NOTE ON HIT-AND-MISS TOPOLOGIES

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This is a continuation of [19]. We characterize first and second countability of the general hit-and-miss hyperspace topology \( \tau_\Delta^+ \) for weakly-\( R_0 \) base spaces. Further, metrizability of \( \tau_\Delta^+ \) is characterized with no preliminary conditions on the base space and the generating family of closed sets and a new proof on uniformizability (i.e. complete regularity) of \( \tau_\Delta^+ \) is given in this general setting, thus generalizing results of [3], [5] and [6].

0. Introduction.

Let \((X, \tau)\) be a topological space and \(CL(X)\) be the nonempty closed subsets of \(X\). Following [2], [3], [5], [16], [17], [19], [20], [21] we will continue to study hit-and-miss hyperspace topologies or \( \Delta \)-topologies on \( CL(X) \), where \( \Delta \) is a fixed subfamily of \( CL(X) \). Two of the most studied hit-and-miss topologies are the Vietoris topology ([14], [13]) and the Fell topology ([7], [13], [17]). In a recent paper [5], Di Maio and Holá have found necessary and sufficient conditions for first and second countability, respectively of the \( \Delta \)-topology \( \tau_\Delta^+ \), if \( X \) is \( T_1 \); more on countability axioms and quasi-uniformizability of \( \tau_\Delta^+ \) was obtained by Holá and Levi in [9], where a characterization of metrizability of \( \tau_\Delta^+ \) is also given for a \( T_1 \) base space \( X \) and \( \Delta \) containing the singletons. Moreover, in [3] (see also [2]),

1991 Mathematics Subject Classification. 54B20, Secondary 54D15, 54E35.

Key words and phrases. Hit-and-miss topologies, countability and separation axioms, metrizability, uniformizability.
Beer and Tamaki characterized uniformizability of $\tau_\Delta^+$ for a Hausdorff $X$ and $\Delta$ containing the singletons.

It is the purpose of this paper to show that quite similar characterizations hold with no preliminary conditions (or with much less restrictive conditions) on $X$ or $\Delta$, respectively. This is achieved by applying techniques and notions from [19] and a completely new approach is employed to characterize complete regularity of $\tau_\Delta^+$.

Note that a characterization of normality of $\tau_\Delta^+$ is not known except for some special cases, like the Vietoris topology ([11], [18]) or the Fell topology ([10]); for some more general results on normality see [6].

1. Notation and terminology.

In the sequel $(X, \tau)$ will be a topological space and $CL(X)$ (resp. $K(X)$) will denote the nonempty closed (resp. nonempty closed compact) subsets of $X$. If $E \subset X$, then $\bar{E}$, $\text{int} E$, $E^c$ will stand for the closure, interior and complement of $E$, respectively in $X$. Put $E^- = \{A \in CL(X); A \cap E \neq \emptyset\}$, $E^+ = \{A \in CL(X); A \subseteq E\}$. In what follows, $\Delta$ will be a fixed but arbitrary nonempty subfamily of $CL(X)$ and for any $\Delta' \subset \Delta$, denote by $\Sigma(\Delta')$ the set of all finite unions of members of $\Delta'$. The hit-and-miss or $\Delta$-topology $\tau_\Delta^+$ for $CL(X)$ has a base all sets of the form $(B^c)^+ \cup \bigcap_{i=1}^n U_i^-$ where $B \in \Sigma(\Delta)$, $U_1, \ldots, U_n \in \tau$ and $n \in \mathbb{N}$ (cf. [3], [17]); this basic element will be denoted by $(U_1, \ldots, U_n)_B^+$ (cf; [20]). If $\Delta = CL(X)$, we obtain the familiar Vietoris topology $\tau_V$, if $\Delta = K(X)$, the Fell topology $\tau_F$.

In accordance with [3], $\Delta$ is said to be a Urysohn family provided whenever $A \in CL(X)$ and $B \in \Delta$ are disjoint, there exists $D \in \Sigma(\Delta)$ such that $B \subset \text{int} D \subset D \subset A^c$. Denote by $\mathcal{F}_\Delta$ the class of all continuous functions $f : X \to [0, 1]$ such that whenever $\inf f < \alpha < \beta < \sup f$, there exists $D \in \Sigma(\Delta)$ with

$$f^{-1}([0, \alpha]) \subset D \subset f^{-1}([0, \beta]),$$

where $f^{-1}(M)$ stands for the preimage of $M \subset [0, 1]$. For $f \in \mathcal{F}_\Delta$ denote by $m_f$ the infimal value functional on $CL(X)$ (cf. [3]) defined by

$$m_f(A) = \inf\{f(x); x \in A\} \text{ for all } A \in CL(X).$$
We will say that $X$ has property $P_\Delta$ provided whenever $A \in CL(X)$ and $x \in A^c_\tau$ there exists $D \in \Delta$ such that $D \subseteq A^c$ and $[x] \cap D \neq \emptyset$ (see [19]). $X$ is called weakly-R$_0$ provided $X$ possesses property $P_{CL(X)}$ or equivalently provided every nonempty difference of $\tau$-open sets contains a nonempty closed subset of $X$ ([19]). Further, $X$ is an R$_0$-space if every open subset of $X$ contains the closure of each of its points ([4]).

We will say that $E \subseteq X$ is c-hemicompact if there exists an increasing sequence of members of $K(X) \cap CL(E)$ which is cofinal in $K(X) \cap CL(E)$. Notions not defined in the paper are used in accordance with [12] (e.g. regular does not include $T_1$).

2. Main results.

First we need some auxiliary material:

**Lemma 2.1.** Let $X$ be weakly-R$_0$, $B, D \in \Sigma(\Delta)$ and $U_1, \ldots, U_n$, $V_1, \ldots, V_m \in \tau$ $(m, n \in \mathbb{N})$. Then the following are equivalent:

(i) $(U_1, \ldots, U_n)_B^+ \subseteq (V_1, \ldots, V_m)_D^+;$

(ii) $B^c \subseteq D^c$ and for every $1 \leq j \leq m$ there exists an $1 \leq i \leq n$ such that $U_i \cap B^c \subseteq V_j \cap D^c$.

**Proof.** Denote $\mathcal{U} = (U_1, \ldots, U_n)_B^+$ and $\mathcal{V} = (V_1, \ldots, V_m)_D^+$. Suppose (i) and choose an $A \in \mathcal{U}$. If $B^c \setminus D^c$ is nonempty, then by the weak-R$_0$ property we can find a nonempty closed set $C \subseteq B^c \setminus D^c$. This implies that $A \cup C \in \mathcal{U} \setminus \mathcal{V}$, which contradicts (i), thus $B^c \subseteq D^c$. Further, if there exists a $1 \leq j \leq m$ such that for each $1 \leq i \leq n$, $\emptyset \neq U_i \cap B^c \setminus V_j \cap D^c$, then we can find a nonempty closed $A_i \subseteq U_i \cap B^c \setminus V_j \cap D^c$, but then $\bigcup_{i=1}^{m} A_i \in \mathcal{U} \setminus \mathcal{V}$, which is a contradiction again, so (ii) holds.

Conversely, suppose (ii) and pick an $A \in \mathcal{U}$. Then $A \subseteq B^c \subseteq D^c$. Further, for every $1 \leq j \leq m$ there is an $1 \leq i \leq n$ such that $U_i \cap B^c \subseteq V_j \cap D^c$, so $A \cap V_j \neq \emptyset$ since $A \cap U_i \neq \emptyset$. It means that $A \in \mathcal{V}$. □

**Lemma 2.2.** If $(CL(X), \tau_\Delta^+)$ is first countable, then every $A \in CL(X)$ is separable.

**Proof.** The proof of Lemma 5.3 in [5] works in every topological space
We can now characterize first countability of the hit-and-miss topology for a weakly-$R_0$ base space $X$ (cf. [5], Theorem 5.4):

**THEOREM 2.3.** Let $X$ be a weakly-$R_0$ space. Then the following are equivalent:

(i) $(\text{Cl}(X), \tau_X^+)$ is first countable;

(ii) $X$ is first countable, every closed set $A \subseteq X$ is separable and there exists a countable family $\Delta_A \subset \Delta$ such that whenever $B \in \Delta$ is disjoint to $A$, then $B \subset D \subset A^c$ for some $D \in \Sigma(\Delta_A)$.

**Proof.** The proof of Theorem 5.4 in [5] can be adopted if Lemma 2.1, Lemma 2.2 and point-closures are used instead of singletons. In the implication (i)$\Rightarrow$(ii) only the proof of first countability of $X$ needs some comments. Let $x \in X$ and put $A_x = \{x\}$. In view of (i) there exist countable families $\Delta_x \subset \Delta$ and $\tau_x \subset \tau$ such that $B_x = \{(U_1, \ldots, U_n)_{\tau_x^+}; B \in \Sigma(\Delta_x), U_1, \ldots, U_n \in \tau_x, n \in \mathbb{N}\}$ forms a countable local base at $A_x$ in $\tau_x^+$. Choose any $\tau$-open neighborhood $U$ of $x$. Then $\mathcal{U} = (U_1, \ldots, U_n)_{\tau_x^+} \subset U^c$ for some $\mathcal{U} \in B_x$, thus by Lemma 2.1, $B^c \cap U \subseteq U$ for $1 \leq i \leq n$ and clearly $x \in B^c \cap U_i$. It means that $\{B^c \cap U; B \in \Sigma(\Delta_x), U \in \tau_x\}$ is a countable local base at $x$. \hfill $\square$

As for second countability of the hit-and-miss topology we have (cf. [5], Theorem 5.13):

**THEOREM 2.4.** Let $X$ be a weakly-$R_0$ space. Then the following are equivalent:

(i) $(\text{Cl}(X), \tau_X^+)\subset \Delta$ is second countable;

(ii) $X$ is second countable and there is a countable family $\Delta' \subset \Delta$ such that whenever $B \in \Delta$ and $A \in \text{Cl}(X)$ are disjoint, then $B \subset D \subset A^c$ for some $D \in \Sigma(\Delta')$.

**Proof.** From (i) we get countable families $\Delta' \subset \Delta$, $\tau' \subset \tau$ such that $\{(U_1, \ldots, U_n)_{\tau'}^+; B \in \Sigma(\Delta'), U_1, \ldots, U_n \in \tau', n \in \mathbb{N}\}$
forms a countable base of $\tau^+_\Delta$. Then $\{B^c \cap U; B \in \Sigma(\Delta'), U \in \tau'\}$ is a countable base for $X$, which easily follows by Lemma 2.1. The rest of the proof is analogous to that of Theorem 5.13 in [5].

It is shown in [19] that regularity and $T_3$-ness of the Vietoris topology are equivalent. We show that it is a general feature of hit-and-miss topologies. First we need the following:

**Lemma 2.5.** The functional $m_f : CL(X) \to [0, 1]$ is $\tau^+_\Delta$-continuous for all $f \in \mathcal{F}_\Delta$.

**Proof.** Choose $f \in \mathcal{F}_\Delta$. Let $\inf f < \alpha < \beta < \sup f$ and $E \in m^{-1}_f((\alpha, \beta))$. Then $\alpha < \inf\{f(x); x \in E\} < \beta$, thus $E \cap f^{-1}((\alpha, \beta)) \neq \emptyset$ and for any $0 < \varepsilon < m_f(E) - \alpha$ we have $f^{-1}([0, \alpha + \varepsilon]) \subset E^c$. Since $f \in \mathcal{F}_\Delta$ we can find a $D \in \Sigma(\Delta)$ such that

$$f^{-1}([0, \alpha + \varepsilon/2]) \subset D \subset f^{-1}([0, \alpha + \varepsilon]),$$

whence $E \subset D^c$. Then $E \in (D^c)^+ \cap (f^{-1}((\alpha, \beta)))^- \subset m_f^{-1}((\alpha, \beta))$. \qed

The following theorem is proved in [3] (Theorem 3.6) for a $T_2$ base space and with $\Delta$ containing the singletons. Here we present a different proof in the completely general setting:

**Theorem 2.6.** The following are equivalent

(i) $(CL(X), \tau^+_\Delta)$ is a Tychonoff space;

(ii) $(CL(X), \tau^+_\Delta)$ is completely regular;

(iii) $(CL(X), \tau^+_\Delta)$ is a $T_3$-space;

(iv) $(CL(X), \tau^+_\Delta)$ is regular;

(v) $X$ has property $P_\Delta$ and $\Delta$ is a Urysohn family.

**Proof.** (v) $\Rightarrow$ (i) In view of Theorem 1 in [19] it suffices to prove that the hyperspace is completely regular. An argument similar to that of in Lemma 3.1 of [3] yields for all $D \in \Delta$ and disjoint $A \in CL(X)$ an $f \in \mathcal{F}_\Delta$ such that $f(D) = 0$ and $f(A) = 1$. Let $A \in CL(X)$ and $\mathcal{U} = (U_1, \ldots, U_n)^+_B$ be a $\tau^+_\Delta$-neighborhood of $A$, where $B \in \Sigma(\Delta)$, $U_1, \ldots, U_n \in \tau$ and $n \in \mathbb{N}$. Then...
Then \( A \subset B^c \) and \( A \cap U_i \neq \emptyset \) for all \( 1 \leq i \leq n \). In virtue of the preceding considerations there exist functions \( f_0, f_1, \ldots, f_n \in \mathcal{F}_\Delta \) such that

\[
\begin{align*}
f_0(B) &= \{0\} \quad \text{and} \quad f_0(A) = \{1\}, \\
f_i(E_i) &= \{0\} \quad \text{and} \quad f_i(U_i^c) = \{1\} \quad \text{for each} \quad 1 \leq i \leq n.
\end{align*}
\]

The by Lemma 2.5, \( m_{f_0}, m_{f_1}, \ldots, m_{f_n} \) are \( 1^- \Delta \)-continuous on \( CL(X) \) so \( F = \max\{1 - m_{f_0}, m_{f_1}, \ldots, m_{f_n}\} \) is \( 1^- \Delta \)-continuous as well. Clearly \( 1 - m_{f_0}(A) = m_{f_1}(A) = \cdots = m_{f_n}(A) = 0 \) so \( F(A) = 0 \). Further if \( E \notin \mathcal{U} \) then either \( E \cap B \neq \emptyset \) or \( E \subset U_i^c \) for some \( 1 \leq i \leq n \). In the first case \( 1 - m_{f_0}(E) = 1 \), whence \( F(E) = 1 \) and in the second case \( m_{f_i}(E) = 1 \), so \( F(E) = 1 \) again.

All the remaining implications follows from Theorem 3 in [19], if regularity of the hyperspace forces \( X \) to have property \( P_\Delta \). Indeed, if \( (CL(X), 1^- \Delta) \) is regular then it is also \( R_0 \), further the hit-and-miss topology is always \( T_0 \) (see [16]) so it is a \( T_1 \)-space (cf. [4], Corollary), which completes the proof by Theorem 1 in [19].

If \( X \) is a Hausdorff space and \( \Delta \) contains the singletons then \( X \) clearly possesses property \( P_\Delta \). Thus the following corollary generalises Theorem 3.6 of [3]:

**Corollary 2.7.** \((CL(X), 1^- \Delta)\) is uniformizable if and only if \( X \) possesses property \( P_\Delta \) and \( \Delta \) is a Urysohn family.

Finally we turn to characterizing metrizability of the hit-and-miss topology:

**Theorem 2.8.** The following are equivalent:

(i) \((CL(X), 1^- \Delta)\) is metrizable;

(ii) \((CL(X), 1^- \Delta)\) is pseudo-metrizable;

(iii) \((CL(X), 1^- \Delta)\) is second countable and regular;

(iv) \( X \) possesses property \( P_\Delta \) and there exists a countable family \( \Delta' \subset \Delta \) such that whenever \( B \in \Delta \) and \( A \in CL(X) \) are disjoint there is a \( D \in \Sigma(\Delta') \) with \( B \subset int D \subset D \subset A^c \).
3. Applications.

In view of our preceding theorems we have:

THEOREM 3.1. (cf. [5], Theorem 5.5) Let $X$ be weakly-$R_0$. Then the following are equivalent:

(i) $(CL(X), \tau_V)$ is first countable;

(ii) every closed subset of $X$ is separable and has a countable base of neighborhoods.

THEOREM 3.2. (cf. [5], Theorem 5.6) Let $X$ be weakly-$R_0$. Then the following are equivalent:

(i) $(CL(X), \tau_F)$ is first countable;

(ii) $X$ is first countable every closed set is separable and every proper open subset is c-hemicompact.

Proof. (i)$\Rightarrow$(ii) The proof of hemicompactness of proper open subsets
of $X$ in [1], Lemma 3.1 is feasible also in our case if closed compact sets are used instead of compact sets and point-closures instead of singletons. Further see our Theorem 2.3. In (ii)⇒(i) the proof of [5], Theorem 5.6 (2)⇒(1) is applicable (using $c$-hemicompactness instead of hemicompactness) along with our Theorem 2.3. □

**Theorem 3.3. (cf. [19]; Theorem 4)** The following are equivalent:

(i) $(CL(X), \tau_V)$ is a Tychonoff space;

(ii) $(CL(X), \tau_V)$ is completely regular;

(iii) $(CL(X), \tau_V)$ is a $T_3$-space;

(iv) $(CL(X), \tau_V)$ is regular;

(v) $(CL(X), \tau_V)$ is uniformizable;

(vi) $X$ is normal and $R_0$.

**Theorem 3.4.** The following are equivalent:

(i) $(CL(X), \tau_F)$ is a Tychonoff space;

(ii) $(CL(X), \tau_F)$ is regular;

(iii) $(CL(X), \tau_F)$ is a Hausdorff space;

(iv) $(CL(X), \tau_F)$ is uniformizable;

(v) $X$ is a locally compact, regular space.

*Proof.* Cf. [17] (Folgerung (a), p. 162) and Theorem 2 of [19]. □

**Theorem 3.5. (cf. [14], Theorem 4.9.7)** The following are equivalent:

(i) $(CL(X), \tau_V)$ is metrizable;

(ii) $X$ is compact and pseudo-metrizable.

*Proof.* (i)⇒(ii) Suppose that $(CL(X), \tau_V)$ is metrizable. Denote by $\tilde{X}$ the quotient space of $X$ induced by identification of points with common closure in $X$. Then in view of Theorem 3 in [15], $(CL(\tilde{X}), \tilde{\tau}_V)$ is homeomorphic to $(CL(X), \tau_V)$, where $\tilde{\tau}_V$ is the Vietoris topology on $CL(\tilde{X})$, $\tilde{\tau}_V$.
consequently it is also metrizable. Further, \((CL(X), \tau_V)\) is a Hausdorff space so by Theorem 2 of [19], \(X\) is regular and hence \(R_0\) as well. Accordingly \(\tilde{X}\) is a \(T_1\)-space, thus by Theorem 4.9.7 of [14], \(\tilde{X}\) is compact, which implies compactness of \(X\) (cf. [15], Theorem 4). Now \(X\) is second countable by Remark 2.9, further it is regular, thus \(X\) is pseudo-metrizable (see [8], p. 167, Exercise 3).

(ii)\(\Rightarrow\)(i) Observe that a pseudo-metrizable space is \(R_0\), hence possesses property \(P_{CL(X)}\) (i.e. weak \(R_0\)-ness). Further by Lemma 2.2, \(X\) is a separable (pseudo-metrizable) space, accordingly second countable as well, which together with compactness and regularity of \(X\) easily yields the second condition of Theorem 2.8 (iv) for \(\Delta = CL(X)\).

THEOREM 3.6. (cf. [1], Theorem 3.4) The following are equivalent:

(i) \((CL(X), \tau_F)\) is metrizable;

(ii) \(X\) is locally compact, regular and second countable.

Proof. (i)\(\Rightarrow\)(ii) According to Remark 2.9, \(X\) is second countable and in virtue of Theorem 3.4, \(X\) is locally compact and regular.

(ii)\(\Rightarrow\)(i) By local compactness plus regularity of \(X\), \(K(X)\) forms a base of neighborhoods for closed compact subsets of \(X\) ([12], p. 146, Theorem 18). Further, second countability of \(X\) yields a countable subfamily of \(K(X)\) which forms also a base of neighborhoods for members of \(K(X)\), thus the second condition of Theorem 2.8 (iv) is fulfilled for \(\Delta = K(X)\). Finally, local compactness and regularity of \(X\) evidently imply property \(P_{K(X)}\), thus Theorem 2.8 applies.

REFERENCES

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