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Developable hyperspaces are metrizable

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Dedicated to Professor S. Naimpally on the occasion of his 70th birthday.

Abstract. developability of hyperspace topologies (locally finite, (bounded) Vietoris, Fell, respectively) on the nonempty closed sets is characterized. Submetrizability and having a G_δ-diagonal in the hyperspace setting is also discussed.

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1. Introduction.

Let $CL(X)$ $(K(X))$ denote the hyperspace of nonempty closed (compact) sets of a T_2 topological space (X, τ). For notions not defined in the paper see [11], [2] and [15].

Historically there have been two hyperspace topologies of particular importance: the Vietoris topology τ_V (see section 1) and the Hausdorff metric topology τ_H, as considered in Michael's fundamental paper on hyperspaces [22]. It is well-known, that $\tau_V = \tau_H$ on $K(X)$ and hence $(K(X), \tau_V)$ is metrizable iff X is. Metrizability of the larger hyperspace $(CL(X), \tau_V)$ is also characterized, it is equivalent to X being compact metrizable [22].

To investigate generalized metric properties of the Vietoris topology, one may start by considering Bing's factorization of metrizability into collectionwise normality (CWN) and Mooreness ([11]). There is an abundance of results on CWN and related properties, e.g. (combined results of Keesling and Velichko from [19], [20], [27]): $(CL(X), \tau_V)$ is CWN (paracompact, normal, resp.) iff X is compact; however, some stronger (hereditary) properties, such as hereditary normality, stratifiability or monotone normality, coincide with metrizability for $(CL(X), \tau_V)$ (cf. [6], [12], [13] and [7]).

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As far as the other half of Bing's Theorem is concerned, Mooreness (developability) has been considered only for $K(X)$; indeed, Mizokami has shown that $(K(X), \tau_V)$ is Moore iff X is [23]. It is one of the purposes of this paper to characterize developability (Mooreness) of the Vietoris topology on $CL(X)$. A good starting point for investigating developability is to look at 1st countability of $(CL(X), \tau_V)$ first, as was done by Holá and Levi [16] while extending an older result of Choban [8]:

Theorem 1.1. *Let X be a T_2 space. The following are equivalent:*

(i) $(CL(X), \tau_V)$ *is 1st countable;*

(ii) X *is perfectly normal, the derived set X' is countably compact, hereditarily separable and of countable character, and $X \setminus X'$ is countable.*

Thus, in general, 1st countability of the Vietoris topology does not guarantee its metrizability (just consider $X = \omega$, the discrete space of non-negative integers); other cases, e.g. if X is dense-in-itself and metrizable, are treated in [10]. In section 2 we will prove that developability and metrizability of $(CL(X), \tau_V)$ always coincide. However, see Remark 3.4 following Theorem 3.3 for a relevant comment.

In fact, after obtaining the same relationship for other well-studied hypertopologies, such as the locally finite, Fell and bounded Vietoris topology, respectively (see section 1 for definitions), as well as for the Wijsman topology (which is 1st countable iff it is metrizable [2]), it seems that the coincidence of developability and metrizability in the hyperspace setting could be established for a broad class of hypertopologies. It remains to be seen if it is the case for hit-and-miss and hit-and-far topologies ([2]) or even for the general hyperspace topology studied in [28], incorporating all the above topologies along with some weak hyperspace topologies, including the Wijsman topology.

Since metrizability is to submetrizability as developability is to having a G_δ-diagonal (see 1.6, 2.2 and 2.5 in [15]), we then turn to studying submetrizability and having a G_δ-diagonal in $CL(X)$. It turns out to be a perfect match for the Fell and the bounded Vietoris topology, respectively, moreover, if X is Morita's M-space, also for the Vietoris topology.

Finally, in Section 4, the above generalized metric properties are discussed on $K(X)$ with the Vietoris and Fell topologies, respectively.

2. Preliminaries.

To describe the hypertopologies we will work with, we need to introduce some notation: for $U \subset X$ put

$$U^+ = \{A \in CL(X) : \ A \subset U\} \ \text{ and } \ U^- = \{A \in CL(X) : \ A \cap U \neq \varnothing\}.$$

Subbase elements of the *Vietoris* (*locally finite*) *topology* τ_V (τ_{lf}) on $CL(X)$ are of the form U^+ with $U \in \tau$ and $\bigcap_{U \in \mathcal{U}} U^-$ with $\mathcal{U} \subset \tau$ finite (locally finite). Note, that for a metrizable X, the supremum of all Hausdorff metric (resp. Wijsman) topologies corresponding to topologically equivalent metrics is the locally finite (resp. Vietoris) topology ([4], [25], [5]).

Another classical hypertopology, the *Fell topology*, has found numerous applications in various fields of mathematics ([21], [1]); it has as a subbase elements of the form U^- and V^+, where $U \in \tau$ and V has a compact complement in X. If (X, d) is a metric space the *bounded Vietoris* topology τ_{bV_d} has as a subbase, elements of the form U^- and V^+, where $U \in \tau$ and the complement of V is a closed bounded set in (X, d).

Proposition 2.1. *Let X be a T_2 space. If $(CL(X), \tau_V)$ or $(CL(X), \tau_{lf})$ is 1st countable, then X is collectionwise normal.*

Proof. In view of Theorem 1.1 (resp. [9]), X is normal and X' is countably compact; thus, if $\mathcal{D}$ is a discrete family of closed subsets of X, only finitely many members $D_0, \dots, D_n \in \mathcal{D}$ intersect X'. Since X is normal, there exist pairwise disjoint open sets $U_0, \dots, U_n$ such that $D_i \subset U_i$ for each $i \leq n$. Denote $\mathcal{D}_0 = \{D_0, \dots, D_n\}$ and observe that $D = \bigcup(\mathcal{D} \setminus \mathcal{D}_0)$ is closed. Consequently, $(\mathcal{D} \setminus \mathcal{D}_0) \cup \{U_0 \setminus D, \dots, U_n \setminus D\}$ is a disjoint open expansion of $\mathcal{D}$. □

Proposition 2.2. *Let X be a T_2 space. If $(CL(X), \tau_V)$ or $(CL(X), \tau_{lf})$ is developable, then X is metrizable and X' is compact.*

Proof. By admissibility of the Vietoris and the locally finite topology, X is a developable space, which is also collectionwise normal by Proposition 2.1; hence, in view of Bing's Theorem ([11]), X is metrizable. Since X' is countably compact, in our case, it is compact. □

3. Developability in $CL(X)$.

Theorem 3.1. *Let X be a T_2 space. The following are equivalent:*

(i) $(CL(X), \tau_{lf})$ *is Moore;*
(ii) $(CL(X), \tau_{lf})$ *is developable;*
(iii) $(CL(X), \tau_{lf})$ *is metrizable;*
(iv) X *is metrizable and* X' *is compact.*

Proof. (ii)⇒(iv) follows from Proposition 2.2 and (iv)⇒(iii) from [4], Theorem 2.3. □

Theorem 3.2. $(CL(\omega), \tau_V)$ *is not developable.*

Proof. Suppose $(CL(\omega), \tau_V)$ is developable. Let $\mathcal{D} = \{\mathcal{U}_n : n \in \omega\}$ be a development of $(CL(\omega), \tau_V)$. Without loss of generality we may suppose that $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for every n. For every $A \in [\omega]^\omega$, define

$$L(A) = \min\{n : \mathrm{St}(A, \mathcal{U}_n) \subset A^+\},$$

and for every $A \in [\omega]^\omega$ and $F \in [A]^{<\omega}$ put

$$\pi(A, F) = \min\{L(B) : F \subset B, B \text{ is a proper infinite subset of } A\}.$$

For every $A \in [\omega]^\omega$, $m = L(A)$, and every $F \in [A]^{<\omega}$ choose $H(A, F) \in [A]^{<\omega}$ such that F is a proper subset of $H(A, F)$ and there is $U \in \mathcal{U}_m$ with $A^+ \cap \bigcap_{p \in H(A,F)} \{p\}^- \subset U$.

For every $A \in [\omega]^\omega$ and every $B, G \in [A]^{<\omega}$ such that $H(A,B) \subset G$ we have $\pi(A,G) > L(A)$: otherwise, $\pi(A,G) \leq L(A)$ and there is an infinite proper subset B of A with $G \subset B$ such that $L(B) \leq L(A)$. Hence, there is $U \in \mathcal{U}_{L(A)}$ such that $\{A,B\} \subset U$. As $\mathcal{U}_{n+1}$ is a refinement of $\mathcal{U}_n$ for every $n \in \omega$, we see that for each $k < L(A)$ there is $U^k \in \mathcal{U}_k$ with $\{A,B\} \subset U^k$. Hence for every $k \leq L(A)$, $\mathrm{St}(B, \mathcal{U}_k)$ is not subset of B^+ (as $A \setminus B \neq \varnothing$) and so $L(B) > L(A)$, a contradiction.

We use an inductive construction now:

- Put $n_0 = \pi(\omega, \varnothing)$. Take $A_0 \in [\omega]^\omega$ such that $L(A_0) = n_0$ and $G_0 \in [\omega]^{<\omega}$ such that there is $U \in \mathcal{U}_{n_0}$ with $A_0^+ \cap_{p \in G_0} \{p\}^- \subset U$. Put $F_0 = \varnothing$.

- Let $F_{j+1} = \bigcup_{i \leq j} G_i$ and $n_{j+1} = \pi(A_j, F_{j+1})$; choose $A_{j+1} \in [\omega]^\omega$ such that $F_{j+1} \subset A_{j+1}$, where A_{j+1} is a proper subset A_j, $L(A_{j+1}) = n_{j+1}$ and put $G_{j+1} = H(A_{j+1}, F_{j+1})$.

Clearly, this construction can be repeated ω-many times. Finally, put $B = \bigcup_{i \in \omega} G_i$ and take $p \in \omega$ such that $n_p > L(B)$. Then $F_p \subset B \subset A_{p-1}$, but $L(B) \geq \pi(A_{p-1}, F_p) = n_p$, a contradiction. □

Theorem 3.3. *Let X be a T_2 space. The following are equivalent:*

(i) $(CL(X), \tau_V)$ *is Moore;*
(ii) $(CL(X), \tau_V)$ *is developable;*
(iii) $(CL(X), \tau_V)$ *is metrizable;*
(iv) X *is compact and metrizable.*

Proof. Only (ii)⇒(iv) needs justification: in view of Proposition 2.2, it suffices to show that every sequence in $X \setminus X'$ has a cluster point in X. Otherwise, $X \setminus X'$ contains a closed copy of ω, thus, $(CL(\omega), \tau_V)$ sits in $(CL(X), \tau_V)$ and is hence developable, a contradiction with Theorem 3.2. □

Remark 3.4. After Ľ. Holá's lecture at Caserta 2001, prof. Arhangel'skiĭ was wondering, whether Theorem 3.3 could be extended to hyperspaces which are σ-spaces (i.e. spaces with a σ-discrete network) (see [15], also for related notions of (strong) Σ-spaces). He was very right. A possible way could be an easy modification of the proof of Theorem 3.2 in effect that $(CL(\omega), \tau_V)$ is not a σ-space and an application of the fact that each countably compact σ-space is compact and metrizable [15]. Fortunately, Popov [26] proved already in 1978 the following

Theorem 3.5. $(CL(\omega), \tau_V)$ *contains the Sorgenfrey line S as a subspace.*

Recall that S is not even a Σ-space. So it follows from Popov's result that if $(CL(X), \tau_V)$ is a Σ-space then X is countably compact. Of course, Theorem 3.2 represents a special case, nevertheless we have decided to keep its proof to make the paper more self-contained. As M-spaces are Σ-spaces, some other information on the Vietoris hyperspaces which are Σ-spaces, may be found in

Proposition 4.9 below. Coming back to the original Arhangel'skiĭ's question, we could reformulate Theorem 3.3:

Theorem 3.6. *Let X be a T_2 space. The following are equivalent:*

(o) $(CL(X), \tau_V)$ *has a σ-discrete network;*
(i) $(CL(X), \tau_V)$ *is Moore;*
(ii) $(CL(X), \tau_V)$ *is developable;*
(iii) $(CL(X), \tau_V)$ *is metrizable;*
(iv) X *is compact and metrizable.*

We proceed with other topologies now:

Theorem 3.7. *Let (X, d) be a metric space. Then the following are equivalent:*

(i) $(CL(X), \tau_{bV_d})$ *is Moore;*
(ii) $(CL(X), \tau_{bV_d})$ *is developable;*
(iii) $(CL(X), \tau_{bV_d})$ *is metrizable;*
(iv) (X, d) *is boundedly compact (i.e. every closed bounded set in (X, d) is compact).*

Proof. Since (iii)⇔(iv) is known, only (ii)⇒(iv) needs some comments: let $B \in CL(X)$ be bounded in (X, d). Developability of $(CL(X), \tau_{bV_d})$ implies that $CL(B)$ equipped with the relative topology τ_{bV_d} on $CL(B)$ is also developable. It is easy to verify that the relative topology τ_{bV_d} on $CL(B)$ coincides with the Vietoris topology τ_V. Thus, $(CL(B), \tau_V)$ is developable and B must be compact by Theorem 3.3. □

Theorem 3.8. *Let X be a T_2 space. The following are equivalent:*

(i) $(CL(X), \tau_F)$ *is Moore;*
(ii) $(CL(X), \tau_F)$ *is developable;*
(iii) $(CL(X), \tau_F)$ *is T_2 and has a G_δ-diagonal;*
(iv) $(CL(X), \tau_F)$ *is submetrizable;*
(v) $(CL(X), \tau_F)$ *is metrizable;*
(vi) X *is hemicompact and metrizable.*

Proof. (ii)⇒(iii) Developability of $(CL(X), \tau_F)$ implies that it has a G_δ-diagonal; moreover, even 1st countability of $(CL(X), \tau_F)$ implies, that X is locally compact ([16]), so, $(CL(X), \tau_F)$ is T_2 by a result of Fell.

(iii)⇒(v) Hausdorffness of $(CL(X), \tau_F)$ implies that $(CL(X), \tau_F)$ is locally compact. Since $(CL(X), \tau_F)$ has a G_δ-diagonal, points of $(CL(X), \tau_F)$ are G_δ; thus, $(CL(X), \tau_F)$ is 1st countable; hence (see [17]) it is paracompact. In summary, $(CL(X), \tau_F)$ is paracompact, locally compact with a G_δ-diagonal and is therefore metrizable.

(v)⇔(vi) is known ([3]) (to prove (vi)⇒(v) realize that every first countable hemicompact Hausdorff space is locally compact). The remaining implications are trivial. □

4. Submetrizability, having a G_δ-diagonal and related properties in $CL(X)$.

The last theorem of the previous section showed that these properties coincide for $(CL(X), \tau_F)$. In what follows, we show that similar relationship holds for the (bounded) Vietoris topology as well.

First, we will see that for the (bounded) Vietoris and locally finite topology, respectively, submetrizability and developability are distinct.

Proposition 4.1.

(i) *If X is a metrizable space, then $(CL(X), \tau_{lf})$ is submetrizable.*
(ii) *If (X, d) is a separable metric space, then $(CL(X), \tau_V)$ and $(CL(X), \tau_{bV_d})$ are submetrizable.*

Proof. (i) If X is a metrizable space, take any compatible metric d on X and consider the Hausdorff metric topology τ_{H_d}. It is known that $\tau_{H_d} \subseteq \tau_{lf}$.

(ii) Since d is a separable metric on X, the Wijsman topology τ_{W_d} is metrizable ([2]). It is known that $\tau_{W_d} \subseteq \tau_{bV_d} \subseteq \tau_V$. □

Corollary 4.2.

(i) *$(CL(X), \tau_V)$ is submetrizable and not developable, if X is non-compact, separable and metrizable.*
(ii) *$(CL(X), \tau_{bV_d})$ is submetrizable and not developable, if (X, d) is a separable metric space which is not boundedly compact.*
(iii) *$(CL(X), \tau_{lf})$ is submetrizable and not developable, if X is a non-compact dense-in-itself metrizable space.*

Proposition 4.3. *Let X be a T_2 space.*

(i) *If the points in $(CL(X), \tau_V)$ are G_δ, then X is hereditarily separable and every closed set in X is G_δ.*
(ii) *If the points in $(CL(X), \tau_F)$ are G_δ, then X is hereditarily separable, every open set in X is σ-compact and every closed set in X is a G_δ-set.*
(iii) *If (X, d) is a metric space, then the points in $(CL(X), \tau_{bV_d})$ are G_δ iff (X, d) is separable.*

Proof. We prove only (i): let $A \subset X$. Since $\overline{A}$ is a G_δ-set in $(CL(X), \tau_V)$, there are τ_V-open sets $\mathcal{G}_n$ such that $\{\overline{A}\} = \bigcap_{n\in\omega} \mathcal{G}_n$. Without loss of generality we can suppose, that for every $n \in \omega$, $\mathcal{G}_n = (G_n)^+ \cap \cap_{l\le n} (U_n^{i_l})^-$, where $G_n, U_n^{i_l}, l \le n$ are open sets in X. For every $n \in \omega$, $l \le n$ choose $a_n^{i_l} \in A \cap U_n^{i_l}$. It is easy to verify that $\overline{\{a_n^{i_l} : l \le n, n \in \omega\}} = \overline{A}$. □

Remark 4.4. Let X be a T_2 space.

(i) If $(CL(X), \tau_V)$ has a G_δ-diagonal, then X is hereditarily separable.
(ii) Points in $(CL(X), \tau_V)$ are G_δ iff every $A \in CL(X)$ is a G_δ-set and has a countable pseudobase (in A).

Proposition 4.5. *Let (X, d) be a metric space. The following are equivalent:*

(i) *$(CL(X), \tau_{bV_d})$ is submetrizable;*

(ii) $(CL(X), \tau_{bV_d})$ *has a* G_δ*-diagonal;*
(iii) (X, d) *is separable.*

Proof. (iii)⇒(i) See Proposition 4.1(ii).
(ii)⇒(iii) Follows from Proposition 4.3(iii). □

Proposition 4.6. *Let* X *be a* $w\Delta$*-space. The following are equivalent:*
(i) $(CL(X), \tau_V)$ *is submetrizable;*
(ii) X *is a separable metrizable space.*

Proof. (ii)⇒(i) See Proposition 4.1(ii).
(i)⇒(ii) Submetrizability of $(CL(X), \tau_V)$ implies its Hausdorffness, so X is regular by a result of Michael [22] and submetrizable, which in turn, being a $w\Delta$-space, is an M-space ([15]). However, an M-space with a G_δ-diagonal is metrizable ([15]). Finally, by Remark 4.4(i), X is separable. □

Proposition 4.7. *Let* X *be an* M*-space. The following are equivalent:*
(i) $(CL(X), \tau_V)$ *is submetrizable;*
(ii) $(CL(X), \tau_V)$ *is* T_2 *and has a* G_δ*-diagonal.*
(iii) X *is a separable metrizable space.*

Proof. ((ii)⇒(i) See Proposition 4.1(ii).
(i)⇒(ii) An M-space with a G_δ-diagonal is metrizable ([15]) and by Remark 4.4(ii), X is separable. □

Proposition 4.8. *Let* X *be a* T_2 *space.*
(i) $(CL(X), \tau_V)$ *is a* $w\Delta$*-space (strict* p*-space,*M*-space, resp.)* $\Rightarrow X$ *is countably compact.*
(ii) X *is countably compact* $\not\Rightarrow (CL(X), \tau_V)$ *is a* $w\Delta$*-space.*

Proof. (i) By Proposition 4.1(ii), $(CL(\omega), \tau_V)$ is submetrizable, so it has a G_δ^*-diagonal ([15]). By a theorem of Hodel ([15]) this implies, that $(CL(\omega), \tau_V)$ is not a $w\Delta$-space (neither is a strict p-space or an M-space, since these properties are stronger than $w\Delta$). On the other hand, if $(CL(X), \tau_V)$ is a $w\Delta$-space and X is not countably compact, then ω sits in X as a closed subset and hence $CL(\omega)$ embeds as a closed subset in $(CL(X), \tau_V)$. Since the $w\Delta$-property is closed hereditary, this would imply that $(CL(\omega), \tau_V)$ is a $w\Delta$-space, a contradiction.
(ii) By Example 2 of [24], there is a countably compact space X, such that $(\mathcal{F}_2(X), \tau_V)$ (= the space of all sets with at most 2 elements) is not a $w\Delta$-space. Since $\mathcal{F}_2(X)$ is a closed set in $(CL(X), \tau_V)$ and the $w\Delta$-property is closed hereditary, $(CL(X), \tau_V)$ is not a $w\Delta$-space. □

A topological space X is *ultracompact* iff every net in X with a countable range has a cluster point. This characterization of ultracompactness is due to Holá and Künzi [18].

Proposition 4.9. *Let* X *be a linearly ordered topological space. The following are equivalent:*
(o) $(CL(X), \tau_V)$ *is countably compact;*

(i) *$(CL(X),\tau_V)$ is an M-space;*
(ii) *$(CL(X),\tau_V)$ is a $w\Delta$-space;*
(iii) *X is countably compact.*

Proof. (ii)⇒(iii) See Proposition 4.8(i).

For the rest of the proof, realize that in linearly ordered topological spaces countable compactness and ultracompactness coincide [14], and X is ultracompact iff $(CL(X),\tau_V)$ is [18]. Further, an ultracompact space is countably compact, which in turn is an M-space. □

Remark 4.10. It can be inferred from the above proof, that if X is ultracompact, then $(CL(X),\tau_V)$ is an M-space ($w\Delta$-space). So, for every non-compact ultracompact space X, $(CL(X),\tau_V)$ is a non-compact M-space ($w\Delta$-space).

Proposition 4.8 also offers another proof of developability of $(CL(X),\tau_V)$:

Proposition 4.11. *Let X be a T_2 space. The following are equivalent:*

(i) *$(CL(X),\tau_V)$ is developable;*
(ii) *$(CL(X),\tau_V)$ is a $w\Delta$-space and X has a G_δ-diagonal.*

Proof. (i)⇒(ii) is clear.

(ii)⇒(i) By Proposition 4.8(i), X is countably compact. Every countably compact space with a G_δ-diagonal is compact and metrizable [15]. Now Theorem 3.3 applies. □

5. Some generalized metric properties in $K(X)$.

Proposition 5.1. *Let X be a T_2 space. The following are equivalent:*

(i) *$(K(X),\tau_V)$ is submetrizable;*
(ii) *X is submetrizable.*

Proof. (i)⇒(ii) It is easy to verify that if $\mathcal{T}$ is a coarser metrizable topology on $(K(X),\tau_V)$, then the relative topology $\mathcal{T}$ on X is coarser than τ.

(ii)⇒(i) Let $\mu \subseteq \tau$ be a metrizable topology. Then $K(X,\tau)$ (= compact sets in τ) $\subseteq K(X,\mu)$. $(K(X,\mu),\mu_V)$ is a metrizable space, where μ_V is the Vietoris topology generated by μ. Thus, also μ_V restricted to $K(X,\tau)$ is a metrizable topology on $K(X,\tau)$. Since $\mu \subseteq \tau$, we have that μ_V restricted on $K(X,\tau)$ is coarser than τ_V. □

The situation in $(K(X),\tau_F)$ is different:

Proposition 5.2. *Let X be a T_2 space. The following are equivalent:*

(i) *$(K(X),\tau_F)$ is Moore;*
(ii) *$(K(X),\tau_F)$ is developable;*
(iii) *$(K(X),\tau_F)$ is T_2 and has a G_δ-diagonal;*
(iv) *$(K(X),\tau_F)$ is submetrizable;*
(v) *$(K(X),\tau_F)$ is metrizable;*
(vi) *X is hemicompact and metrizable.*

Proof. (iii)⇒(vi) First we show that if $(K(X), \tau_F)$ is T_2, then X is locally compact. Suppose that X fails to be locally compact. Let $x \in X$ be such that for every $U \in \mathcal{B}(x)$ and $K \in K(X)$ there is some $x_{U,K} \in U \backslash K$ (here $\mathcal{B}(x)$ stands for a base of neighborhoods of x). Let $y \in X$ be a point different from x. It is easy to verify that the net of compact sets $\{\{x_{U,K}, y\} : U \in \mathcal{B}(x), K \in K(X)\}$ converges both to $\{x, y\}$ and to $\{y\}$ in $(K(X), \tau_F)$, a contradiction.

Since $(K(X), \tau_F)$ has a G_δ-diagonal, points of $(K(X), \tau_F)$ are G_δ and X has a G_δ-diagonal. Thus, X is σ-compact, locally compact with a G_δ-diagonal, i.e. it must be hemicompact, by [11] 3.8.C (b), and metrizable.

(vi)⇒(v) If X is hemicompact and metrizable, then $(CL(X), \tau_F)$ is metrizable [2]; thus, $(K(X), \tau_F)$ is also metrizable.

(v)⇒(iv), (iv)⇒(iii) and (i)⇒(ii) are trivial.

(ii)⇒(iii) It follows from [3], that 1st countability of $(K(X), \tau_F)$ implies hemicompactness of X and hence its local compactness (since X must be first countable). This in turn is equivalent to Hausdorffness of $(K(X), \tau_F)$.

(iii)⇒(i) By the above, (iii) is equivalent to metrizability of $(K(X), \tau_F)$. (i)⇒(ii) is trivial □

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