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In this thesis we study the nonlinear eigenvalue problem of the form

$$
\begin{aligned}
-u''(x) &= \lambda f(u(x)) \quad \text{for } 0 < x < 1 \\
u(0) &= 0 = u(1).
\end{aligned}
\tag{1}
$$

We shall first discuss certain types of multiplicity results starting with the existence of two or three solutions which occur surprisingly often for semipositone problems arising in diverse applications. We develop necessary and sufficient conditions for these cases as well as the conditions sufficient for an arbitrary number of solutions. Both the semipositone problem and the problem where $f(0) = 0$ are considered. We also examine nonlinearities f with a variety of behaviors including superlinear, sublinear and bounded growth. In addition, we examine cases in which f has an arbitrary number of zero points. Finally we conclude with a discussion of how to construct example functions exhibiting the behaviors of the nonlinearities considered in this thesis.

MULTIPLE POSITIVE SOLUTIONS
FOR SEMIPOSITONE PROBLEMS

by
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CHAPTER I

INTRODUCTION

In this thesis we study the nonlinear eigenvalue problem of the form

$$\begin{aligned} -\triangle u(x) = &\ \lambda f(u(x)), \quad x \in \Omega \\ u(x) = 0, &\quad x \in \partial\Omega, \end{aligned}$$

where $\lambda > 0$ is a parameter, Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with a smooth boundary $\partial\Omega$, $\triangle u = \Sigma_{i=1}^{N} \frac{\partial^2 u}{\partial x_i^2}$ is the Laplacian operator.

A particular class of these problems where $f(u) > 0$, referred to as positone problems, has been studied extensively for the past 50 years. The term "positone" originates from the fact that the early problems studied had the nonlinearity f being positive and monotone. By semipositone we mean a semilinear equation where the nonlinearity f is negative at the origin. Semipositone problems have more recently attracted considerable interest from the mathematical community. The study of semipositone problems was initiated by Castro and Shivaji in [5], where, for dimension $N = 1$, they considered the autonomous Dirichlet boundary value problem

$$\begin{aligned} -u''(x) = \lambda f(u(x)) \quad \text{for } 0 < x < 1 \\ u(0) = 0 = u(1). \end{aligned} \tag{2}$$

In that paper they studied the cases when the nonlinearity f is

a) convex and superlinear at infinity

b) concave and sublinear at infinity

c) initially concave and later convex and superlinear

By "sublinear at infinity" we mean that the growth rate of the nonlinearity $f(u)$ for large u is "less than linear", which for our purposes we can take to mean that $\lim_{u\to\infty}\frac{u}{f(u)} = \infty$. Similarly, by "superlinear at infinity" we mean that the growth rate of the nonlinearity $f(u)$ for large u is "more than linear", which for our purposes we can take to mean that $\lim_{u\to\infty}\frac{u}{f(u)} = 0$.

For the positone case, Rabinowitz [6] showed that if we let $S = \{(\lambda, u) : \lambda \geq 0\}$ then S is a continuum (i.e., set of connected points) in $\mathbb{R}^+ \times C(\Omega)$ joining $(0,0)$ to ∞. The nature of the continuum depends clearly on the nonlinearity f. Keller and Cohen in [7], and Crandall and Rabinowutz [8] studied the case when f is convex and showed that if f is sufficiently convex then the function $\frac{u}{f(u)}$ has a local maximum and thus the continuum of solutions must bend back on itself. They applied their results to the nonlinearity $f(u) = e^u$ which arises in nonlinear heat conduction. Laetsch and Cohen [4] considered the case when f is concave and in particular proved that if the function $\frac{u}{f(u)}$ is non-decreasing, then (2) has a unique solution for every $\lambda > 0$. We will see that the function $\frac{u}{f(u)}$ plays a crucial role in our analysis of semipositone problems. This work gave rise to the use of "bifurcation" curves relating the parameter λ to $\sup|u|$ as shown in Figure 1. The bifurcation curve shows graphically the ranges of λ for which we have no solutions, one solution, or multiple solutions. For $N = 1$ Laetsch [3] developed a quadrature method in [3] to study the existence of positive solutions to (2) with respect to the parameter λ.

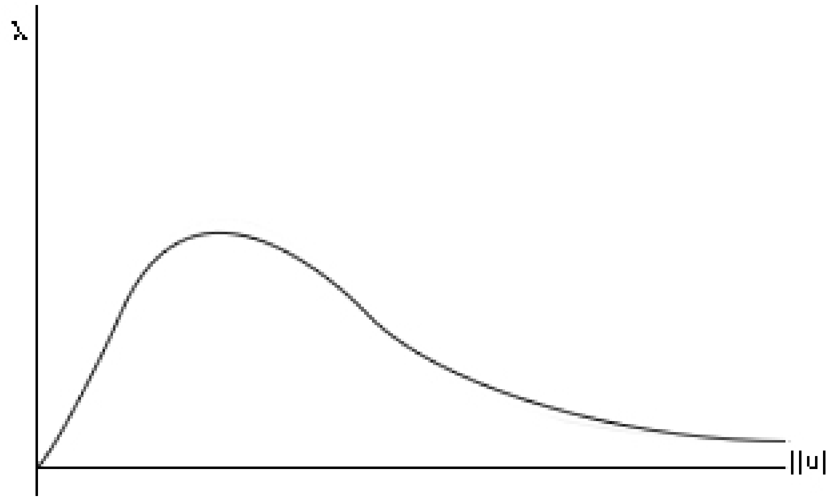


Figure 1: Bifurcation curve

In this thesis, multiplicity results for these cases were derived independently and then extended for the Dirichlet boundary value problem (2) where the nonlinearity f is defined on $[0, r]$ and satisfies

$(f1)$ $f \in \mathcal{C}^2([0, r])$

$(f2)$ $f(0) \leq 0$

$(f3)$ $f(u) < 0$ for $0 < u < \tau$, $f(\tau) = 0$ and $f(u) > 0$ for $u > \tau$.

We shall first discuss certain types of multiplicity results starting with the existence of two or three solutions which occur surprisingly often for semipositone problems arising in diverse applications. We develop necessary and sufficient conditions for these cases as well as the conditions sufficient for an arbitrary number of solutions. Both the semipositone problem and the problem where $f(0) = 0$ as represented by condition $(f2)$ are considered. We also examine nonlinearities f with a variety of behaviors including superlinear, sublinear and bounded growth. In addition, we examine cases in which f has an arbitrary number of zero points. Finally we conclude with a discussion of how to construct example functions exhibiting the behaviors of the nonlinearities considered in this thesis.

CHAPTER II

QUADRATURE METHOD

The quadrature method developed by Laetsch [3] gives an explicit formula for the bifurcation curve which determines (numerically at least) the shape of the bifurcation curve. Using this formula we can also determine sufficient conditions that guarantee two or more solutions to the BVP (2). We will discuss the quadrature method below for completeness. Throughout our discussion we will assume $f(1)-(f3)$ unless otherwise specified and impose additional conditions when needed.

Derivation of Bifurcation Curve

The following theorem is crucial in the development of the quadrature method for autonomous differential equations.

Theorem 1: *Let $g \in \mathcal{C}^2([0,r])$ and u be any solution of the autonomous boundary value problem*

$$-u''(x) = g(u(x)), \quad 0 < x < 1$$
$$u(0) = 0 = u(1),$$

and suppose $x_0 \in (0,1)$ is such that $u'(x_0) = 0$. Then $u(x_0 - x) = u(x_0 + x)$ for every $x \in [0, \tilde{x}]$, where $\tilde{x} = min\{x_0, 1 - x_0\}$.

Proof:

Let $w_1(x) = u(x_0 - x)$ and $w_2(x) = u(x_0 + x)$ for $x \in [0, \tilde{x}]$. Then $-w_1''(x) = -u''(x_0 - x) = \lambda g(u(x_0 - x)) = \lambda g(w_1(x))$ for $x \in [0, \tilde{x}]$ and

$w_1(0) = u(x_0),\ w_1'(0) = -u'(x_0) = 0.$

Similarly,

$-w_2''(x) = \lambda g(w_2(x))$ for $x \in [0, \tilde{x}],\ \ w_2(0) = u(x_0),\ \ w_2'(0) = -u'(x_0) = 0.$

Thus w_1 and w_2 are both solutions of the initial value problem

$$\begin{aligned} -v''(x) &= g(v) \\ v(0) = u(x_0),\ v'(0) &= 0 \end{aligned} \tag{3}$$

By Picard's Theorem $w_1 = w_2$ which implies that $u(x_0 - x) = u(x_0 + x)$ for $x \in [0, \tilde{x}]$. Thus, the solution u is symmetric about x_0.

QED

Let u be any non-negative solution of (2) with n local maximums $u(\frac{2i+1}{2n})$ for $i = 0, 1, ...n-1$ and $u'(\frac{2i+1}{2n}) = 0$. By Theorem 1, u is periodic in [0,1] with period $\frac{1}{n}$, $n = 1, 2, ...$. The interesting aspects, for our purpose, are that all of the local maximums have the same value, say ρ, and that $u(x)$ is convex whenever $u < \tau$ $(f(u) < 0)$. Figure 2 is an example of such a function u for $n = 5$.

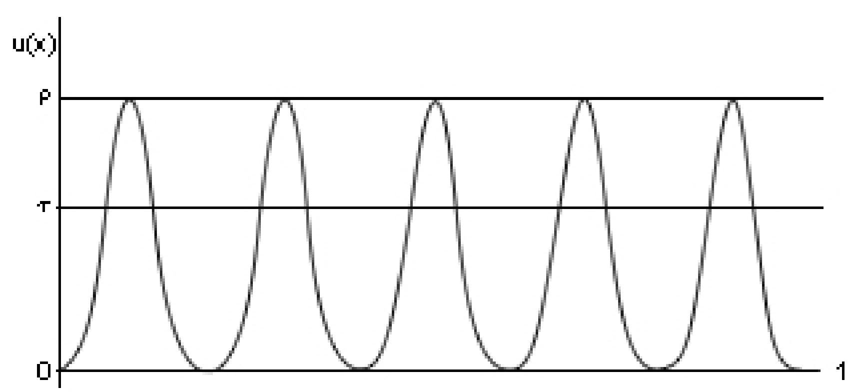


Figure 2: $u(x)$ for $n = 5$

Now we describe the quadrature method where Theorem 1 makes it sufficient to consider the interval $\left[0, \frac{1}{2n}\right]$.

Multiplying the differential equation of the boundary value problem (2) by $u'(x)$, we get

$$-u''(x)u'(x) = \lambda f(u(x))u'(x)$$

and integrating we have

$$-\frac{[u'(x)]^2}{2} = \lambda F(u(x)) + C, \text{ where } F(s) = \int_0^s f(t)dt.$$

Let $\sup_{x\in(0,1)} |u| = u(\frac{1}{2n}) = \rho$. Then $u'(\frac{1}{2n}) = 0$ and hence $C = -\lambda F(\rho)$.

Thus, we obtain

$$u'(x) = \sqrt{2\lambda[F(\rho) - F(u)]} \qquad x \in \left[0, \frac{1}{2n}\right].$$

Integrating on $(0, x)$, we have

$$\int_{u(0)=0}^{u(x)} \frac{du}{\sqrt{F(\rho) - F(u)}} = \sqrt{2\lambda}x\,, \qquad x \in \left[0, \frac{1}{2n}\right].$$

Substituting $x = \frac{1}{2n}$, we obtain

$$\sqrt{\lambda(\rho)} = n\sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho) - F(s)}} \tag{4}$$

$u(x)$ is given by

$$\sqrt{2\lambda}x = \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \ \text{x} \in \left(0, \frac{1}{2n}\right) \tag{5}$$

Let $\sigma > 0$ be such that $F(\sigma) = 0$. Notice that for ρ in $(0, \sigma)$ there will be some s in $[0, \sigma]$ such that $F(\rho) - F(s) < 0$. Hence, for (4) and (5) to be defined we need $\rho \geq \sigma$ (see Figure 3).

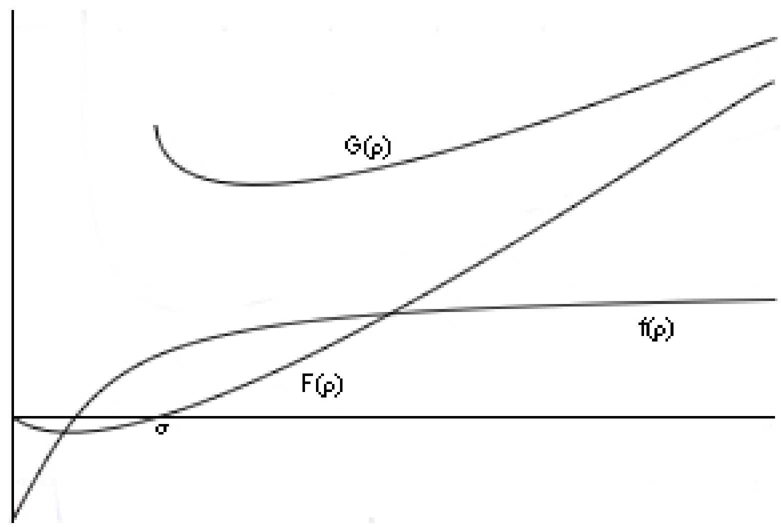


Figure 3: No real-valued solutions for $\rho < \sigma$

Conversely, it is straightforward to check that, for a given $\rho \in (\sigma, r)$, if λ is defined by (4) and $u(x)$ is defined by (5), then u is a non-negative solution of (2) with $\sup_{x\in(0,1)} |u| = \rho$.

However, in this thesis we are only interested in positive (rather than non-negative) solutions of (2), so letting $n = 1$ we get the important function

$$G(\rho) = \sqrt{\lambda(\rho)} = \sqrt{2}\int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}}\,, \tag{6}$$

which gives an explicit formulas for the bifurcation curve.

In this case, Figure 4 shows the shape of u which is given by

$$\sqrt{2\lambda}x = \int_0^{u(x)} \frac{ds}{\sqrt{F(\rho) - F(s)}} \;\; x \in \left(0, \frac{1}{2}\right). \tag{7}$$

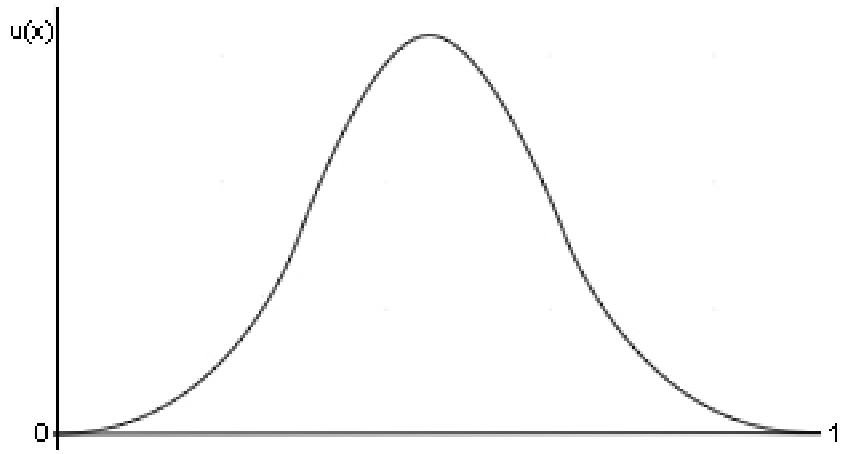


Figure 4: $u(x)$ for $n = 1$

The following theorems summarize the discussion above.

Theorem 1.1: *If $\rho \in (\sigma, r)$, there exists a unique $\lambda > 0$ such that (2) has a non-negative solution u satisfying $\|u\|_\infty = \rho$. For any $\rho \in (\sigma, r)$ the corresponding λ is given by (4) and $u(x)$ is given by (5) for $0 \leq x \leq \frac{1}{2n}$.*

Theorem 1.2: *If $\rho \in (\sigma, r)$, there exists a unique $\lambda > 0$ such that (2) has a positive solution u satisfying $\|u\|_\infty = \rho$. For any $\rho \in (\sigma, r)$ the corresponding λ is given by (6) and $u(x)$ is given by (7) for $0 \leq x \leq \frac{1}{2}$.*

Continuity of $G(\rho)$

It is straightforward to show that $G(\rho)$ is a continuous function. Indeed, let $\epsilon > 0$ be given and without loss of generality suppose that $\rho_1 > \rho_2 \geq \sigma$. Then

$$|G(\rho_1) - G(\rho_2)| = \left| \sqrt{2} \int_0^{\rho_1} \frac{ds}{\sqrt{F(\rho_1) - F(s)}} - \sqrt{2} \int_0^{\rho_2} \frac{ds}{\sqrt{F(\rho_2) - F(s)}} \right|$$

$$\leq \sqrt{2} \left| \int_0^{\rho_1} \frac{ds}{\sqrt{F(\rho_1) - F(s)}} - \int_0^{\rho_2} \frac{ds}{\sqrt{F(\rho_1) - F(s)}} \right|$$

$$= \sqrt{2}\left|\int_{\rho_2}^{\rho_1} \frac{ds}{\sqrt{F(\rho_1) - F(s)}}\right|.$$

By the Mean Value Theorem, there exists an $\eta \in (s, \rho_1)$ such that $F(\rho_1) - F(s) = F'(\eta)(\rho_1 - s)$. Since $F(\eta) = \int_0^\eta f(t)dt$ we have that $F'(\eta) = f(\eta)$.

Thus,

$$\sqrt{2}\left|\int_{\rho_2}^{\rho_1} \frac{ds}{\sqrt{F(\rho_1) - F(s)}}\right|$$

$$= \sqrt{2}\left|\int_{\rho_2}^{\rho_1} \frac{ds}{\sqrt{f(\eta)(\rho_1 - s)}}\right|$$

$$\leq \sqrt{2}\left|\int_{\rho_2}^{\rho_1} \frac{ds}{\sqrt{h_\eta(\rho_1 - s)}}\right| \quad \text{where } h_\eta = \min(f(\eta)) \text{ and } \ \eta \in (s, \rho_1)$$

$$= \frac{\sqrt{2}}{\sqrt{h_\eta}}\left|\int_{\rho_2}^{\rho_1} \frac{ds}{\sqrt{(\rho_1 - s)}}\right|$$

$$= \frac{\sqrt{2}}{\sqrt{h_\eta}} 2\sqrt{\rho_1 - \rho_2}\,.$$

This implies

$$|G(\rho_1) - G(\rho_2)| < \epsilon \quad \text{if } \ \rho_1 - \rho_2 < \frac{\epsilon^2 h_\eta}{8}\,.$$

So, $G(\rho) = \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho_1) - F(s)}}$ is uniformly continuous for $\rho \geq \sigma$.

QED.

Bifurcation Curve for Multiple Solutions

Observe that all solutions of (2) correspond to points of the curve $\rho \to G(\rho)$, which gives the full bifurcation diagram for (2). If the non-linearity f is given, a graph of $G(\rho)$ can be obtained numerically. Hence, whether or not (2) has two, three or more solutions is determined by the shape of $G(\rho)$. As shown in Figure 5, $G(\rho)$ being decreasing for ρ small and ρ large and also having a local minimum followed by a local maximum (wave-shaped) corresponds to three positive solutions on some λ interval

$[\lambda_1, \lambda_2]$.

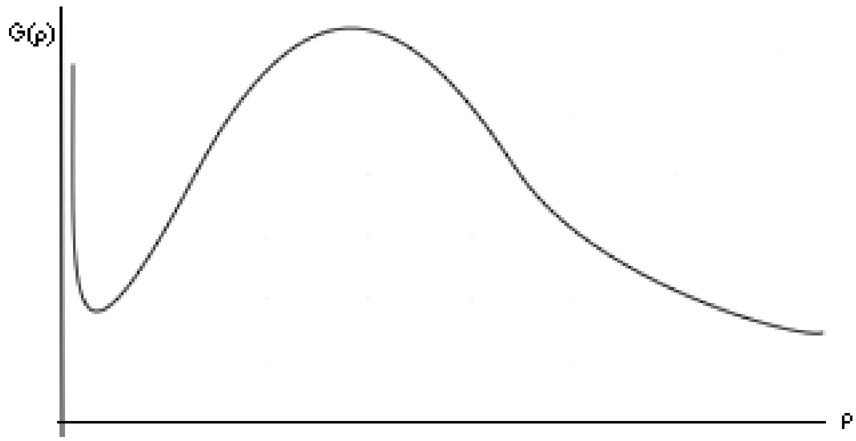


Figure 5: 3 solution, wave-shaped bifurcation curve

Behavior of $G(\rho)$ near σ

The behavior of $G(\rho)$ near σ turns out to be a crucial factor in determing the number of solutions. We will see that this behavior also depends on the value of $f(0)$. Considering only positive solutions, recall that

$$G(\rho) = \sqrt{2} \int_0^{\rho} \frac{ds}{\sqrt{F(\rho) - F(s)}} \text{ where } F(s) = \int_0^s f(t)dt\,.$$

Lemma 1: *If $f(0) < 0$ then $lim_{\rho \to \sigma^+} G(\rho) < \infty$.*

Proof:

$F(0) = 0$ and since $F^{'}(s) = f(s)$ and $f(0) < 0$ we have that $F^{'}(0) < 0$. Then there exists an $\omega \in (0, \sigma)$ such that for some $m > 0$, $-ms > F(s)$ for $s \in (0, \omega)$. Also, $ns - b > F(s)$ for some $n > 0$ and $b > 0$, $s \in (\omega, \sigma)$ and $F^{'}(s) > n$ for $s \in [\sigma, \sigma + \epsilon]$ for some $\epsilon > 0$ (see Figure 6) .

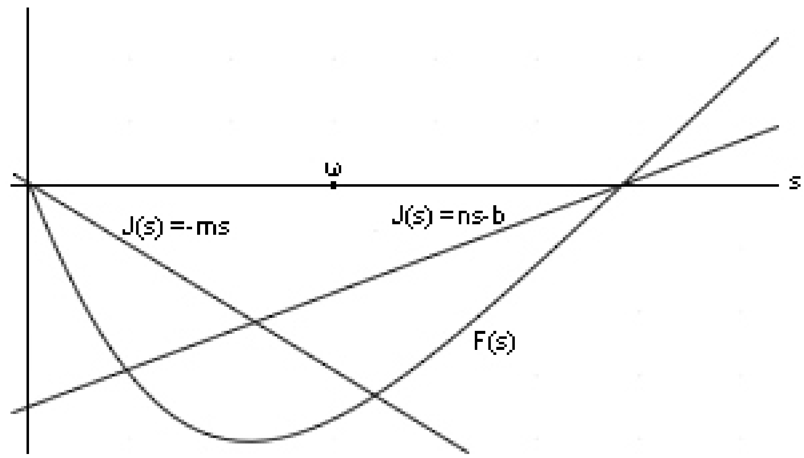


Figure 6: Function $J(s)$

Thus, defining

$$J(s) = \begin{cases} -ms, & 0 \le s \le \omega \\ ns - b, & \omega < s \le \rho \text{ and } \rho \in [\sigma, \sigma+\epsilon]. \end{cases}$$

we have that

$$J(\rho) - J(s) \le F(\rho) - F(s) \text{ for } \rho \in (\sigma, \sigma+\epsilon] \text{ and } 0 \le s \le \rho\,.$$

This implies that

$$G(\rho) = \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{F(\rho)-F(s)}} \le \sqrt{2}\int_0^\rho \frac{ds}{\sqrt{J(\rho)-J(s)}}$$

$$= \sqrt{2}\int_0^\omega \frac{ds}{\sqrt{n\rho - b + ms}} + \sqrt{2}\int_\omega^\rho \frac{ds}{\sqrt{n\rho - ns}}$$

$$= \frac{2\sqrt{2}}{m}\left[\sqrt{n\rho - b + m\omega} - \sqrt{n\rho - b}\right] + \frac{2\sqrt{2}}{n}\sqrt{n(\rho-\omega)}\,.$$

Noting that $n\sigma - b = 0$and that there exists $k \in (0,1)$ such that $\omega = k\sigma$, we have that

$$\lim_{\rho\to\sigma^+}\left[\frac{2\sqrt{2}}{m}\left[\sqrt{n\rho-b+m\omega}-\sqrt{n\rho-b}\right]+\frac{2\sqrt{2}}{n}\sqrt{n(\rho-\omega)}\right]$$

$$=\left[\frac{2\sqrt{2}}{m}\left[\sqrt{n\sigma-b+mk\sigma}-\sqrt{n\sigma-b}\right]+\frac{2\sqrt{2}}{n}\sqrt{n(\sigma-k\sigma)}\right]\quad(0<k<1)$$

$$=2\sqrt{2}\left[\frac{1}{m}\left[\sqrt{mk\sigma}\right]+\frac{1}{n}\sqrt{n\sigma(1-k)}\right]<\infty\,.$$

Therefore, $\lim_{\rho\to\sigma^+}G(\rho)<\infty$ as desired.

QED

Lemma 2: *(Lemma 3.1* [2]*) If* $f(0)=0$ *then* $lim_{\rho\to\sigma^+}G(\rho)=\infty$.

Proof:

We have that

$$G(\rho)=\sqrt{2}\int_0^\rho\frac{ds}{\sqrt{F(\rho)-F(s)}}$$

$$\geq\sqrt{2}\int_0^\epsilon\frac{ds}{\sqrt{F(\rho)-F(s)}}\quad 0<\epsilon<\rho$$

Let $k(s)=F(\rho)-F(s)$. Then $k^{'}(s)=-f(s)$ and $k^{''}(s)=-f^{'}(s)$. Assume that ϵ is sufficiently small so that $f(s)<0$ and $f^{'}(s)<0$ for $s\in(0,\epsilon]$ and $\rho\in(\sigma,\sigma+1]$. Then there exists $a>0$ such that $k(s)\leq F(\rho)+a^2s^2$. Then

$$G(\rho)\geq\sqrt{2}\int_0^\epsilon\frac{ds}{\sqrt{F(\rho)+a^2s^2}}$$

$$=\frac{\sqrt{2}}{a}\int_0^\epsilon\frac{ds}{\sqrt{s^2+\frac{F(\rho)}{a^2}}}$$

$$= \frac{\sqrt{2}}{a} \ln \left[s + \sqrt{s^2 + \frac{F(\rho)}{a^2}} \right]_0^{\epsilon}$$

$$= \frac{\sqrt{2}}{a} \left[\ln \left\{ \epsilon + \sqrt{\epsilon^2 + \frac{F(\rho)}{a^2}} \right\} - \ln \sqrt{\frac{F(\rho)}{a^2}} \right]$$

Therefore, $\lim_{\rho \to \sigma^+} G(\rho) = \infty$ since $F(\sigma) = 0$.

QED

Derivative of $G(\rho)$

In computing the derivative of $G(\rho)$ we will need to use a consequence of Leibniz's formula,

$$\frac{dG(\rho)}{d\rho} = \int_{v(\rho)}^{w(\rho)} \frac{\partial f}{\partial \rho} ds + f[w(\rho), \rho] \frac{dw}{d\rho} - f[v(\rho), \rho] \frac{dv}{d\rho} \,,$$

(named after the German Mathematician Gottfried Wilhelm Leibniz (1646-1716)) which shows that if the integration limits are not functions of the independent variable, then the derivative of an integral is simply the integral of the derivative.
After a change of variable which gives us constant limits of integration, the derivative of (4) can now be straightforwardly computed ...

$$\frac{dG(\rho)}{d\rho} = \frac{d}{d\rho} \left[\sqrt{2}\rho \int_0^1 \frac{ds}{\sqrt{F(\rho) - F(\rho s)}} \right]$$

$$= \sqrt{2}\rho \frac{d}{d\rho} \left[\int_0^1 \frac{ds}{\sqrt{F(\rho) - F(\rho s)}} \right] + \sqrt{2} \int_0^1 \frac{ds}{\sqrt{F(\rho) - F(\rho s)}}$$

which , by Leibniz's rule above

$$= \sqrt{2}\rho \int_0^1 \frac{\partial}{\partial \rho} \left[\frac{ds}{\sqrt{F(\rho) - F(\rho s)}} \right] + \sqrt{2} \int_0^1 \frac{ds}{\sqrt{F(\rho) - F(\rho s)}}$$

$$= -\frac{\sqrt{2}}{2}\rho\int_0^1\left[\frac{ds}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}\frac{\partial}{\partial\rho}(F(\rho)-F(\rho s))\right]+\sqrt{2}\int_0^1\frac{ds}{\sqrt{F(\rho)-F(\rho s)}}$$

$$= -\frac{\sqrt{2}}{2}\rho\int_0^1\left[\frac{ds}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}\left(\frac{d}{d\rho}F(\rho)-\frac{\partial}{\partial\rho}F(\rho s)\right)\right]+\sqrt{2}\int_0^1\frac{ds}{\sqrt{F(\rho)-F(\rho s)}}$$

$$= -\frac{\sqrt{2}}{2}\rho\int_0^1\left[\frac{ds}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}\left(\frac{d}{d\rho}\int_0^{\rho}f(t)dt-\frac{\partial}{\partial\rho}\int_0^{\rho s}f(t)dt\right)\right]+\sqrt{2}\int_0^1\frac{ds}{\sqrt{F(\rho)-F(\rho s)}}$$

$$= -\frac{\sqrt{2}}{2}\rho\int_0^1\left[\frac{f(\rho)-sf(s\rho)}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}ds\right]+\sqrt{2}\int_0^1\frac{F(\rho)-F(\rho s)}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}ds$$

$$= \sqrt{2}\left(\int_0^1\left[\frac{-\frac{1}{2}\rho f(\rho)+\frac{1}{2}\rho s f(s\rho)}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}ds\right]+\int_0^1\frac{F(\rho)-F(\rho s)}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}ds\right)$$

$$= \sqrt{2}\int_0^1\left[\frac{-\frac{1}{2}\rho f(\rho)+\frac{1}{2}\rho s f(s\rho)+F(\rho)-F(\rho s)}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}ds\right]$$

$$= \sqrt{2}\int_0^1\left[\frac{\left(F(\rho)-\frac{1}{2}\rho f(\rho)\right)-\left(F(\rho s)-\frac{1}{2}\rho s f(s\rho)\right)}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}ds\right]$$

$$= \sqrt{2}\int_0^1\left[\frac{H(\rho)-H(s\rho)}{(F(\rho)-F(\rho s))^{\frac{3}{2}}}ds\right]\quad \text{where } H(\rho)=F(\rho)-\frac{1}{2}\rho f(\rho)$$

and thus we have that

$$G'(\rho)=\frac{\sqrt{2}}{\rho}\int_0^{\rho}\frac{H(\rho)-H(s)}{(F(\rho)-F(s))^{\frac{3}{2}}}ds \tag{8}$$

QED

Since $F(\rho)-F(s)>0$ for all $s\in(0,\rho)$ and $\sigma<\rho$, we can see from (8) that sufficient conditions for $G'(\rho)$ to be positive and negative respectively are that

$$H(\rho)-H(s)>0,\quad \forall s\ \ 0<s<\rho,\ \sigma<\rho\,, \tag{9}$$

$$\text{and}\quad H(\rho)-H(s)<0,\quad \forall s\ \ 0<s<\rho,\ \sigma<\rho \tag{10}$$

respectively.

Slope of $G(\rho)$ near σ

Lemma 3: *If $f(0) \leq 0$ then $lim_{\rho \to \sigma^+} G^{'}(\rho) < 0$.*

Proof:

If $f(0) = 0$ then $\lim_{\rho \to \sigma^+} G(\rho) = \infty$ and hence $\lim_{\rho \to \sigma^+} G(\rho) < 0$

Recall that if $f(0) < 0$ then $G(\sigma) < \infty$ so it is not immediately obvious that $\lim_{\rho \to \sigma^+} G(\rho) < 0$ as it was above. However, for sufficiently small $\epsilon > 0$ we have that $F^{'}(s) > 0$ for $\sigma \leq s < \sigma + \epsilon$. Thus $G(\sigma + s) - G(\sigma) < 0$ for $\sigma < s < \sigma + \epsilon$ and hence $G^{'}(\rho) < 0$ for ρ near σ .

QED

CHAPTER III
MULTIPLICITY RESULTS

It is difficult to draw general conclusions about the behavior of the bifurcation curve (6), and hence the number and range of solutions of the boundary value problem (2) because of the strong dependence on the nature of the nonlinearity f. However, if we make certain simple and straightforward assumptions about the behavior of f then we can derive some conclusions about the number of solutions of the boundary value problem (2). Before we examine specific cases, each with particular assumptions about the behavior of f (and hence $\frac{u}{f(u)}$) let's investigate how the nature of f affects the shape of the bifurcation curve.

First, notice that $\frac{d}{du}(\frac{u}{f(u)}) = \frac{d}{du}\frac{f(u)-uf'(u)}{f(u)^2}<0$ for $0 < u < \tau$ and will remain negative for $u > \tau$ unless f becomes sufficiently concave so that $\frac{u}{f(u)}$ reaches a local minimum (see Figure 7).

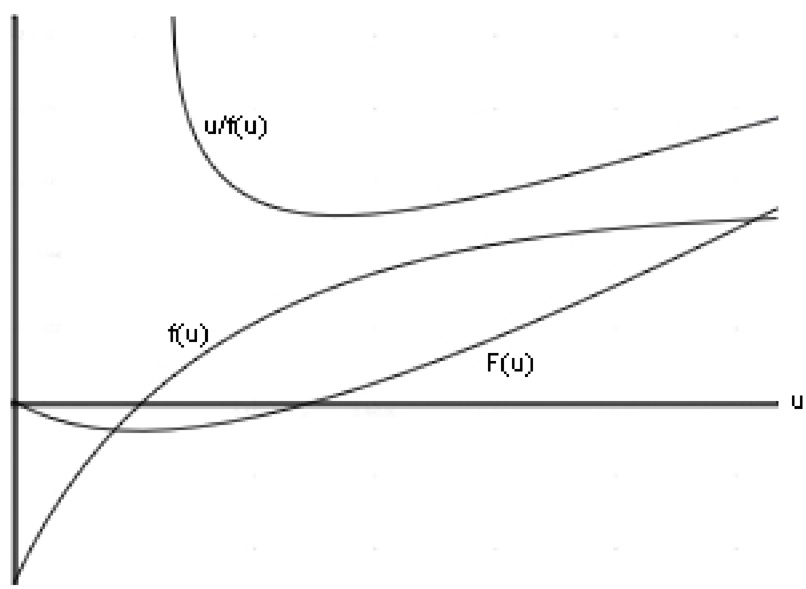


Figure 7: Concave f

Recall that $H(u) = F(u) - \frac{1}{2}uf(u))$ and so $H'(u) = \frac{1}{2}(f(u) - uf'(u))$ and $H''(u) = -uf''(u)$. Since $\frac{d}{du}(\frac{u}{f(u)}) = \frac{f(u)-uf'(u)}{f(u)^2}$ the sign of $H'(u)$ corresponds to the sign of $\frac{u}{f(u)}$ and we have that $H(u)$ and $\frac{u}{f(u)}$ are convex when f is concave and concave when f is convex.

Two Solutions

From the previous observations we see that if f is sufficiently concave then $\frac{\rho}{f(\rho)}$ and hence $H(\rho)$ will have a local minimum and continue to increase until $H(\rho) > 0$, at which point condition (9) is satisfied and we know that $G(\rho)$ has changed sign from negative to positive as shown in Figure 8 . These observations are more formally expressed in the following theorem which gives necessary and sufficient conditions for the bifurcation curve to be U-shaped, representing at least two solutions for a range of λ.

Theorem 4: *Let* $f(0) < 0$, $f(\rho) < 0$ for $0 < \rho < \tau$, $f(\tau) = 0$ and $f(\rho) > 0$ for $\rho > \tau$.

(n1) If the bifurcation curve of (2) represents at least two solutions (roughly U-shaped) then $(f(\rho) - \rho f'(\rho) > 0$ *for some* $\rho > 0$.

(s1) If $F(\rho) - \frac{1}{2}\rho f(\rho) > 0$ *for some* $\rho > \sigma$ *then (2) has at least two solutions for some range of* λ *(the bifurcation curve is roughly U-shaped).*

Proof:

Since $H'(\rho) = f(\rho) - \rho f'(\rho) < 0$ for small ρ, if $H'(\rho) < 0$ for all $\rho > 0$ then condition (10) guarantees that the bifurcation curve would be a strictly decreasing function and would not be U-shaped.

However, if $H(\rho)$ becomes greater than zero for some $\rho > \sigma$ then condition (9) is satisfied, the slope of the bifurcation curve will become positive, and since by Lemma 3 $G'(\sigma) < 0$, the curve will U-shaped.

QED

These conditions can be summarized by saying that a necessary condition for a U-shaped bifurcation curve is that $H(\rho)$ is somewhere increasing and a sufficient condition is that $H(\rho)$ exceeds 0 at some $\rho > \sigma$. If these conditions are satisfied then the bifurcation curve will be U-shaped and the boundary value problem (2) will have at least two solutions as shown in Figure 9.

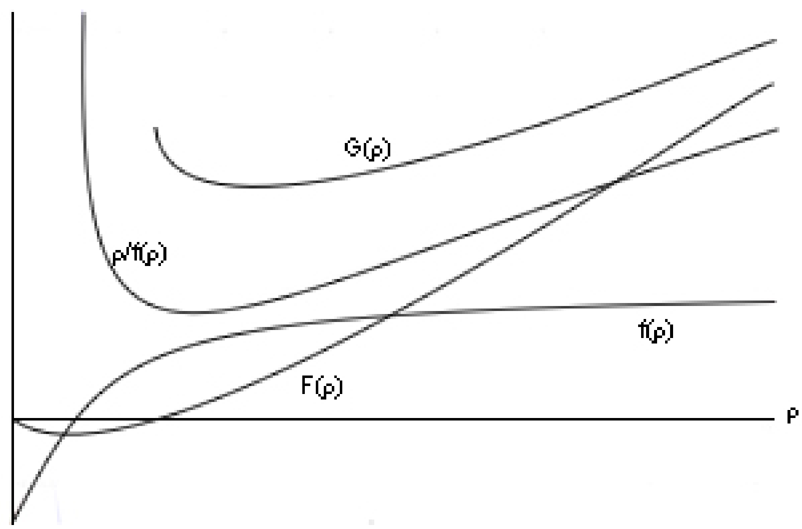


Figure 8: $G(\rho)$ vs $H(\rho)$

Three Solutions

We saw in the proof of the two solution case (Theorem 4) that if f is sufficiently concave then $\frac{\rho}{f(\rho)}$ and hence $H(\rho)$ will have a local minimum and continue to increase until $H(\rho) > 0$, at which point condition (9) is satisfied and $G^{'}(\rho)$ changes sign from negative to positive. Now, if f, having been concave, becomes sufficiently convex then $\frac{\rho}{f(\rho)}$ will reach a local maximum and hence $H'(\rho)$ will change sign again, this time from positive to negative. If, after having turned downward, $\frac{\rho}{f(\rho)}$ and $H(\rho)$ continue to decrease, $H(\rho)$ will eventually become less than its previous local minimum. At

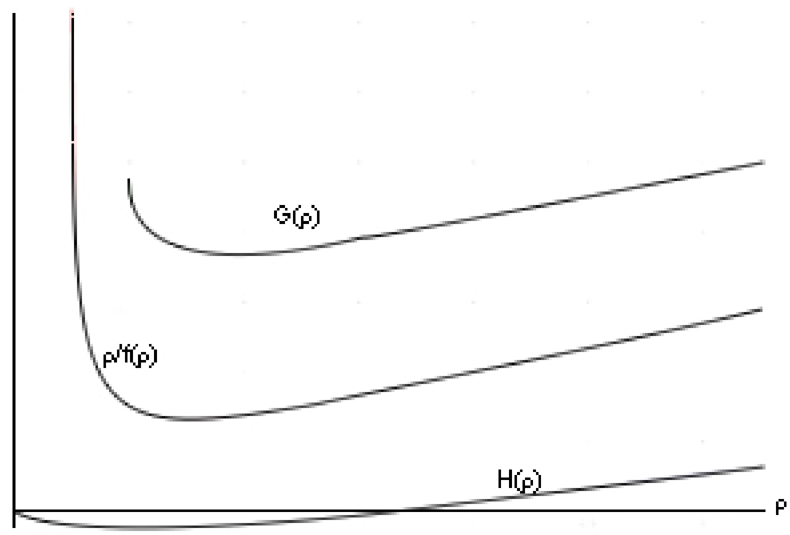


Figure 9: U-shaped bifurcation curve

that point condition (10) is satisfied which implies that $G^{'}(\rho)$ changes sign again, this time from positive to negative as shown in Figure 10.

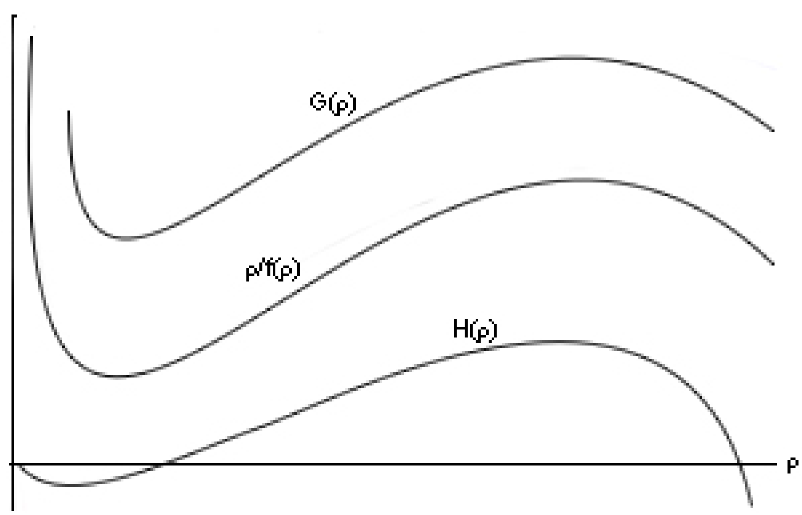


Figure 10: $G^{'}(\rho)$ same sign as $H(u)$

If $f(\rho)$ continues sufficiently convex such that $\frac{\rho}{f(\rho)}$, $H(\rho)$ and the bifurcation curve continue to decrease until we have that $G(\rho) < G(\sigma)$ (which may be immediately if

$G(\rho)$ at its local maximum is less than $G(\sigma)$ as is always the case when $f(0) = 0$) then the bifurcation curve will become wave-shaped and the boundary value problem (2) has at least three solutions for some range of λ. This outcome is guaranteed if $\lim_{\rho \to r} f(\rho) = \infty$ such that $\frac{\rho}{f(\rho)} \to 0$ (as for superlinear $f(\rho)$) since $G(\rho) > 0$ and $G^{'}(\rho) < 0$ as $\rho \to \infty$. This situation is shown on Figure 11 where $\frac{\rho}{f(\rho)} \to 0$ after it has reached one local maximum.

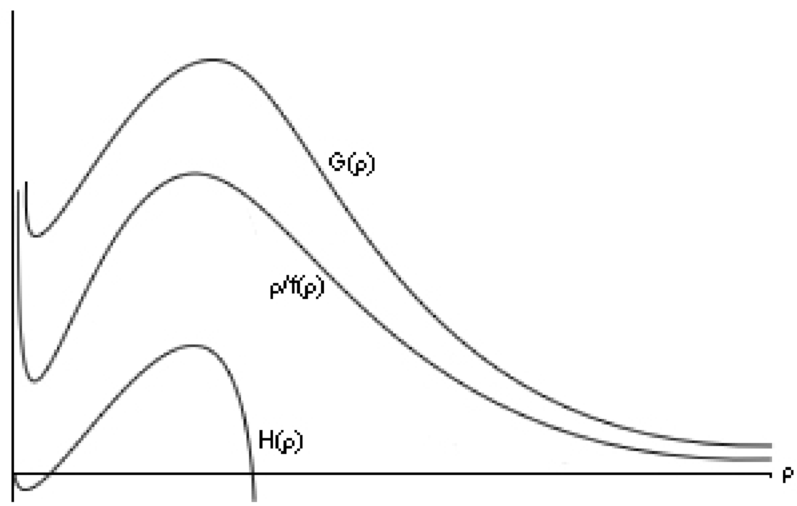


Figure 11: $\frac{\rho}{f(\rho)} \to 0$

These observations are formalized in the following theorem.

Theorem 5: *Let* $f(0) < 0, \quad f(\rho) < 0$ for $0 < \rho < \tau$, $f(\tau) = 0$ and $f(\rho) > 0$ for $\rho > \tau$.

$(n1)$ *If the bifurcation curve of (2) represents at least three solutions (roughly wave-shaped) then* $f(\rho) - \rho f'(\rho) > 0$ for some $\rho > 0$.

$(n2)$ *If the bifurcation curve of (2) represents at least three solutions (roughly wave-shaped) then* $F(\rho) - \frac{1}{2}\rho f(\rho) > 0$ for some $\rho > \sigma$.

(s1) *If* $\lim_{\rho\to\infty}\frac{\rho}{f(\rho)} = 0$ *then (2) has at least three solutions for some range of* λ *(the bifurcation curve is roughly wave-shaped).*

Proof:

The proof is similar to that for Theorem 4 above. Conditions $(n1)$ and $(n2)$ and Lemma 3 guarantee that the bifurcation curve will be U-shaped. In addition, $\frac{\rho}{f(\rho)} \to 0$ implies that the slope eventually changes sign again and the bifurcation curve approaches zero thereby making it roughly wave-shaped (see Figure 12).

QED

These conditions can be summarized by saying that necessary conditions for a wave-shaped bifurcation curve are that $H(\rho)$ is somewhere increasing and that $H(\rho)$ exceeds 0 at some point and a sufficient condition is that $\frac{\rho}{f(\rho)} \to 0$ as $\rho \to \infty$.

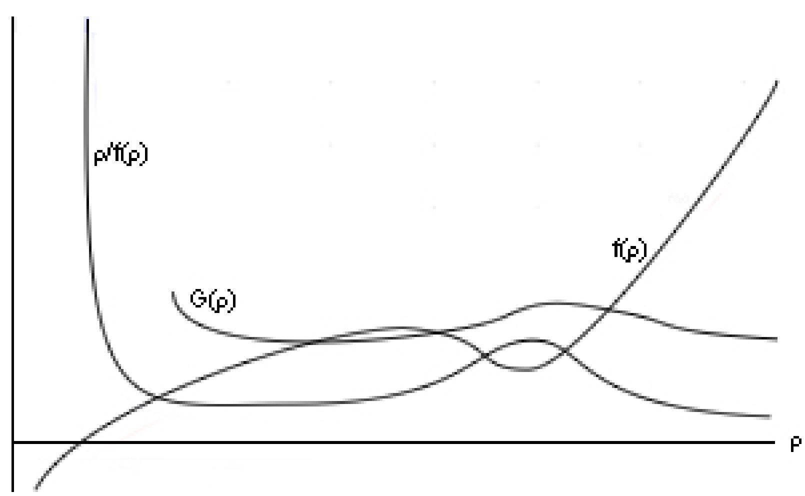


Figure 12: Wave-shaped bifurcation curve

In the situation described above, a requirement for a wave-shaped curve is that $\frac{\rho}{f(\rho)} \to 0$ which will be the case if f is superlinear. However, if f is linear, asymptotically sublinear, or bounded then $\lim_{\rho\to\infty}\frac{\rho}{f(\rho)} = 0$ and we cannot ensure that the bifurcation curve will be wave-shaped. It may very well be wave-shaped but that will depend on the particular intermediate behavior of f.

More than Three Solutions

If f behaves, with respect to slope and concavity, such that $\frac{u}{f(u)}$ has a sequence of increasing local maximums and decreasing local minimums then we can get an arbitrary number of solutions to (2) for some range of λ. This behavior is reflected in the following theorem.

Theorem 6: *Suppose* $f(0) < 0,\ \ f(\rho) < 0$ for $0 < \rho < \tau,\ f(\tau) = 0$ and $f(\rho) > 0$ for $\rho > \tau$ *and* $\frac{u}{f(u)}$ *has* m *local minimums at* $u = m_0, m_1, m_2$*...and* n *local maximums at* $u = n_0, n_1, n_2$*... respectively. Also suppose that* $H(m_i) < H(m_{i-1})$ *such that* $G(m_i) < G(\sigma)$ *and* $H(n_i) > H(n_{i-1})$ *such that* $G(m_i) > G(\sigma)$.

If $\frac{\rho}{f(\rho)} \to 0$ *or* $\frac{\rho}{f(\rho)} \to \infty$ *then (2) will have* $m + n + 1$ *solutions.*

Proof:

Lemma 3 ensures that $G'(\sigma) < 0$. The second hypothesis ensures that conditions (9) and (10) are satisfied for each pair of local minimums and maximums respectively, which means that the slope of the bifurcation curve will alternate direction. In addition, since each maximum and minimum of $G(\rho)$ occurs above or below $G(\sigma)$ respectively it is easy to see that a horizontal line on the graph at $G(\sigma)$ will intersect the bifurcation curve after each slope direction change.

QED

Figure 13 shows an example of a bifurcation curve indicating 6 solutions to (2) for a range of λ.

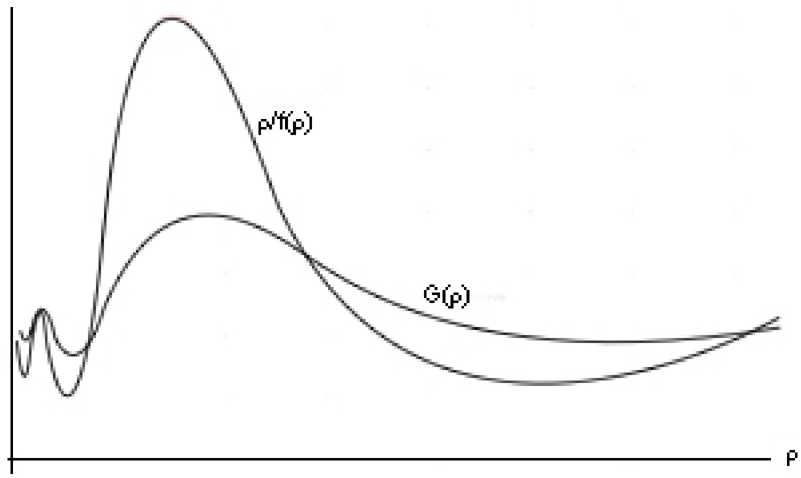


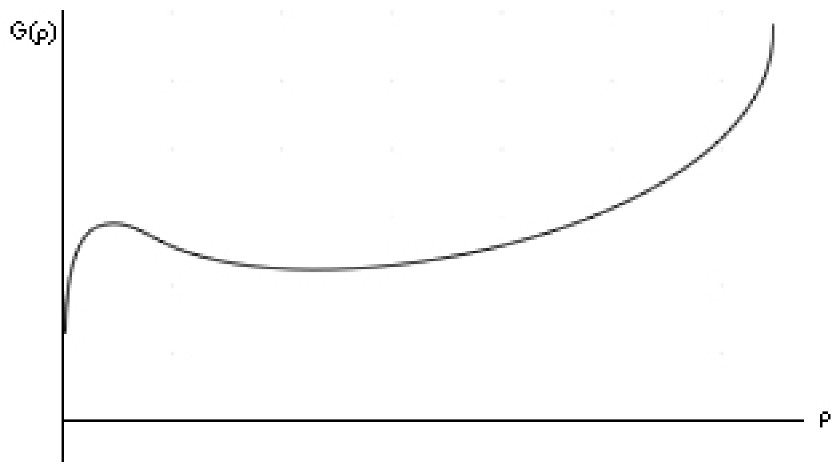
Figure 13: More than 3 solutions

Other Characteristics

In this section we derive another set of necessary and sufficient conditions for a U-shaped bifurcation curve when $f(u) \leq C$ for $u \geq 0$ and $C > 0$ (f bounded above).

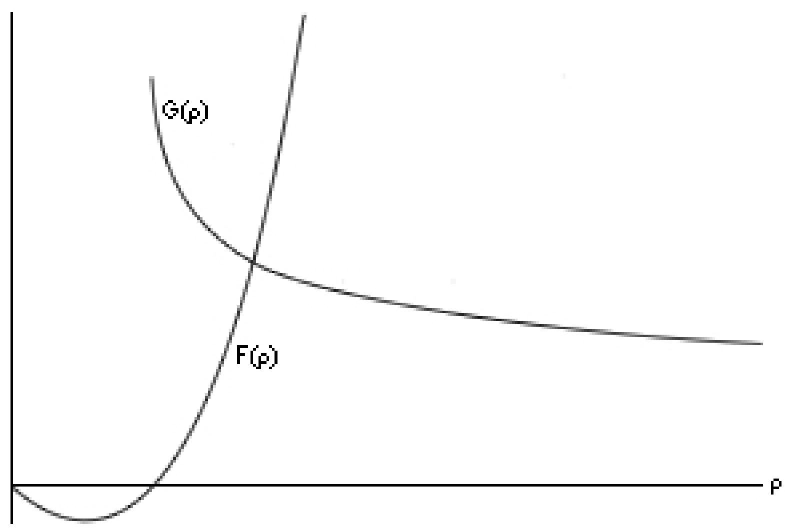
Condition $(f2)$ says that $f(0) \leq 0$ but for our purpose here suppose $f < 0$. Let's now consider a positone version of the BVP problem (2) where we replace f by $f_p(u) = f(u) - 2f(0) > 0$ and hence $F_p(s) = \int_0^s f_p(t)dt = \int_0^s f(t)dt - 2sf(0)$.

Suppose that the bifurcation curve $G_p(\rho)$ for this positone version of the original problem is wave-shaped. It is clear that an wave-shaped bifurcation curve for the translated problem corresponds to $G_p(\rho)$ being increasing close to 0 and for ρ large and also possessing a local maximum at some $\rho = \rho_0$ followed by a local minimum at some $\rho = \rho_1$. In addition, the fact that f is bounded above together with results in Laetsch [3] ensure that $\lim_{u\to\infty} G_p(\rho) = \infty$. Figure 14 below shows a numerical plot of $G_p(\rho)$ for the particular positone function $f_p(u) = 1 + u + u - .05u + 1$.



$$f_p(u) = 1 + u + u - .05u + 1$$

Figure 14: Bifurcation curve of positone problem



$$f(u) = 1 + u + u^2 - .05u - 3$$

Figure 15: Bifurcation curve of semipositone problem

Notice that with respect to the semipositone problem, for ρ in $[0, \sigma]$ there will be some s in $[0, \sigma]$ such that $F(\rho) - F(s) < 0$. Hence, as mentioned earlier, there will be no real solutions to (2) for $0 < \rho < \sigma$. Figure 15 shows this behavior for f, a translated, semipositone version of f_p where $f(u) = 1 + u + u^2 - .05u - 3$.

Let's now examine the relationship between the bifurcation curve of the original semipositone problem and the translated positone problem i.e., how $G(\rho)$ is related to $G_p(\rho)$.

Because $F(\rho) - F(\rho s) = F_p(\rho) - F_p(\rho s) + \rho f(0)(1-s) < F_p(\rho) - F_p(\rho s)\ \ \forall s\ \ 0 < s < 1$ we have that

$$G_p(\rho) = \sqrt{2}\rho \int_0^1 \frac{ds}{\sqrt{F_p(\rho) - F_p(\rho s)}} < \sqrt{2}\rho \int_0^1 \frac{ds}{\sqrt{F(\rho) - F(\rho s)}} = G(\rho) \tag{11}$$

which means that the bifurcation curve of the positone problem is less than the bifurcation curve of the semipositone problem. Since the bifurcation curve of the positone problem is wave-shaped, the slope of $G_p(\rho)$ is initially negative and $G_p(\rho) \to \infty$ then (11) ensures that the semipositone curve will be U-shaped.

The following theorem of Brown, Ibrahim, and Shivaji [1] gives necessary and sufficient conditions for the bifurcation curve for the positone problem

$$\begin{gathered} -u''(x) = \lambda f_p(u(x)) \quad \text{for } 0 < x < 1 \ \text{ where } f_p = f - 2f(0) \\ u(0) = 0 = u(1) \end{gathered} \tag{12}$$

where $f_p = f - 2f(0)$ to be wave-shaped.

Recall that $H(\rho) = F(\rho) - \frac{1}{2}\rho f(\rho)$ and $H'(u) = \frac{1}{2}(f(u) - uf'(u))$.

> ***Theorem 7:*** *Suppose* $f(0) < 0,\ \ f(\rho) < 0$ for $0 < \rho < \tau,\ \ f(\tau) = 0$ and $f(\rho) > 0$ for $\rho > \tau$ *and* f *is bounded above* [?].
>
> *(a) If the bifurcation curve of (12) is wave-shaped, then* $H_p'(\rho) = f(\rho) - \rho f'(\rho) - 2f(0) < 0$ *for some* $\rho > 0$.

(b) If there exists a $\rho_0 > 0$ *such that* $H_p(\rho_0) = F(\rho_0) - \frac{1}{2}\rho_0 f(\rho_0) - \rho_0 f(0) < 0$ *then (12) has at least three solutions for a certain range of* λ *(i.e., the bifurcation curve is roughly wave-shaped).*

Theorem 7 together with the previous remarks results in the following theorem giving necessary and sufficient conditions for (2) to have a U-shaped bifurcation curve. Theorem 8 below simply says that if the bifurcation curve of the positone problem (12) is wave-shaped then the bifurcation curve of the corresponding semipositone problem (2) will be U-shaped.

Theorem 8: *Suppose* $f(0) < 0$, $f(\rho) < 0$ for $0 < \rho < \tau$, $f(\tau) = 0$ and $f(\rho) > 0$ for $\rho > \tau$ *and* f *is bounded above.*

(a) If the bifurcation curve of (2) is U-shaped, then $f(\rho) - \rho f'(\rho) - 2(f(0) < 0$ *for some* $\rho > 0$.

(b) If there exists a $\rho_0 > 0$ *such that* $F(\rho_0) - \frac{1}{2}\rho_0 f(\rho_0) - \rho_0 f(0) < 0$ *then (2) has at least two solutions for a certain range of* λ *(i.e., the bifurcation curve is roughly U-shaped).*

Proof:

Since $f(0) < 0$, the BVP (12) is a positone problem. Suppose the bifurcation curve $G_p(\rho)$ of (12) is wave-shaped and $G_p(\rho) \to \infty$ as $\rho \to \infty$. Then because the bifurcation curve $G(\rho)$ of (2) is greater than $G_p(\rho)$ for all ρ (see (11) above) and $G'(\sigma) < 0$ then $G'(\rho)$ will eventually become positive. Hence, $G(\rho)$ will be roughly U-shaped. Since Theorem 7 provides the necessary and sufficient conditions for $G_p(\rho)$ to be wave-shaped then satisfaction of the same conditions guarantees that $G(\rho)$ is U-shaped.

QED

These conditions could be summarized by saying that a necessary condition for a U-shaped bifurcation curve is that $H_p(\rho)$ is somewhere decreasing and sufficient conditions are that $H_p(\rho)$ decreases until its function values become negative.

Figure 16 shows the relationship between the bifurcation curves for the positone and semipositone versions of the BVP.

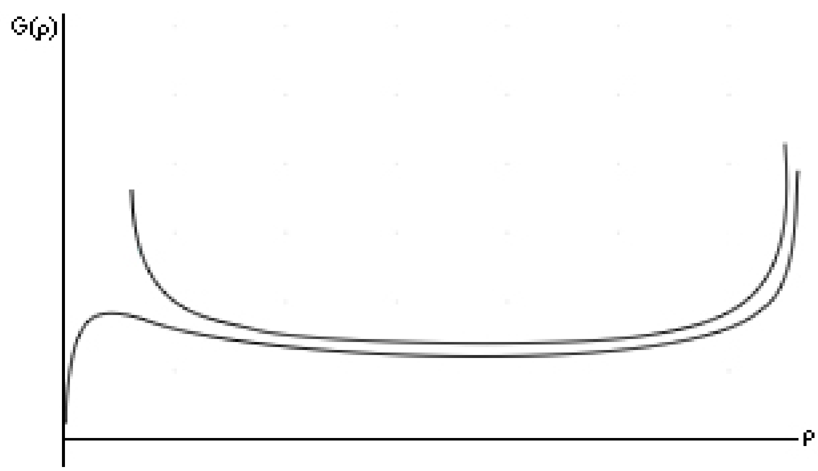


Figure 16: Bifurcation curves for related positone & semipositone problems

The numerical plots of the semipositone bifurcation curve used in our example showed a smooth U-shaped curve with exactly one bend. We have not proved , however, that the curve has only one bend i.e., that there cannot be another bend somewhere in the interval $[\rho_1, \infty]$. To do so would probably require a careful analysis and comparison of $\frac{d^2G(\rho)}{d\rho^2}$ and $\frac{d^2G_p(\rho)}{d\rho^2}$ but we do not have a reasonably tractable formula for either.

Multiple Solutions for f with Multiple Zeros

In this section we consider the case where f is initially negative but then may have an aribitrary number of positive and negative regions as shown in Figure 17.

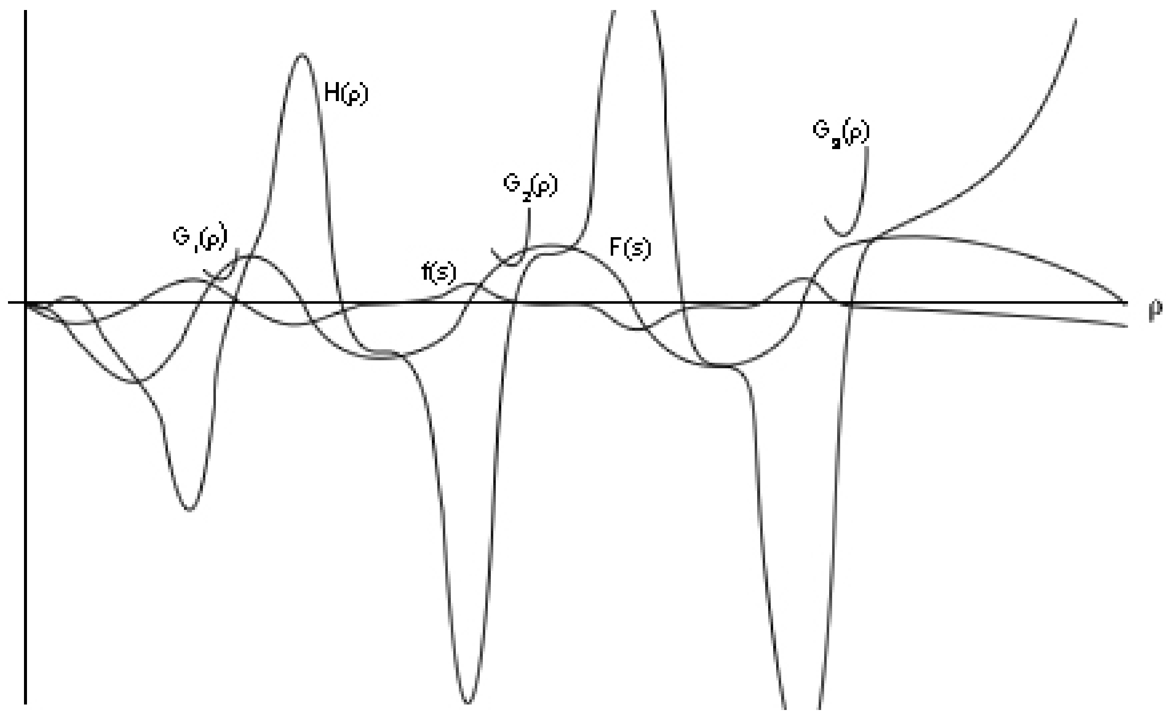


Figure 17: f with multiple zeros

Suppose f is defined on $[0, r]$ and satisfies the following conditions.

$(f5)$ $f \in \mathcal{C}^2([0, r])$

$(f6)$ $f(0) \leq 0$

$(f7)$ $f(u) < 0$ for $0 < u < \tau,\ f(\tau) = 0$ and there exists $C > 0$ such that $f(u) < C$ for all u

$(f8)$ $f(u)$ has $m \geq 1$ zero points

In addition to the conditions above it will be useful to define a couple of new objects which we call a *Quadrature Point and Quadrature Pair* respectively.

Quadrature Point

Let $F(m_1)$, $F(m_2)$, ... , $F(m_n)$ be the sequence of local minimums of $F(s)$ where

$s \in [0, r]$ and $0 < m_1 < m_2 < \ ... \ < m_n$.

Let $f(z_1)$, $f(z_2)$, ... , $f(z_m)$ be the sequence of zero points of $f(s)$ where $s \in [0, r]$ and $0 < z_1 < z_2 < \ ... \ < z_m$.

For each m_i the smallest $s > m_i$, call it b_i, such that $F(b_i) > F(s)$ for all $s \in [0,\ b_i]$ is defined as a *Quadrature Point* of f.

Quadrature Pair

Let b_i, b_2, ..., b_p $(p \leq n)$ be the ordered sequence of distinct Quadrature Points of f.

Let $H(h_1)$, $H(h_2)$, ... , $H(h_q)$ be the sequence of local minimums of $H(\rho)$ where $\rho \in [0, r]$ and $h_1 < h_2 < \ ... \ < h_q$.

Let $H(h^1)$, $H(h^2)$, ... , $H(h^s)$ be the sequence of local maximums of $H(\rho)$ where $\rho \in [0, r]$ and $h^1 < h^2 < \ ... \ < h^s$.

For each h_i, let p^i be the smallest $\rho > h_i$ such that $H(p^i) > H(\rho)$ for all $\rho \in [0,\ p^i]$.

For each h^i, let p_i be the smallest $\rho > h^i$ such that $H(p_i) < H(\rho)$ for all $\rho \in [0,\ p_i]$.

Let $(p^1, p_1, p^2, p_2, \ldots, , p_i, p^i, \ldots, p_t)$ be the ordered sequence of p_i and p^i values.

For each interval $[b_i,\ z_j]$, where z_j is the first zero point of f greater than b_i , any pair $(p^i,\ p_i)) \in [b_i,\ z_j]$ is defined as a *Quadrature Pair* of f.

Using these definitions and conditions (9) and (10) the following postulate is readily suggested and is offered without formal proof.

Conjecture 1: *Let f satisfy (f5)-(f8)*

If $f(0) \leq 0$ then

a) if f has n quadrature points then the BVP (2) has at least 2 solutions for at least n intervals of λ.

b) if f has n quadrature points then the BVP (2) has at least $n-1$ solutions for some range of λ.

c) for any interval $[b_i,\, z_j]$ that contains one or more quadrature pairs, the BVP (2) has at least 3 solutions for some range of λ where $\rho \subset [b_i, z_j](\rho = sup\,|u|)$.

If $f(0) = 0$ then

d) if f has n quadrature points then the BVP (2) has at least 2 solutions for at least n intervals of λ.

e) if f has n quadrature points then the BVP (2) has at least n solutions for some range of λ.

f) for any interval $[b_i,\, z_j]$ that contains one or more quadrature pairs, the BVP (2) has at least 4 solutions for some range of λ and range of $\rho \subset [b_i, z_j]$.

The number of solutions differs with the value of $f(0)$ because, as we showed earlier in Lemmas 1 and 2, that if $f(0) = 0$ then $G_1(\rho) \to \infty$ as $\rho \to b_1^+$ and if $f(0) < 0$ then $G_1(\rho) < \infty$.

Let's look at a numerical plot of a actual example to illustrate the application of Conjecture 1.

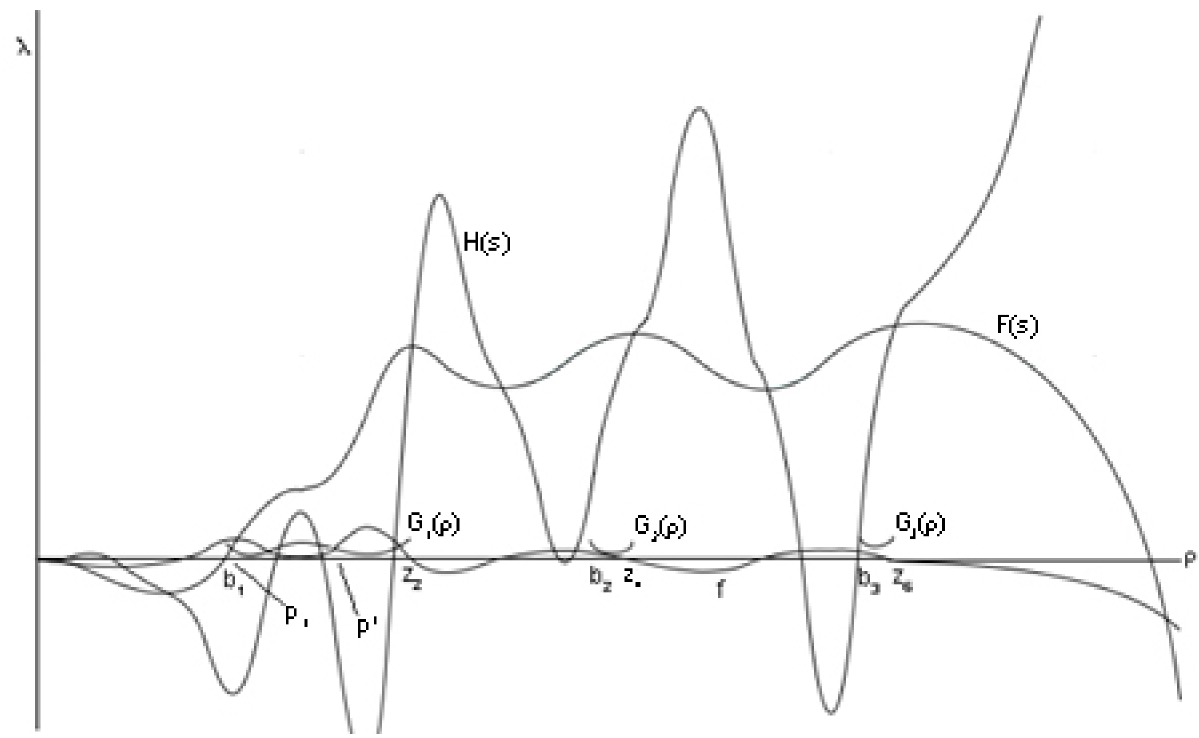


Figure 18: Another f with multiple zeros

Figure 18 above shows an f with 6 zeros and $f(0) < 0$ (although the fact that $f(0) < 0$ is not readily apparent from the diagram). The relevant zeros of f are labeled z_2, z_4 and z_6. The points b_1, b_2 and b_3 are quadrature points since $F(b_1), F(b_2)$ and $F(b_3)$ are each greater than all previous $F(\rho)$ values respectively. Conjecture 1 says that we should have at least 2 solutions for at least 3 ranges of λ which we can readily see is the case with the three bifurcation curves G_1, G_2 and G_3. However, whereas G_2 and G_3 are simple U-shaped curves, notice that G_1 is more complex and looks as if there might be 4 solutions for a range of λ. We can see that in the interval $[b_1, z_2]$ that there are two points p_1 and p^1 such that $H(p_1)$ is greater than all previous $H(\rho)$ and $H(p^1)$ is less than all previous $H(s)$. Thus, according to Conjecture 1 we have at least 3 solutions for some range of λ and some range of $\rho \in [b_1, z_2]$. But why only 3 solutions and not 4 as the picture suggests? Well, while there certainly could be and probably are 4 solutions for the particular example depicted in Figure 18, it could be the case that for other f functions that the local maximum (middle hump) and second local minimum are both greater than the value of $G(b_1)$ such that no horizontal line could intersect G_1 four times. But we can be sure that some horizontal line can intersect G_1 at least three times. However, if $f(0) = 0$ then $G_1(\rho) \to \infty$ as $\rho \to b_1^+$ and some horizontal line would intersect G_1 four times, and as Conjecture 1 indicates, there would be 4 solutions for some range of λ for $\rho \subset [b_1, z_2]$.

Although we have not explicitly stated it, a version of Lemma 2 translated to $f(u - z_i)$ would imply that for each $G_{i>1}$we have that $G_i(\rho) \to \infty$ as $\rho \to b_i^+$. If it was also the case that $f(0) = 0$ (as it may appear in the diagram) then because the G_1 curve approaches infinity as ρ approaches b_1 from the right then some horizontal line would intersect each G_i curve at least once implying at least n solutions for some range of λ. But, if $f(0) < 0$ then $G_1(\rho) < \infty$ and we cannot guarantee that some horizontal line would intersect the G_1 curve in addition to all the $G_{i>1}$ curves at least once. Hence, we can only ensure $n - 1$ solutions if $f(0) < 0$.

A Final Note on Conjecture 1

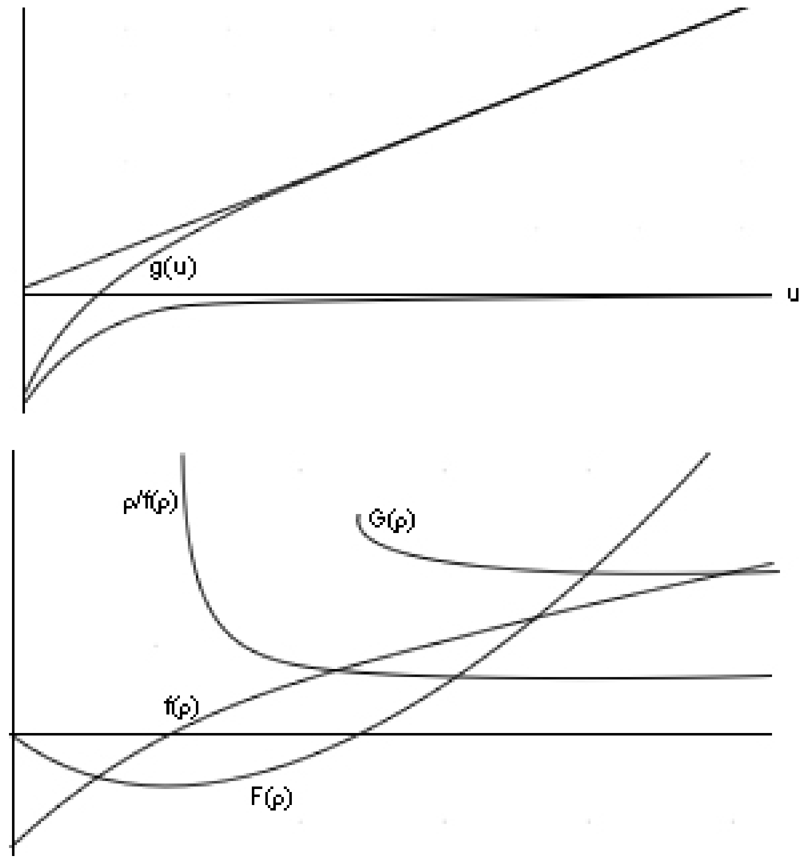
Although we know that the slope of each $G_i(\rho)$ becomes positive as it approaches a zero point of f from the left, we have not investigated whether or not $\lim_{\rho \to z_i^-} G_i(\rho) = \infty$. If this turned out to be the case then we could modify Conjecture 1 (b) and (e) to say that there would be $2n - 1$ and $2n$ solutions for some range of λ respectively.

CHAPTER IV

CONSTRUCTING EXAMPLE FUNCTIONS

We can construct $f(u)$ with the desired characteristics and behavior in a fairly straightforward way. The process consists in summing several simpler component functions each of which contributes to a particular desired behavior of the composite function. In fact, most of the examples in this thesis were created in this way. Let's look at these behaviors and the associated component functions that can be used to implement them.

To illustrate the process, let's create an example of a f function for which the BVP (2) has at least 3 solutions. From previous discussions we know that we need $f(0) \leq 0$ and $f(u) > 0$ for u greater than some positive value of u (we call it τ). With a little thought and experimentation we can easily produce a function that demonstrates this particular behavior. It would also be nice if the function had adjustable parameters so that we can alter the slope and concavity of f in order to "tweak" the behavior as needed. This tweaking partially accounts for the particular values of the parameters of our example function. One such example is the function $g(u) = \frac{1}{e^{u-1}} + 0.7(u-4) + 3$ which is a straight line superimposed on the reciprocal of an exponential function and is shown in Figure 19 below. So let's let $f(u) = g(u)$.

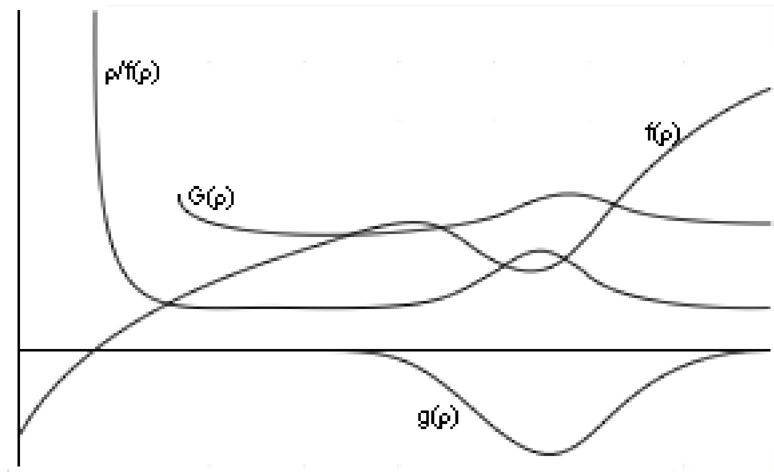


$$f(u) = g(u) = \frac{1}{e^{u-1}} + 0.7(u-4) + 3$$

Figure 19: $f(0) = 0$ and $f(u) > 0$ for $u > \tau$

We also know that we want f to be sufficiently concave for some initial range of u so that $\frac{u}{f(u)}$ and hence $H(u)$ will reach a local minimum (recall that the slopes of $\frac{u}{f(u)}$ and $H(u)$ have the same sign). We determined earlier that if $f(u)$ is "sufficiently" concave (after it becomes positive) then $H(u)$ will become greater than $H(0) = 0$, at which point we know that the bifurcation curve G has changed direction from negative to positive. The trick is to somehow make $f(u)$ sufficiently concave but only over a particular and limited range. Once the bifurcation curve turns upward we are guaranteed the existence of 2 solutions to the BVP (2). But in order to ensure the existence of 3 solutions it is necessary that $\frac{u}{f(u)}$ and hence $H(u)$ change sign again, this time from positive to negative. So, if in addition to making f concave enough so that G' becomes positve we can then make f become "sufficiently" convex so that $\frac{u}{f(u)}$ reaches a local maximum then maybe we can get 3 solutions. The idea is to somehow make f sufficiently concave over a particular limited range and then convex over different and limited range. We don't want to unduly affect the shape of g to the left and right since we already have the desired behavior in the region to the left near σ and we may want to make different but unrelated changes further to the right. So, how can we change the concavity of f over a limited range of u leaving it relatively unaffected to either side of the area of interest? We need a mechanism that offers some control over the slope and concavity of f over a selected limited range of u. One way to do this would be to superimpose another function on g that has a smooth "valley" or downward bump in the desired range of u and is zero everywhere else. One such function that might do the trick is the "bell" curve of the Normal distribution in statistics, i.e., $\frac{e^{-\frac{(u-\mu)^2}{2\omega^2}}}{\sqrt{2\pi\omega^2}}$ where the parameter μ determines the position of the center of the bell curve and the parameter ω controls the width of the bell. Figure 20 shows what happens when we add one of these bell-curves, $h(u) = -8\frac{e^{-\frac{(u-8)^2}{2(1^2)}}}{\sqrt{2\pi(2(1^2))}}$, to the function f. Fortunately, as can be seen, we can get the desired change in direction of the bifurcation curve. So, at this point in the construction of the desired finction let $f(s) = g(s) + h(s)$.

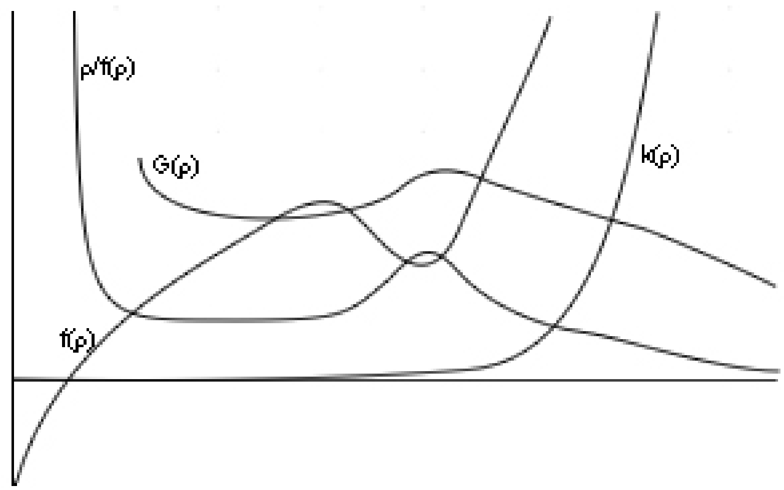
If we went no further with our construction, we can only be guaranteed that the BVP (2) would have 2 solutions for a range of λ. We might have 3 solutions as is the case in Figure 20 but that would just be an accident of construction. It could be the case that $\frac{u}{f(u)}$ and hence $H(u)$ do indeed change sign from positive to negative but that $H(u)$ never becomes less than its previous minimum, in which case we can't invoke



$$f(u) = g(u) + h(u) = \frac{1}{e^{u-1}} + 0.7(u-4) + 3 + -8\frac{e^{-\frac{(u-8)^2}{2(1^2)}}}{\sqrt{2\pi(2(1^2))}}$$

Figure 20: Bifurcation curve turns up

condition (10) to ensure that the bifurcation curve turns downward and becomes less than G. However, if $\frac{u}{f(u)}$ approached zero as $u \to \infty$ then we could be certain that $H(u)$ (and hence G) would continue to decrease until the bifurcation curve can be intersected 3 times by a horizontal line after it continues until $G(\rho) < G(\sigma)$. So, if we can get f to approach ∞ superlinearly then we can make $\frac{u}{f(u)} \to 0$ and thereby ensure that the BVP (2) has at least 3 solutions (in this example exactly 3). The problem is how to modify f without messing up the shape of f to the left where we already have the behavior we want.. As before, we can superimpose the right kind of function on f to get the desired outcome. As can be seen from Figure 21, one such function is $k(u) = 0.08(u+1)^{\frac{10(u+1)}{(u+1)+2}}$. The numeric parameters can be varied to control the detailed shape of $k(u)$ but the important characteristic is the fact that it is essentially 0 until it "goes exponential" which leaves f unaffected for u to the left of our region of interest. Superimposing $k(u)$ on the current $f(u)$ will cause the resulting function to approach ∞ superlinearly and hence $\frac{u}{f(u)}$ to approach 0, thereby ensuring that the bifurcation curve can be intersected by some horizontal line 3 times which implies 3 solutions to the boundary value problem (2). This situation is shown in Figure 21 where the constructed function is $f(u) = g(u) + h(u) + k(u) = \left[\frac{1}{e^{u-1}} + 0.7(u-4) + 3\right] + \left[-8\frac{e^{-\frac{(u-8)^2}{2(1^2)}}}{\sqrt{2\pi(2(1^2))}}\right] + \left[0.08(u+1)^{\frac{10(u+1)}{(u+1)+2}}\right]$.



$$f(u) = g(u)+h(u)+k(u) = \left[\frac{1}{e^{u-1}} + 0.7(u-4) + 3\right] + \left[-8\frac{e^{-\frac{(u-8)^2}{2(1^2)}}}{\sqrt{2\pi(2(1^2))}}\right] + \left[0.08(u+1)^{\frac{10(u+1)}{(u+1)+2}}\right]$$

Figure 21: Bifurcation curve turns down again

Thus we have produced a concrete example of a function with the shape and behavior we want. We have produced other example functions that don't involve e but were finite sums of more complicated polynomials.

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