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MORE ON PRODUCTS OF BAIRE SPACES

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Abstract. New results on the Baire product problem are presented. It is shown that an arbitrary product of almost locally ccc Baire spaces is Baire; moreover, the product of a Baire space and a 1st countable space which is β-unfavorable in the strong Choquet game is Baire.

1. introduction

A topological space is a *Baire space* provided countable intersections of dense open subsets are dense [16]. If the product $X \times Y$ is Baire, then X, Y must be Baire; however, the converse is not true in general. Indeed, *Oxtoby* [23] constructed, under CH, a Baire space with a non-Baire square, and various absolute examples followed (see [6], [10], [25], [26]). As a result, there has been a considerable effort to find various completeness properties for the coordinate spaces to get Baireness of the product (cf. [20], [12], [23], [16], [1], [32], [10], [25], [11], [34], [5], [22], [21]). There have been two successful approaches in solving the product problem: given Baire spaces X, Y, either one adds some condition to Y (such as 2nd countability [23], the uK-U property [11], having a countable-in-itself π-base [34]), or strengthens completeness of Y (to Čech-completeness, (strong) α-favorability [1],[32], or more recently, to hereditary Baireness [5],[22],[21]). It is the purpose of this paper to generalize these product theorems, as well as, show how a new fairly weak completeness property of *β-unfavorability in the strong Choquet game* [27],[30],[30] can be added to the list of spaces giving a Baire product.

Since Baire spaces can be characterized via the Banach-Mazur game, it is not surprising that topological games have been applied to attack the Baire product problem. Our results continue in this line of research (precise definitions will be given in the next section); in the games two players take countably many turns in choosing objects from a topological space X: in the *strong Choquet game* [4, 17] player β starts, and always chooses an open set V and a point $x \in V$, then player α responds by choosing an open set U such that $x \in U \subseteq V$; α wins if the intersection of the chosen open sets is nonempty, otherwise, β wins.

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The strong Choquet game provides a useful unifying platform for studying completeness - type properties, as the following two celebrated theorems demonstrate in a metrizable space X:

- $Ch(X)$ is α-favorable if and only if X is completely metrizable [4],
- $Ch(X)$ is β-unfavorable if and only if X is hereditarily Baire (i.e. the nonempty *closed* subspaces of X are Baire) [8, 30, 27].

The *Banach-Mazur game* $BM(X)$ [16] (also called the *Choquet game* [17]) is played as $Ch(X)$, except that both β, α choose open sets only. In a topological space X, $BM(X)$ is β-unfavorable iff X is a *Baire space* [24, 19, 29]; consequently, if $BM(X)$ is α-favorable, then X is a Baire space.

To put our results in perspective, recall that $X \times Y$ is a Baire space if X is a Baire topological space and

- either Y is a topological space such that $BM(Y)$ is α-favorable (in particular, if $Ch(Y)$ is α-favorable) [32],
- or Y is a hereditarily Baire space which is metrizable [22], or more generally, 1st countable T_3 space [21].

Since there are spaces which are α-favorable in the strong Choquet game but are not hereditarily Baire (the Moore line), as well as metric hereditarily Baire spaces, which are not α-favorable in the Banach-Mazur game (a Bernstein set), being β-unfavorable in the strong Choquet game is distinct from both hereditary Baireness as well as being α-favorable in the strong Choquet game, thus, it is natural to ask the status of this property in the Baire product problem. Our main result in Section 3 (Theorem 3.2) implies the following:

Theorem 1.1. *Let X be a Baire space, Y be a 1st countable topological space such that $Ch(Y)$ is β-unfavorable. Then $X \times Y$ is a Baire space.*

The proof works for finite products, but it does not naturally extend to infinite products, so we will separately consider the infinite product case in Section 4, using the idea of a *Krom space* ([18],[14]), to obtain:

Theorem 1.2. *Let I be an index set, and X_i be an almost locally ccc Baire space for each $i \in I$. Then $\prod_i X_i$ is a Baire space.*

2. Preliminaries

Unless otherwise noted, all spaces are topological spaces. As usual, ω denotes the non-negative integers, and $n \geq 1$ will be considered as sets of predecessors $n = \{0, \dots, n-1\}$. Let $\mathcal{B}$ be a base for a topological space X, and denote

$$\mathcal{E} = \mathcal{E}(X) = \mathcal{E}(X, \mathcal{B}) = \{(x, U) \in X \times \mathcal{B} : x \in U\}.$$

In the *strong Choquet game* $Ch(X)$ players β and α alternate in choosing $(x_n, V_n) \in \mathcal{E}$ and $U_n \in \mathcal{B}$, respectively, with β choosing first, so that for each $n < \omega$, $x_n \in U_n \subseteq V_n$, and $V_{n+1} \subseteq U_n$. The play

$$(x_0, V_0), U_0, \ldots, (x_n, V_n), U_n, \ldots$$

is won by α, if $\bigcap_n U_n (= \bigcap_n V_n) \neq \emptyset$; otherwise, β wins.

A *strategy* in $Ch(X)$ for α (resp. β) is a function $\sigma : \mathcal{E}^{<\omega} \to \mathcal{B}$ (resp. $\sigma : \mathcal{B}^{<\omega} \to \mathcal{E}$) such that $x_n \in \sigma((x_0, V_0), \ldots, (x_n, V_n)) \subseteq V_n$ for all $((x_0, V_0), \ldots, (x_n, V_n)) \in \mathcal{E}^{<\omega}$ (resp. $\sigma(\emptyset) = (x_0, V_0)$ and $V_n \subseteq U_{n-1}$, where $\sigma(U_0, \ldots, U_{n-1}) = (x_n, V_n)$, for all $(U_0, \ldots, U_{n-1}) \in \mathcal{B}^n$, $n \geq 1$). A strategy σ for α (resp. β) is a *winning strategy*, if α (resp. β) wins every run of $Ch(X)$ compatible with σ, i.e. such that $\sigma((x_0, V_0), \ldots, (x_n, V_n)) = U_n$ for all $n < \omega$ (resp. $\sigma(\emptyset) = (x_0, V_0)$ and $\sigma(U_0, \ldots, U_{n-1}) = (x_n, V_n)$ for all $n \geq 1$). We will say that $Ch(X)$ *is* α*-,* β*-favorable*, respectively, provided α, resp. β has a winning strategy in $Ch(X)$.

The *Banach-Mazur game* $BM(X)$ [16] is played similarly to $Ch(X)$, the only difference is that both β, α choose open sets from a fixed π-base of X. Winning strategies, α-, and β-favorability of $BM(X)$ can be defined analogously to $Ch(X)$.

In the *Gruenhage game* $G(X)$ [15] given a point $x \in X$, at the n-th round Player I picks an open neighborhood U_n of x, and Player II chooses $x_n \in U_n$. Player I wins if the sequence $(x_n)_n$ converges to x, otherwise, Player II wins; x is a W*-point* [28], provided Player I has a winning strategy $W_x : X^{<\omega} \to \{\text{open neighborhoods of } x\}$ in $G(X)$ at x.

Given a topological space (X, τ), consider the ultrametric space τ^ω, where τ has the discrete topology. For every $n < \omega$ denote

$$\downarrow \tau^n = \{f \in (\tau \setminus \{\emptyset\})^n : f(k+1) \subseteq f(k) \text{ whenever } k \leq n\}, \text{ and}$$
$$\downarrow \tau^\omega = \{f \in (\tau \setminus \{\emptyset\})^\omega : f(k+1) \subseteq f(k) \text{ whenever } k < \omega\}.$$

The *Krom space* [18, 14] of X is defined as

$$\mathcal{K}(X) = \{f \in \downarrow \tau^\omega : \bigcap_n f(n) \neq \emptyset\}.$$

Note that a base of neighborhoods at $f \in \mathcal{K}(X)$ is $\{[f \restriction_{n+1}] : n < \omega\}$, where

$$[f \restriction_{n+1}] = \{g \in \mathcal{K}(X) : g \restriction_{n+1} = f \restriction_{n+1}\}.$$

Put differently, a base for $\mathcal{K}(X)$ is $\{[f] : f \in \bigcup_n \downarrow \tau^n\}$ where, if $n < \omega$ and $f \in \downarrow \tau^n$, then

$$[f] = \{g \in \mathcal{K}(X) : g \restriction_{n+1} = f\}.$$

Given a base $\mathcal{B}$ for X, we will also consider the following subspace of $\mathcal{K}(X)$:

$$\mathcal{K}^0_{\mathcal{B}}(X) = \{f \in \mathcal{K}(X) \cap \mathcal{B}^\omega : (f(n))_n \text{ is a neighborhood base at each } x \in \bigcap_n f(n)\}.$$

Krom's Theorem [18, Theorem 3] states that for topological spaces X, Y, $X \times Y$ is a Baire space iff $X \times \mathcal{K}(Y)$ is a Baire space iff $\mathcal{K}(X) \times \mathcal{K}(Y)$ is a Baire space.

3. Finite Baire products

We will say that a space is *almost locally ccc*, provided every open set contains an open ccc subspace. This property is strictly weaker than being *almost locally uK-U* [34] (see [11, Examples 1,2]), as well as *having a countable-in-itself* π*-base* [34] (termed *locally countable pseudo-base* in [23]), which are known to produce Baire products (see [23, Theorem 2], [11, Property 1] [34, Proposition 4]). Since these properties all coincide in Baire metric spaces (see [34, Proposition 3]), a simple observation about Krom spaces immediately yields a generalization of these Baire product theorems:

Theorem 3.1. *Let* X, Y *be a Baire spaces, and* Y *be almost locally ccc. Then* $X \times Y$ *is a Baire space.*

Proof. First note that $\mathcal{K}(Y)$ has a countable-in-itself π-base: indeed, let $f \in\downarrow \tau^n$ for some $n < \omega$, choose $U \subset f(n)$ which is ccc, and define $f_0 = f^\frown U$. Consider a pairwise disjoint open partition $\{[g] : g \in J\}$ of $[f_0]$, where $J \subset \bigcup_{n<\omega} \downarrow \tau^n$. For each $g \in J$ let $n_g < \omega$ be such that $g \in\downarrow \tau^{n_g}$; then $\{g(n_g) : g \in J\}$ is a pairwise disjoint open partition of U, which must be countable, and so is $\{[g] : g \in J\}$; thus, $\mathcal{K}(Y)$ is an almost locally ccc metric space, and so it has a countable-in-itself π-base.

It follows from Krom's theorem that $\mathcal{K}(Y)$ is a Baire space, moreover, by [23, Theorem 2], $X \times \mathcal{K}(Y)$ is a Baire space, which it turn implies $X \times Y$ is a Baire space by Krom's theorem. $\square$

An approach involving the strong Choquet game yields a different kind of generalization of Baire product theorems (cf. [1], [32],[22],[21],[5]):

Theorem 3.2. *Let* X *be a Baire space and* Y *have a dense set of* W*-points and* $Ch(Y)$ *be* β*-unfavorable. Then* $X \times Y$ *is a Baire space.*

Proof. Denote by τ_X, τ_Y the nonempty open subsets of X, Y, respectively. Let $\{O_n : n < \omega\}$ be a decreasing sequence of dense open subsets of $X \times Y$, and pick $U \in \tau_X, V \in \tau_Y$. If $y \in Y$ is a W-point, denote by W_y a winning strategy for the open-set picker in the Gruenhage game at y.

Define a tree $T \subset \omega^{<\omega}$ as follows: let $T_0 = \{\emptyset\}$ be the root of the tree, and $T_1 = \{(0)\}$ its first level; further, given level T_n for some $n \geq 1$, and $t = (t_0, \dots, t_k) \in T_n$, define the immediate successors of t as $t^- = (t_0, \dots, t_k, 0)$ and $t^+ = (t_0, \dots, t_k+1)$, and put $T_{n+1} = \{t^-, t^+ : t \in T_n\}$. It follow that each $t \in T \setminus T_0$ has a source $s_t \in T$, which is the immediate predecessor of the node where the last minus-branching occurs before t, so if $t \in T_n$ for some $n \geq 1$, and $s_t \in T_k$ for some $0 \leq k < n$, then $t = s_t^\frown(n-k-1)$.

We will define a strategy σ_X for β in $BM(X)$: first, pick a W-point $y_\emptyset \in V$ and denote $V_\emptyset = V$. Then choose $U_0 \in \tau_X$, $V_{(0)} \in \tau_Y$ so that $U_0 \times V_{(0)} \subseteq O_0 \cap U \times V$, pick a W-point $y_{(0)} \in V_{(0)}$ and put $\sigma_X(\emptyset) = U_0$. Given $A_0 \in \tau_X, A_0 \subseteq U_0$, find $U_1 \in \tau_X$

and $V_t \in \tau_Y$ for each $t \in T_2$ so that

$$U_1 \times V_{(0,0)} \subseteq O_1 \cap [A_0 \times V_{(0)}]$$
$$U_1 \times V_{(1)} \subseteq O_1 \cap [A_0 \times V_\emptyset \cap W_{y_\emptyset}(y_{(0)})];$$

moreover, pick a W-point $y_t \in V_t$ for each $t \in T_2$, and put $\sigma_X(A_0) = U_1$.

Assume that for some $n \geq 1$, and given $A_0, \ldots, A_{n-1} \in \tau_X$, we have constructed $U_n \in \tau_X$ along with $(y_t, V_t) \in \mathcal{E}(Y)$ for each $t \in T_{n+1}$ so that each y_t is a W-point,

$$U_n = \sigma_X(A_0, \ldots, A_{n-1}),$$

and for each $t \in T_n$

$$U_n \times V_{t^-} \subseteq O_n \cap [A_{n-1} \cap V_t], \tag{1}$$

$$U_n \times V_{t^+} \subseteq O_n \cap [A_{n-1} \times V_{s_t} \cap W_{y_{s_t}}(y_{s_t^\frown 0}, \ldots, y_t)]. \tag{2}$$

Let $A_n \in \tau_X, A_n \subseteq U_n$ be given, and denote $T_{n+1} = \{t_1, \ldots, t_{2^n}\}$. Using density of O_{n+1} repeatedly, we can define a decreasing sequence $\{H_i \in \tau_X : i \leq 2^{n+1}\}$, where $H_0 = A_n$, as well as $V_{t_i^-}, V_{t_i^+} \in \tau_Y$ so that for all $1 \leq i \leq 2^n$

$$H_i \times V_{t_i^-} \subseteq O_{n+1} \cap [H_{i-1} \times V_{t_i}],$$
$$H_{i+2^n} \times V_{t_i^+} \subseteq O_{n+1} \cap [H_{i+2^n-1} \times V_{s_{t_i}} \cap W_{y_{s_{t_i}}}(y_{s_{t_i}^\frown 0}, \ldots, y_{t_i})].$$

Then for $U_{n+1} = H_{2^{n+1}}$ and each $t \in T_{n+1}$ we have

$$U_{n+1} \times V_{t^-} \subseteq O_{n+1} \cap [A_n \cap V_t]$$
$$U_{n+1} \times V_{t^+} \subseteq O_{n+1} \cap [A_n \times V_{s_t} \cap W_{y_{s_t}}(y_{s_t^\frown 0}, \ldots, y_t)].$$

Pick a W-point $y_t \in V_t$ for each $t \in T_{n+2}$, and define $\sigma_X(A_0, \ldots, A_n) = U_{n+1}$, which concludes the definition of σ_X. Notice that, by (2),

$$(y_{t^\frown k})_k \text{ converges to } y_t \text{ for each } t \in T. \tag{3}$$

Since X is a Baire space, there is run $U_0, A_0, \ldots, U_n, A_n, \ldots$ of $BM(X)$ compatible with σ_X that β looses, i.e. there is some $x \in \bigcap_n U_n$.

We will define a strategy σ_Y for β in $Ch(Y)$. First, put $\sigma_Y(\emptyset) = (z_0, W_0)$, where $z_0 = y_\emptyset$, and $W_0 = V_\emptyset$. Let $B_0 \in \tau_Y$ with $z_0 \in B_0 \subseteq W_0$ be given. Using (3), we can define

$$k_{B_0} = \min\{k \geq 0 : y_{(k)} \in B_0\},\ z_1 = y_{(k_{B_0})},\ W_1 = B_0 \cap V_{(k_{B_0})},$$

and put $\sigma_Y(B_0) = (z_1, W_1)$. Assume that $\sigma_Y(B_0, \ldots, B_{n-1}) = (z_n, W_n) \in \mathcal{E}(Y)$ have been defined for some $n \geq 1$ and $B_0, \ldots, B_{n-1} \in \tau_Y$ so that

$$z_n = y_{(k_{B_0}, \ldots, k_{B_{n-1}})},\ \text{and } W_n = B_{n-1} \cap V_{(k_{B_0}, \ldots, k_{B_{n-1}})}$$

for appropriate $k_{B_i} \geq i$ for each $i < n$. Let $B_n \in \tau_Y$ be such that $z_n \subseteq B_n \subseteq W_n$, then for $t = (k_{B_0}, \ldots, k_{B_{n-1}})$, $(y_{t^\frown k})_k$ converges to $z_n = y_t$ by (3), so we can define

$$k_{B_n} = \min\{k \geq n : y_{t^\frown k} \in B_n\},\ z_{n+1} = y_{t^\frown k_{B_n}},\ W_{n+1} = B_n \cap V_{t^\frown k_{B_n}},$$

and put $\sigma_Y(B_0, \ldots, B_n) = (z_{n+1}, W_{n+1})$.

Since σ_Y cannot be a winning strategy for β in $Ch(Y)$, there is a run

$$(z_0, W_0), B_0, (z_1, W_1), \ldots, B_n, (z_{n+1}, W_{n+1}), \ldots$$

of $Ch(Y)$ compatible with σ_Y that β looses. Then there is some $y \in \bigcap_n W_n \subseteq \bigcap_n V_{(k_{B_0},\ldots,k_{B_n})}$, so (1) and (2) imply that $(x, y) \in U \times V \cap \bigcap_n O_n$, and we are done. □

The proof of Theorem 1.1 immediately follows, which in turn implies the following (recall, that a space is R_0 [7], when every open subset contains the closure of each of its points):

Corollary 3.3. *Let X be a Baire space, and Y a 1st countable hereditarily Baire R_0-space. Then $X \times Y$ is a Baire space.*

Proof. It suffices to note that a 1st countable hereditarily Baire R_0-space Y is β-unfavorable in $Ch(Y)$ by [35, Corollary 3.8.]; thus, Theorem 1.1 applies. □

Remark 3.4. Some of the results in [21] are similar in flavor to the above results, in particular, [21, Theorem 4.4] states, that *if X is a Baire space, and Y is a T_3-space possessing a rich family $\mathcal{F}$ of Baire spaces (i.e. $\mathcal{F}$ consists of nonempty separable closed Baire subspaces of X such that if $Y \subset X$ is separable, then $Y \subseteq F$ for some $F \in \mathcal{F}$, moreover, $\overline{\bigcup_{n<\omega} F_n} \in \mathcal{F}$ whenever $\{F_n : n < \omega\} \subset \mathcal{F}$), then $X \times Y$ is a Baire space.* The next example shows that our results are different (although overlapping), since spaces that are β-unfavorable in the strong Choquet game are not directly connected to spaces having rich Baire families. Indeed, there exists a T_1-space X with no rich Baire family which is α-favorable in $Ch(X)$: to see this, let $\mathbb{Q}$ be the rationals and D an uncountable set. Define $X = \mathbb{Q} \cup D$, let elements of D be isolated, and a neighborhood base at $q \in \mathbb{Q}$ be of the form $I \cup D \setminus C$, where $I \subseteq \mathbb{Q}$ is a Euclidean open neighborhood of q, and $C \subset D$ is countable. Then

- X is strongly α-favorable: define a tactic σ for α in $Ch(X)$ as follows

$$\sigma(x, V) = \begin{cases} \{x\}, & \text{if } x \in D, \\ V, & \text{if } x \in \mathbb{Q}. \end{cases}$$

Each run of $Ch(X)$ compatible with σ contains an element of D in the intersection, so σ is a winning tactic for α.

- X has no rich Baire family: indeed, for every separable $S \supseteq \mathbb{Q}$ we have $S = \mathbb{Q} \cup C$ for some countable $C \subset D$. It follows that if $I \subseteq \mathbb{Q}$ is a Euclidean open neighborhood of some $q \in \mathbb{Q}$, then it is also an open set in S (since $I = S \cap (I \cup D \setminus C)$), and of the 1st category in S, thus, S is not a Baire space.

Remark 3.5. It is known that hereditary Baireness is not a stand-alone topological property that gives a Baire product since, under (CH), there is a hereditarily Baire space with a non-Baire square [31]; however, it is an open question weather $X \times Y$ is Baire if X is Baire and $Ch(Y)$ is β-unfavorable.

4. Infinite Baire products

The following is the arbitrary product version of Krom's Theorem:

Theorem 4.1. *Let I be an index set. Then $\prod_{i\in I} X_i$ is a Baire space if and only if $\prod_{i\in I} \mathcal{K}(X_i)$ is a Baire space.*

Proof. Denote $X = \prod_{i\in I} X_i$ and $X^* = \prod_{i\in I} \mathcal{K}(X_i)$.

• Assume that β has a winning strategy σ in $BM(X)$, and define a strategy σ^* for β in $BM(X^*)$ as follows: if $\sigma(\emptyset) = \prod_{i\in I_0} V_{0,i} \times \prod_{i\notin I_0} X_i$ for some finite $I_0 \subset I$ and $V_{0,i} \in \mathcal{B}_i$, define

$$\sigma^*(\emptyset) = \prod_{i\in I_0} V^*_{0,i} \times \prod_{i\notin I_0} \mathcal{K}(X_i), \text{ where } V^*_{0,i} = [(V_{0,i})]$$

If $U^*_0 \subseteq \sigma^*(\emptyset)$ is α's response in $BM(X^*)$, then $U^*_0 = \prod_{i\in J_0} U^*_{0,i} \times \prod_{i\notin J_0} \mathcal{K}(X_i)$ for some finite $J_0 \supset I_0$, and for all $i \in J_0$, $U^*_{0,i} = [(U_{0,i}(0), \dots, U_{0,i}(m_{0,i}))]$ for some decreasing X_i-open $U_{0,i}(0), \dots, U_{0,i}(m_{0,i})$ and $m_{0,i} \geq 0$, where $U_{0,i}(0) = V_{0,i}$ for all $i \in I_0$. Denote $U_0 = \prod_{i\in J_0} U_{0,i}(m_{0,i}) \times \prod_{i\notin J_0} X_i$ and let

$$\sigma(U_0) = \prod_{i\in I_1} V_{1,i} \times \prod_{i\notin I_1} X_i,$$

where $I_1 \supseteq J_0$ is finite, $V_{1,i} \in \mathcal{B}_i$ for each $i \in I_1$ and $V_{1,i} \subseteq U_{0,i}(m_{0,i})$ whenever $i \in J_0$. Define

$$\sigma^*(U^*_0) = \prod_{i\in I_1} V^*_{1,i} \times \prod_{i\notin I_1} \mathcal{K}(X_i), \text{ where}$$

$$V^*_{1,i} = \begin{cases} [(U_{0,i}(0), \dots, U_{0,i}(m_{0,i}), V_{1,i})], & \text{if } i \in J_0, \\ [(V_{1,i})], & \text{if } i \in I_1 \setminus J_0. \end{cases}$$

Proceeding inductively, we can define σ^* so that whenever $k < \omega$, and

$$U^*_k = \prod_{i\in J_k} U^*_{k,i} \times \prod_{i\notin J_k} \mathcal{K}(X_i)$$

is given for some finite J_k, and for all $i \in J_k$, $U^*_{k,i} = [(U_{k,i}(0), \dots, U_{k,i}(m_{k,i}))]$ for decreasing X_i-open $U_{k,i}(0), \dots, U_{k,i}(m_{k,i})$ and $m_{k,i} \geq 0$, then

$$\sigma^*(U^*_0, \dots, U^*_k) = \prod_{i\in I_{k+1}} V^*_{k+1,i} \times \prod_{i\notin I_{k+1}} \mathcal{K}(X_i)$$

have been chosen, where $I_{k+1} \supseteq J_k$ is finite, and

$$V^*_{k+1,i} = \begin{cases} [(U_{k,i}(0), \dots, U_{k,i}(m_{k,i}), V_{k+1,i})], & \text{if } i \in J_k, \\ [(V_{k+1,i})], & \text{if } i \in I_{k+1} \setminus J_k \end{cases}$$

is such that

$$\prod_{i\in I_{k+1}} V_{k+1,i} \times \prod_{i\notin I_{k+1}} X_i = \sigma\left(U_0, \dots U_k\right).$$

where $U_j = \prod_{i\in J_j} U_{j,i}(m_{j,i}) \times \prod_{i\notin J_j} X_i$ for all $j \le k$. We will show that σ^* is a winning strategy for β in $MB(X^*)$: indeed, take a run of $MB(X^*)$

$$\sigma^*(\emptyset), U_0^*, \dots, U_n^*, \sigma^*(U_0^*, \dots, U_n^*), \dots$$

compatible with σ^*, and assume there is some $f \in \bigcap_n \sigma^*(U_0^*, \dots, U_n^*)$. Then for each $i \in I$, $f(i) \in \mathcal{K}(X_i)$ so we can pick some $x_i \in \bigcap_n f(i)(n)$. Moreover, if $i \in I_k$ for a given $k < \omega$, then $x_i \in \bigcap_{n\ge k} V_{n,i}$, so $(x_i)_{i\in I} \in \prod_{i\in I_k} V_{k,i} \times \prod_{i\notin I_{k+1}} X_i$, thus, $(x_i)_{i\in I} \in \bigcap_n \sigma(U_0, \dots U_k)$ which is impossible, since σ is a winning strategy for β in $BM(X)$.

• Assume that β has a winning strategy σ^* in $BM(X^*)$, and define a strategy σ for β in $BM(X)$ as follows: if $\sigma^*(\emptyset) = \prod_{i\in I_0} V_{0,i}^* \times \prod_{i\notin I_0} \mathcal{K}(X_i)$, where for all $i \in I_0$, $V_{0,i}^* = [(V_{0,i}(0), \dots, V_{0,i}(m_{0,i}))]$, define $\sigma(\emptyset) = \prod_{i\in I_0} V_{0,i}(m_{0,i}) \times \prod_{i\notin I_0} X_i$. Let $U_0 = \prod_{i\in J_0} U_{0,i} \times \prod_{i\notin J_0} X_i$ be α's response in $BM(X)$. Then $J_0 \supseteq I_0$ is finite and $U_{0,i} \subseteq V_{0,i}(m_{0,i})$ for all $i \in I_0$. Define

$$U_{0,i}^* = \begin{cases} [(V_{0,i}(0), \dots, V_{0,i}(m_{0,i}), U_{0,i})], & \text{for all } i \in I_0, \\ [(U_{0,i})], & \text{for all } i \in J_0 \setminus I_0, \end{cases}$$

and let

$$\sigma^*\left(\prod_{i\in J_0} U_{0,i}^* \times \prod_{i\notin J_0} \mathcal{K}(X_i)\right) = \prod_{i\in I_1} V_{1,i}^* \times \prod_{i\notin I_1} \mathcal{K}(X_i),$$

where $V_{1,i}^* = [(V_{1,i}(0), \dots, V_{1,i}(m_{1,i}))]$ whenever $i \in I_1$. Define

$$\sigma(U_0) = \prod_{i\in I_1} V_{1,i}(m_{1,i}) \times \prod_{i\notin I_1} X_i.$$

Proceeding inductively, assume that whenever $k \ge 1$, and $j < k$, then $\sigma(U_0, \dots, U_j) = \prod_{i\in I_{j+1}} V_{j,i}(m_{j,i}) \times \prod_{i\notin I_{j+1}} X_i$ is defined, and let $U_k = \prod_{i\in J_k} U_{k,i} \times \prod_{i\notin J_k} X_i$ be α's next step in $BM(X)$. Then $J_k \supseteq I_k$ is finite and $U_{k,i} \subseteq V_{k,i}(m_{k,i})$ for all $i \in I_k$. Define

$$U_{k,i}^* = \begin{cases} [(V_{k,i}(0), \dots, V_{k,i}(m_{k,i}), U_{k,i})], & \text{for all } i \in I_k, \\ [(U_{k,i})], & \text{for all } i \in J_k \setminus I_k, \end{cases}$$

and let

$$\sigma^*\left(\prod_{i\in J_k} U_{k,i}^* \times \prod_{i\notin J_k} \mathcal{K}(X_i)\right) = \prod_{i\in I_{k+1}} V_{k+1,i}^* \times \prod_{i\notin I_{k+1}} \mathcal{K}(X_i),$$

where $V^*_{k+1,i} = [(V_{k+1,i}(0), \ldots, V_{k+1,i}(m_{k+1,i}))]$ whenever $i \in I_{k+1}$. Define

$$\sigma(U_0, \ldots, U_k) = \prod_{i \in I_{k+1}} V_{k+1,i}(m_{k+1,i}) \times \prod_{i \notin I_{k+1}} X_i.$$

We will show that σ is a winning strategy for β in $BM(X)$: take a run

$$\sigma(\emptyset), U_0, \ldots, U_n, \sigma(U_0, \ldots, U_n), \ldots$$

compatible with σ, and assume there is some $(x_i)_{i \in I} \in \bigcap_n \sigma(U_0, \ldots, U_n)$. For all $k < \omega$ and $i \in I_k$, define a decreasing sequence of X_i-open sets $f(i)$ so that $f(i) \restriction_n = (V_{n,i}(0), \ldots, V_{n,i}(m_{n,i}))$ for all $n < \omega$, moreover, if $i \in I \setminus \bigcup_k I_k$, put $f(i) = (X_i)_{n<\omega}$. Then for each $i \in I$, $x_i \in \bigcap_n f(i)(n)$, so $f(i) \in \mathcal{K}(X_i)$, thus, $f \in X^*$. Moreover, $f \in \bigcap_n \sigma^*(U^*_0, \ldots, U^*_n)$, which is impossible, since the run

$$\sigma^*(\emptyset), U^*_0, \ldots, U^*_n, \sigma^*(U^*_0, \ldots, U^*_n), \ldots$$

is compatible with σ^*. □

As a consequence, we have

Proof of Theorem 1.2. Since $\mathcal{K}(X_i)$ is a Baire space with a countable-in-itself π-base (see the proof of Theorem 3.1), then $\prod_{i \in I} \mathcal{K}(X_i)$ is a Baire space by [34, Theorem 5], and so is $\prod_i X_i$ by our Theorem 4.1. □

Recall that X has a *base of countable order* (BCO) $\mathcal{B}$ [33], provided each strictly decreasing sequence of members of $\mathcal{B}$ containing some $x \in X$ forms a base of neighborhoods at x.

Theorem 4.2. *Let I be an index set, and for each $i \in I$, X_i be an R_0 hereditarily Baire space with a BCO . Then $\prod_i X_i$ is a Baire space.*

Proof. For each $i \in I$, choose a BCO $\mathcal{B}_i$ for X_i and prove that $\mathcal{K}^0_{\mathcal{B}_i}(X_i)$ *is a dense hereditarily Baire subspace of* $\mathcal{K}(X_i)$: as for density, take a decreasing $h \in \mathcal{B}^k_i$, $k < \omega$, choose $g \in \mathcal{K}^0_{\mathcal{B}_i}(X_i)$ with $g(0) \subset h(k)$, and define

$$f(m) = \begin{cases} h(m), & \text{if } m \le k, \\ g(m-k-1), & \text{if } m > k. \end{cases}$$

Then $f \in [h] \cap \mathcal{K}^0_{\mathcal{B}_i}(X_i)$, so $\mathcal{K}^0_{\mathcal{B}_i}(X_i)$ is dense in $\mathcal{K}(X_i)$,

To show that $\mathcal{K}^0_{\mathcal{B}_i}(X_i)$ is a hereditarily Baire space, we will use that, by [35, Corollary 3.9], in spaces with a BCO, hereditary Baireness is equivalent to β-unfavorability in the strong Choquet game: indeed, assume that σ^*_i is a winning strategy for β in $Ch(\mathcal{K}^0_{\mathcal{B}_i}(X))$, and define a strategy σ_i for β in $Ch(X_i)$ as follows: if $\sigma^*_i(\emptyset) = (f_0, V^*_0)$ for some $f_0 \in \mathcal{K}^0_{\mathcal{B}_i}(X)$ and $V^*_0 = [f_0 \restriction_{m_0}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X)$, where $m_0 \ge 1$, then pick $x_0 \in \bigcap_n f_0(n)$, choose $V_0 \in \mathcal{B}_i$ so that $x_0 \in V_0 \subsetneq f_0(m_0 - 1)$, if $f_0(m_0 - 1)$ is not a singleton, and $V_0 = f_0(m_0 - 1)$, if $f_0(m_0 - 1)$ is a singleton, and define $\sigma_i(\emptyset) = (x_0, V_0)$. If $x_0 \in U_0 \subseteq V_0$ for some $U_0 \in \mathcal{B}_i$, let $n_0 \ge m_0$ be such that

$f_0(n_0) \subseteq U_0$, and consider $\sigma_i^*([f_0 \restriction_{n_0+1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X)) = (f_1, V_1^*)$, where $f_1 \in \mathcal{K}^0_{\mathcal{B}_i}(X)$ and $V_1^* = [f_1 \restriction_{m_1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X)$ for some $m_1 \geq n_0 + 1$. Pick $x_1 \in \bigcap_n f_1(n)$, choose $V_1 \in \mathcal{B}_i$ so that $x_1 \in V_1 \subsetneq f_1(m_1 - 1)$, if$f_1(m_1 - 1)$ is not a singleton, and $V_1 = f_1(m_1 - 1)$, if $f_1(m_1 - 1)$ is a singleton, and define $\sigma_i(U_0) = (x_1, V_1)$. Proceeding inductively, assume that for a given $k \geq 1$ and all $j < k$, $\sigma_i(U_0, \ldots, U_j) = (x_{j+1}, V_{j+1})$ has been defined, along with $V_{j+1}^* = [f_{j+1} \restriction_{m_{j+1}}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X)$ and $m_{j+1} > n_j \geq m_j$ so that $f_j(n_j) \subseteq U_j$ and $\sigma_i^*([f_0 \restriction_{n_0+1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X), \ldots, [f_j \restriction_{n_j+1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X)) = (f_{j+1}, V_{j+1}^*)$, where V_j is either a singleton or a proper subset of $f_j(m_j - 1)$. Let $U_k \in \mathcal{B}_i$ be such that $x_k \in U_k \subseteq V_k$, and find $n_k \geq m_k$ such that $f_k(n_k) \subseteq U_k$ Consider

$$\sigma_i^*([f_0 \restriction_{n_0+1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X), \ldots, [f_k \restriction_{n_k+1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X)) = (f_{k+1}, V_{k+1}^*),$$

where $f_{k+1} \in \mathcal{K}^0_{\mathcal{B}_i}(X)$ and $V_{k+1}^* = [f_{k+1} \restriction_{m_{k+1}}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X)$ for some $m_{k+1} \geq n_k + 1$. Pick $x_{k+1} \in \bigcap_n f_{k+1}(n)$, choose $V_{k+1} \in \mathcal{B}_i$ so that $x_{k+1} \in V_{k+1} \subsetneq f_{k+1}(m_{k+1} - 1)$, if $f_{k+1}(m_{k+1} - 1)$ is not a singleton, and $V_{k+1} = f_{k+1}(m_{k+1} - 1)$, if $f_{k+1}(m_{k+1} - 1)$ is a singleton, and put $\sigma_i(U_0, \ldots, U_k) = (x_{k+1}, V_{k+1})$. We will show that σ_i is a winning strategy for β in $Ch(X_i)$: indeed, let

$$(x_0, V_0), U_0, \ldots, (x_k, V_k), U_k, \ldots$$

be a run of $Ch(X_i)$ compatible with σ_i, and assume $\bigcap_k V_k \neq \emptyset$. Define $f \in\downarrow \mathcal{B}_i^\omega$ as follows: for all $k < \omega$ and $m_{k-1} \leq p < m_k$ put $f(p) = f_k(p)$ (for completeness, let $m_{-1} = 0$). Then $\bigcap_p f(p) = \bigcap_k f(m_k - 1) = \bigcap_k f_k(m_k - 1) \supseteq \bigcap_k V_k$, so $f \in \mathcal{K}^0_{\mathcal{B}_i}(X)$, since by the construction of σ_i, $\{V_n : n < \omega\}$ is either a strictly decreasing sequence of elements of $\mathcal{B}_i$, or a singleton. Moreover, $f \restriction_{m_k} = f_k \restriction_{m_k}$, thus, $f \in \bigcap_k V_k^*$, which is impossible, since

$$(f_0, V_0^*), [f_0 \restriction_{n_0+1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X), \ldots, (f_k, V_k^*), [f_k \restriction_{n_k+1}] \cap \mathcal{K}^0_{\mathcal{B}_i}(X), \ldots$$

is a run of $Ch(\mathcal{K}^0_{\mathcal{B}_i}(X))$ compatible with σ_i^*.

It now follows from [5, Theorem 1.1] that $\prod_{i \in I} \mathcal{K}^0_{\mathcal{B}_i}(X_i)$ is a Baire space, which is also dense in $\prod_{i \in I} \mathcal{K}(X_i)$, which it turn implies that $\prod_{i \in I} X_i$ is a Baire space by Theorem 4.1. $\square$

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