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NOTE ON HIT-AND-MISS TOPOLOGIES

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This is a continuation of [19]. We characterize first and second countability of the general hit-and-miss hyperspace topology τ_Δ^+ for weakly-R_0 base spaces. Further, metrizability of τ_Δ^+ is characterized with no preliminary conditions on the base space and the generating family of closed sets and a new proof on uniformizability (i.e. complete regularity) of τ_Δ^+ is given in this general setting, thus generalizing results of [3], [5] and [6].

0. Introduction.

Let (X, τ) be a topological space and $CL(X)$ be the nonempty closed subsets of X. Following [2], [3], [5], [16], [17], [19], [20], [21] we will continue to study hit-and-miss hyperspace topologies or Δ-topologies on $CL(X)$, where Δ is a fixed subfamily of $CL(X)$. Two of the most studied hit-and-miss topologies are the Vietoris topology ([14], [13]) and the Fell topology ([7], [13], [17]). In a recent paper [5], *Di Maio* and *Holá* have found necessary and sufficient conditions for first and second countability, respectively of the Δ-topology τ_Δ^+, if X is T_1; more on countability axioms and quasi-uniformizability of τ_Δ^+ was obtained by *Holá* and *Levi* in [9], where a characterization of metrizability of τ_Δ^+ is also given for a T_1 base space X and Δ containing the singletons. Moreover, in [3] (see also [2]),

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Beer and *Tamaki* characterized unifomizability of τ_Δ^+ for a Hausdorff X and Δ containing the singletons.

It is the purpose of this paper to show that quite similar characterizations hold with no preliminary conditions (or with much less restrictive conditions) on X or Δ, respectively. This is achieved by applying techniques and notions from [19] and a completely new approach is employed to characterize complete regularity of τ_Δ^+.

Note that a characterization of normality of τ_Δ^+ is not known except for some special cases, like the Vietoris topology ([11], [18]) or the Fell topology ([10]); for some more general results on normality see [6].

1. Notation and terminology.

In the sequel (X, τ) will be a topological space and $CL(X)$ (resp. $K(X)$) will denote the nonempty closed (resp. nonempty closed compact) subsets of X. If $E \subset X$, then $\bar{E}$, $\operatorname{int} E$, E^c will stand for the closure, interior and complement of E, respectively in X. Put $E^- = \{A \in CL(X);\ A \cap E \neq \emptyset\}$, $E^+ = \{A \in CL(X);\ A \subset E\}$. In what follows, Δ will be a fixed but arbitrary nonempty subfamily of $CL(X)$ and for any $\Delta' \subset \Delta$, denote by $\Sigma(\Delta')$ the set of all finite unions of members of Δ'. The *hit-and-miss* or Δ-*topology* τ_Δ^+ for $CL(X)$ has a base all sets of the form $(B^c)^+ \cup \bigcap_{i=1}^{n} U_i^-$ where $B \in \Sigma(\Delta)$, $U_1, \dots, U_n \in \tau$ and $n \in \mathbb{N}$ (cf. [3], [17]); this basic element will be denoted by $(U_1, \dots, U_n)_B^+$ (cf; [20]). If $\Delta = CL(X)$, we obtain the familiar Vietoris topology τ_V, if $\Delta = K(X)$, the Fell topology τ_F.

In accordance with [3], Δ is said to be a *Urysohn family* provided whenever $A \in CL(X)$ and $B \in \Delta$ are disjoint, there exists $D \in \Sigma(\Delta)$ such that $B \subset \operatorname{int} D \subset D \subset A^c$. Denote by $\mathcal{F}_\Delta$ the class of all continuous functions $f : X \to [0,1]$ such that whenever $\inf f < \alpha < \beta < \sup f$, there exists $D \in \Sigma(\Delta)$ with

$$f^{-1}([0,\alpha]) \subset D \subset f^{-1}([0,\beta]),$$

where $f^{-1}(M)$ stands for the preimage of $M \subset [0,1]$. For $f \in \mathcal{F}_\Delta$ denote by m_f the *infimal value functional* on $CL(X)$ (cf. [3]) defined by

$$m_f(A) = \inf\{f(x); x \in A\} \quad \text{for all} \quad A \in CL(X).$$

We will say that X *has property* P_Δ provided whenever $A \in CL(X)$ and $x \in A^c$ there exists $D \in \Delta$ such that $D \subset A^c$ and $\overline{\{x\}} \cap D \neq \emptyset$ (see [19]). X is called *weakly*-R_0 provided X possesses property $P_{CL(X)}$ or equivalently provided every nonempty difference of τ-open sets contains a nonempty closed subset of X ([19]). Further, X is an R_0*-space* if every open subset of X contains the closure of each of its points ([4]).

We will say that $E \subset X$ is *c-hemicompact* if there exists an increasing sequence of members of $K(X) \cap CL(E)$ which is cofinal in $K(X) \cap CL(E)$. Notions not defined in the paper are used in accordance with [12] (e.g. regular does not include T_1).

2. Main results.

First we need some auxiliary material:

LEMMA 2.1. *Let* X *be weakly*-R_0, $B, D \in \Sigma(\Delta)$ *and* $U_1, \dots, U_n$, $V_1, \dots, V_m \in \tau$ $(m, n \in \mathbb{N})$. *Then the following are equivalent:*

(i) $(U_1, \dots, U_n)^+_B \subset (V_1, \dots, V_m)^+_D$;

(ii) $B^c \subset D^c$ *and for every* $1 \leq j \leq m$ *there exists an* $1 \leq i \leq n$ *such that* $U_i \cap B^c \subset V_j \cap D^c$.

Proof. Denote $\mathcal{U} = (U_1, \dots, U_n)^+_B$ and $\mathcal{V} = (V_1, \dots, V_m)^+_D$. Suppose (i) and choose an $A \in \mathcal{U}$. If $B^c \backslash D^c$ is nonempty, then by the weak-R_0 property we can find a nonempty closed set $C \subset B^c \backslash D^c$. This implies that $A \cup C \in \mathcal{U} \backslash \mathcal{V}$, which contradicts (i), thus $B^c \subset D^c$. Further, if there exists a $1 \leq j \leq m$ such that for each $1 \leq i \leq n$, $\emptyset \neq U_i \cap B^c \backslash V_j \cap D^c$, then we can find a nonempty closed $A_i \subset U_i \cap B^c \backslash V_j \cap D^c$, but then $\bigcup_{i=1}^{m} A_i \in \mathcal{U} \backslash \mathcal{V}$, which is a contradiction again, so (ii) holds.

Conversely, suppose (ii) and pick an $A \in \mathcal{U}$. Then $A \subset B^c \subset D^c$. Further, for every $1 \leq j \leq m$ there is an $1 \leq i \leq n$ such that $U_i \cap B^c \subset V_j \cap D^c$, so $A \cap V_j \neq \emptyset$ since $A \cap U_i \neq \emptyset$. It means that $A \in \mathcal{V}$.□

LEMMA 2.2. *If* $(CL(X), \tau^+_\Delta)$ *is first countable, then every* $A \in CL(X)$ *is separable.*

Proof. The proof of Lemma 5.3 in [5] works in every topological space if point-closures are used instead of singletons. □

We can now characterize first countability of the hit-and-miss topology for a weakly-R_0 base space X (cf. [5], Theorem 5.4):

THEOREM 2.3. *Let X be a weakly-R_0 space. Then the following are equivalent:*

(i) $(Cl(X), \tau_\Delta^+)$ *is first countable;*

(ii) X *is first countable, every closed set* $A \subset X$ *is separable and there exists a countable family* $\Delta_A \subset \Delta$ *such that whenever* $B \in \Delta$ *is disjoint to* A*, then* $B \subset D \subset A^c$ *for some* $D \in \Sigma(\Delta_A)$.

Proof. The proof of Theorem 5.4 in [5] can be abopted if Lemma 2.1, Lemma 2.2 and point-closures are used insetad of singletons. In the implication (i)⇒(ii) only the proof of first countability of X needs some comments. Let $x \in X$ and put $A_x = \overline{\{x\}}$. In view of (i) there exist countable families $\Delta_x \subset \Delta$ and $\tau_x \subset \tau$ such that $\mathcal{B}_x = \{(U_1, \ldots, U_n)_B^+;\ B \in \Sigma(\Delta_x), U_1, \ldots, U_n \in \tau_x, n \in \mathbb{N}\}$ forms a countable local base at A_x in τ_Δ^+. Choose any τ-open neighborhood U of x. Then $\mathcal{U} = (U_1, \ldots, U_n)_B^+ \subset U^-$ for some $\mathcal{U} \in \mathcal{B}_x$, thus by Lemma 2.1, $B^c \cap U_i \subset U$ for an $1 \le i \le n$ and clearly $x \in B^c \cap U_i$. It means that $\{B^c \cap U;\ B \in \Sigma(\Delta_x), U \in \tau_x\}$ is a countable local base at x. □

As for second countability of the hit-and-miss topology we have (cf. [5], Theorem 5.13):

THEOREM 2.4. *Let X be a weakly-R_0 space. Then the following are equivalent:*

(i) $(CL(X), \tau_\Delta^+)$ *is second countable;*

(ii) X *is second countable and there is a countable family* $\Delta' \subset \Delta$ *such that whenever* $B \in \Delta$ *and* $A \in CL(X)$ *are disjoint, then* $B \subset D \subset A^c$ *for some* $D \in \Sigma(\Delta')$.

Proof. From (i) we get countable families $\Delta' \subset \Delta$, $\tau' \subset \tau$ such that

$$\{(U_1, \ldots, U_n)_B^+;\ B \in \Sigma(\Delta'), U_1, \ldots, U_n \in \tau', n \in \mathbb{N}\}$$

forms a countable base of τ_Δ^+. Then $\{B^c \cap U; B \in \Sigma(\Delta'), U \in \tau'\}$ is a countable base for X, which easily follows by Lemma 2.1. The rest of the proof is analoguous to that of Theorem 5.13 in [5]. □

It is shown in [19] that regularity and T_3-ness of the Vietoris topology are equivalent. We show that it is a general feature of hit-and-miss topologies. First we need the following:

LEMMA 2.5. *The functional* $m_f : CL(X) \to [0, 1]$ *is* τ_Δ^+*-continuous for all* $f \in \mathcal{F}_\Delta$.

Proof. Choose $f \in \mathcal{F}_\Delta$. Let $\inf f < \alpha < \beta < \sup f$ and $E \in m_f^{-1}((\alpha, \beta))$. Then $\alpha < \inf\{f(x); x \in E\} < \beta$, thus $E \cap f^{-1}((\alpha, \beta)) \neq \emptyset$ and for any $0 < \varepsilon < m_f(E) - \alpha$ we have $f^{-1}([0, \alpha + \varepsilon]) \subset E^c$. Since $f \in \mathcal{F}_\Delta$ we can find a $D \in \Sigma(\Delta)$ such that

$$f^{-1}([0, \alpha + \varepsilon/2]) \subset D \subset f^{-1}([0, \alpha + \varepsilon]),$$

whence $E \subset D^c$. Then $E \in (D^c)^+ \cap (f^{-1}((\alpha, \beta)))^- \subset m_f^{-1}((\alpha, \beta))$. □

The following theorem is proved in [3] (Theorem 3.6) for a T_2 base space and with Δ containing the singletons. Here we present a different proof in the completely general setting:

THEOREM 2.6. *The following are equivalent*

(i) $(CL(X), \tau_\Delta^+)$ *is a Tychonoff space;*

(ii) $(CL(X), \tau_\Delta^+)$ *is completely regular;*

iii) $(CL(X), \tau_\Delta^+)$ *is a* T_3*-space;*

(iv) $(CL(X), \tau_\Delta^+)$ *is regular;*

(v) X *has property* P_Δ *and* Δ *is a Urysohn family.*

Proof. (v)⇒(i) In view of Theorem 1 in [19] it suffices to prove that the hyperspace is completely regular. An argument similar to that of in Lemma 3.1 of [3] yields for all $D \in \Delta$ and disjoint $A \in CL(X)$ an $f \in \mathcal{F}_\Delta$ such that $f(D) = 0$ and $f(A) = 1$. Let $A \in CL(X)$ and $\mathcal{U} = (U_1, \ldots, U_n)_B^+$ be a τ_Δ^+-neighborhood of A, where $B \in \Sigma(\Delta)$, $U_1, \ldots, U_n \in \tau$ and $n \in \mathbb{N}$.

Then $A \subset B^c$ and $A \cap U_i \neq \emptyset$ for all $1 \le i \le n$. In virtue of the preceding considerations there exist functions $f_0, f_1, \ldots, f_n \in \mathcal{F}_\Delta$ such that

$$f_0(B) = \{0\} \quad \text{and} \quad f_0(A) = \{1\},$$

$$f_i(E_i) = \{0\} \quad \text{and} \quad f_i(U_i^c) = \{1\} \quad \text{for each} \quad 1 \le i \le n.$$

The by Lemma 2.5, $m_{f_0}, m_{f_1}, \ldots, m_{f_n}$ are τ_Δ^+-continuous on $CL(X)$ so $F = \max\{1 - m_{f_0}, m_{f_1}, \ldots, m_{f_n}\}$ is τ_Δ^+-continuous as well. Clearly $1 - m_{f_0}(A) = m_{f_1}(A) = \cdots = m_{f_n}(A) = 0$ so $F(A) = 0$. Further if $E \notin \mathcal{U}$ then either $E \cap B \neq \emptyset$ or $E \subset U_i^c$ for some $1 \le i \le n$. In the first case $1 - m_{f_0}(E) = 1$, whence $F(E) = 1$ and in the second case $m_{f_i}(E) = 1$, so $F(E) = 1$ again.

All the remaining implications follows from Theorem 3 in [19], if regularity of the hyperspace forces X to have property P_Δ. Indeed, if $(CL(X), \tau_\Delta^+)$ is regular then it is also R_0, further the hit-and-miss topology is always T_0 (see [16]) so it is a T_1-space (cf. [4], Corollary), which completes the proof by Theorem 1 in [19]. □

If X is a Hausdorff space and Δ contains the singletons then X clearly possesses property P_Δ. Thus the following corollary generalises Theorem 3.6 of [3]:

COROLLARY 2.7. *$(CL(X), \tau_\Delta^+)$ is uniformizable if and only if X possesses property P_Δ and Δ is a Urysohn family.*

Finally we turn to characterizing metrizability of the hit-and-miss topology:

THEOREM 2.8. *The following are equivalent:*

(i) *$(CL(X), \tau_\Delta^+)$ is metrizable;*

(ii) *$(CL(X), \tau_\Delta^+)$ is pseudo-metrizable;*

(iii) *$(CL(X), \tau_\Delta^+)$ is second countable and regular,*

(iv) *X possesses property P_Δ and there exists a countable family $\Delta' \subset \Delta$ such that whenever $B \in \Delta$ and $A \in CL(X)$ are disjoint there is a $D \in \Sigma(\Delta')$ with $B \subset \operatorname{int} D \subset D \subset A^c$.*

Proof. The equivalence (i)⇔(ii) follows from Theorem 2.8. For (i)⇒(iii) see Proposition 5.18(1)⇒(2) in [5], further our Lemma 2.2 and use point-closures instead of singletons. Now suppose (iii). Regularity of $(CL(X), \tau_\Delta^+)$ implies by Lemma 2(ii) of [19] that X is weakly-R_0, so our Theorem 2.4 and Theorem 2.6 implies (iv) similarly as in [5] (Theorem 5.19 (1)⇒(2)). Finally, if we assume (iv) then according to Theorem 2.6, $(CL(X), \tau_\Delta^+)$ is a T_3-space, consequently by Lemma 2(ii) of [19], X is weakly-R_0 so if X was second countable then in view of Theorem 2.4 the Δ-topology would be second countable and the Urysohn Metrization Theorem would yield (i). Hence, it remains to justify that the countable family $\mathcal{B} = \{int D; D \in \Sigma(\Delta')\}$ is a base for (X, τ). Indeed, if U is a nonempty τ-open set and $x \in U$, then by property P_Δ there exists $B \in \Delta$ with $B \subset U$ and $B \cap \overline{\{x\}} \neq \emptyset$. In virtue of the second condition of (iv) we can find $D \in \Sigma(\Delta')$ such that $B \subset int D \subset D \subset U$ (we can assume that $U \neq X$). Then $x \in \operatorname{int} D \subset U$. □.

Remark 2.9. It is inferable from the proof of the preceding theorem that metrizability of $(CL(X), \tau_\Delta^+)$ always forces second countability on X.

3. Applications.

In view of our preceding theorems we have:

THEOREM 3.1. (cf. [5], Theorem 5.5) *Let X be weakly-R_0. Then the following are equivalent:*

(i) $(CL(X), \tau_V)$ is first countable;

(ii) every closed subset of X is separable and has a countable base of neighborhoods.

THEOREM 3.2. (cf. [5], Theorem 5.6) *Let X be weakly-R_0. Then the following are equivalent:*

(i) $(CL(X), \tau_F)$ s first countable;

(ii) X is first countable every closed set is separable and every proper open subset is c-hemicompact.

Proof. (i)⇒(ii) The proof of hemicompactness of proper open subsets of X in [1], Lemma 3.1 is feasible also in our case if closed compact sets are used instead of compact sets and point-closures instead of singletons. Further see our Theorem 2.3. In (ii)⇒(i) the proof of [5], Theorem 5.6 (2)⇒(1) is applicable (using c-hemicompactness instead of hemicompactness) along with our Theorem 2.3. □

THEOREM 3.3. (cf. [19]; Theorem 4) *The following are equivalent:*

(i) $(CL(X), \tau_V)$ *is a Tychonoff space;*

(ii) $(CL(X), \tau_V)$ *is completely regular;*

(iii) $(CL(X), \tau_V)$ *is a* T_3*-space;*

(iv) $(CL(X), \tau_V)$ *is regular;*

(v) $(CL(X), \tau_V)$ *is uniformizable;*

(vi) X *is normal and* R_0.

THEOREM 3.4. *The following are equivalent:*

(i) $(CL(X), \tau_F)$ *is a Tychonoff space;*

(ii) $(CL(X), \tau_F)$ *is regular;*

(iii) $(CL(X), \tau_F)$ *is a Hausdorff space;*

(iv) $(CL(X), \tau_F)$ *is uniformizable;*

(v) X *is a locally compact, regular space.*

Proof. Cf. [17] (Folgerung (a), p. 162) and Theorem 2 of [19]. □

THEOREM 3.5. (cf. [14], Theorem 4.9.7) *The following are equivalent:*

(i) $(CL(X), \tau_V)$ *is metrizable;*

(ii) X *is compact and pseudo-metrizable.*

Proof. (i)⇒(ii) Suppose that $(CL(X), \tau_V)$ is metrizable. Denote by $\tilde{X}$ the quotient space of X induced by identification of points with common closure in X. Then in view of Theorem 3 in [15], $(CL(\tilde{X}), \tilde{\tau}_V)$ is homeomorphic to $(CL(X), \tau_V)$, where $\tilde{\tau}_V$ is the Vietoris topology on $CL(\tilde{X})$, consequently it is also metrizable. Further, $(CL(X), \tau_V)$ is a Hausdorff space so by Theorem 2 of [19], X is regular and hence R_0 as well. Accordingly $\tilde{X}$ is a T_1-space, thus by Theorem 4.9.7 of [14], $\tilde{X}$ is compact, which implies compactness of X (cf. [15], Theorem 4). Now X is second countable by Remark 2.9, further it is regular, thus X is pseudo-metrizable (see [8], p. 167, Exercise 3).

(ii)⇒(i) Observe that a pseudo-metrizable space is R_0, hence possesses property $P_{CL(X)}$ (i.e. weak R_0-ness). Further by Lemma 2.2, X is a separable (pseudo-metrizable) space, accordingly second countable as well, which together with compactness and regularity of X easily yields the second condition of Theorem 2.8 (iv) for $\Delta = CL(X)$. □

THEOREM 3.6. (cf. [1], Theorem 3.4) *The following are equivalent:*

(i) $(CL(X), \tau_F)$ *is metrizable;*

(ii) X *is locally compact, regular and second countable.*

Proof. (i)⇒(ii) According to Remark 2.9, X is second countable and in virtue of Theorem 3.4, X is locally compact and regular.

(ii)⇒(i) By local compactness plus regularity of X, $K(X)$ forms a base of neighborhoods for closed compact subsets of X ([12], p. 146, Theorem 18). Further, second countability of X yields a countable subfamily of $K(X)$ which forms also a base of neighboroods for members of $K(X)$, thus the second condition of Theorem 2.8 (iv) is fulfilled for $\Delta = K(X)$. Finally, local compactness and regularity of X evidently imply property $P_{K(X)}$, thus Theorem 2.8 applies. □

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