ON BAIRENESS OF THE WIJSMAN HYPERSPACE

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Abstract. Baireness of the Wijsman hyperspace topology is characterized for a metrizable base space with a countable-in-itself $\pi$-base; further, a separable 1st category metric space is constructed with a Baire Wijsman hyperspace.

1. Introduction

There has been a considerable effort in exploring various completeness properties of the Wijsman hyperspace topology $w_d$, i.e. the weak topology on the nonempty closed subsets of the metric space $(X, d)$ generated by the distance functionals viewed as functions of set argument [17]. It was first shown by Effros [10], that a Polish space admits a metric for which the Wijsman topology is Polish; later, Beer showed [2],[3], that given a separable complete metric base space, the corresponding Wijsman hyperspace is Polish. Finally, Costantini demonstrated in [6], that Polish base spaces always generate Polish Wijsman topologies (a short proof, using the so-called strong Choquet game, was found by the author in [19]). As a related result, note that the Wijsman hyperspace is analytic iff $X$ is analytic [1].

Beer asked, whether complete metrizability of $X$ alone (without separability) is equivalent to some completeness property of the Wijsman hyperspace. Costantini [7] showed that a natural candidate, Čech-completeness, is not the right property; on the other side, complete metrizability of $X$ guarantees Baireness [18], even strong $\alpha$-favorability [19], of the Wijsman hyperspace regardless of the underlying metric on $X$. It is also known, that less than complete metrizability of $X$ - e.g. having a dense completely metrizable subspace [20] or being a separable Baire space [18], respectively - guarantees Baireness of the Wijsman topology; however, $w_d$ may be non-hereditarily Baire, even if $X$ is separable, hereditarily Baire and has a dense completely metrizable subspace [20], or $X$ is completely metrizable [9], respectively.

It is the purpose of this paper to continue in this research by characterizing Baireness of the Wijsman hyperspace for almost locally separable metrizable spaces.

A space is almost locally separable, provided the set of points of local separability is dense. In a metrizable space, this is equivalent to having a countable-in-itself $\pi$-base, i.e. a $\pi$-base, each element of which contains only countably many elements of the $\pi$-base [21] (cf. locally countable pseudo-base of Oxtoby [14]). A topological space is a Baire space, provided

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countable collections of dense open subsets have a dense intersection or, equivalently, if nonempty open sets are of 2nd Baire category [12].

Let $CL(X)$ stand for the space of nonempty closed subsets of a topological space $(X, \tau)$ (the so-called hyperspace of $X$), and for $M \subseteq X$ define

\[ M^- = \{ A \in CL(X) : A \cap M \neq \emptyset \}, \]

\[ M^+ = \{ A \in CL(X) : A \subseteq M \}. \]

We will write $M^c$, $\overline{M}$ for the complement and closure, respectively, of $M$ in $X$. If $(X, \tau)$ is a metrizable topological space with a compatible metric $d$, denote by $S(x, \varepsilon)$ (resp. $B(x, \varepsilon)$) the open (resp. closed) ball of radius $\varepsilon > 0$ about $x \in X$, and put $S(M, \varepsilon) = \bigcup_{m \in M} S(m, \varepsilon)$ for the open $\varepsilon$-hull of $M \subseteq X$. Denote by $B(X)$ the collection of finite unions of closed balls.

The Wijsman topology $w_d$ on $CL(X)$ is the weak topology generated by the distance functionals $d(x, A) = \inf\{d(x, a) : a \in A\}$ ($x \in X, A \in CL(X)$) viewed as functionals of set argument. It is easy to show that subbase elements of $w_d$ are of the form $U^-$ and \{ $A \in CL(X) : d(x, A) > \varepsilon$, \}, where $U \in \tau$ and $x \in X$ and $\varepsilon > 0$. The Wijsman topology is a fundamental tool in the construction of the lattice of hyperspace topologies, since many of these arise as suprema and infima, respectively of appropriate Wijsman topologies [4],[8].

The ball proximal topology $bp_d$ has subbase elements of the form $U^-$ and \{ $A \in CL(X) : d(x, A) > \varepsilon$, \}, where $U \in \tau$ and $x \in X$ and $\varepsilon > 0$. It coincides with the Wijsman topology when $X$ is a normed space (for a characterization of this coincidence see [11]). Moreover, $bp_d$ is Baire if and only if $w_d$ is [18]. As we will see, there is an even simpler hypertopology on $CL(X)$ with this “Baire connection”, the so-called ball topology.

The ball topology $b_d$ has subbase elements of the form $U^-$ and $(B^c)^+$, where $U \in \tau$ and $B \in B(X)$. It is not hard to show that the collection

\[ B = \{(B^c)^+ \cap \bigcap_{i \leq n} S(x_i^+, r^+) : B \in B(X), r > 0, n \in \omega, \]

the $S(x_i^+, r^+)$'s are pairwise disjoint and miss $B$}

is a base for $b_d$. The ball topology is a hit-and-miss topology, like the well-known Vietoris or Fell topologies [3]; $b_d$ may be non-regular [11], so it is certainly not always equal to the (completely regular) Wijsman topology; however, we have:

**Theorem 1.1.** The following are equivalent:

(i) $(CL(X), w_d)$ is a Baire space,

(ii) $(CL(X), b_d)$ is a Baire space.

**Proof.** If $U_w = \bigcap_{i \leq n} U_i^+ \cap \bigcap_{j \leq m} \{ A \in CL(X) : d(x_j, A) > \varepsilon_j \} \in w_d$ is nonempty, $r = \min\{ \frac{d(x_j, A) - \varepsilon_j}{2} : j \leq m \}$, and

\[ U_b = \bigcup_{i \leq n} U_i^- \cap \bigcap_{j \leq m} ((B(x_j, \varepsilon_j + r))^c)^+ \in b_d, \]

then $\emptyset \neq U_b \subseteq U_w$. 

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Conversely, if \( \mathcal{U}_b = \bigcap_{i \leq n} U_i^c \cap \bigcap_{j \leq m} \{ (B(x_j, \varepsilon_j))^c \in b_d \} \) is nonempty, and \( a_i \in U_i \) so that \( d(x_j, a_i) > \varepsilon_j \) for all \( i, j \), then \( r = \min_{i,j} d(x_j, a_i) > \varepsilon_j \) for all \( j \), and for
\[
U_w = \bigcap_{i \leq n} U_i^c \cap \bigcap_{j \leq m} \{ A \in CL(X) : d(x_j, A) > \frac{\varepsilon_j + r}{2} \} \in w_d
\]
we have \( \{ a_0, \ldots, a_n \} \in U_w \subseteq \mathcal{U}_b \). The theorem now follows by [16], Proposition 3.4. \( \square \) \( \square \)

2. Main Results

We need some auxiliary material first: natural numbers will be viewed as sets of predecessors, \( \Delta \subseteq CL(X) \) will be closed under finite unions, bold symbols will denote notions related to the product space \( X = X^\omega \) endowed with the so-called pinched-cube topology [15] \( \tau = \tau(\Delta) \) having the base
\[
\mathcal{B} = \{ \prod_{i \leq n} (U_i) \times (B^c)^{\omega \setminus (n+1)} : B \in \Delta, n \in \omega, U_i \in \tau, U_i \subseteq B^c \ \forall \ i \leq n \}.
\]
If \( J \subseteq \omega \), the projection map \( \pi_J : (X, \tau) \to X^J \) is continuous and open. If \( C \subseteq X, J \subseteq \omega \) and \( x \in X^J \), denote \( C[x] = C \cap \pi_J^{-1}(x) \); further, if \( \mathcal{C} \) is a collection of subsets of \( X \), put \( \mathcal{C}[x] = \{ C \in \mathcal{C} : C[x] \neq \emptyset \} \).

The proof of the next theorem is a modification of analogous results about the Tychonoff and box products of Baire spaces from [21], we sketch the proof for completeness:

**Theorem 2.1.** If \( (X, \tau) \) is a topological Baire space with a countable-in-itself \( \pi \)-base, then \( (X, \tau) \) is a Baire space.

**Proof.** Let \( \mathcal{P} \) be a countable-in-itself \( \pi \)-base of \( X \). Let \( \{ G_n \}_n \) be a decreasing sequence of dense open subsets of \( (X, \tau) \) and fix a nonempty \( \tau \)-open \( V \). Choose \( V_0 \in \mathcal{B} \) with \( V_0 \subseteq V \cap G_0 \), \( \pi_1(V_0) \in \mathcal{P} \) and put \( \mathcal{B}_0 = \{ V_0 \} \). By induction, we can define \( \mathcal{B}_i \subseteq \mathcal{B} \) for each \( i \geq 1 \) so that \( \mathcal{B}_i = \bigcup_{B \in \mathcal{B}_{i-1}} \mathcal{B}_i(B) \), where for all \( B \in \mathcal{B}_{i-1}, \mathcal{B}_i(B) \) is a maximal collection such that

1. \( A \subseteq B \cap G_i \) for each \( A \in \mathcal{B}_i(B) \),
2. \( \pi_{i+1}(A) \in \{ \prod_{k \leq i} P_k : (P_0, \ldots, P_i) \in \mathcal{P}^{i+1} \} \) for each \( A \in \mathcal{B}_i(B) \),
3. \( \{ \pi_i(A) : A \in \mathcal{B}_i(B) \} \) is pairwise disjoint.

Notice, that each \( \mathcal{B}_i(B) \) is countable, since, by (2), \( \pi_i(B) \) has ccc for each \( i \geq 1 \) and \( B \in \mathcal{B}_{i-1} \). For \( i \geq 1 \) denote
\[
\mathcal{U}_{i} = \bigcup_{B \in \mathcal{B}_i} \{ U \in \mathcal{B} : \pi_{\omega \setminus i}(U) = \pi_{\omega \setminus i}(B) \text{ and } U \subseteq B \}.
\]
For each \( B \in \mathcal{B}_1(V_0) \) define
\[
Y_B^{(0)} = \{ x \in X : B[x] \neq \emptyset \text{ and } \forall U \in \mathcal{U}_2(U \subseteq B \Rightarrow U[x] = \emptyset) \}.
\]

**Claim 2.1.1.** \( Y_B^{(0)} \) is nowhere dense in \( X \) for each \( B \in \mathcal{B}_1(V_0) \).
Claim 2.1.2. \( Y^{(n+1)}_B \) is nowhere dense in \( X \) for each \( B \in \mathcal{B}_{n+2}(V_{n+1})[x_0, \ldots, x_n] \).

(Indeed, if \( Y^{(n+1)}_B \) is dense in some \( U \in \tau \), then \( \mathcal{B} \cap \pi_{(n+2) \setminus (n+1)}^{-1}(U) \neq \emptyset \), and by maximality of \( \mathcal{B}_{n+3}(B) \), there exists \( E \in \mathcal{B}_{n+3}(B) \) such that \( U = E \cap \pi_{(n+2) \setminus (n+1)}^{-1}(U) \neq \emptyset \). Then \( U \in \mathcal{U}_{n+3}, U \subseteq B, \) and \( \pi_{(n+2) \setminus (n+1)}(U) \) is a nonempty open subset of \( U \); thus, it intersects with \( Y^{(n+1)}_B \), say, in \( x \). Finally, \( x \in \pi_{(n+2) \setminus (n+1)}(U) \) means that \( U[x] \neq \emptyset \); on the other hand, \( x \in Y^{(n+1)}_B \) implies \( U[x] = \emptyset \).

Now, \( X \) is a Baire space so, by Claim 2.1.2, we can find

\[ x_{n+1} \in W_{n+1} \setminus \bigcup \{ Y^{(n+1)}_B : B \in \mathcal{B}_{n+2}(V_{n+1})[x_0, \ldots, x_n] \}. \]
so (5) is satisfied for \( i \leq n + 1 \). By induction, we have constructed \( x = (x_n)_{n \in \omega} \in X \), and a sequence \( \{V_n \in B : n \in \omega\} \) with \( V_{n+1} \in B_{n+1}(V_n)[x_0, \ldots, x_n] \) for all \( n \in \omega \). Then \( x \in V_n \subseteq V \cap G_n \) for each \( n \in \omega \), so \( V \cap \bigcap_n G_n \neq \emptyset \), and \((X, \tau)\) is a Baire space. \( \square \) \( \square \)

**Theorem 2.2.** Let \( X \) be metrizable with a compatible metric \( d \).

If \((X, \tau(B(X)))\) is a Baire space, then \((CL(X), b_d)\) is a Baire space.

**Proof.** The set

\[
S(X) = \{A \in CL(X) : A \text{ separable}\}
\]

is dense in \((CL(X), b_d)\), since even the set of finite subsets of \( X \) is; thus, we only need to prove that \((S(X), b_d |_{S(X)})\) is a Baire space. Define \( \varphi : (X, \tau) \to (S(X), b_d |_{S(X)}) \) via

\[
\varphi(x) = \{x_k : k \in \omega\}, \text{ where } x = (x_k)_k \in X.
\]

We will be done, if we show that \( \varphi \) is continuous and feebly open (i.e. the interior of \( \varphi(V) \) is nonempty for each nonempty \( V \in \tau \)), since Baire spaces are invariant of these mappings (see [12] or [13]).

To see continuity, take a basic \( V = (B^c)^+ \cap \bigcap_{i \leq m} V_i^- \in B \), where \( B = \bigcup_{j \leq p} B(z_j, \varepsilon_j) \subseteq B(X) \). If \( x = (x_k)_k \in \varphi^{-1}(V) \), then \( \varphi(x) \in V \), so there exists a \( k_i \in \omega \) with \( x_{k_i} \in V_i \) for each \( i \leq m \). We can find some \( \delta_j > \varepsilon_j \) so that \( d(z_j, f(x)) > \delta_j \) for each \( j \leq p \), as well as neighborhoods \( U_i \) of \( x_{k_i} \) (for all \( i \leq m \)) that are subsets of \( U = (\bigcup_{j \leq p} B(z_j, \delta_j))^c \). Then \( U = (\prod_{i \leq m} U_i) \times U^{\omega \setminus (m+1)} \in \mathcal{B} \), and \( x \in U \subseteq \varphi^{-1}(V) \).

To justify feebly openness of \( \varphi \), take a nonempty \( V = (\prod_{i \leq m} V_i^c) \times (B^c)^{\omega \setminus (m+1)} \in \mathcal{B} \), where \( B = \bigcup_{j \leq p} B(z_j, \varepsilon_j) \subseteq B(X) \). Denote \( V = (B^c)^+ \cap \bigcap_{i \leq m} V_i^- \subseteq \mathcal{B} \), and take an \( A \in \mathcal{V} \cap S(X) \).

If \( C = \{c_k : k \in \omega\} \) is a countable dense subset of \( X \), where \( c_i \in C \cap V_i \) for each \( i \leq m \), then \( c = (c_k)_k \in V \), so \( A = \varphi(c) \in \varphi(V) \). Consequently, \( \emptyset \neq \mathcal{V} \cap S(X) \subseteq \varphi(V) \). \( \square \) \( \square \)

**Theorem 2.3.** If \( X \) is almost locally separable, then the following are equivalent:

(i) \((CL(X), w_d)\) is Baire for each compatible metric \( d \) on \( X \),

(ii) \((CL(X), b_{pd})\) is Baire for every compatible metric \( d \) on \( X \),

(iii) \((CL(X), b_d)\) is Baire for every compatible metric \( d \) on \( X \),

(iv) \( X \) is a Baire space.

**Proof.** (i)\( \Leftrightarrow \) (ii)\( \Leftrightarrow \) (iii) follows from Theorem 1.1, and (iv)\( \Rightarrow \) (iii) from Theorems 2.1 and 2.2.

(iii)\( \Rightarrow \) (iv) If \( X \) is a Baire space, it has a nonempty open 1st category subset \( U \), and so there exists a separable closed 1st category subset \( C \) of \( U \) with a nonempty interior \( \operatorname{int} C \).

If \( d_0 \) is a compatible totally bounded metric on \( C \), then by a theorem of Bing [5], it can be extended to a compatible metric \( d \) on \( X \). Let \( C_n \) be an increasing sequence of closed nowhere dense sets such that \( C = \bigcup_{n \in \omega} C_n \). By total boundedness of \( d \) on \( C \), there exists a finite set \( F_{k,n} \subseteq C_n \) for each \( n \in \omega, k \geq 1 \), such that \( C_n \subseteq S(F_{k,n}, \frac{1}{k}) \). Define

\[
G_n = \bigcup_{k \geq 1} \bigcup_{x \in F_{k,n}} (B(x, \frac{1}{k})^c)^+.
\]
Then \( \mathcal{G}_n \) is clearly \( b_d \)-open. Moreover, it is also dense in \((CL(X), b_d)\): indeed, let \( \mathcal{V} = (B^c)^+ \cap \bigcap_{i \leq m} V_i^- \in \mathcal{B} \). For each \( i \leq m \) there exist \( k_i \geq 1 \) and \( v_i \in V_i \setminus S(C_n, \frac{1}{k_i}) \), since otherwise, \( V_i \subseteq \bigcap_{k \geq 1} S(C_n, \frac{1}{k}) = C_n \) for some \( i \leq m \), which would contradict nowhere density of \( C_n \). If \( k = \max\{k_i : i \leq m\} \) and \( A = \{v_0, \ldots, v_m\} \), then \( A \cap S(F_{k,n}, \frac{1}{k}) = \emptyset \), so \( A \in \mathcal{V} \cap \mathcal{G}_n \). To conclude, notice that \( \bigcap_{n \in \omega} \mathcal{G}_n \) is disjoint to \((int\mathcal{C})^- \). \( \square \quad \square \)

Since the Wijsman hyperspace is metrizable iff \( X \) is separable \([3]\), we have

**Corollary 2.4.** The following are equivalent:

(i) \((CL(X), w_d)\) is a metrizable Baire space for each compatible metric \( d \) on \( X \),
(ii) \( X \) is a separable Baire space.

In light of Theorem 2.3, it is natural to ask whether Baireness of just a single Wijsman topology \( w_d \) \((\text{b}_{b_d}, \text{b}_d\text{, respectively})\) implies Baireness of \( X \). The following example (given by R. Pol) shows, that this is not the case:

**Example 2.5.** There exists a separable 1st category metric space with a Baire Wijsman \((\text{ball proximal, ball, resp.})\) hyperspace.

**Proof.** Consider \( \omega^\omega \) with the Baire metric

\[
e(x, y) = 1/\min\{n : x(n) \neq y(n)\}
\]

and its 1st category subset \( \omega^{<\omega} \) of sequences eventually equal to zero. Then the product \( X = \omega^{<\omega} \times \omega^\omega \) is a separable, 1st category space endowed with the metric \( d((x_0, x_1), (y_0, y_1)) = \max\{e(x_0, y_0), e(x_1, y_1)\} \).

We claim that \((CL(X), b_d)\) is a Baire space: let \( p : X \to \omega^{<\omega} \) be the projection onto the first axis. Let \( \mathcal{G}_1 \supseteq \mathcal{G}_2 \supseteq \ldots \) be dense open sets in \((CL(X), b_d)\), and \( \mathcal{U} \in \mathcal{B} \). Inductively, we will define \( \mathcal{U}_i \in \mathcal{B} \) with \( \mathcal{U}_i \subseteq \mathcal{G}_i \), and finite sets \( F_i \in \mathcal{U}_i \) such that for each \( u \in F_i \) there is \( u^* \in F_{i+1} \) with \( p(u) = p(u^*) \) and \( d(u, u^*) < \frac{1}{i} \).

Let \( \mathcal{U}_i \in \mathcal{B} \) be a nonempty subset of \( \mathcal{G}_i \), and choose a finite set \( F_i \in \mathcal{U}_i \). Suppose that \( \mathcal{U}_i \) and \( F_i \) have been defined for some \( i \geq 1 \).

\[
m_i = \max\{n : p(u)(n) \neq 0 \text{ for some } u \in F_i\},
\]

and \( u^* \in S(u, \frac{1}{m_i}) \) for some \( u \in F_i \), then \( p(u) = p(u^*) \). Since \( \mathcal{G}_{i+1} \) is dense and \( F_i \in \mathcal{U}_i \cap \bigcap_{u \in F_i} S(u, \frac{1}{m_i}) \), we can find \( \mathcal{U}_{i+1} \in \mathcal{B} \) with

\[
\mathcal{U}_{i+1} \subseteq \mathcal{G}_{i+1} \cap \mathcal{U}_i \cap \bigcap_{u \in F_i} S(u, \frac{1}{m_i}),
\]

and choose a finite \( F_{i+1} \in \mathcal{U}_{i+1} \). Assume that

\[
\mathcal{U}_i = (B^c)^+ \cap \bigcap_{u \in F_i} V_u^-, \quad u \in F_i.
\]

For any \( u \in F_i \), the sequence \( u, u^*, u^{**}, \ldots \) is Cauchy in \( \{p(u)\} \times \omega^\omega \), so it converges to some \( u^\infty \in S(u, \frac{1}{m_i}) \subseteq V_u \). By the definition of \( \mathcal{B} \), the set \( \{u^\infty : u \in \bigcup_{i \geq 1} F_i\} \) misses the clopen \( B_i \).
for each $i \geq 1$, so

$$\emptyset \neq \left\{u^\infty : u \in \bigcup_{i \geq 1} F_i\right\} \subseteq \bigcap_{i \geq 1} U_i \subseteq U \cap \bigcap_{i \geq 1} G_i;$$

thus, $(CL(X), b_d)$ is a Baire space.

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References


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