ON BAIRENESS OF THE WIJSMAN HYPERSPACE

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Abstract. Baireness of the Wijsman hyperspace topology is characterized for a metrizable base space with a countable-in-itself π-base; further, a separable 1st category metric space is constructed with a Baire Wijsman hyperspace.

1. Introduction

There has been a considerable effort in exploring various completeness properties of the Wijsman hyperspace topology w_d, i.e. the weak topology on the nonempty closed subsets of the metric space (X, d) generated by the distance functionals viewed as functions of set argument [17]. It was first shown by *Effros* [10], that a Polish space admits a metric for which the Wijsman topology is Polish; later, *Beer* showed [2],[3], that given a separable *complete* metric base space, the corresponding Wijsman hyperspace is Polish. Finally, *Costantini* demonstrated in [6], that Polish base spaces always generate Polish Wijsman topologies (a short proof, using the so-called strong Choquet game, was found by the author in [19]). As a related result, note that the Wijsman hyperspace is analytic iff X is analytic [1].

Beer asked, whether complete metrizability of X alone (without separability) is equivalent to some completeness property of the Wijsman hyperspace. *Costantini* [7] showed that a natural candidate, Čech-completeness, is not the right property; on the other side, complete metrizability of X guarantees Baireness [18], even strong α-favorability [19], of the Wijsman hyperspace regardless of the underlying metric on X. It is also known, that less than complete metrizability of X - e.g. having a dense completely metrizable subspace [20] or being a separable Baire space [18], respectively - guarantees Baireness of the Wijsman topology; however, w_d may be non-hereditarily Baire, even if X is separable, hereditarily Baire and has a dense completely metrizable subspace [20], or X is completely metrizable [9], respectively.

It is the purpose of this paper to continue in this research by characterizing Baireness of the Wijsman hyperspace for almost locally separable metrizable spaces.

A space is *almost locally separable*, provided the set of points of local separability is dense. In a metrizable space, this is equivalent to having a *countable-in-itself π-base*, i.e. a π-base, each element of which contains only countably many elements of the π-base [21] (cf. locally countable pseudo-base of Oxtoby [14]). A topological space is a *Baire space*, provided

2000 *Mathematics Subject Classification.* Primary 54B20; Secondary 54E52.

Key words and phrases. Wijsman topology, ball (proximal) topology, Baire space, countable-in-itself π-base.

countable collections of dense open subsets have a dense intersection or, equivalently, if nonempty open sets are of 2nd Baire category [12].

Let $CL(X)$ stand for the space of nonempty closed subsets of a topological space (X, τ) (the so-called *hyperspace* of X), and for $M \subseteq X$ define

$$M^- = \{A \in CL(X) : A \cap M \neq \emptyset\},$$
$$M^+ = \{A \in CL(X) : A \subseteq M\}.$$

We will write M^c, $\overline{M}$ for the complement and closure, respectively, of M in X. If (X, τ) is a metrizable topological space with a compatible metric d, denote by $S(x, \varepsilon)$ (resp. $B(x, \varepsilon)$) the open (resp. closed) ball of radius $\varepsilon > 0$ about $x \in X$, and put $S(M, \varepsilon) = \bigcup_{m \in M} S(m, \varepsilon)$ for the open ε-hull of $M \subseteq X$. Denote by $B(X)$ the collection of finite unions of closed balls.

The *Wijsman topology* w_d on $CL(X)$ is the weak topology generated by the distance functionals $d(x, A) = \inf\{d(x, a); a \in A\}$ $(x \in X, A \in CL(X))$ viewed as functionals of set argument. It is easy to show that subbase elements of w_d are of the form U^- and $\{A \in CL(X) : d(x, A) > \varepsilon\}$, where $U \in \tau, x \in X$ and $\varepsilon > 0$. The Wijsman topology is a fundamental tool in the construction of the lattice of hyperspace topologies, since many of these arise as suprema and infima, respectively of appropriate Wijsman topologies [4],[8].

The *ball proximal topology* bp_d has subbase elements of the form U^- and $\{A \in CL(X) : \inf\{d(a, b) : (a, b) \in A \times B\} > \varepsilon\}$, where $U \in \tau$, $B \in \Delta$, $\varepsilon > 0$; it coincides with the Wijsman topology when X is a normed space (for a characterization of this coincidence see [11]). Moreover, bp_d is Baire if and only if w_d is [18]. As we will see, there is an even simpler hypertopology on $CL(X)$ with this “Baire connection”, the so-called ball topology.

The *ball topology* b_d has subbase elements of the form U^- and $(B^c)^+$, where $U \in \tau$ and $B \in B(X)$. It is not hard to show that the collection

$$\mathcal{B} = \{(B^c)^+ \cap \bigcap_{i \leq n} S(x_i, r)^- : B \in B(X), r > 0, n \in \omega, \text{ the } S(x_i, r)\text{'s are pairwise disjoint and miss } B\}$$

is a base for b_d. The ball topology is a hit-and-miss topology, like the well-known Vietoris or Fell topologies [3]; b_d may be non-regular [11], so it is certainly not always equal to the (completely regular) Wijsman topology, however, we have:

Theorem 1.1. *The following are equivalent:*

(i) $(CL(X), w_d)$ *is a Baire space,*
(ii) $(CL(X), b_d)$ *is a Baire space.*

Proof. If $\mathcal{U}_w = \bigcap_{i \leq n} U_i^- \cap \bigcap_{j \leq m} \{A \in CL(X) : d(x_j, A) > \varepsilon_j\} \in w_d$ is nonempty, $r = \min\{\frac{d(x_j, A) - \varepsilon_j}{2} : j \leq m\}$, and

$$\mathcal{U}_b = \bigcap_{i \leq n} U_i^- \cap \bigcap_{j \leq m} ((B(x_j, \varepsilon_j + r))^c)^+ \in b_d,$$

then $\emptyset \neq \mathcal{U}_b \subseteq \mathcal{U}_w$.

Conversely, if $\mathcal{U}_b = \bigcap_{i\leq n} U_i^- \cap \bigcap_{j\leq m}((B(x_j,\varepsilon_j))^c)^+ \in b_d$ is nonempty, and $a_i \in U_i$ so that $d(x_j, a_i) > \varepsilon_j$ for all i, j, then $r = \min_{i,j} d(x_j, a_i) > \varepsilon_j$ for all j, and for

$$\mathcal{U}_w = \bigcap_{i\leq n} U_i^- \cap \bigcap_{j\leq m} \{A \in CL(X) : d(x_j, A) > \frac{\varepsilon_j + r}{2}\} \in w_d$$

we have $\{a_0, \dots, a_n\} \in \mathcal{U}_w \subseteq \mathcal{U}_b$. The theorem now follows by [16], Proposition 3.4. □ □

2. Main Results

We need some auxiliary material first: natural numbers will be viewed as sets of predecessors, $\Delta \subseteq CL(X)$ will be closed under finite unions, bold symbols will denote notions related to the product space $\mathbf{X} = X^\omega$ endowed with the so-called *pinched-cube topology* [15] $\boldsymbol{\tau} = \boldsymbol{\tau}(\Delta)$ having the base

$$\boldsymbol{\mathcal{B}} = \{(\prod_{i\leq n} U_i) \times (B^c)^{\omega\setminus(n+1)} : B \in \Delta, n \in \omega,\ U_i \in \tau, U_i \subseteq B^c\ \forall\, i \leq n\}.$$

If $J \subseteq \omega$, the projection map $\pi_J : (\mathbf{X}, \boldsymbol{\tau}) \to X^J$ is continuous and open. If $\mathbf{C} \subseteq \mathbf{X}$, $J \subseteq \omega$ and $x \in X^J$, denote $\mathbf{C}[x] = \mathbf{C} \cap \pi_J^{-1}(x)$; further, if $\boldsymbol{\mathcal{C}}$ is a collection of subsets of $\mathbf{X}$, put $\boldsymbol{\mathcal{C}}[x] = \{\mathbf{C} \in \boldsymbol{\mathcal{C}} : \mathbf{C}[x] \neq \emptyset\}$.

The proof of the next theorem is a modification of analogous results about the Tychonoff and box products of Baire spaces from [21], we sketch the proof for completeness:

Theorem 2.1. *If (X, τ) is a topological Baire space with a countable-in-itself π-base, then $(\mathbf{X}, \boldsymbol{\tau})$ is a Baire space.*

Proof. Let $\mathcal{P}$ be a countable-in-itself π-base of X. Let $\{\mathbf{G}_n\}_n$ be a decreasing sequence of dense open subsets of $(\mathbf{X}, \boldsymbol{\tau})$ and fix a nonempty $\boldsymbol{\tau}$-open $\mathbf{V}$. Choose $\mathbf{V}_0 \in \boldsymbol{\mathcal{B}}$ with $\mathbf{V}_0 \subseteq \mathbf{V} \cap \mathbf{G}_0$, $\pi_1(\mathbf{V}_0) \in \mathcal{P}$ and put $\boldsymbol{\mathcal{B}}_0 = \{\mathbf{V}_0\}$. By induction, we can define $\boldsymbol{\mathcal{B}}_i \subseteq \boldsymbol{\mathcal{B}}$ for each $i \geq 1$ so that $\boldsymbol{\mathcal{B}}_i = \bigcup_{\mathbf{B}\in \boldsymbol{\mathcal{B}}_{i-1}} \boldsymbol{\mathcal{B}}_i(\mathbf{B})$, where for all $\mathbf{B} \in \boldsymbol{\mathcal{B}}_{i-1}$, $\boldsymbol{\mathcal{B}}_i(\mathbf{B})$ is a maximal collection such that

(1) $\mathbf{A} \subseteq \mathbf{B} \cap \mathbf{G}_i$ for each $\mathbf{A} \in \boldsymbol{\mathcal{B}}_i(\mathbf{B})$,
(2) $\pi_{i+1}(\mathbf{A}) \in \{\prod_{k\leq i} P_k : (P_0, \dots, P_i) \in \mathcal{P}^{i+1}\}$ for each $\mathbf{A} \in \boldsymbol{\mathcal{B}}_i(\mathbf{B})$,
(3) $\{\pi_i(\mathbf{A}) : \mathbf{A} \in \boldsymbol{\mathcal{B}}_i(\mathbf{B})\}$ is pairwise disjoint.

Notice, that each $\boldsymbol{\mathcal{B}}_i(\mathbf{B})$ is countable, since, by (2), $\pi_i(\mathbf{B})$ has ccc for each $i \geq 1$ and $\mathbf{B} \in \boldsymbol{\mathcal{B}}_{i-1}$. For $i \geq 1$ denote

$$\boldsymbol{\mathcal{U}}_i = \bigcup_{\mathbf{B}\in \boldsymbol{\mathcal{B}}_i} \{\mathbf{U} \in \boldsymbol{\mathcal{B}} : \pi_{\omega\setminus i}(\mathbf{U}) = \pi_{\omega\setminus i}(\mathbf{B}) \text{ and } \mathbf{U} \subseteq \mathbf{B}\}.$$

For each $\mathbf{B} \in \boldsymbol{\mathcal{B}}_1(\mathbf{V}_0)$ define

$$Y_{\mathbf{B}}^{(0)} = \{x \in X : \mathbf{B}[x] \neq \emptyset \text{ and } \forall \mathbf{U} \in \boldsymbol{\mathcal{U}}_2(\mathbf{U} \subseteq \mathbf{B} \Rightarrow \mathbf{U}[x] = \emptyset)\}.$$

Claim 2.1.1. *$Y_{\mathbf{B}}^{(0)}$ is nowhere dense in X for each $\mathbf{B} \in \boldsymbol{\mathcal{B}}_1(\mathbf{V}_0)$.*

(Assume by contradiction that for some $\mathbf{B} \in \mathcal{B}_1(\mathbf{V}_0)$, $Y^{(0)}_{\mathbf{B}}$ is dense in some $U \in \tau$. Then $\mathbf{B} \cap \pi_1^{-1}(U) \neq \emptyset$, and by maximality of $\mathcal{B}_2(\mathbf{B})$, there exists $\mathbf{E} \in \mathcal{B}_2(\mathbf{B})$ such that $\mathbf{U} = \mathbf{E} \cap \pi_1^{-1}(U) \neq \emptyset$. Then $\mathbf{U} \in \mathcal{U}_2$, $\mathbf{U} \subseteq \mathbf{B}$, and $\pi_1(\mathbf{U})$ is a nonempty open subset of U; thus, it intersects with $Y^{(0)}_{\mathbf{B}}$, say, in x. To get a contradiction, note that $x \in \pi_1(\mathbf{U})$ means $\mathbf{U}[x] \neq \emptyset$; on the other hand, $x \in Y^{(0)}_{\mathbf{B}}$ implies $\mathbf{U}[x] = \emptyset$.)

Define $W_0 = \pi_1(\bigcup \mathcal{B}_1(\mathbf{V}_0))$. Since X is a Baire space, by Claim 2.1.1, we can choose some

$$x_0 \in W_0 \setminus \bigcup \{Y^{(0)}_{\mathbf{B}} : \mathbf{B} \in \mathcal{B}_1(\mathbf{V}_0)\}.$$

Then, by (3), there is a unique $\mathbf{V}_1 \in \mathcal{B}_1(\mathbf{V}_0)[x_0]$; further, for each $\mathbf{B} \in \mathcal{B}_1(\mathbf{V}_0)[x_0]$ there is a $\mathbf{U} \in \mathcal{U}_2$ with $\mathbf{U} \subseteq \mathbf{B}$ and $\mathbf{U}[x_0] \neq \emptyset$.

By induction, assume that for all $i \leq n$ ($n \in \omega$), $\mathbf{V}_i \in \mathcal{B}_i$, and

$$x_i \in W_i = \pi_{(i+1)\setminus i}(\bigcup \mathcal{B}_{i+1}(\mathbf{V}_i)[x_0, \dots, x_{i-1}])$$

have been defined (when $i = 0$, $\mathcal{B}_1(\mathbf{V}_0)[x_0, x_{-1}]$) is meant to be $\mathcal{B}_1(\mathbf{V}_0)$) so that

(4) $\forall 1 \leq i \leq n\, (\mathbf{V}_i \in \mathcal{B}_i(\mathbf{V}_{i-1})[x_0, \dots, x_{i-1}])$,

(5) $\forall i \leq n\, \forall \mathbf{B} \in \mathcal{B}_{i+1}(\mathbf{V}_i)[x_0, \dots, x_i]\ \exists \mathbf{U} \in \mathcal{U}_{i+2}\ (\mathbf{U} \subseteq \mathbf{B}$ and $\mathbf{U}[x_0, \dots, x_i] \neq \emptyset)$.

Since $x_n \in W_n$, and, by (3), $\{\pi_{(n+1)\setminus n}(\mathbf{A}) : \mathbf{A} \in \mathcal{B}_{n+1}(\mathbf{V}_n)[x_0, \dots, x_{n-1}]\}$ is pairwise disjoint, there is a unique $\mathbf{V}_{n+1} \in \mathcal{B}_{n+1}(\mathbf{V}_n)[x_0, \dots, x_{n-1}]$ with $\mathbf{V}_{n+1}[x_0, \dots, x_n] \neq \emptyset$; thus, (4) holds for $1 \leq i \leq n+1$.

Moreover, (5) implies that there is some $\mathbf{U} \in \mathcal{U}_{n+2}$ with $\mathbf{U} \subseteq \mathbf{V}_{n+1}$ and $\mathbf{U}[x_0, \dots, x_n] \neq \emptyset$. We can also find $\mathbf{B} \in \mathcal{B}_{n+2}$ with $\mathbf{U} \subseteq \mathbf{B}$ and $\pi_{\omega\setminus(n+2)}(\mathbf{U}) = \pi_{\omega\setminus(n+2)}(\mathbf{B})$. By (1), $\mathbf{B} \subseteq \mathbf{G}_{n+2}$, and since $\mathbf{B}$ intersects $\mathbf{V}_{n+1}$, it follows by (3), that $\mathbf{B} \subseteq \mathbf{V}_{n+1}$; thus, $\mathbf{B} \in \mathcal{B}_{n+2}(\mathbf{V}_{n+1})[x_0, \dots, x_n]$, so

$$W_{n+1} = \pi_{(n+2)\setminus(n+1)}(\bigcup \mathcal{B}_{n+2}(\mathbf{V}_{n+1})[x_0, \dots, x_n])$$

is a nonempty open subset of X. For each $\mathbf{B} \in \mathcal{B}_{n+2}(\mathbf{V}_{n+1})[x_0, \dots, x_n]$ define

$$Y^{(n+1)}_{\mathbf{B}} = \{x \in X : \mathbf{B}[x_0, \dots, x_n, x] \neq \emptyset \text{ and } \forall \mathbf{U} \in \mathcal{U}_{n+3}(\mathbf{U} \subseteq \mathbf{B} \Rightarrow \mathbf{U}[x_0, \dots, x_n, x] = \emptyset)\}.$$

Claim 2.1.2. *$Y^{(n+1)}_{\mathbf{B}}$ is nowhere dense in X for each $\mathbf{B} \in \mathcal{B}_{n+2}(\mathbf{V}_{n+1})[x_0, \dots, x_n]$.*

(Indeed, if $Y^{(n+1)}_{\mathbf{B}}$ is dense in some $U \in \tau$, then $\mathbf{B} \cap \pi^{-1}_{(n+2)\setminus(n+1)}(U) \neq \emptyset$, and by maximality of $\mathcal{B}_{n+3}(\mathbf{B})$, there exists $\mathbf{E} \in \mathcal{B}_{n+3}(\mathbf{B})$ such that $\mathbf{U} = \mathbf{E} \cap \pi^{-1}_{(n+2)\setminus(n+1)}(U) \neq \emptyset$. Then $\mathbf{U} \in \mathcal{U}_{n+3}$, $\mathbf{U} \subseteq \mathbf{B}$, and $\pi_{(n+2)\setminus(n+1)}(\mathbf{U})$ is a nonempty open subset of U; thus, it intersects with $Y^{(n+1)}_{\mathbf{B}}$, say, in x. Finally, $x \in \pi_{(n+2)\setminus(n+1)}(\mathbf{U})$ means that $\mathbf{U}[x] \neq \emptyset$; on the other hand, $x \in Y^{(n+1)}_{\mathbf{B}}$ implies $\mathbf{U}[x] = \emptyset$.)

Now, X is a Baire space so, by Claim 2.1.2, we can find

$$x_{n+1} \in W_{n+1} \setminus \bigcup \{Y^{(n+1)}_{\mathbf{B}} : \mathbf{B} \in \mathcal{B}_{n+2}(\mathbf{V}_{n+1})[x_0, \dots, x_n]\},$$

so (5) is satisfied for $i \le n+1$. By induction, we have constructed $\mathbf{x} = (x_n)_{n\in\omega} \in \mathbf{X}$, and a sequence $\{\mathbf{V}_n \in \boldsymbol{\mathcal{B}} : n \in \omega\}$ with $\mathbf{V}_{n+1} \in \boldsymbol{\mathcal{B}}_{n+1}(\mathbf{V}_n)[x_0,\dots,x_n]$ for all $n \in \omega$. Then $\mathbf{x} \in \mathbf{V}_n \subseteq \mathbf{V} \cap \mathbf{G}_n$ for each $n \in \omega$, so $\mathbf{V} \cap \bigcap_n \mathbf{G}_n \neq \emptyset$, and $(\mathbf{X}, \boldsymbol{\tau})$ is a Baire space. □ □

Theorem 2.2. *Let X be metrizable with a compatible metric d.*
If $(\mathbf{X}, \boldsymbol{\tau}(B(X)))$ is a Baire space, then $(CL(X), b_d)$ is a Baire space.

Proof. The set

$$S(X) = \{A \in CL(X) : \ A \text{ separable}\}$$

is dense in $(CL(X), b_d)$, since even the set of finite subsets of X is; thus, we only need to prove that $(S(X), b_d\restriction_{S(X)})$ is a Baire space. Define $\varphi : (\mathbf{X}, \boldsymbol{\tau}) \to (S(X), b_d\restriction_{S(X)})$ via

$$\varphi(\mathbf{x}) = \overline{\{x_k : \ k \in \omega\}}, \text{ where } \mathbf{x} = (x_k)_k \in \mathbf{X}.$$

We will be done, if we show that φ is continuous and feebly open (i.e. the interior of $\varphi(\mathbf{V})$ is nonempty for each nonempty $\mathbf{V} \in \boldsymbol{\tau}$), since Baire spaces are invariant of these mappings (see [12] or [13]).

To see continuity, take a basic $\mathcal{V} = (B^c)^+ \cap \bigcap_{i\le m} V_i^- \in \mathcal{B}$, where $B = \bigcup_{j\le p} B(z_j, \varepsilon_j) \in B(X)$. If $\mathbf{x} = (x_k)_k \in \varphi^{-1}(\mathcal{V})$, then $\varphi(\mathbf{x}) \in \mathcal{V}$, so there exists a $k_i \in \omega$ with $x_{k_i} \in V_i$ for each $i \le m$. We can find some $\delta_j > \varepsilon_j$ so that $d(z_j, f(\mathbf{x})) > \delta_j$ for each $j \le p$, as well as neighborhoods U_i of x_{k_i} (for all $i \le m$) that are subsets of $U = (\bigcup_{j\le p} B(z_j, \delta_j))^c$. Then $\mathbf{U} = (\prod_{i\le m} U_i) \times U^{\omega\setminus(m+1)} \in \boldsymbol{\mathcal{B}}$, and $\mathbf{x} \in \mathbf{U} \subseteq \varphi^{-1}(\mathcal{V})$.

To justify feeble openness of φ, take a nonempty $\mathbf{V} = (\prod_{i\le m} V_i) \times (B^c)^{\omega\setminus(m+1)} \in \boldsymbol{\mathcal{B}}$, where $B = \bigcup_{j\le p} B(z_j, \varepsilon_j) \in B(X)$. Denote $\mathcal{V} = (B^c)^+ \cap \bigcap_{i\le m} V_i^- \in \mathcal{B}$, and take an $A \in \mathcal{V} \cap S(X)$.

If $C = \{c_k : k \in \omega\}$ is a countable dense subset of A, where $c_i \in C \cap V_i$ for each $i \le m$, then $\mathbf{c} = (c_k)_k \in \mathbf{V}$, so $A = \varphi(\mathbf{c}) \in \varphi(\mathbf{V})$. Consequently, $\emptyset \neq \mathcal{V} \cap S(X) \subseteq \varphi(\mathbf{V})$. □ □

Theorem 2.3. *If X is almost locally separable, then the following are equivalent:*

(i) *$(CL(X), w_d)$ is Baire for each compatible metric d on X,*
(ii) *$(CL(X), bp_d)$ is Baire for every compatible metric d on X,*
(iii) *$(CL(X), b_d)$ is Baire for every compatible metric d on X,*
(iv) *X is a Baire space.*

Proof. (i)⇔(ii)⇔(iii) follows from Theorem 1.1, and (iv)⇒(iii) from Theorems 2.1 and 2.2

(iii)⇒(iv) If X is not a Baire space, it has a nonempty open 1st category subset U, and so there exists a separable closed 1st category subset C of U with a nonempty interior $intC$. If d_0 is a compatible totally bounded metric on C, then by a theorem of Bing [5], it can be extended to a compatible metric d on X. Let C_n be an increasing sequence of closed nowhere dense sets such that $C = \bigcup_{n\in\omega} C_n$. By total boundedness of d on C, there exists a finite set $F_{k,n} \subseteq C_n$ for each $n \in \omega, k \ge 1$, such that $C_n \subseteq S(F_{k,n}, \frac{1}{k})$. Define

$$\mathcal{G}_n = \bigcup_{k\ge 1} \bigcup_{x\in F_{k,n}} (B(x, \tfrac{1}{k})^c)^+.$$

Then $\mathcal{G}_n$ is clearly b_d-open. Moreover, it is also dense in $(CL(X), b_d)$: indeed, let $\mathcal{V} = (B^c)^+ \cap \bigcap_{i \le m} V_i^- \in \mathcal{B}$. For each $i \le m$ there exist $k_i \ge 1$ and $v_i \in V_i \setminus S(C_n, \frac{1}{k_i})$, since otherwise, $V_i \subseteq \bigcap_{k \ge 1} S(C_n, \frac{1}{k}) = C_n$ for some $i \le m$, which would contradict nowhere density of C_n. If $k = \max\{k_i : i \le m\}$ and $A = \{v_0, \ldots, v_m\}$, then $A \cap S(F_{k,n}, \frac{1}{k}) = \emptyset$, so $A \in \mathcal{V} \cap \mathcal{G}_n$. To conclude, notice that $\bigcap_{n \in \omega} \mathcal{G}_n$ is disjoint to $(intC)^-$. □ □

Since the Wijsman hyperspace is metrizable iff X is separable [3], we have

Corollary 2.4. *The following are equivalent:*

(i) $(CL(X), w_d)$ *is a metrizable Baire space for each compatible metric* d *on* X*,*
(ii) X *is a separable Baire space.*

In light of Theorem 2.3, it is natural to ask whether Baireness of just a single Wijsman topology w_d (bp_d, b_d, respectively) implies Baireness of X. The following example (given by *R. Pol*) shows, that this is not the case:

Example 2.5. *There exists a separable 1st category metric space with a Baire Wijsman (ball proximal, ball, resp.) hyperspace.*

Proof. Consider ω^ω with the Baire metric

$$e(x, y) = 1/\min\{n : \ x(n) \neq y(n)\}$$

and its 1st category subset $\omega^{<\omega}$ of sequences eventually equal to zero. Then the product $X = \omega^{<\omega} \times \omega^\omega$ is a separable, 1st category space endowed with the metric $d((x_0, x_1), (y_0, y_1)) = \max\{e(x_0, y_0), e(x_1, y_1))\}$.

We claim that $(CL(X), b_d)$ is a Baire space: let $p : X \to \omega^{<\omega}$ be the projection onto the first axis. Let $\mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \ldots$ be dense open sets in $(CL(X), b_d)$, and $\mathcal{U} \in \mathcal{B}$. Inductively, we will define $\mathcal{U}_i \in \mathcal{B}$ with $\mathcal{U}_i \subseteq \mathcal{G}_i$, and finite sets $F_i \in \mathcal{U}_i$ such that for each $u \in F_i$ there is $u^\star \in F_{i+1}$ with $p(u) = p(u^\star)$ and $d(u, u^\star) < \frac{1}{i}$.

Let $\mathcal{U}_1 \in \mathcal{B}$ be a nonempty subset of $\mathcal{G}_1$, and choose a finite set $F_1 \in \mathcal{U}_1$. Suppose that $\mathcal{U}_i$ and F_i have been defined for some $i \ge 1$. If

$$m_i \ge \max\{n : p(u)(n) \neq 0 \text{ for some } u \in F_i\},$$

and $u' \in S(u, \frac{1}{m_i})$ for some $u \in F_i$, then $p(u) = p(u')$. Since $\mathcal{G}_{i+1}$ is dense and $F_i \in \mathcal{U}_i \cap \bigcap_{u \in F_i} S(u, \frac{1}{m_i})$, we can find $\mathcal{U}_{i+1} \in \mathcal{B}$ with

$$\mathcal{U}_{i+1} \subseteq \mathcal{G}_{i+1} \cap \mathcal{U}_i \cap \bigcap_{u \in F_i} S(u, \tfrac{1}{m_i}),$$

and choose a finite $F_{i+1} \in \mathcal{U}_{i+1}$. Assume that

$$\mathcal{U}_i = (B_i^c)^+ \cap \bigcap_{u \in F_i} V_u^- \in \mathcal{B}.$$

For any $u \in F_i$, the sequence $u, u^\star, u^{\star\star}, \ldots$ is Cauchy in $\{p(u)\} \times \omega^\omega$, so it converges to some $u^\infty \in S(u, \frac{1}{m_i}) \subseteq V_u$. By the definition of $\mathcal{B}$, the set $\{u^\infty : u \in \bigcup_{i \ge 1} F_i\}$ misses the clopen B_i

for each $i \geq 1$, so

$$\emptyset \neq \overline{\{u^\infty : u \in \bigcup_{i\geq 1} F_i)\}} \in \bigcap_{i\geq 1} \mathcal{U}_i \subseteq \mathcal{U} \cap \bigcap_{i\geq 1} \mathcal{G}_i;$$

thus, $(CL(X), b_d)$ is a Baire space. □ □

Acknowledgements. The author wishes to thank *Roman Pol* for communicating Example 2.5 to him, as well as the referee for the useful comments concerning Theorem 2.1 that greatly improved the paper.

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