On complete metrizability of hyperspaces

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- If d is not bounded, H_d is an infinite-valued distance, which generates the *Hausdorff metric topology* τ_{H_d} on $CL(X)$; moreover, $d' = \min\{1, d\}$ is, an equivalent to d, bounded metric on X and $\tau_{H_{d'}} = \tau_{H_d}$.

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$$X = \mathbb{R}^\omega \setminus \{x \in \mathbb{R}^\omega : x(0) \neq 0 \text{ and } x(k) = 0 \text{ for all } k > 0\}.$$

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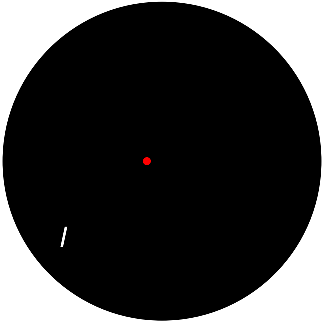

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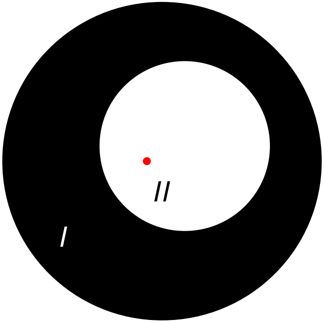

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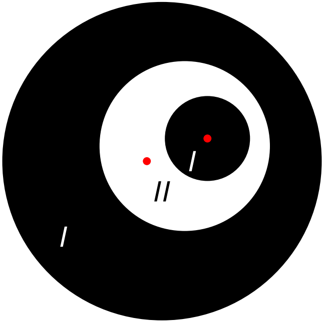

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There exists a completely metrizable space X with compatible metrics d, d' so that

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Theorem (Zs.)

Let (X, d) be any metric space. If Player I has NO w.s. in $Ch(CL(X), \tau_{H_d})$ (i.e. $(CL(X), \tau_{H_d})$ is hereditarily Baire),

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