DEVELOPABILITY AND RELATED PROPERTIES OF THE GENERALIZED COMPACT-OPEN TOPOLOGY

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Abstract. Developability and related properties (like weak developability, G_δ-diagonal, G_δ^*-diagonal, submetrizability) of the generalized compact-open topology τ_C on partial continuous functions $\mathcal{P}$ with closed domains in X and values in Y are studied. First countability of $(\mathcal{P}, \tau_C)$ is also characterized. New results are obtained on weak developability, submetrizability, and first countability for the classical compact-open topology on the space of continuous functions with a general range space Y.

1. Introduction and preliminaries

Perhaps the first to consider a topological structure on the space of partial maps was Zaremba [40] in 1936. Later, in 1955, Kuratowski [27] studied the Hausdorff metric topology on the space of partial maps with compact domains.

The generalized compact-open topology τ_C on the space of partial continuous functions with closed domains was introduced by J. Back in [5] in connection with investigating utility functions emerging in mathematical economics. It also proved to be a useful tool in studying convergence of dynamic programming models [39], [29], as well as in applications to the theory of differential equations [8]. This new interest in τ_C complements the attention paid to spaces of partial maps in the past [40], [27], [28], [1], [2], [7], [36], and more recently in [15], [38], [26], [9], [10], [12], [13], [21], [22], [23]. The Hausdorff metric topology on the space of partial maps with closed domains was studied in [11].

Various topological properties of τ_C have already been established, e.g. separation axioms in [17], complete metrizability in [18], [23] and other completeness type properties in [21], [23] and [35], respectively; also in [12], [13], the authors study topological properties of spaces of partial maps in a more general setting.

Continuing the research started in [17],[35],[21],[23], in the present paper we will focus on some generalized metric properties, and first countability of the generalized compact-open topology, as well as of the classical compact-open topology.

Unless otherwise noted, all spaces are nontrivial Hausdorff spaces. If X is a topological space, then B^c, int$B$, and $\overline{B}$ will stand for the complement, interior and closure of $B \subseteq X$, respectively. Denote by $CL(X)$ the family of nonempty closed subsets of X, and by $K(X)$ the nonempty compact subsets of X. For any $B \in CL(X)$ and a topological space Y, $C(B, Y)$ will stand for the space of continuous

2000 *Mathematics Subject Classification.* Primary 54C35; Secondary 54E18, 54E50, 54D20, 54B20, 54C60, 54D99.

Key words and phrases. Generalized compact-open topology, Fell topology, compact-open topology, 1st countable spaces, (weakly) developable spaces, spaces with a G_δ-diagonal.

This work was supported by the Science and Technology Assistance Agency under the contract No. APVT-51-006904.

functions from B to Y. A partial map is a pair (B, f) such that $B \in CL(X)$, and $f \in C(B, Y)$. Denote by $\mathcal{P} = \mathcal{P}(X, Y)$ the family of all partial maps.

The so-called *generalized compact-open topology* τ_C on $\mathcal{P}$ [21] is the topology having subbase elements of the form

$$[U] = \{(B, f) \in \mathcal{P} : B \cap U \neq \emptyset\},$$
$$[K : V] = \{(B, f) \in \mathcal{P} : f(K \cap B) \subseteq V\},$$

where U is open in X, $K \in K(X)$, and V is an open (possibly empty) subset of Y.

The *compact-open topology* τ_{CO} on $C(X, Y)$ [14], [30] has subbase elements of the form

$$[K, V] = \{f \in C(X, Y) : f(K) \subseteq V\},$$

where $K \in K(X)$, and $V \subseteq Y$ is open.

Denote by τ_F the *Fell topology* on $CL(X)$ [6], [25] having subbase elements of the form

$$U^- = \{A \in CL(X) : A \cap U \neq \emptyset\}, \text{ and } (K^c)^+ = \{A \in CL(X) : A \subseteq K^c\}$$

with U open in X, and $K \in K(X)$. If we replace the compact set K by a closed set we obtain subbase elements for the classical *Vietoris topology* [6].

The following proposition shows the relationship between the above mentioned topologies, and will be helpful for our analysis.

Proposition 1.1 ([18], [23])**.**
(1) *X, and $(CL(X), \tau_F)$ embed in $(\mathcal{P}, \tau_C)$; further $(CL(X), \tau_F)$ embeds as a closed set in $(\mathcal{P}, \tau_C)$, if X is locally compact.*
(2) *Y, and $(C(X, Y), \tau_{CO})$ embed as closed subsets in $(\mathcal{P}, \tau_C)$.*

Let X be a *hemicompact* space (i.e. in $K(X)$ ordered by inclusion, there exists a countable cofinal subfamily [14]). If X is also locally compact, fix a cofinal sequence $\{C_n\}$ of compacts that is strongly increasing (i.e. $C_n \subseteq \operatorname{int} C_{n+1}$).

Suppose now that X is a hemicompact metrizable space with a compatible metric d, and Y is Hausdorff. Denote by $S(x, r)$ the open ball with center x, and radius r. Let $n \in \omega$. For a collection $\mathcal{V}$ of open sets in Y, a finite collection $\mathcal{U}$ of open balls of radius at most $\frac{1}{n}$ that are subsets of C_{n+1}, and $\varphi : \mathcal{U} \to \mathcal{V}$, the set

$$H_n(\mathcal{V}, \mathcal{U}, \varphi) = [C_n \setminus \cup\mathcal{U} : \emptyset] \cap \bigcap_{U \in \mathcal{U}} ([U] \cap [\overline{U} : \varphi(U)])$$

is open in $(\mathcal{P}, \tau_C)$. Put $\mathcal{H}_n(\mathcal{V}) = \{H_n(\mathcal{V}, \mathcal{U}, \varphi) : \mathcal{U}, \varphi\}$.

Lemma 1.2. *Let X be a hemicompact metrizable space, and $\mathcal{V}$ an open cover of Y. Then $\mathcal{H}_n(\mathcal{V})$ is an open cover of $\mathcal{P}$ for each $n \in \omega$.*

Proof. Let $(B, f) \in \mathcal{P}$. If $B \cap C_n = \emptyset$, put $\mathcal{U} = \emptyset$ and $\varphi = \emptyset$, then $(B, f) \in H_n(\mathcal{V}, \mathcal{U}, \varphi) \in \mathcal{H}_n(\mathcal{V})$. If $B \cap C_n \neq \emptyset$, then by continuity of f, and compactness of $B \cap C_n$, there exists a finite family $\mathcal{U}$ of open balls of radius $\leq \frac{1}{n}$ that are subsets of C_{n+1} such that $B \cap C_n \subseteq \cup\mathcal{U}$, and for all $U \in \mathcal{U}$ there is $V_U \in \mathcal{V}$ with $f(B \cap \overline{U}) \subseteq V_U$. If $\varphi(U) = V_U$ for all $U \in \mathcal{U}$, then $(B, f) \in H_n(\mathcal{V}, \mathcal{U}, \varphi) \in \mathcal{H}_n(\mathcal{V})$. □

A space X is *almost σ-compact*, provided there is $C_n \in K(X)$ with $X = \overline{\bigcup_{n \in \omega} C_n}$ (see [30]). If $T = \bigoplus_n C_n$ is the topological sum, and $p : T \to X$ is the natural map, define the function

$$p^* : (C(X, Y), \tau_{CO}) \to (C(T, Y), \tau_{CO}) \text{ via } p^*(f) = f \circ p.$$

Proposition 1.3. *Let Y be a topological space.*

(1) *If X is almost σ-compact, then p^* is a continuous injection.*
(2) *If X is hemicompact, then p^* is an embedding.*
(3) *$(C(T,Y),\tau_{CO})$ is homeomorphic to $\Pi_n(C(C_n,Y),\tau_{CO})$.*

Proof. (1) p is almost onto (i.e. its image is a dense subset of its range [30]), so [30, Theorem 2.2.6(a), Corollary 2.2.8(b)] applies.

(2) p is k-covering (i.e. for each $K \in K(X)$ there is $L \in K(T)$ with $K \subseteq p(L)$), so [30, Corollary 2.2.8(b)] applies.

(3) See [30, Corollary 2.4.7] □

2. G_δ-diagonal and related properties

A topological space Y is *submetrizable*, if it has a coarser metrizable topology; further, Y has a *G_δ-diagonal* (*G^*_δ-diagonal*, resp.), provided there is sequence $\mathcal{V}_m$ of open covers of Y such that $\{y\} = \bigcap_m \mathrm{St}(y,\mathcal{V}_m)$ ($\{y\} = \bigcap_m \overline{\mathrm{St}(y,\mathcal{V}_m)}$, resp.) for each $y \in Y$, where $\mathrm{St}(y,\mathcal{V}_m) = \bigcup\{V \in \mathcal{V}_m : y \in V\}$ (see [16]). Finally, Y has a *regular G_δ-diagonal*, provided there is a sequence $\mathcal{V}_m$ of open covers of Y such that if $y_0, y_1 \in Y, y_0 \neq y_1$, then there exists $m \in \omega$ and open sets W_0, W_1 containing y_0, y_1 respectively such that no member of $\mathcal{V}_m$ intersects both W_0, W_1 [41]. These notions are related as follows:

$$\text{submetrizable} \Rightarrow \text{regular } G_\delta\text{-diagonal} \Rightarrow G^*_\delta\text{-diagonal} \Rightarrow G_\delta\text{-diagonal}.$$

Submetrizable spaces, spaces with a regular G_δ-diagonal, and with a G^*_δ-diagonal, respectively, are Hausdorff.

Theorem 2.1. *The following are equivalent.*

(1) *$(\mathcal{P},\tau_C)$ is submetrizable (with a regular G_δ-diagonal, with a G^*_δ-diagonal, T_2 with a G_δ-diagonal, resp.),*
(2) *X is hemicompact, metrizable, and Y is submetrizable (with a regular G_δ-diagonal, with a G^*_δ-diagonal, T_2 with a G_δ-diagonal, resp.).*

Proof. (1) $\Rightarrow$ (2) $(CL(X),\tau_F)$, and Y are submetrizable (with a regular G_δ-diagonal, with a G^*_δ-diagonal, T_2 with a G_δ-diagonal, resp.), since they embed in $(\mathcal{P},\tau_C)$. It follows that X is hemicompact, and metrizable [20, Theorem 7].

(2) $\Rightarrow$ (1) Let X be a hemicompact metrizable space.

• *Submetrizability of $\mathcal{P}$*: if Y is submetrizable, then there exists a topology τ' on Y, which is weaker than the original topology τ on Y, such that (Y,τ') is metrizable. Then by [18, Theorem 2.4], $(\mathcal{P}(X,(Y,\tau')),\tau_C)$ is metrizable, and hence, $(\mathcal{P}(X,(Y,\tau)),\tau_C)$ is submetrizable.

Let $\{\mathcal{V}_m\}_m$ be a sequence of open covers of Y satisfying the regular G_δ-diagonal (G^*_δ-diagonal, G_δ-diagonal, resp.) property. By Lemma 1.2, $\{\mathcal{H}_n(\mathcal{V}_m) : n,m \in \omega\}$ is a sequence of open covers of $(\mathcal{P},\tau_C)$, and we will show that it is a regular G_δ-diagonal (G^*_δ-diagonal, G_δ-diagonal, resp.) sequence.

• *Regular G_δ-diagonal property of $\mathcal{P}$*: let $(B,f),(D,g) \in \mathcal{P}$ be distinct. Assume first that $B \neq D$, say, there is some $x \in B \setminus D$ (the argument is identical, if $x \in D \setminus B$). Find n so that $\overline{S(x,\frac{1}{n})} \subseteq \mathrm{int}C_n \setminus D$. Then $\mathcal{W}_0 = [S(x,\frac{1}{3n})]$, and $\mathcal{W}_1 = [\overline{S(x,\frac{1}{n})} : \emptyset]$ are $\mathcal{P}$-neighborhoods of $(B,f),(D,g)$, respectively. If some

$H_n = H_{3n}(\mathcal{V}_1,\mathcal{U},\varphi) \in \mathcal{H}_{3n}(\mathcal{V}_1)$ hits $\mathcal{W}_0$, choose $(E,h) \in H_n \cap \mathcal{W}_0$. Let $e \in E \cap S(x,\frac{1}{3n}) \subseteq C_n$, then there is $U \in \mathcal{U}$ with $e \in U$, so for all $u \in U$ we have

$$d(x,u) \leq d(x,e) + d(e,u) < \tfrac{1}{3n} + \mathrm{diam}(U) \leq \tfrac{1}{3n} + \tfrac{2}{3n} = \tfrac{1}{n};$$

thus, $U \subseteq S(x,\frac{1}{n})$, which implies that H_n misses $\mathcal{W}_1$.

Now assume that $B = D$, but $f(x) \neq g(x)$ for some $x \in B$, and choose Y-open neighborhoods W_0, W_1 of $f(x), g(x)$, respectively, and $m \in \omega$ such that no member of $\mathcal{V}_m$ hits both W_0, W_1. Find $n \in \omega$ so that $S(x,\frac{1}{n}) \subseteq C_n$,

$$f(B \cap \overline{S(x,\tfrac{1}{n})}) \subseteq W_0, \text{ and } g(D \cap \overline{S(x,\tfrac{1}{n})}) \subseteq W_1.$$

Then

$$\mathcal{W}_0 = [S(x,\tfrac{1}{3n})] \cap [\overline{S(x,\tfrac{1}{n})} : W_0], \text{ and } \mathcal{W}_1 = [S(x,\tfrac{1}{3n})] \cap [\overline{S(x,\tfrac{1}{n})} : W_1]$$

are $\mathcal{P}$-neighborhoods of $(B,f),(D,g)$, respectively. If some $H_n = H_{3n}(\mathcal{V}_m,\mathcal{U},\varphi) \in \mathcal{H}_{3n}(\mathcal{V}_m)$ hits $\mathcal{W}_0$, choose $(E,h) \in H_n \cap \mathcal{W}_0$. Let $e \in E \cap S(x,\frac{1}{3n}) \subseteq C_n$, then there is $U \in \mathcal{U}$ with $e \in U$, so (as above) $U \subseteq S(x,\frac{1}{n})$; thus,

$$h(e) \in h(E \cap \overline{U}) \subseteq \varphi(U) \cap W_0,$$

hence, $\varphi(U) \in \mathcal{V}_m$ will not hit W_1, which implies that H_n misses $\mathcal{W}_1$.

- G^*_δ*-diagonal property of* $\mathcal{P}$: let $D_0 = (B_0, f_0) \in \mathcal{P}$, and

$$D \in \bigcap_{n,m\in\omega} \overline{\mathrm{St}(D_0,\mathcal{H}_n(\mathcal{V}_m))}, \text{ where } D = (B,f).$$

It suffices to prove that $D = D_0$:

Claim. $B = B_0$

Suppose there is $x \in B \setminus B_0$. Let n be such that $x \in C_n$, and $B_0 \cap C_n \neq \emptyset$. Let $k > n$ be such that $S(x,\frac{1}{k}) \subseteq \mathrm{int}C_{n+1} \setminus B_0$. Since $D \in \overline{\mathrm{St}(D_0,\mathcal{H}_{4k}(\mathcal{V}_1)}$, there is

$$(H,g) \in [S(x,1/4k)] \cap \mathrm{St}(D_0,\mathcal{H}_{4k}(\mathcal{V}_1)),$$

and hence some $z \in H \cap S(x,\frac{1}{4k})$. Further, there is a finite family $\mathcal{U}$ of open balls of radius at most $\frac{1}{4k}$, and $\varphi : \mathcal{U} \to \mathcal{V}_1$ such that $(H,g), D_0 \in H_{4k}(\mathcal{V}_1,\mathcal{U},\varphi)$. But then there is $U \in \mathcal{U}$ with $z \in U$ and $U \cap B_0 \neq \emptyset$, which is a contradiction, since for $b \in B_0 \cap U$ we have $d(z,b) \leq \mathrm{diam}(U) \leq \frac{1}{2k}$, so $d(x,b) \leq d(x,z) + d(z,b) \leq \frac{3}{4k} < \frac{1}{k}$; on the other side, $S(x,\frac{1}{k}) \subseteq B_0^c$, so $d(x,b) \geq \frac{1}{k}$.

Now suppose $x \in B_0 \setminus B$, and L is an open set with compact closure such that $x \in L \subseteq \overline{L} \subseteq B^c$; then $[\overline{L} : \emptyset]$ is a τ_C-neighborhood of D. There is $n \in \omega$ and $k > n$ such that $x \in C_n$, and $S(x,\frac{1}{k}) \subseteq L$. Since $D \in \overline{\mathrm{St}(D_0,\mathcal{H}_{3k}(\mathcal{V}_1))}$, there is $(H,g) \in [\overline{L} : \emptyset] \cap \mathrm{St}(D_0,\mathcal{H}_{3k}(\mathcal{V}_1))$, so there is a finite family $\mathcal{U}$ of open balls of radius at most $\frac{1}{3k}$, and $\varphi : \mathcal{U} \to \mathcal{V}_1$ such that $D_0, (H,g) \in H_{3k}(\mathcal{V}_1,\mathcal{U},\varphi)$. It follows that for some $U \in \mathcal{U}$, $x \in U$ and $U \cap H \neq \emptyset$, say, $h \in U \cap H$. Then $d(x,h) \leq \mathrm{diam}(U) \leq \frac{2}{3k} < \frac{1}{k}$, so $h \in \overline{L}$, which is impossible since $(H,g) \in [\overline{L} : \emptyset]$.

Claim. $f_0 = f$.

Suppose $f_0(x) \neq f(x)$ for some $x \in B = B_0$. Then there is $m \in \omega$ with $f(x) \notin \overline{\mathrm{St}(f_0(x),\mathcal{V}_m)}$. We can find an X-open set O with compact closure such that $x \in O$, and $f(\overline{O} \cap B) \subseteq Y \setminus \overline{\mathrm{St}(f_0(x),\mathcal{V}_m)}$. Let $n \in \omega$ be such that $x \in C_n$, and $S(x,\frac{3}{n}) \subseteq O$. Then $[\overline{O} : Y \setminus \overline{\mathrm{St}(f_0(x),\mathcal{V}_m)}]$ is a τ_C-neighborhood of D, so there is

$$(H,g) \in [\overline{O} : Y \setminus \overline{\mathrm{St}(f_0(x),\mathcal{V}_m)}] \cap \mathrm{St}(D_0,\mathcal{H}_n(\mathcal{V}_m)).$$

Then we can find a finite family $\mathcal{U}$ of open balls of radius at most $\frac{1}{n}$, and $\varphi : \mathcal{U} \to \mathcal{V}_m$ such that $(H, g), D_0 \in H_n(\mathcal{V}_m, \mathcal{U}, \varphi)$. Hence, there is $U \in \mathcal{U}$ with $x \in U$ and $U \cap H \neq \emptyset$ such that $f_0(x) \in \varphi(U)$, and $g(\overline{U} \cap H) \subseteq \varphi(U)$; thus, $g(\overline{U} \cap H) \subseteq \mathrm{St}(f_0(x), \mathcal{V}_m)$. On the other side, if $h \in \overline{U} \cap H$, then $d(x, h) \leq \mathrm{diam}(\overline{U}) \leq \frac{2}{n}$, so $\overline{U} \cap H \subseteq S(x, \frac{3}{n}) \subseteq O$; thus, $g(\overline{U} \cap H) \subseteq g(\overline{O} \cap H) \subseteq (\overline{\mathrm{St}(f_0(x), \mathcal{V}_m)})^c$.

• *G_δ-diagonal property of $\mathcal{P}$*: let $D_0 = (B_0, f_0) \in \mathcal{P}$ and

$$D \in \bigcap_{n,m \in \omega} \mathrm{St}(D_0, \mathcal{H}_n(\mathcal{V}_m)), \text{ where } D = (B, f).$$

We will show that $D = D_0$: let $x \in B$. We can find an n such that $x \in C_n$, and $B_0 \cap C_n \neq \emptyset$. Fix $m \in \omega, m \geq n$. Then there is a finite family $\mathcal{U}$ of open balls of radius at most $\frac{1}{m}$ that are subsets of C_{m+1}, and a function $\varphi : \mathcal{U} \to \mathcal{V}_m$ such that $D, D_0 \in H_m(\mathcal{V}_m, \mathcal{U}, \varphi)$. Then there exists $U_m \in \mathcal{U}$ with $x \in U_m$, and $B_0 \cap U_m \neq \emptyset$ such that $f(x) \in \varphi(U_m)$, and $f_0(B_0 \cap \overline{U}_m) \subseteq \varphi(U_m)$. Then $\{U_m\}_m$ is a local base at x, thus, $\{x\} = \bigcap_m B_0 \cap \overline{U}_m$. It follows, that $f(x) \in \bigcap_m \mathrm{St}(f_0(x), \mathcal{V}_m) = \{f_0(x)\}$, so $D \subseteq D_0$. It is also true, that $D_0 \in \bigcap_{n,m} \mathrm{St}(D, \mathcal{H}_n(\mathcal{V}_m))$, so we can argue as above to get $D_0 \subseteq D$; hence, $\{D_0\} = \bigcap_{n,m \in \omega} \mathrm{St}(D_0, \mathcal{H}_n(\mathcal{V}_m))$. □

It was proved in [37] that, if X is compact and Y has a regular G_δ-diagonal (G_δ^*-diagonal, G_δ-diagonal, resp.), then $(C(X, Y), \tau_{CO})$ has a regular G_δ-diagonal (G_δ^*-diagonal, G_δ-diagonal, resp.); then, in [34], the same was proved for an almost σ-compact X. In our next result we give another proof, and also show that if X is an almost σ-compact space, and Y is submetrizable, then $(C(X, Y), \tau_{CO})$ is submetrizable. For $Y = \mathbb{R}$, the results concerning submetrizability, and the G_δ-diagonal property, were proved in [30].

Theorem 2.2. *Let X be an almost σ-compact space, and Y be submetrizable (have a regular G_δ-diagonal, G_δ^*-diagonal, G_δ-diagonal, resp.). Then $(C(X, Y), \tau_{CO})$ is submetrizable (has a regular G_δ-diagonal, G_δ^*-diagonal, G_δ-diagonal, resp.).*

Proof. Let $X, T = \bigoplus_n C_n$, and $p : T \to X$ be as in Proposition 1.3(1), and Y have a regular G_δ-diagonal (G_δ^*-diagonal, G_δ-diagonal, resp.). Since these diagonal properties are countably productive, $(C(T, Y), \tau_{CO})$ has them by Proposition 1.3(3), so by Proposition 1.3(1), $(C(X, Y), \tau_{CO})$ has a coarser topology having these diagonal properties; thus, $(C(X, Y), \tau_{CO})$ itself has them.

Let (Y, τ) be submetrizable, and $\tau' \subseteq \tau$ be a metrizable topology on Y. Then $(C(T, (Y, \tau')), \tau_{CO})$ is metrizable by [30, Exercise IV.9.1(a)], let α be this metrizable topology on $C(T, (Y, \tau'))$ that is coarser than the original $(C(T, (Y, \tau)), \tau_{CO})$. The family $\beta = \{(p^*)^{-1}(U) : U \in \alpha\}$ is a topology on $C(X, Y)$ coarser than τ_{CO}. The mapping $p^* : (C(X, Y), \beta) \to (p^*(C(X, Y), \alpha)$ is a homeomorphism, so $(C(X, Y), \beta)$ is metrizable, and $(C(X, (Y, \tau)), \tau_{CO})$ is submetrizable. □

3. Developability and related properties

Let Y be a topological space. A sequence $\{\mathcal{V}_n\}$ of open covers of Y is called a *(weak) development*, if for every $y \in Y$ and $\{V_n\}$ such that $y \in V_n \in \mathcal{V}_n$ for every n, the sequence $\{V_n\}$ (resp. $\{\bigcap_{i \leq n} V_i\}$) is a base at y. A space with a (weak) development is called *(weakly) developable*; a *Moore space* is a regular developable space. A sequence $\{\mathcal{V}_n\}$ of open covers of Y is called a *weak k-development*, provided for each $K \in K(Y)$, and every finite $\mathcal{W}_n \subseteq \mathcal{V}_n$ such that $K \subseteq \bigcup \mathcal{W}_n$, and $K \cap W \neq \emptyset$ for every $W \in \mathcal{W}_n$, the sequence $\{\bigcap_{i \leq n}(\bigcup \mathcal{W}_i)\}$ is a base at K. A space with a weak k-development is called *weakly k-developable*.

Observe, that a T_1 weakly developable space has a G_δ-diagonal, and developable, as well as, weakly k-developable spaces are weakly developable. On the other side, there are weakly k-developable spaces which are not developable [4], as well as developable Hausdorff spaces that are not weakly k-developable [3].

Theorem 3.1. *Let X be a hemicompact metrizable space, and Y a weakly k-developable space. Then $(\mathcal{P}, \tau_C)$ is weakly developable.*

Proof. Let $\{\mathcal{V}_n\}$ be a weak k-development of Y, and, without loss of generality, suppose that $\mathcal{V}_{n+1}$ is a refinement of $\mathcal{V}_n$ for every $n \in \omega$. We claim that $\{\mathcal{H}_n(\mathcal{V}_n)\}$ is a weak development in $(\mathcal{P}, \tau_C)$: that $\mathcal{H}_n(\mathcal{V}_n)$ is an open cover of $(\mathcal{P}, \tau_C)$ for every $n \in \omega$, follows from Lemma 1.2.

Let $(B, f) \in (\mathcal{P}, \tau_C)$, and $H_n = H_n(\mathcal{V}_n, \mathcal{U}_n, \varphi_n) \in \mathcal{H}_n(\mathcal{V}_n)$ be such that $(B, f) \in H_n$ for every $n \in \omega$. To prove that $\bigcap_{i \leq n} H_n$ is a base at (B, f), first choose an X-open G with $(B, f) \in [G]$, and pick some $b \in B \cap G$. There is $n \in \omega$ such that $b \in C_n$, and $S(b, 1/n) \subseteq G$. Now, $(B, f) \in H_{3n}$, and since $b \in C_n$, we can find a $U \in \mathcal{U}_{3n}$ with $b \in U$; thus, if $(C, g) \in H_{3n}$, and $c \in C \cap U$, then $d(b, c) \leq \operatorname{diam} U \leq \frac{2}{3n} < \frac{1}{n}$, so $C \cap G \neq \emptyset$, which implies, that $(B, f) \in H_{3n} \subseteq [G]$.

Now let $(B, f) \in [K : V]$, where $K \in K(X)$, V is Y-open, and assume $B \cap K = \emptyset$. Then $\operatorname{dist}(K, B) > 0$, so we can find $n \in \omega$ such that $K \subseteq C_n$, and $\operatorname{dist}(K, B) > 2/n$. To show that $H_n \subseteq [K : V]$, choose $(C, g) \in H_n$. If $\mathcal{U}_n = \emptyset$, then $C \cap K = \emptyset$, and we are done. If $\mathcal{U}_n \neq \emptyset$, assume that $K \cap U \neq \emptyset$ for some $U \in \mathcal{U}_n$, and find $k \in K \cap U, b \in B \cap U$. Then $\operatorname{dist}(K, B) \leq d(k, b) \leq \operatorname{diam} U \leq 2/n$, which is impossible, so $K \cap (\bigcup \mathcal{U}_n) = \emptyset$; thus, $K \subseteq C_n \setminus \bigcup \mathcal{U}_n$, and again, $C \cap K = \emptyset$.

Finally, suppose $B \cap K \neq \emptyset$. Then $f(B \cap K) \subseteq V$, so by compactness of K, there is $\delta > 0$ such that $f(B \cap \overline{S(K, \delta)}) \subseteq V$, and $\overline{S(K, \delta)}$ is compact, where $S(K, \delta) = \bigcup_{x \in K} S(x, \delta)$. Choose $n_0 \in \omega$ so that $\frac{2}{n_0} < \delta$ and $\overline{S(K, \delta)} \subseteq C_{n_0}$. For $n \geq n_0$, there is a finite collection $\emptyset \neq \mathcal{U}'_n \subseteq \mathcal{U}_n$ such that $U \cap \overline{S(K, \delta)} \cap B \neq \emptyset$ for all $U \in \mathcal{U}'_n$; put $\mathcal{W}_n = \{\varphi_n(U) : U \in \mathcal{U}'_n\}$.

Then $f(\overline{S(K, \delta)} \cap B) \subseteq \bigcup \mathcal{W}_n$, and $f(\overline{S(K, \delta)} \cap B) \cap W \neq \emptyset$ for every $W \in \mathcal{W}_n$. For $n < n_0$, $\mathcal{V}_{n_0}$ is a refinement of $\mathcal{V}_n$, so for each $W \in \mathcal{W}_{n_0}$ there is $V_W \in \mathcal{V}_n$ with $W \subseteq V_W$; put $\mathcal{W}_n = \{V_W : W \in \mathcal{W}_{n_0}\}$. Since $\{\mathcal{V}_n\}$ is a weak k-development, there is $k > n_0$ such that

$$f(\overline{S(K, \delta)} \cap B) \subseteq \bigcap_{n \leq k} (\bigcup \mathcal{W}_n) \subseteq V.$$

Let $n_0 \leq n \leq k$, and choose $(C, g) \in H_n$. If $C \cap K = \emptyset$, we are done, so suppose $\emptyset \neq C \cap K \subseteq C \cap C_n$. If $c \in C \cap K$, then $c \in U \in \mathcal{U}_n$, and if $b \in B \cap U$, then $d(c, b) \leq \operatorname{diam}(U) \leq \frac{2}{n} < \delta$, so $U \in \mathcal{U}'_n$.

It follows, that $\mathcal{U}''_n = \{U \in \mathcal{U}_n : U \cap K \cap C \neq \emptyset\} \subseteq \mathcal{U}'_n$, so

$$g(C \cap K) \subseteq \bigcup \{g(\overline{U} \cap C) : U \in \mathcal{U}''_n\} \subseteq \bigcup \{\varphi_n(U) : U \in \mathcal{U}''_n\} \subseteq \bigcup \mathcal{W}_n.$$

Now, if $(C, g) \in \bigcap_{n \leq k} H_n$, then

$$g(C \cap K) \subseteq \bigcap_{n_0 \leq n \leq k} (\bigcup \mathcal{W}_n) = \bigcap_{n \leq k} (\bigcup \mathcal{W}_n) \subseteq V,$$

so $(C, g) \in [K : V]$. □

Remark 3.2. If $(\mathcal{P},\tau_C)$ is a weakly developable T_2 space, then $(CL(X),\tau_F)$ is, too (Proposition 1.1); thus, $(CL(X),\tau_F)$ is T_2 with a G_δ-diagonal, hence, X is a hemicompact metrizable space by [20, Theorem 7].

Theorem 3.3. *Let X be a hemicompact space, and Y a weakly k-developable space. Then $(C(X,Y),\tau_{CO})$ is weakly developable.*

Proof. Let $\{C_n\}$ be a cofinal family in $K(X)$. Given $n\in\omega$, a collection $\mathcal{V}$ of Y-open sets, a finite collection $\mathcal{U}$ of open sets in C_n covering C_n, and $\varphi:\mathcal{U}\to\mathcal{V}$, the set

$$G_n(\mathcal{V},\mathcal{U},\varphi)=\bigcap_{U\in\mathcal{U}}[\overline{U},\varphi(U)]$$

is open in $(C(X,Y),\tau_{CO})$. Let $\{\mathcal{V}_n\}$ be a weak k-development of Y such that $\mathcal{V}_{n+1}$ refines $\mathcal{V}_n$ for every $n\in\omega$. Then

$$\mathcal{G}_n(\mathcal{V}_n)=\{G_n(\mathcal{V}_n,\mathcal{U},\varphi):\mathcal{U},\varphi\}$$

is an open cover of $(C(X,Y),\tau_{CO})$ for each n: let $f\in C(X,Y)$, and $n\in\omega$ be fixed. Then there is a finite collection $\mathcal{V}_n'\subseteq\mathcal{V}_n$ that covers the compact $f(C_n)$. By regularity of C_n, for each $V\in\mathcal{V}_n'$ and $x\in C_n\cap f^{-1}(V)$, find a C_n-open $U(x)$ such that $x\in U(x)\subseteq\overline{U(x)}\subseteq C_n\cap f^{-1}(V)$, and choose a finite subcover $\mathcal{U}$ of $\{U(x):x\in C_n\cap f^{-1}(\bigcup\mathcal{V}_n')\}$. Then for each $U\in\mathcal{U}$, there is $V_U\in\mathcal{V}_n'$ with $f(\overline{U})\subseteq V_U$, so we can define $\varphi(U)=V_U$. Clearly, $f\in G_n(\mathcal{V}_n,\mathcal{U},\varphi)$.

To prove that $\{\mathcal{G}_n(\mathcal{V}_n)\}$ is a weak development, take $f\in C(X,Y)$, and $G_n=G_n(\mathcal{V}_n,\mathcal{U}_n,\varphi_n)\in\mathcal{G}_n(\mathcal{V}_n)$ such that $f\in G_n$ for every $n\in\omega$. Consider $[K,V]\in\tau_{CO}$ with $f\in[K,V]$, and choose n_0 so that $K\subseteq C_{n_0}$.

For $n\geq n_0$, let $\mathcal{U}_n'\subseteq\mathcal{U}_n$ be such that $K\subseteq\bigcup\mathcal{U}_n'$, and $K\cap U\neq\emptyset$ for all $U\in\mathcal{U}_n'$, and put $\mathcal{W}_n=\{\varphi_n(U):U\in\mathcal{U}_n'\}$. For $n<n_0$, $\mathcal{V}_{n_0}$ is a refinement of $\mathcal{V}_n$, so for each $W\in\mathcal{W}_{n_0}$, there is $V_W\in\mathcal{V}_n$ with $W\subseteq V_W$; put $\mathcal{W}_n=\{V_W:W\in\mathcal{W}_{n_0}\}$.

Observe, that for all $n\geq n_0$, $f(K)\subseteq\bigcup\mathcal{W}_n$ and $f(K)\cap W\neq\emptyset$ for all $W\in\mathcal{W}_n$, so by weak k-developability of Y, there is $k>n_0$ such that

$$f(K)\subseteq\bigcap_{n\leq k}(\bigcup\mathcal{W}_n)\subseteq V.$$

Let $n_0\leq n\leq k$, and $g\in G_n$. Then $g(\overline{U})\subseteq\varphi_n(U)$ for each $U\in\mathcal{U}_n$; thus,

$$g(K)\subseteq\bigcup\{g(\overline{U}):U\in\mathcal{U}_n'\}\subseteq\bigcup\mathcal{W}_n.$$

It follows, that if $g\in\bigcap_{n\leq k}G_n$, then

$$g(K)\subseteq\bigcap_{n_0\leq n\leq k}(\bigcup\mathcal{W}_n)=\bigcap_{n\leq k}(\bigcup\mathcal{W}_n)\subseteq V,$$

so $g\in[K,V]$, hence, $f\in\bigcap_{n\leq k}G_n\subseteq[K,V]$. □

As for developability of $(C(X,Y),\tau_{CO})$, it is known that *if X is hemicompact with metrizable compacts, and Y is developable, then $(C(X,Y),\tau_{CO})$ is developable* [33]; further, *if X is hemicompact, and Y is a Moore space with a regular G_δ-diagonal, then $(C(X,Y),\tau_{CO})$ is a Moore space* [37]. The following question from [34] seems to be still open:

Problem 3.4. *Let X be a compact space, and Y a Moore space. Is $(C(X,Y),\tau_{CO})$ a Moore space?*

In the last part of the paragraph, we have a result about developability of $(\mathcal{P},\tau_C)$:

Theorem 3.5. *Let X be a topological sum of a countable family of compact metrizable spaces, and Y be developable. Then $(\mathcal{P}, \tau_C)$ is developable.*

Proof. First assume that X is compact. Then the generalized compact-open topology τ_C, and the Vietoris topology τ_V coincide on $\mathcal{P}$. Since $X \times Y$ is developable, also $(K(X \times Y), \tau_V)$ is developable [32], so $\mathcal{P} \subseteq K(X \times Y)$ is developable.

Now, let $X = \bigoplus_{n\in\omega} C_n$, where C_n is a metrizable compact for each n. Consider $\mathcal{P}_n = \mathcal{P}(C_n, Y) \cup \{\emptyset\}$, with the topology $\tau'_C = \tau_C \cup \{\{\emptyset\}\}$. If $(B, f) \in \mathcal{P}$, define $(D_n)_n \in \Pi_n \mathcal{P}_n$ so that

$$D_n = \begin{cases} (B_n, f_n), & \text{if } B_n = B \cap C_n \neq \emptyset,\ f_n = f\restriction_{B_n}, \\ \emptyset, & \text{if } B \cap C_n = \emptyset, \end{cases}$$

and put $\psi(B, f) = (D_n)_n$. It is not hard to show, that ψ is a homeomorphism, so $(\mathcal{P}, \tau_C)$ is developable, since $\Pi_n \mathcal{P}_n$ is (as a countable product of developable spaces). □

4. First countability and related properties

Theorem 4.1. *The following are equivalent:*

1. *points in $(\mathcal{P}, \tau_C)$ are G_δ,*
2. *X-open sets are σ-compact, each $A \in CL(X)$ has a countable π-base (in A), and points in Y are G_δ.*

Proof. (1)⇒(2) Points in $(CL(X), \tau_F)$ and Y are G_δ, since they embed in $(\mathcal{P}, \tau_C)$. Then, by [20, Proposition 4.3(ii)], open sets in X are σ-compact; further, let $A \in CL(X)$, and $\mathcal{B}_n = ((K_n)^c)^+ \cap \bigcap_{i\in I_n} U_i^n$ be basic τ_F-open sets such that $\{A\} = \bigcap_{n\in\omega} \mathcal{B}_n$. If $\emptyset \neq U \subseteq A$ is open in A, and for all n, and $i \in I_n$, there exists $a_i^n \in U_i^n \cap A \setminus U$, then $A \setminus U \in \bigcap_{n\in\omega} \mathcal{B}_n$, which is a contradiction. It follows, that $\{A \cap U_i^n : n \in \omega, i \in I_n\}$ is a countable π-base in A.

(2)⇒(1) Let $(B_0, f_0) \in \mathcal{P}$, $B_0 \neq X$, and $B_0^c = \bigcup_n K_n$ for some $K_n \in K(X)$. Let $\{U_n\}$ be a countable sequence of X-open sets such that $\{B_0 \cap U_n\}$ is a π-base for B_0; then B_0 is also separable, so we can find a countable set C dense in B_0. Finally, for each $c \in C$, choose a sequence $\mathcal{G}(c)$ of Y-open sets intersecting in $f_0(c)$. Consider the collection

$$\mathcal{G} = \{[K_n : \emptyset] \cap [U_k] \cap [\{c\} : V] : n, k \in \omega, c \in C, V \in \mathcal{G}(c)\},$$

and take a $(B, f) \in \bigcap \mathcal{G}$. We will show that $(B, f) = (B_0, f_0)$: assume that there is $x \in B \setminus B_0$. Choose an n so that $x \in K_n$, then $(B, f) \notin [K_n : \emptyset]$, which is impossible, so $B \subseteq B_0$. Conversely, if $B_0 \cap B^c \neq \emptyset$, there is some k such that $B_0 \cap U_k \subseteq B_0 \cap B^c$, so $(B, f) \notin [U_k]$, which is a contradiction again, so $B_0 \subseteq B$; thus, $B = B_0$.

Now, for each $c \in C$, and $V \in \mathcal{G}(c)$, $f(c) \in V$, so $f(c) \in \bigcap \mathcal{G}(c) = \{f_0(c)\}$. This means that f, f_0 are identical on the dense set C, so by continuity, $f = f_0$.

If $B_0 = X$, we can choose $\mathcal{G} = \{[U_k] \cap [\{c\} : V] : k \in \omega, c \in C, V \in \mathcal{G}(c)\}$, and the above argument still works. □

Since $(CL(X), \tau_F)$ is embedded in $(\mathcal{P}, \tau_C)$, 1st countability of $(\mathcal{P}, \tau_C)$ implies that of $(CL(X), \tau_F)$. Conversely, if Y is locally convex and completely metrizable, then 1st countability of $(CL(X), \tau_F)$ implies complete metrizability of $(C(X, Y), \tau_{CO})$

(through results of [19], [31]), and the restriction mapping $\eta : (CL(X), \tau_F) \times (C(X,Y), \tau_{CO}) \to (\mathcal{P}, \tau_C)$, defined as $\eta((B,f)) = (B, f \restriction B)$, is continuous, open and onto (see [18], [21]). Thus, $(\mathcal{P}, \tau_C)$ is 1st countable if and only if $(CL(X), \tau_F)$ is. We can strengthen this result as follows:

Theorem 4.2. *Let Y be a space where compact sets are both metrizable, and of countable character. The following are equivalent:*

(1) *$(\mathcal{P}, \tau_C)$ is 1st countable;*
(2) *$(CL(X), \tau_F)$ is 1st countable;*
(3) *X is 1st countable, the open sets in X are hemicompact, and every $B \in CL(X)$ is separable.*

Proof. Since $(CL(X), \tau_F)$ is embedded in $(\mathcal{P}, \tau_C)$ we have (1)⇒(2); for (2)⇔(3) see [19].

(3)⇒(1) Let $(B,f) \in \mathcal{P}$, and $C \subseteq B$ be a countable set dense in B. For every $c \in C$, let $\mathcal{B}(c)$ be a countable base of neighborhoods at c. Since B is hemicompact, we can find a cofinal subfamily $\{B_n\}$ in $K(B)$. Let $n \in \omega$. Then $f(B_n)$ is compact and metrizable, so there is a countable base $\{O_n^m\}$ in $f(B_n)$. Let $\mathcal{G}(\overline{O_n^m})$ be a countable base of neighborhoods of the compact $\overline{O_n^m}$ for every $n, m \in \omega$. Enumerate the countable set $\bigcup_{m,n} \mathcal{G}(\overline{O_n^m})$ as $\{V_n\}$.

Let $\{U_n\}_n$ be a sequence of X-open sets such that $U_n \cap B = f^{-1}(V_n)$. Finally, let $\{K_n^i : i \in \omega\} \subseteq K(X)$ be a cofinal family in $K(U_n)$, and $\{D_m : m \in \omega\} \subseteq K(X)$ an increasing cofinal family in $K(B^c)$. We claim that the sets of the form

$$[D_m : \emptyset] \cap \bigcap_{c \in C'} [G_c] \cap \bigcap_{(i,n,s) \in I \times N \times S} [K_n^i : V_s],$$

where $C' \in [C]^{<\omega}$, $I, N, S \in [\omega]^{<\omega}$, and $G_c \in \mathcal{B}(c)$, form a τ_C-open base of neighborhoods at (B,f) (the symbol $[T]^{<\omega}$ stands for the set of finite subsets of T).

Indeed, if U is X-open and $(B,f) \in [U]$, then $(B,f) \in [G_c] \subseteq [U]$ for some $c \in C$, and $G_c \in \mathcal{B}(c)$ such that $G_c \subseteq U$. Further, if $(B,f) \in [K : \emptyset]$ for some $K \in K(X)$, then $(B,f) \in [D_m : \emptyset] \subseteq [K : \emptyset]$ for some $m \in \omega$.

Now, let $(B,f) \in [K : V]$, where $V \subseteq Y$ is nonempty open, and $K \in K(X)$ such that $K \cap B \neq \emptyset$. Then $f(B \cap K) \subseteq V$, and there is $n \in \omega$ with $K \cap B \subseteq B_n$, so $f(K \cap B) \subseteq f(B_n) \cap V$. There are $O_n^{m_0}, \ldots, O_n^{m_j}$ such that

$$f(K \cap B) \subseteq \bigcup_{i \le j} \overline{O_n^{m_i}} \subseteq f(B_n) \cap V \subseteq V,$$

therefore, we can also find some $N \in [\omega]^{<\omega}$ so that $f(K \cap B) \subseteq \bigcup_{n \in N} V_n \subseteq V$. Observe, that for each $x \in K \cap B$ there is $n \in N$ with $f(x) \in V_n$ and an X-open neighborhood W_x of x such that $K \cap \overline{W_x} \subseteq U_n$, and $f(\overline{W_x} \cap K \cap B) \subseteq V_n$. Compactness of $K \cap B$ guarantees the existence of $P \in [\omega]^{<\omega}$ such that $K \cap B \subseteq \bigcup_{p \in P} W_{x_p}$ for some $x_p \in K \cap B$.

Now, $K \setminus \bigcup_{p \in P} W_{x_p} \subseteq B^c$, so there is a D_m with $K \setminus \bigcup_{p \in P} W_{x_p} \subseteq D_m$. For every $p \in P$ we can find an $n_p \in N$ so that $K \cap \overline{W_{x_p}} \subseteq U_{n_p}$, hence $K \cap \overline{W_{x_p}} \subseteq K_{n_p}^{i_p} \subseteq U_{n_p}$ for some $i_p \in \omega$. It follows, that $(B,f) \in [D_m : \emptyset] \cap \bigcap_{p \in P} [K_{n_p}^{i_p} : V_{n_p}] \subseteq [K : V]$. □

Since every weakly developable space is first countable, it follows, by Theorem 3.3, that $(C(X,Y), \tau_{CO})$ is first countable, if X is a hemicompact space, and Y is weakly k-developable. In the next theorem, we will extend this for a Y in which compact sets are both metrizable, and of countable character. Note that in weakly k-developable spaces compacts are metrizable, and of countable character, but the converse is not true (ω_1 with the order topology is a counterexample [3]).

Theorem 4.3. *Let X be a hemicompact space, and Y be a space where compact sets are both metrizable, and of countable character. Then $(C(X,Y),\tau_{CO})$ is 1st countable.*

Proof. Assume first, that X is compact. Let $f \in C(X,Y)$, and $\{O_n\}$ be a countable base in the metrizable compact $f(X)$. Then the compact $\overline{O_n}$ has a countable base of neighborhoods $\{V_n^m\}_m$ for every $n \in \omega$. By regularity of $f(X)$, find a Y-open subset W_n^m of V_n^m such that

$$\overline{O_n} \subseteq f(X) \cap W_n^m \subseteq f(X) \cap \overline{W_n^m} \subseteq V_n^m,$$

and put $K_n^m = \overline{f^{-1}(W_n^m)}$ for every m, n. We claim, that

$$\{ \bigcap_{(n,m)\in F} [K_n^m, V_n^m] : F \in [\omega \times \omega]^{<\omega} \}$$

is a countable τ_{CO}-open base of neighborhoods at f. Indeed, let $[K,V]$ be a τ_{CO}-open neighborhood of $f \in C(X,Y)$. Then $f(K) \subseteq V$, and we can find finite collections $\{O_{n_i} : i \leq k\}$ and $\{V_{n_i}^{m_i} : i \leq k\}$ such that

$$f(K) \subseteq \bigcup_{i\leq k} \overline{O_{n_i}} \subseteq \bigcup_{i\leq k} V_{n_i}^{m_i} \subseteq V.$$

Then $f(K_{n_i}^{m_i}) \subseteq f(X) \cap \overline{W_{n_i}^{m_i}} \subseteq V_{n_i}^{m_i}$, so $f \in [K_{n_i}^{m_i}, V_{n_i}^{m_i}]$ for all $i \leq k$; further, if $g \in \bigcap_{i\leq k}[K_{n_i}^{m_i}, V_{n_i}^{m_i}]$, then $g(K) \subseteq \bigcup_{i\leq k} g(K_{n_i}^{m_i}) \subseteq \bigcup_{i\leq k} V_{n_i}^{m_i} \subseteq V$, so $g \in [K,V]$.

For a hemicompact X, the theorem now follows from Proposition 1.3(2),(3). □

Remark 4.4. It was proved in [30, Theorem 4.4.2] that, if $(C(X,\mathbb{R}),\tau_{CO})$ is 1st countable, then X is hemicompact.

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