \to \mathcal{E}(Y_s, \mathcal{D}_s)$ for each $s \in S$ such that for every $s \in S$ and $x \in X_s$ there is some $y \in Y_s$ with $\varphi_s(x, X_s) = (y, Y_s)$. Let $G = \Gamma(X, \mathcal{B})$ and $H = \Gamma(Y, \mathcal{D})$ be the strong Choquet games on the product spaces $X = \prod_{s\in S} X_s$ and $Y = \prod_{s\in S} Y_s$, respectively, with the respective Tychonoff product canonical bases $\mathcal{B}$ and $\mathcal{D}$. Define the mapping $\varphi : \mathcal{E}(X, \mathcal{B}) \to \mathcal{E}(Y, \mathcal{D})$ as follows: put $\varphi(\boldsymbol{x}, \prod_{s\in S} U_s) = (\boldsymbol{y}, \prod_{s\in S} V_s)$, where $U_s = X_s$ for all but finitely many $s \in S$ and $(\boldsymbol{y}(s), V_s) = \varphi_s(\boldsymbol{x}(s), U_s)$ for all $s \in S$. It follows that

Proposition 2.2. *The mapping φ is a game morphism between G and H.*

Recall that a topological space is a *Baire space*, provided the intersection of any countable collection of open dense subsets of X is dense in X; further, X is *hereditarily Baire*, provided every nonempty *closed* subspace is a Baire space.

Theorem D. *Consider the following properties for a topological space (X, τ):*

(a) *X is hereditarily Baire,*

(b) *$\Gamma(X)$ is not β-favorable,*

(c) *X has no closed countable dense-in-itself subsets.*

Then

(i) *for a 1st countable regular space X, $(a) \Leftrightarrow (c) \Rightarrow (b)$;*

(ii) *for a Moore space X, $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.*

Proof. (i) See [8], Corollaire 3.7 and Proposition 2.7.

(ii) As for $(b) \Rightarrow (a)$, observe that G_δ subspaces of a space X such that $\Gamma(X)$ is not β-favorable are Baire spaces [8, Corollaire 2.3]; further, closed subsets of a Moore space are G_δ-sets. □

3. The strong Choquet game and $K(X)$

For any $\boldsymbol{U} = \langle U_0, \dots, U_n\rangle \in \mathcal{B}^v$ (respectively $\boldsymbol{U} = \langle U_0, \dots, U_n\rangle^\star \in \mathcal{B}^\star$) denote $\widetilde{\boldsymbol{U}} = \langle \overline{U}_0, \dots, \overline{U}_n\rangle$ (respectively $\widetilde{\boldsymbol{U}} = \langle \overline{U}_0, \dots, \overline{U}_n\rangle^\star$), which is the τ^v-closure (respectively τ^*-closure) of $\boldsymbol{U}$ (see [21, Lemma 2.3.2]), so $\widetilde{\boldsymbol{U}}$ is well-defined.

We can define a game $\Gamma^\star$ on $\mathcal{E}^\star = \mathcal{E}(X^\star, \mathcal{B}^\star)$ as follows:

$$(\boldsymbol{x}, \boldsymbol{U}) <_\alpha (\boldsymbol{x}', \boldsymbol{U}') \iff \boldsymbol{x} = \boldsymbol{x}' \text{ and } \widetilde{\boldsymbol{U}} \subset \boldsymbol{U}',$$
$$(\boldsymbol{x}, \boldsymbol{U}) <_\beta (\boldsymbol{x}', \boldsymbol{U}') \iff \boldsymbol{U} \subset \boldsymbol{U}'.$$

Moreover, define a game Γ^v on $\mathcal{E}^v = \mathcal{E}(K(X), \mathcal{B}^v)$ as follows:

$$(A, \boldsymbol{U}) <_\alpha (A', \boldsymbol{U}') \iff A = A' \text{ and } \widetilde{\boldsymbol{U}} \subset \boldsymbol{U}',$$
$$(A, \boldsymbol{U}) <_\beta (A', \boldsymbol{U}') \iff \boldsymbol{U} \subset \boldsymbol{U}'.$$

Remark 3.1. Observe that if X is a regular space then in view of Proposition 2.1, $\Gamma^\star$ is equivalent to $\Gamma(X^\star, \mathcal{B}^\star)$ and Γ^v is equivalent to $\Gamma(K(X), \mathcal{B}^v)$, respectively.

To every $\boldsymbol{U} = \langle U_0, \dots, U_n\rangle^\star$ we can assign $\boldsymbol{U}^v = \langle U_0, \dots, U_n\rangle$. This assignment is well-defined, since

$$\boldsymbol{U} \subset \boldsymbol{V} \Longrightarrow \boldsymbol{U}^v \subset \boldsymbol{V}^v, \tag{1}$$

where $\boldsymbol{U} = \langle U_0, \dots, U_n\rangle^\star$ and $\boldsymbol{V} = \langle V_0, \dots, V_m\rangle^\star$. Indeed, if $A \in \boldsymbol{U}^v$ we can find an $\boldsymbol{x}_{(A,\boldsymbol{U})} \in X^\star$ such that $\boldsymbol{x}_{(A,\boldsymbol{U})}(i) \in U_i$ for each $i \leqslant n$ and $\overline{\boldsymbol{x}}_{(A,\boldsymbol{U})} \subset A$. Then $\boldsymbol{x}_{(A,\boldsymbol{U})} \in \boldsymbol{U} \subset \boldsymbol{V}$, whence $A \in \boldsymbol{V}^v$, since $A \subset \bigcup_{i \leqslant n} U_i \subset \bigcup_{j \leqslant m} V_j$. Now define the mapping $\varphi : \mathcal{E}^\star \to \mathcal{E}^v$ via

$$\varphi(\boldsymbol{x}, \boldsymbol{U}) = (\overline{\boldsymbol{x}}, \boldsymbol{U}^v).$$

Theorem 3.2. *Suppose that the compact subsets of a regular, 1st countable space X are separable. Then φ is a game morphism of $\Gamma^\star$ onto Γ^v.*

Proof.

• (M1): It suffices to use (1) and that for $\boldsymbol{U}, \boldsymbol{V} \in \mathcal{B}^\star$, $\widetilde{\boldsymbol{U}} \subset \boldsymbol{V}$ implies $\widetilde{\boldsymbol{U}}^v \subset \boldsymbol{V}^v$, which can be shown similarly to (1).

• (M2) for α: assume that $(A, \boldsymbol{V}) <_\alpha \varphi(\boldsymbol{x}, \boldsymbol{U})$ for some $(A, \boldsymbol{V}) \in \mathcal{E}^v$ and $(\boldsymbol{x}, \boldsymbol{U}) \in \mathcal{E}^\star$. Then

$$A = \overline{\boldsymbol{x}} \quad \text{and} \quad \widetilde{\boldsymbol{V}} \subset \boldsymbol{U}^v. \tag{2}$$

Denote $\boldsymbol{V} = \langle V_0, \dots, V_n\rangle$ and $\boldsymbol{U} = \langle U_0, \dots, U_m\rangle^\star$. It is not hard to show that $\widetilde{\boldsymbol{V}} \subset \boldsymbol{U}^v$ if and only if

$$\bigcup_{k \leqslant n} \overline{V_k} \subset \bigcup_{j \leqslant m} U_j \quad \text{and } \forall j \leqslant m\ \exists k \leqslant n\colon \overline{V_k} \subset U_j. \tag{3}$$

Observe by (2), that $A = \overline{\boldsymbol{x}} \in \boldsymbol{V}$, so for all $i \in \omega$ there exists a $k \leqslant n$ with $\boldsymbol{x}(i) \in V_k$ and, for all $k \leqslant n$, there is some $i \in \omega$ with $\boldsymbol{x}(i) \in V_k$. Therefore, for each $j \leqslant m$ we can define a nonempty open U'_j such that

$$\boldsymbol{x}(j) \in U'_j \subset \overline{U'_j} \subset U_j \cap \bigcap_{\boldsymbol{x}(j) \in V_k} V_k.$$

Furthermore, if $k \leqslant n$ is such that V_k does not participate in the definition of any of U'_j for $j \leqslant m$, then we can find the smallest $i_k > m$ such that $\boldsymbol{x}(i_k) \in V_k$. Denote by p the maximum of these i_k and for each $m < j \leqslant p$ put $U'_j = \bigcap_{\boldsymbol{x}(j) \in V_k} V_k$ and let $U'_{p+1} = \bigcup_{k \leqslant n} V_k$. If $\boldsymbol{U}' = \langle U'_0, \ldots, U'_{p+1} \rangle^\star$, then $\boldsymbol{x} \in \boldsymbol{U}'$; thus, $(\boldsymbol{x}, \boldsymbol{U}') \in \mathcal{E}^\star$. Moreover, in virtue of (3), $\bigcup_{k \leqslant n} \overline{V_k} \subset \bigcup_{j \leqslant m} U_j$ and clearly $\overline{U'_j} \subset U_j$ for all $j \leqslant m$, whence $(\boldsymbol{x}, \boldsymbol{U}') <_\alpha (\boldsymbol{x}, \boldsymbol{U})$.

Finally, $\bigcup_{l \leqslant p} U'_l \subset \bigcup_{k \leqslant n} V_k$ and for all $k \leqslant n$ there exists $l \leqslant p$ such that $U'_l \subset V_k$, which means that $(\boldsymbol{U}')^v \subset \boldsymbol{V}$. Consequently,

$$\varphi(\boldsymbol{x}, \boldsymbol{U}') = \left(\overline{\boldsymbol{x}}, (\boldsymbol{U}')^v\right) <_\beta (A, \boldsymbol{V}).$$

• (M2) for β: assume that $(A, \boldsymbol{V}) <_\beta \varphi(\boldsymbol{x}, \boldsymbol{U})$ and adopt the notation from the previous case. Then $\boldsymbol{V} \subset \boldsymbol{U}^v$ and hence, without loss of generality, we may assume that $m \leqslant n$ and $V_j \subset U_j$ for all $j \leqslant m$. Let $a_i \in A \cap V_i$ and $V'_i \in \tau$ be such that $a_i \in V'_i \subset \overline{V'_i} \subset V_i$ for each $i \leqslant n$. Also, by compactness of A, we can find $V'_{n+1} \in \tau$ with $A \subset V'_{n+1} \subset \overline{V'_{n+1}} \subset \bigcup_{i \leqslant n} V_i$ and denote $\boldsymbol{U}' = \langle V'_0, \ldots, V'_{n+1} \rangle^\star$. Since A is separable, we can find $\boldsymbol{x}_{(A,\boldsymbol{U}')} \in X^\star$ such that $\boldsymbol{x}_{(A,\boldsymbol{U}')}(i) = a_i$ for each $i \leqslant n$ and $\overline{\boldsymbol{x}}_{(A,\boldsymbol{U}')} = A$. Then $(\boldsymbol{x}_{(A,\boldsymbol{U}')}, \boldsymbol{U}') \in \mathcal{E}^\star$, further,

$$(\boldsymbol{x}_{(A,\boldsymbol{U}')}, \boldsymbol{U}') <_\beta (\boldsymbol{x}, \boldsymbol{U}) \quad \text{and} \quad \varphi(\boldsymbol{x}_{(A,\boldsymbol{U}')}, \boldsymbol{U}') = \left(\overline{\boldsymbol{x}}_{(A,\boldsymbol{U}')}, (\boldsymbol{U}')^v\right) <_\alpha (A, \boldsymbol{V}).$$

• (M2′): By (M4) it suffices to assure that

$$\exists (\boldsymbol{x}, \boldsymbol{W}) \in \mathcal{E}^\star \ \forall (A, \boldsymbol{V}) \in \mathcal{E}^v\colon\ (A, \boldsymbol{V}) <_\beta \varphi(\boldsymbol{x}, \boldsymbol{W}).$$

This can be done for $\boldsymbol{W} = X^\star$ and any $\boldsymbol{x} \in X^\star$.

• (M3): consider a run $\{(\boldsymbol{x}_n, \boldsymbol{U}_n)\colon n \in \omega\}$ in $\Gamma^\star$. Assume that α wins this run, i.e., that there exists an $\boldsymbol{x} \in \bigcap_{n \in \omega} \boldsymbol{U}_n$. Then clearly $\overline{\boldsymbol{x}} \in \bigcap_{n \in \omega} (\boldsymbol{U}_n)^v$, so α wins the run $\{\varphi(\boldsymbol{x}_n, \boldsymbol{U}_n)\colon n \in \omega\} = \{(\overline{\boldsymbol{x}}_n, (\boldsymbol{U}_n)^v)\colon n \in \omega\}$ in Γ^v. Conversely, if α wins $\{\varphi(\boldsymbol{x}_n, \boldsymbol{U}_n)\colon n \in \omega\}$ in Γ^v, then we get some $A \in \bigcap_{n \in \omega} (\boldsymbol{U}_n)^v$. Denote $\boldsymbol{U}_n = \langle U^n_0, \ldots, U^n_{m_n} \rangle^\star$, where without loss of generality assume that $m_{n+1} > m_n$ for all $n \in \omega$. Since for each even n, $\widetilde{\boldsymbol{U}}_{n+1} \subset \boldsymbol{U}_n$, we have $\overline{U^{n+1}_i} \subset U^n_i$ for each $i \leqslant m_n$. Now $A \in (\boldsymbol{U}_{n+1})^v$, so there exists $x^n_i \in U^{n+1}_i \cap A$ for all $i \leqslant m_n$. By compactness of A we get an $x_i \in A$, which is the limit of some subsequence of $\{x^n_i\colon n \in \omega\}$. Then for all $i \leqslant m_n$, $x_i \in \overline{U^{n+1}_i} \subset U^n_i$. Define $\boldsymbol{x} \in X^\omega$ via $\boldsymbol{x}(i) = x_i$ for all $i \in \omega$. It is not hard to see that $\overline{\boldsymbol{x}} \subset A$ is compact and $\boldsymbol{x} \in \bigcap_{n \in \omega} \boldsymbol{U}_n$. It means that α wins the run $\{(\boldsymbol{x}_n, \boldsymbol{U}_n)\colon n \in \omega\}$ in $\Gamma^\star$. □

In view of Theorem 3.2, Theorem DSR and Remark 3.1 we get:

Theorem 3.3. *Suppose that the compact subsets of a regular, 1st countable space X are separable and $\gamma \in \{\alpha, \beta\}$. Then the following are equivalent*:

(i) $\Gamma(K(X))$ *is γ-favorable*;

(ii) $\Gamma(X^\star)$ *is γ-favorable.*

4. Hereditary Baireness of $K(X)$

Theorem 4.1. *Let (X,τ) be a Moore space. The following are equivalent*:

(i) $(K(X),\tau^v)$ *is hereditarily Baire*;

(ii) $\Gamma(X^\star)$ *is not β-favorable.*

Proof. In a Moore space all the conditions of Theorem 3.3 are satisfied, so Theorem D(ii) and the fact that $K(X)$ is a Moore space if and only if X is [23] yield the desired result. □

Remark 4.2. Hereditary Baireness of a Hausdorff space X is necessary for hereditary Baireness of $(K(X),\tau^v)$ (since X embeds as a closed subspace in $K(X)$), however, it is not sufficient. Indeed, if we take the hereditarily Baire (separable) metric space X of [2] having a non-hereditarily Baire square X^2, then by Theorem D(ii), player β has a winning strategy σ in the strong Choquet game on X^2. This strategy σ generates a winning strategy for β on $X^\star$: indeed, it suffices for β to follow what σ dictates on the *first two coordinate spaces*. Consequently, $\Gamma(X^\star)$ is β-favorable and $K(X)$ is not hereditarily Baire by Theorem 4.1.

Another way of showing it is by using that the set T of at most two-element subsets of X is a closed subspace of $K(X)$ and the natural mapping of X^2 onto T is perfect; hence T is not hereditarily Baire, since a regular space, which is a perfect preimage of a hereditarily Baire metric space is itself hereditarily Baire (see even more generally [6, Théorème 2.1] or [32, Theorem 5.1]).

Theorem 4.3. *Let (X,τ) be metrizable. The following are equivalent*:

(i) $(K(X),\tau^v)$ *is hereditarily Baire*;

(ii) $(X^\star,\tau^\star)$ *is hereditarily Baire.*

Proof. See Theorem 4.1, Lemma 1.1 and Theorem D(ii). □

Theorem 4.4. *Suppose that $\{X_k\}_{k\in I}$ is an at most countable collection of Moore spaces. Then the following are equivalent*:

(i) $\prod_{k\in I} K(X_k)$ *is hereditarily Baire*;

(ii) $K(\prod_{k\in I} X_k)$ *is hereditarily Baire.*

Proof. $K(X_k)$ is a Moore space for all $k\in I$ [23], so $\prod_{k\in I} K(X_k)$ is also a Moore space. Therefore, by Theorem 3.3, Proposition 2.2 and Theorem D(ii), $\prod_{k\in I} K(X_k)$ is hereditarily Baire if and only if $\Gamma(\prod_{k\in I} X_k^\star, \prod_{k\in I}\tau^\star(X_k))$ is not β-favorable. Since $X=\prod_{k\in I} X_k$ is a Moore space, as well as $K(X)$, in view of Theorem 4.1 it is enough to prove that $(\prod_{k\in I} X_k^\star, \prod_{k\in I}\tau^\star(X_k))$ is homeomorphic to $(X^\star,\tau^\star(X))$.

Indeed, it is a routine to prove, that the mapping, assigning to each $(\boldsymbol{x}_k)_k\in\prod_{k\in I} X_k^\star$ (where $\boldsymbol{x}_k=(x_{k,i})_{i\in\omega}\in X_k^\star$ for all $k\in I$) the element $((x_{k,i})_{k\in I})_{i\in\omega}\in X^\star$ is a homeomorphism. □

Remark 4.5. An application of the previous theorem in the space of continuous partial maps with compact domains $\mathcal{P}_K$ (studied in [18] or more recently in [12,19]) is exhibited in the upcoming paper [13], where various completeness properties of $\mathcal{P}_K$, including hereditary Baireness and Čech-completeness, are investigated.

Theorem 4.6. *Suppose that (X,τ) is a regular space and the compact subsets of X are separable and of countable character. Then the following are equivalent*:

(i) $(K(X),\tau^v)$ *is hereditarily Baire*;

(ii) $(K(Y),\tau^v)$ *is hereditarily Baire for each separable closed subspace $Y \subset X$.*

Proof. Observe that $K(Y)$ is closed in $(K(X),\tau^v)$ for a closed $Y \subset X$, whence (i) ⇒ (ii) follows. Now notice that by [22, Theorem 3], $(K(X),\tau^v)$ is 1st countable and by [21, Section 4], it is regular. To see (ii) ⇒ (i), take a countable closed subset $\mathcal{F}$ of $(K(X),\tau^v)$ and consider the closed separable set $Y = \overline{\bigcup \mathcal{F}} \subset X$. Then in view of (ii), $(K(Y),\tau^v)$ is hereditarily Baire and is a closed subspace of $(K(X),\tau^v)$; thus, by Theorem D(i), $\mathcal{F} \subset K(Y)$ is not dense-in-itself, consequently $(K(X),\tau^v)$ is hereditarily Baire by Theorem D(i). □

The following improves Proposition 5 of [3]:

Corollary 4.7. *Suppose that (X,τ) is a Tychonoff space and the compact subsets of X are separable and of countable character. If the separable closed subspaces of X are consonant, then $(K(X),\tau^v)$ is hereditarily Baire.*

Proof. Let Y be a separable closed subspace of X. Then by Proposition 5 of [3], $(K(Y),\tau^v)$ is hereditarily Baire and the above Theorem 4.6 applies. □

Corollary 4.8. *Let X be a metrizable space with completely metrizable separable closed subspaces. Then $(K(X),\tau^v)$ is hereditarily Baire.*

Proof. It suffices to note, that in a metrizable space all the conditions of the previous corollary are satisfied; further, completely metrizable spaces are consonant [11, Theorem 4.1]. □

Consider the product space ω_1^ω, where the first uncountable ordinal ω_1 is endowed with the discrete topology. Let $E = \{f \in \omega_1^\omega\colon\ f \text{ is strictly increasing}\}$. Denote by $\mathcal{L}$ the infinite countable limit ordinals. For each $\xi \in \mathcal{L}$ pick a sequence $x_\xi \in E$ such that $\sup\{\operatorname{ran} x_\xi\} = \xi$ and denote $E_0 = \{x_\xi\colon\ \xi \in \mathcal{L}\}$.

It was proved in [26, Lemma 3], under ZFC, that $Z = E \setminus E_0$ is a non-completely metrizable space each separable closed subspace of which is completely metrizable. Therefore the following theorem is a consequence of Corollary 4.8:

Theorem 4.9. (ZFC) *There exists a metrizable, non-completely metrizable space X such that $(K(X),\tau^v)$ is hereditarily Baire.*

Remark 4.10. By a classical theorem of Hurewicz (i.e., Theorem D (a) ⇔ (c)—see [14] or more generally [8,28]), Z is hereditarily Baire. It is non-separable and hence not analytic;

further, it was observed by F. van Engelen (see his review of [26] in Zentralblatt für Mathematik—Zbl. 861.54030), that Z is not co-analytic. □

By weakening the conditions on X we still get a result on Baireness of $K(X)$:

Theorem 4.11. *Suppose that X is a 1st-countable, regular space such that the compact subsets are separable and $K(Y)$ is hereditarily Baire for each separable closed subspace $Y \subset X$. Then $(K(X), \tau^v)$ is a Baire space.*

Proof. The finite subsets of X form a dense 1st countable subspace of the regular space $K(X)$. Now, an argument analogous to that of in Theorem 4.6 shows, that all separable closed subspaces of $(K(X), \tau^v)$ are Baire spaces, which proves by [8, Corollaire 3.5], that $(K(X), \tau^v)$ is itself a Baire space. □

5. A result on consonance

It is our aim in what follows, to construct a non-consonant, metrizable space with completely metrizable separable closed subspaces. The construction uses CH; in general, the problem is open.

Given a 1st countable Hausdorff space X, let $\mathcal{B}$ be a base for X and denote by $T \subset \mathcal{B}^\omega$ the set of all sequences $\{U_n \in \mathcal{B}\colon n \in \omega\}$ corresponding to a neighborhood system for some point in X. Endow T with the topology inherited from the product topology on $\mathcal{B}^\omega$, with $\mathcal{B}$ having the discrete topology.

Theorem 5.1. *Let X be a 1st countable, compact space. The following are equivalent*:
- (i) *T is consonant*;
- (ii) *X is metrizable*;
- (iii) *T is completely metrizable.*

Proof. (iii) $\Rightarrow$ (i) is proven in [11, Theorem 4.1].

(i) $\Rightarrow$ (ii) The map $f: T \to X$ defined via $f((U_n)_n) = \bigcap_{n\in\omega} U_n$ is open, continuous and onto, hence, by Corollary 8 in [4], f is compact covering (i.e., for each compact $A \subset X$ there is a compact $B \subset T$ such that $f(B) = A$). Consequently, there is a compact set $T_0 \subset T$, such that $f(T_0) = X$, whence $f\restriction_{T_0}$ is a perfect mapping onto the compact space X. It implies, that X is metrizable.

(ii) $\Rightarrow$ (iii) Fix a compatible metric d on X and let T' be the set of all centered sequences $\{U_n \in \mathcal{B}\colon n \in \omega\}$ satisfying $\lim_{n\to\infty} \delta(\bigcap_{i\leqslant n} U_i) = 0$, where $\delta(A)$ denotes the diameter of A. Since X is compact, it is not hard to see that $T' = T$, further T' is a G_δ subset of $\mathcal{B}^\omega$, hence, T is completely metrizable. □

Theorem 5.2. (CH) *There is a metrizable non-consonant space each separable closed subspace of which is completely metrizable.*

Proof. Let X be the Hausdorff, compact, 1st countable, hereditarily Lindelöf, non-metrizable space with no isolated points constructed by Kunen under CH in [16]. Then the space T from the previous theorem is not consonant, since X is not metrizable. Also note, that the closed separable subspaces of X are metrizable.

If A is a countable subset of T, then $\overline{A}$ is completely metrizable. Indeed, let d be some compatible metric on $K = \overline{f(A)}$, where f is the mapping from the proof of Theorem 5.1. Since X is perfectly normal, we can find a sequence $\{W_n\colon n \in \omega\}$ of open subsets of X such that $\bigcap_{n\in\omega} W_n = K$.

For all $n, k, j \in \omega$, the set $\mathcal{W}_{n,k,j}$ of all sequences $(U_i)_i \in \mathcal{B}^\omega$ for which there exists $m \in \omega$ such that $\emptyset \neq \overline{U_m} \subset W_n \cap \bigcap_{i\leqslant k} U_i$ and $\delta(K \cap U_m) \leqslant 1/j$, is open in $\mathcal{B}^\omega$. To argue that

$$f^{-1}(K) = \bigcap_{n,k,j\in\omega} \mathcal{W}_{n,k,j}, \tag{4}$$

take some $(U_i)_i \in f^{-1}(K)$ first. Then $(U_i)_i$ is a neighborhood system of some $x \in K$, so by regularity of X, for each $n, k, j \in \omega$ we can find some $m \in \omega$ such that $x \in U_m \subset \overline{U_m} \subset W_n \cap \bigcap_{i\leqslant k} U_i$ and $K \cap U_m$ is contained in the open d-ball of radius $1/(2j)$ about x and hence $\delta(K \cap U_m) \leqslant 1/j$.

Conversely, assume that $(U_i)_i \in \bigcap_{n,k,j\in\omega} \mathcal{W}_{n,k,j}$. Since X is compact, $(K \cap U_i)_i$ intersects in a singleton $x \in K$, which is the only element of $\bigcap_i U_i$, because $\bigcap_i U_i \subset \bigcap_n W_n = K$. Since X is compact and 1st countable, it follows that $(U_i)_i$ is a neighborhood system for x and hence $(U_i)_i \in f^{-1}(K)$.

It is clear now by (4), that $f^{-1}(K)$ is a G_δ subset of $\mathcal{B}^\omega$; on the other hand, $\overline{A} \subset f^{-1}(K)$, hence, $\overline{A}$ is completely metrizable. $\square$

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