On complete metrizability of hyperspaces

László Zsilinszky

The University of North Carolina at Pembroke
Pembroke, NC 28372, USA

IX Iberoamerican Conference on Topology and its Applications
Almería, Spain
June 24–27, 2014
Hausdorff metric topology

- $(X, d) \ldots$ metric space
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$ ... the completion of $(X, d)$
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$ ... the completion of $(X, d)$
- $CL(X)$ ... the nonempty closed subsets of $X$
Hausdorff metric topology

- \((X, d)\) ... metric space
- \((\tilde{X}, \tilde{d})\) ... the completion of \((X, d)\)
- \(CL(X)\) ... the nonempty closed subsets of \(X\) (hyperspace)
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$ ... the completion of $(X, d)$
- $CL(X)$ ... the nonempty closed subsets of $X$ (hyperspace)
- $B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ ... open $\varepsilon$-hull of $A$
(\(X, d\)) ... metric space

(\(\tilde{X}, \tilde{d}\)) ... the completion of \((X, d)\)

\(CL(X)\) ... the nonempty closed subsets of \(X\) (hyperspace)

\(B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}\) ... open \(\varepsilon\)-hull of \(A\)

**Definition**

The *Hausdorff distance* \(H_d\) on \(CL(X)\) is defined as
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$ ... the completion of $(X, d)$
- $CL(X)$ ... the nonempty closed subsets of $X$ (hyperspace)
- $B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ ... open $\varepsilon$-hull of $A$

**Definition**

The *Hausdorff distance* $H_d$ on $CL(X)$ is defined as

$$H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}$$
Hausdorff metric topology

- \((X, d)\) ... metric space
- \((\tilde{X}, \tilde{d})\) ... the completion of \((X, d)\)
- \(CL(X)\) ... the nonempty closed subsets of \(X\) (hyperspace)
- \(B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}\) ... open \(\varepsilon\)-hull of \(A\)

**Definition**

The *Hausdorff distance* \(H_d\) on \(CL(X)\) is defined as

\[
H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}
\]

- If \(d\) is bounded,
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$ ... the completion of $(X, d)$
- $CL(X)$ ... the nonempty closed subsets of $X$ (hyperspace)
- $B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ ... open $\varepsilon$-hull of $A$

**Definition**

The **Hausdorff distance** $H_d$ on $CL(X)$ is defined as

$$H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}$$

- If $d$ is bounded, $H_d$ is a metric on $CL(X)$. 
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$ ... the completion of $(X, d)$
- $CL(X)$ ... the nonempty closed subsets of $X$ (hyperspace)
- $B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ ... open $\varepsilon$-hull of $A$

**Definition**

The **Hausdorff distance** $H_d$ on $CL(X)$ is defined as

$$H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}$$

- If $d$ is bounded, $H_d$ is a metric on $CL(X)$.
- If $d$ is not bounded,
Hausdorff metric topology

- \((X, d)\) ... metric space
- \((\tilde{X}, \tilde{d})\) ... the completion of \((X, d)\)
- \(CL(X)\) ... the nonempty closed subsets of \(X\) (hyperspace)
- \(B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}\) ... open \(\varepsilon\)-hull of \(A\)

**Definition**

The **Hausdorff distance** \(H_d\) on \(CL(X)\) is defined as

\[
H_d(A_0, A_1) = \inf \{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}
\]

- If \(d\) is bounded, \(H_d\) is a metric on \(CL(X)\).
- If \(d\) is not bounded, \(H_d\) is an infinite-valued distance,
Hausdorff metric topology

- \((X, d)\) ... metric space
- \((\tilde{X}, \tilde{d})\) ... the completion of \((X, d)\)
- \(CL(X)\) ... the nonempty closed subsets of \(X\) (hyperspace)
- \(B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}\) ... open \(\varepsilon\)-hull of \(A\)

**Definition**

The Hausdorff distance \(H_d\) on \(CL(X)\) is defined as

\[H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}\]

- If \(d\) is bounded, \(H_d\) is a metric on \(CL(X)\).
- If \(d\) is not bounded, \(H_d\) is an infinite-valued distance, which generates the Hausdorff metric topology \(\tau_{H_d}\) on \(CL(X)\);
Hausdorff metric topology

- $(X, d)$ ... metric space
- $(\tilde{X}, \tilde{d})$ ... the completion of $(X, d)$
- $CL(X)$ ... the nonempty closed subsets of $X$ (hyperspace)
- $B_d(A, \varepsilon) = \{x \in X : d(x, A) < \varepsilon\}$ ... open $\varepsilon$-hull of $A$

**Definition**

The *Hausdorff distance* $H_d$ on $CL(X)$ is defined as

$$H_d(A_0, A_1) = \inf\{\varepsilon > 0 : A_0 \subseteq B_d(A_1, \varepsilon) \text{ and } A_1 \subseteq B_d(A_0, \varepsilon)\}$$

- If $d$ is bounded, $H_d$ is a metric on $CL(X)$.
- If $d$ is not bounded, $H_d$ is an infinite-valued distance, which generates the *Hausdorff metric topology* $\tau_{H_d}$ on $CL(X)$; moreover, $d' = \min\{1, d\}$ is, an equivalent to $d$, bounded metric on $X$ and $\tau_{H_{d'}} = \tau_{H_d}$. 
Known properties of $\mathcal{H}_d$

$\mathcal{H}_d$ of $(X,d)$ bounded $\Rightarrow$ $(\mathcal{CL}(X),\mathcal{H}_d)$ complete

$(X,d)$ complete $\Rightarrow$ $(\mathcal{CL}(X),\mathcal{H}_d)$ completely metrizable

$(\mathcal{CL}(X),\mathcal{T}_{\mathcal{H}_d})$ completely metrizable $\Rightarrow$ $(X,d)$ completely metrizable

$\mathcal{T}_{\mathcal{H}_d} = \mathcal{T}_{\mathcal{H}_d}' \iff d,d'$ are uniformly equivalent metrics on $X$

$(X,d)$ completely metrizable $\Rightarrow$ $(\mathcal{CL}(X),\mathcal{T}_{\mathcal{H}_d})$ Polish

$(\mathcal{CL}(X),\mathcal{T}_{\mathcal{H}_d})$ Polish $\Rightarrow$ $(\mathcal{CL}(X),\mathcal{T}_{\mathcal{H}_d})$ completely metrizable

Saint Raymond, Costantini, Kubiś
Known properties of $\mathcal{T}_{H_d}$

- $(X, d)$ bounded + complete
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded $+$ complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable

$(X, d) = (X, d')$ uniformly equivalent metrics on $X$ $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ Polish (Effros) $(X, d)$ Polish $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'}$
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'}$ $\iff d, d'$ are uniformly equivalent metrics on $X$
Known properties of $\tau_{Hd}$

- $(X, d)$ bounded $+$ complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{Hd})$ completely metrizable
- $(CL(X), \tau_{Hd})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{Hd} = \tau_{Hd'}$ $\iff$ $d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'} \iff d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'} \Leftrightarrow d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(X, d)$ completely metrizable $+$ totally bounded
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable

- $\tau_{H_d} = \tau_{H_d'}$ $\iff$ $d, d'$ are uniformly equivalent metrics on $X$

- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable

- $(X, d)$ completely metrizable + totally bounded $\iff$ $(CL(X), \tau_{H_d})$ Polish
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'}$ $\iff$ $d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(X, d)$ completely metrizable + totally bounded $\iff (CL(X), \tau_{H_d})$ Polish (Effros)
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'} \iff d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(X, d)$ completely metrizable + totally bounded $\iff (CL(X), \tau_{H_d})$ Polish (Effros)
- $(X, d)$ Polish
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d}' \iff d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(X, d)$ completely metrizable + totally bounded $\iff (CL(X), \tau_{H_d})$ Polish (Effros)
- $(X, d)$ Polish $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'} \iff d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(X, d)$ completely metrizable + totally bounded $\iff (CL(X), \tau_{H_d})$ Polish (Effros)
- $(X, d)$ Polish $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable (Saint Raymond,
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\implies (CL(X), H_d)$ complete
- $(X, d)$ complete $\implies (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\implies (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d'}$ $\iff$ $d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\implies (CL(X), \tau_{H_d})$ completely metrizable
- $(X, d)$ completely metrizable + totally bounded $\iff (CL(X), \tau_{H_d})$ Polish (Effros)
- $(X, d)$ Polish $\implies (CL(X), \tau_{H_d})$ completely metrizable (Saint Raymond, Costantini,
Known properties of $\tau_{H_d}$

- $(X, d)$ bounded + complete $\Rightarrow (CL(X), H_d)$ complete
- $(X, d)$ complete $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(CL(X), \tau_{H_d})$ completely metrizable $\Rightarrow (X, d)$ completely metrizable
- $\tau_{H_d} = \tau_{H_d}' \iff d, d'$ are uniformly equivalent metrics on $X$
- $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable
- $(X, d)$ completely metrizable + totally bounded $\iff (CL(X), \tau_{H_d})$ Polish (Effros)
- $(X, d)$ Polish $\Rightarrow (CL(X), \tau_{H_d})$ completely metrizable (Saint Raymond, Costantini, Kubiš)
New results on $\tau_{Hd}$
New results on $\tau_{Hd}$

- Key observation (L’ubica Holá):
  $(X, d)$ completely metrizable
New results on $\tau_{Hd}$

- Key observation (L’ubica Holá):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable
New results on $\tau_{Hd}$

- Key observation (L’ubica Holá):

$(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$

$(CL(X), \tau_{Hd})$ completely metrizable

Example (Zs.)

There exists a bounded metric space $(X, d)$ such that

1. $(X, d)$ is completely metrizable,
2. $CL(X, H_d)$ contains a closed copy of the rationals; in particular, $(CL(X), H_d)$ is not completely metrizable,
3. $(CL(X), H_d)$ is a Baire space.

Consider $(R^\omega, d)$, where $R$ has the discrete topology, and $d$ is the Baire metric $d(x, y) = 1/\min\{n + 1: x(n) \neq y(n)\}$ for $x, y \in R^\omega$. The space is $X = R^\omega \setminus \{x \in R^\omega: x(0) \neq 0 \text{ and } x(k) = 0 \text{ for all } k > 0\}$. 
New results on $\tau_{Hd}$

- Key observation (**L’ubica Holá**): $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow (CL(X), \tau_{Hd})$ completely metrizable

**Example (Zs.)**

There exists a bounded metric space $(X, d)$ such that
New results on $\tau_{Hd}$

- Key observation (L'ubica Holá):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$
  $(CL(X), \tau_{Hd})$ completely metrizable

Example (Zs.)

There exists a bounded metric space $(X, d)$ such that

1. $(X, d)$ is completely metrizable,
New results on $\tau_{H_d}$

- Key observation (L’ubica Holá):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$
  $(\text{CL}(X), \tau_{H_d})$ completely metrizable

Example (Zs.)

There exists a bounded metric space $(X, d)$ such that
1. $(X, d)$ is completely metrizable,
2. $(\text{CL}(X), H_d)$ contains a closed copy of the rationals;
New results on $\tau_{H_d}$

- Key observation (L’ubica Holá):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$
  $(CL(X), \tau_{H_d})$ completely metrizable

**Example (Zs.)**

There exists a bounded metric space $(X, d)$ such that

1. $(X, d)$ is completely metrizable,
2. $(CL(X), H_d)$ contains a closed copy of the rationals; in particular, $(CL(X), H_d)$ is not completely metrizable,
New results on $\tau_{H_d}$

- Key observation (L’ubica Holá):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$
  $(CL(X), \tau_{H_d})$ completely metrizable

Example (Zs.)
There exists a bounded metric space $(X, d)$ such that

1. $(X, d)$ is completely metrizable,
2. $(CL(X), H_d)$ contains a closed copy of the rationals; in particular, $(CL(X), H_d)$ is not completely metrizable,
3. $(CL(X), H_d)$ is a Baire space.
New results on $\tau_{H_d}$

- Key observation (L’ubica Holá):
  \((X, d)\) completely metrizable and \(\tilde{X} \setminus X\) separable \(\Rightarrow\)
  \((CL(X), \tau_{H_d})\) completely metrizable

Example (Zs.)

There exists a bounded metric space \((X, d)\) such that

1. \((X, d)\) is completely metrizable,
2. \((CL(X), H_d)\) contains a closed copy of the rationals; in particular, \((CL(X), H_d)\) is not completely metrizable,
3. \((CL(X), H_d)\) is a Baire space.

Consider \((\mathbb{R}^\omega, d)\),
New results on $\tau_{Hd}$

- Key observation (L’ubica Holá):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$
  $(\text{CL}(X), \tau_{Hd})$ completely metrizable

Example (Zs.)

There exists a bounded metric space $(X, d)$ such that

1. $(X, d)$ is completely metrizable,
2. $(\text{CL}(X), H_d)$ contains a closed copy of the rationals; in particular, $(\text{CL}(X), H_d)$ is not completely metrizable,
3. $(\text{CL}(X), H_d)$ is a Baire space.

Consider $(\mathbb{R}^\omega, d)$, where $\mathbb{R}$ has the discrete topology,
New results on $\tau_{H_d}$

- Key observation (L'ubica Holá):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$
  $(CL(X), \tau_{H_d})$ completely metrizable

Example (Zs.)

There exists a bounded metric space $(X, d)$ such that
1. $(X, d)$ is completely metrizable,
2. $(CL(X), H_d)$ contains a closed copy of the rationals; in particular, $(CL(X), H_d)$ is not completely metrizable,
3. $(CL(X), H_d)$ is a Baire space.

Consider $(\mathbb{R}^\omega, d)$, where $\mathbb{R}$ has the discrete topology, and $d$ is the Baire metric.
New results on $\tau_{Hd}$

- Key observation (**L'ubica Holá**):
  $(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$
  $(CL(X), \tau_{Hd})$ completely metrizable

**Example (Zs.)**

There exists a bounded metric space $(X, d)$ such that

1. $(X, d)$ is completely metrizable,
2. $(CL(X), H_d)$ contains a closed copy of the rationals; in particular, $(CL(X), H_d)$ is not completely metrizable,
3. $(CL(X), H_d)$ is a Baire space.

Consider $(\mathbb{R}^\omega, d)$, where $\mathbb{R}$ has the discrete topology, and $d$ is the Baire metric

$$d(x, y) = \frac{1}{\min\{n+1: x(n) \neq y(n)\}}$$

for $x, y \in \mathbb{R}^\omega$. 
New results on $\tau_{H_d}$

- Key observation (L‘ubica Holá): 

$(X, d)$ completely metrizable and $\tilde{X} \setminus X$ separable $\Rightarrow$ 

$(CL(X), \tau_{H_d})$ completely metrizable

Example (Zs.)

There exists a bounded metric space $(X, d)$ such that

1. $(X, d)$ is completely metrizable,
2. $(CL(X), H_d)$ contains a closed copy of the rationals; in particular, $(CL(X), H_d)$ is not completely metrizable,
3. $(CL(X), H_d)$ is a Baire space.

Consider $(\mathbb{R}^\omega, d)$, where $\mathbb{R}$ has the discrete topology, and $d$ is the Baire metric

$$d(x, y) = \frac{1}{\min\{n+1: x(n) \neq y(n)\}}$$

for $x, y \in \mathbb{R}^\omega$. The space is

$$X = \mathbb{R}^\omega \setminus \{x \in \mathbb{R}^\omega : x(0) \neq 0 \text{ and } x(k) = 0 \text{ for all } k > 0\}.$$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$

$z_0 \in V_0$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$

$z_0 \in U_0 \subseteq V_0$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$

$z_1 \in V_1 \subseteq U_0 \subseteq V_0$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$

$z_1 \in U_1 \subseteq V_1 \subseteq U_0 \subseteq V_0$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$

$\cdots \subseteq U_1 \subseteq V_1 \subseteq U_0 \subseteq V_0$

Player I wins, if $\bigcap_n U_n = \emptyset$

Player II wins, if $\bigcap_n U_n \neq \emptyset$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$

$$\cdots \subseteq U_1 \subseteq V_1 \subseteq U_0 \subseteq V_0$$

- Player I wins, if $\bigcap_n U_n = \emptyset$
- Player II wins, if $\bigcap_n U_n \neq \emptyset$
Strong Choquet game $Ch(Z)$ on a metrizable $Z$

$\cdots \subseteq U_1 \subseteq V_1 \subseteq U_0 \subseteq V_0$

• Player I wins, if $\bigcap_n U_n = \emptyset$
• Player II wins, if $\bigcap_n U_n \neq \emptyset$
Strong Choquet game

1. (Choquet)
Strong Choquet game

(Choquet)

Player II has a w.s. in $Ch(Z) \iff Z$ is completely metrizable

Example (rephrased)
There exists a completely metrizable space $X$ with compatible metrics $d, d'$ so that
Player II has a w.s. in $Ch(\text{CL}(X), H_d) \iff Z$ is completely metrizable
Strong Choquet game

1. (Choquet)  
   Player II has a w.s. in $Ch(Z) \iff Z$ is completely metrizable

2. (Telgársely, Porada, Debs)
Strong Choquet game

1. (Choquet) Player II has a w.s. in $Ch(Z) \iff Z$ is completely metrizable

2. (Telgársky, Porada, Debs) Player I has a w.s. in $Ch(Z) \iff Z$ contains a closed copy of $\mathbb{Q}$
Strong Choquet game

1. (Choquet)
   Player II has a w.s. in $Ch(Z) \iff Z$ is completely metrizable

2. (Telgársky, Porada, Debs)
   Player I has a w.s. in $Ch(Z) \iff Z$ contains a closed copy of $\mathbb{Q}$
   $\iff Z$ is not hereditarily Baire

Example (rephrased)
There exists a completely metrizable space $X$ with compatible metrics $d, d'$ so that
Player II has a w.s. in $Ch(CL(X), H_d)$
Player I has a w.s. in $Ch(CL(X), H_{d'})$
Strong Choquet game

1. (Choquet) Player II has a w.s. in \( Ch(Z) \iff Z \) is completely metrizable

2. (Telgársky, Porada, Debs) Player I has a w.s. in \( Ch(Z) \iff Z \) contains a closed copy of \( \mathbb{Q} \)
   \( \iff Z \) is not hereditarily Baire

Example (rephrased)

There exists a completely metrizable space \( X \) with compatible metrics \( d, d' \) so that
Strong Choquet game

1. (Choquet) Player II has a w.s. in $Ch(Z) \iff Z$ is completely metrizable

2. (Telgársky, Porada, Debs) Player I has a w.s. in $Ch(Z) \iff Z$ contains a closed copy of $\mathbb{Q}$
   $\iff Z$ is not hereditarily Baire

Example (rephrased)

There exists a completely metrizable space $X$ with compatible metrics $d, d'$ so that

- Player II has a w.s. in $Ch(CL(X), H_d)$
Strong Choquet game

1. (Choquet)
   Player II has a w.s. in $Ch(Z) \iff Z$ is completely metrizable

2. (Telgársky, Porada, Debs)
   Player I has a w.s. in $Ch(Z) \iff Z$ contains a closed copy of $\mathbb{Q}$
   $\iff Z$ is not hereditarily Baire

Example (rephrased)

There exists a completely metrizable space $X$ with compatible metrics $d, d'$ so that

- Player II has a w.s. in $Ch(CL(X), H_d)$
- Player I has a w.s. in $Ch(CL(X), H_{d'}$)
Main results about $\tau_{H_d}$

Theorem (Zs.)

Let $(X, d)$ be any metric space. If Player I has NO w.s. in $\text{Ch}(\text{CL}(X), \tau_{H_d})$ (i.e. $(\text{CL}(X), \tau_{H_d})$ is hereditarily Baire), then $(\tilde{X} \setminus X, \tilde{d})$ is separable.

Corrolary

The following are equivalent:
1. $(\text{CL}(X), \tau_{H_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Main results about $\tau_{H_d}$

Theorem (Zs.)

Let $(X, d)$ be any metric space.

Corollary

The following are equivalent:

1. $(CL(X), \tau_{H_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Main results about $\tau_{H_d}$

**Theorem (Zs.)**

Let $(X, d)$ be any metric space. If Player I has NO w.s. in $Ch(CL(X), \tau_{H_d})$, then $(\tilde{X} \setminus X, \tilde{d})$ is separable.

**Corollary**

The following are equivalent:

1. $Ch(CL(X), \tau_{H_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Main results about $\tau_{H_d}$

**Theorem (Zs.)**

Let $(X, d)$ be any metric space. If Player I has NO w.s. in $\text{Ch} (\text{CL}(X), \tau_{H_d})$ (i.e. $(\text{CL}(X), \tau_{H_d})$ is hereditarily Baire),
Main results about $\tau_{H_d}$

**Theorem (Zs.)**

Let $(X, d)$ be any metric space. If Player I has NO w.s. in $Ch(CL(X), \tau_{H_d})$ (i.e. $(CL(X), \tau_{H_d})$ is hereditarily Baire), then $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Main results about $\tau_{H_d}$

**Theorem (Zs.)**

Let $(X, d)$ be any metric space. If Player I has NO w.s. in $\text{Ch}(\text{CL}(X), \tau_{H_d})$ (i.e. $(\text{CL}(X), \tau_{H_d})$ is hereditarily Baire), then $(\tilde{X} \setminus X, \tilde{d})$ is separable.

**Corrolary**

The following are equivalent:

1. $(\text{CL}(X), \tau_{H_d})$ is completely metrizable,
Main results about $\mathcal{T}_{H_d}$

**Theorem (Zs.)**

Let $(X, d)$ be any metric space. If Player I has NO w.s. in $Ch(\text{CL}(X), \mathcal{T}_{H_d})$ (i.e. $(\text{CL}(X), \mathcal{T}_{H_d})$ is hereditarily Baire), then $(\tilde{X} \setminus X, \tilde{d})$ is separable.

**Corollary**

The following are equivalent:

1. $(\text{CL}(X), \mathcal{T}_{H_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Main results about $\tau_{H_d}$

**Theorem (Zs.)**

Let $(X, d)$ be any metric space. If Player I has NO w.s. in $Ch(CL(X), \tau_{H_d})$ (i.e. $(CL(X), \tau_{H_d})$ is hereditarily Baire), then $(\tilde{X} \setminus X, \tilde{d})$ is separable.

**Corrolary**

The following are equivalent:

1. $(CL(X), \tau_{H_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Definition
Fix $x_0 \in X$. A local $\tau_{AW_d}$-base at $A \in \text{CL}(X)$ consists of the sets $\text{AW}_d(A, n) = \{ C \in \text{CL}(X) : B_d(x_0, n) \cap A \subseteq B_d(C, 1) \text{ and } B_d(x_0, n) \cap C \subseteq B_d(A, 1) \}$, where $n \geq 1$.

Theorem (Zs.)
The following are equivalent:
1. $(\text{CL}(X), \tau_{AW_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
**Definition**

Fix $x_0 \in X$. A local $\tau_{AW_d}$-base at $A \in CL(X)$ consists of the sets

$$AW_d(A, n) = \{ C \in CL(X) : B_d(x_0, n) \cap A \subseteq B_d(C, \frac{1}{n}) \text{ and } B_d(x_0, n) \cap C \subseteq B_d(A, \frac{1}{n}) \},$$

where $n \geq 1$. 
Definition
Fix $x_0 \in X$. A local $\tau_{AW_d}$-base at $A \in CL(X)$ consists of the sets

$$AW_d(A, n) = \{ C \in CL(X) : B_d(x_0, n) \cap A \subseteq B_d(C, \frac{1}{n}) \text{ and } B_d(x_0, n) \cap C \subseteq B_d(A, \frac{1}{n}) \},$$

where $n \geq 1$.

Theorem (Zs.)
The following are equivalent:

1. $(CL(X), \tau_{AW_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
**Definition**

Fix $x_0 \in X$. A local $\tau_{AW_d}$-base at $A \in CL(X)$ consists of the sets

$$AW_d(A, n) = \{ C \in CL(X) : B_d(x_0, n) \cap A \subseteq B_d(C, \frac{1}{n}) \text{ and } B_d(x_0, n) \cap C \subseteq B_d(A, \frac{1}{n}) \},$$

where $n \geq 1$.

**Theorem (Zs.)**

The following are equivalent:

1. $(CL(X), \tau_{AW_d})$ is completely metrizable,
2. $(X, d)$ is completely metrizable and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Topology of bornological convergence $\tau_S$

Definition

Let $S$ be an ideal of subsets of $X$; the so-called $\tau_S$-convergence on $\text{CL}(X)$ is defined via the neighborhood filter at $A \in \text{CL}(X)$ of the form

$$\left\{ C \in \text{CL}(X) : S \cap A \subseteq \text{Bd}(C,\varepsilon) \text{ and } S \cap C \subseteq \text{Bd}(A,\varepsilon) \right\},$$

where $S \in S$, $\varepsilon > 0$. When $S$ is stable under small enlargements, $S$ is topological. Let $\tau_S$ be this topology of bornological convergence.

Theorem (Zs.)

The following are equivalent:

1. $(\text{CL}(X), \tau_S)$ is completely metrizable,
2. $S$ is a bornology, stable under small enlargements, with a countable base, and $(X, d)$ is completely metrizable, and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Definition

Let $S$ be an ideal of subsets of $X$;

Theorem (Zs.)

The following are equivalent:

1. $(\text{CL}(X), \tau_S)$ is completely metrizable,
2. $S$ is a bornology, stable under small enlargements, with a countable base, and $(X, d)$ is completely metrizable, and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $CL(X)$ is defined via the neighborhood filter at $A \in CL(X)$.
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $CL(X)$ is defined via the neighborhood filter at $A \in CL(X)$ of the form $\{ C \in CL(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon) \}$, where $S \in S$, $\varepsilon > 0$. When $S$ is stable under small enlargements, $S$ is topological. Let $\tau_S$ be this topology of bornological convergence.

Theorem (Zs.)

The following are equivalent:

1. $(CL(X), \tau_S)$ is completely metrizable,
2. $S$ is a bornology, stable under small enlargements, with a countable base, and $(X, d)$ is completely metrizable, and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $\text{CL}(X)$ is defined via the neighborhood filter at $A \in \text{CL}(X)$ of the form \( \{ C \in \text{CL}(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon) \} \), where $S \in S$, $\varepsilon > 0$.

When $S$ is stable under small enlargements,
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $CL(X)$ is defined via the neighborhood filter at $A \in CL(X)$ of the form $\{C \in CL(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon)\}$, where $S \in S$, $\varepsilon > 0$.

When $S$ is *stable under small enlargements*, $S$ is topological.
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $CL(X)$ is defined via the neighborhood filter at $A \in CL(X)$ of the form $\{ C \in CL(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon) \}$, where $S \in S$, $\varepsilon > 0$.

When $S$ is stable under small enlargements, $S$ is topological. Let $\tau_S$ be this topology of bornological convergence.

Theorem (Zs.)

The following are equivalent:

1. $(CL(X), \tau_S)$ is completely metrizable,
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $\text{CL}(X)$ is defined via the neighborhood filter at $A \in \text{CL}(X)$ of the form $\{C \in \text{CL}(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon)\}$, where $S \in S, \varepsilon > 0$.

When $S$ is \textit{stable under small enlargements}, $S$ is topological. Let $\tau_S$ be this topology of bornological convergence.

Theorem (Zs.)

\textit{The following are equivalent:}

1. $(\text{CL}(X), \tau_S)$ is completely metrizable,
2. $S$ is a bornology,
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $CL(X)$ is defined via the neighborhood filter at $A \in CL(X)$ of the form $\{ C \in CL(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon) \}$, where $S \in S$, $\varepsilon > 0$.

When $S$ is stable under small enlargements, $S$ is topological. Let $\tau_S$ be this topology of bornological convergence.

Theorem (Zs.)

The following are equivalent:

1. $(CL(X), \tau_S)$ is completely metrizable,
2. $S$ is a bornology, stable under small enlargements,
## Topology of bornological convergence $\mathcal{T}_S$

### Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $\text{CL}(X)$ is defined via the neighborhood filter at $A \in \text{CL}(X)$ of the form

$$\{ C \in \text{CL}(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon) \},$$

where $S \in S$, $\varepsilon > 0$.

When $S$ is *stable under small enlargements*, $S$ is topological. Let $\mathcal{T}_S$ be this topology of bornological convergence.

### Theorem (Zs.)

The following are equivalent:

1. $(\text{CL}(X), \mathcal{T}_S)$ is completely metrizable,
2. $S$ is a bornology, stable under small enlargements, with a countable base, and
Definition

Let $S$ be an ideal of subsets of $X$; the so-called $S$-convergence on $CL(X)$ is defined via the neighborhood filter at $A \in CL(X)$ of the form \( \{ C \in CL(X) : S \cap A \subseteq B_d(C, \varepsilon) \text{ and } S \cap C \subseteq B_d(A, \varepsilon) \} \), where $S \in S$, $\varepsilon > 0$.

When $S$ is stable under small enlargements, $S$ is topological. Let $\tau_S$ be this topology of bornological convergence.

Theorem (Zs.)

The following are equivalent:

1. \((CL(X), \tau_S)\) is completely metrizable,
2. $S$ is a bornology, stable under small enlargements, with a countable base, and
   - $(X, d)$ is completely metrizable, and $(\tilde{X} \setminus X, \tilde{d})$ is separable.
Open problems

1. $(X, d)$ completely metrizable $\Rightarrow (\mathcal{C}(X), \tau_{Hd})$ Baire space

2. $(\mathcal{C}(X), \tau_{Hd})$ hereditarily Baire $\Rightarrow (X, d)$ completely metrizable
Open problems

1. $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ Baire space
Open problems

1. $(X, d)$ completely metrizable $\Rightarrow (CL(X), \tau_{H_d})$ Baire space
2. $(CL(X), \tau_{H_d})$ hereditarily Baire $\Rightarrow ? (X, d)$ completely metrizable