COMPLEMENTED SUBSPACES OF LOCALLY CONVEX DIRECT SUMS OF BANACH SPACES

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ABSTRACT. We show that a complemented subspace of a locally convex direct sum of an uncountable collection of Banach spaces is a locally convex direct sum of complemented subspaces of countable subsums. As a corollary we prove that a complemented subspace of a locally convex direct sum of an arbitrary collection of \( \ell_1(\Gamma) \)-spaces is isomorphic to a locally convex direct sum of \( \ell_1(\Gamma) \)-spaces.

1. INTRODUCTION

In 1960 A. Pelczynski proved [4] that complemented subspaces of \( \ell_1 \) are isomorphic to \( \ell_1 \). In [3] G. Köthe generalized this result to the non-separable case. Later, while answering Köthe’s question about precise description of projective spaces in the category of (LB)-spaces, P. Domański showed [2] that complemented subspaces of locally convex direct sums of countable collections of \( \ell_1(\Gamma) \)-spaces have the same structure, i.e. are isomorphic to locally convex direct sums of countable collections of \( \ell_1(\Gamma) \)-spaces.

Below we complete this series of statements by showing (Corollary 2.3) that countability assumption in Domański’s result is not essential. More precisely, we prove that complemented subspaces of a locally convex direct sums of arbitrary collections of \( \ell_1(\Gamma) \)-spaces are isomorphic to locally convex direct sums of \( \ell_1(\Gamma) \)-spaces. This is obtained as a corollary of our main result (Theorem 2.2) stating that complemented subspaces of locally convex direct sums of arbitrary collections of Banach spaces are isomorphic to locally convex direct sums of complemented subspaces of countable subsums.

2. RESULTS

Below we work with locally convex direct sums \( \bigoplus \{B_t : t \in T\} \) of uncountable collections of Banach spaces \( B_t \), \( t \in T \). Recall that if \( S \subseteq R \subseteq T \), then \( \bigoplus \{B_t : t \in S\} \) can be canonically identified with the subspace

\[ \left\{ \{x_t : t \in R\} \in \bigoplus \{B_t : t \in R\} : x_t = 0 \text{ for each } t \in R - S \right\} \]

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of $\bigoplus\{B_t: t \in R\}$. The corresponding inclusion is denoted by $i^B_S$. The following statement is used in the proof of Theorem 2.2.

**Proposition 2.1.** Let $r: \bigoplus\{B_t: t \in T\} \rightarrow \bigoplus\{B_t: t \in T\}$ be a continuous linear map of a locally convex direct sum of an uncountable collection of Banach spaces into itself. Let also $A$ be a countable subset of $T$. Then there exists a countable subset $S \subseteq T$ such that $A \subseteq S$ and $r(\bigoplus\{B_t: t \in S\}) \subseteq \bigoplus\{B_t: t \in S\}$.

**Proof.** Let $\exp_\omega T$ denote the set of all countable subsets of the indexing set $T$. Consider the following relation

$$L = \left\{(A, C) \in (\exp_\omega T)^2: A \subseteq C \text{ and } r\left(\bigoplus\{B_t: t \in A\}\right) \subseteq \bigoplus\{B_t: t \in C\}\right\}.$$  

We need to verify the following three properties of the above defined relation.

**Existence.** If $A \in \exp_\omega T$, then there exists $C \in \exp_\omega T$ such that $(A, C) \in L$.

**Proof.** First of all let us make the following observation.

**Claim.** For each $j \in T$ there exists a finite subset $C_j \subseteq T$ such that $r(B_j) \subseteq \bigoplus\{B_t: t \in C_j\}$.

**Proof of Claim.** The unit ball $K = \{x \in B_j: ||x||_j \leq 1\}$ (here $|| \cdot ||_j$ denotes the norm of the Banach space $B_j$) being bounded in $B_j$ is, by [5, Theorem 6.3], bounded in $\bigoplus\{B_t: t \in T\}$. Continuity of $r$ guarantees that $r(K)$ is also bounded in $\bigoplus\{B_t: t \in T\}$. Applying [5, Theorem 6.3] once again, we conclude that there exists a finite subset $C_j \subseteq T$ such that $r(K) \subseteq \bigoplus\{B_t: t \in C_j\}$. Finally the linearity of $r$ implies that $r(B_j) \subseteq \bigoplus\{B_t: t \in C_j\}$ and proves the Claim.

Let now $A \in \exp_\omega T$. For each $j \in A$, according to Claim, there exists a finite subset $C_j \subseteq T$ such that $r(B_j) \subseteq \bigoplus\{B_t: t \in C_j\}$. Without loss of generality we may assume that $A \subseteq C_j$ for each $j \in A$. Let $C = \cup\{C_j: j \in A\}$. Clearly $C$ is countable, $A \subseteq C$ and $r(B_j) \subseteq \bigoplus\{B_t: t \in C\}$ for each $j \in A$. This guarantees that $r(\bigoplus\{B_t: t \in A\}) \subseteq \bigoplus\{B_t: t \in C\}$ and shows that $(A, C) \in L$.

**Majorantness.** If $(A, C) \in L$, $D \in \exp_\omega T$ and $C \subseteq D$, then $(A, D) \in L$.

**Proof.** Condition $(A, C) \in L$ implies that $r(\bigoplus\{B_t: t \in A\}) \subseteq \bigoplus\{B_t: t \in C\}$. The inclusion $C \subseteq D$ implies that $\bigoplus\{B_t: t \in D\} \subseteq \bigoplus\{B_t: t \in D\}$. Consequently $r(\bigoplus\{B_t: t \in A\}) \subseteq \bigoplus\{B_t: t \in C\} \subseteq \bigoplus\{B_t: t \in D\}$, which means that $(A, D) \in L$.

**$\omega$-closeness.** Suppose that $(A_i, C) \in L$ and $A_i \subseteq A_{i+1}$ for each $i \in \omega$. Then $(A, C) \in L$, where $A = \cup\{A_i: i \in \omega\}$.

**Proof.** Consider the following inductive sequence
Choose a continuous linear map $r$. Let us agree that a subset $S$ of the countable sum $\bigoplus \{B_t : t \in A_i\}$ (horizontal arrows represent canonical inclusions). Since $r\left(\bigoplus \{B_t : t \in A_i\}\right) \subseteq \bigoplus \{B_t : t \in C\}$ for each $i \in \omega$ (assumption $(A_i, C) \in \mathcal{L}$), it follows that
\[
r\left(\bigoplus \{B_t : t \in A\}\right) = r\left(\lim_{i \to \omega} \bigoplus \{B_t : t \in A_i\}, i^{A_i+1}, \in \omega\right) \subseteq \bigoplus \{B_t : t \in C\}.
\]
This obviously means that $(A, C) \in \mathcal{L}$ as required.

According to [1, Proposition 1.1.29] the set of $\mathcal{L}$-reflexive elements of $\exp_\omega T$ is cofinal in $\exp_\omega T$. An element $S \in \exp_\omega T$ is $\mathcal{L}$-reflexive if $(S, S) \in \mathcal{L}$. In our situation this means that the given countable subset $A$ of $T$ is contained in a larger countable subset $S$ such that $r\left(\bigoplus \{B_t : t \in S\}\right) \subseteq \bigoplus \{B_t : t \in S\}$. Proof is completed.

**Theorem 2.2.** Let $T$ be an uncountable set. A complemented subspace of a locally convex direct sum $\bigoplus \{B_t : t \in T\}$ of Banach spaces $B_t, t \in T$, is isomorphic to a locally convex direct sum $\bigoplus \{F_j : j \in J\}$, where $F_j$ is a complemented subspace of the countable sum $\bigoplus \{B_t : t \in T_j\}$ where $|T_j| = \omega$ for each $j \in J$.

**Proof.** Let $X$ be a complemented subspace of the sum $B = \bigoplus \{B_t : t \in T\}$. Choose a continuous linear map $r : B \to X$ such that $r(x) = x$ for each $x \in X$. Let us agree that a subset $S \subseteq T$ is called $r$-admissible if $r\left(\bigoplus \{B_t : t \in S\}\right) \subseteq \bigoplus \{B_t : t \in S\}$.

For a subset $S \subseteq T$, let $X_S = r\left(\bigoplus \{B_t : t \in S\}\right)$.

**Claim 1.** If $S \subseteq T$ is an $r$-admissible, then $X_S = X \cap \left(\bigoplus \{B_t : t \in S\}\right)$.

**Proof.** Indeed, if $y \in X_S$, then there exists a point $x \in \bigoplus \{B_t : t \in S\}$ such that $r(x) = y$. Since $S$ is $r$-admissible, it follows that
\[
y = r(x) \in r\left(\bigoplus \{B_t : t \in S\}\right) \subseteq \bigoplus \{B_t : t \in S\}.
\]
Clearly, $y \in X$. This shows that $X_S \subseteq X \cap \left(\bigoplus \{B_t : t \in S\}\right)$.

Conversely, if $y \in X \cap \left(\bigoplus \{B_t : t \in S\}\right)$, then $y \in X$ and hence, by the property of $r$, $y = r(y)$. Since $y \in \bigoplus \{B_t : t \in S\}$, it follows that $y = r(y) \in r\left(\bigoplus \{B_t : t \in S\}\right) = X_S$. 

Claim 2. The union of an arbitrary collection of $r$-admissible subsets of $T$ is $r$-admissible.

Proof. Straightforward verification based on the definition of the $r$-admissibility.

Claim 3. Every countable subset of $T$ is contained in a countable $r$-admissible subset of $T$.

Proof. This follows from Proposition 2.1 applied to the map $r$.

Claim 4. If $S \subseteq T$ is an $r$-admissible subset of $T$, then $r_S(x) = x$ for each point $x \in X_S$, where $r_S = r \left| \left( \bigoplus \{B_t : t \in S\} : \bigoplus \{B_t : t \in S\} \to X_S \right) \right.$

Proof. This follows from the corresponding property of the map $r$.

Before we state the next property of $r$-admissible sets note that if $S \subseteq R \subseteq T$, then the map

$$\pi^R_S : \bigoplus \{B_t : t \in R\} \to \bigoplus \{B_t : t \in S\},$$

defined by letting

$$\pi^R_S (\{x_t : t \in R\}) = \begin{cases} x_t, & \text{if } t \in S \\ 0, & \text{if } t \in R - S, \end{cases}$$

is continuous and linear.

Claim 5. Let $S$ and $R$ are $r$-admissible subsets of $T$ and $S \subseteq R$. Then $X_S$ is a complemented subspace in $X_R$ and $X_R / X_S$ is a complemented subspace in $\bigoplus \{B_t : t \in R - S\}$.

Proof. Consider the following commutative diagram

$$\begin{array}{ccc}
\bigoplus \{B_t : t \in R\} & \xrightarrow{r_R} & X_R \\
\downarrow \pi^R_{R-S} & & \downarrow p \\
\bigoplus \{B_t : t \in R - S\} & \to & \bigoplus \{B_t : t \in R\} / \bigoplus \{B_t : t \in S\} \xrightarrow{q} X_R / X_S
\end{array}$$

in which $p$ is the canonical map and $q$ is defined on cosets by letting (recall that $r_R (\bigoplus \{B_t : t \in S\}) = X_S$)

$$q \left( x + \bigoplus \{B_t : t \in S\} \right) = r_R(x) + X_S \text{ for each } x \in \bigoplus \{B_t : t \in R\}.$$
j(x + XS) = i(x) + ⊔{B_t: t ∈ S} for each x ∈ XR.

Note that q ∘ j = id_{XR/XS} (this follows from the equality r_R ∘ i = id_{XR}). In particular, this shows that XR/XS is isomorphic to a complemented subspace of ⊔{B_t: t ∈ R − S}.

Finally consider the composition r_R ∘ i_{R−S} ∘ j: XR/XS → XR and note that

\[ p ∘ (r_R ∘ i_{R−S} ∘ j) = p ∘ r_R ∘ i_{R−S} ∘ j = q ∘ π_{R−S} ∘ i_{R−S} ∘ j = q ∘ \text{id} ∘ j = id_{XR/XS}. \]

This shows that XS is a complemented subspace of XR and completes the proof of Claim 5.

Let |T| = τ. Then we can write T = \{t_α: α < τ\}. Since the collection of countable r-admissible subsets of T is cofinal in exp_α T (see Claim 3), each element t_α ∈ T is contained in a countable r-admissible subset A_α ⊆ T. According to Claim 2, the set T_α = \bigcup\{A_β: β ≤ α\} is r-admissible for each α < τ. Consider the inductive system \(S = \{X_α, i_α^{α+1}, τ\}\), where X_α = X_{T_α} = \bigcap r(\bigoplus B_t: t ∈ T_α) (see Claim 1) and i_α^{α+1}: X_α → X_{α+1} denotes the natural inclusion for each α < τ. For a limit ordinal number β < τ the space X_β is isomorphic to the limit space of the direct system \(\{X_α, i_α^{α+1}, α < β\}\) (verification of this fact is based on Claim 4 coupled with the fact that \(\bigoplus B_t: t ∈ T_β\) is isomorphic to the limit of the direct system \(\bigoplus X_α, i_α^{α+1}, α < β\}\}).

In particular, X is isomorphic to the limit of the inductive system \(\{X_α, i_α^{α+1}, α < τ\}\).

For each α < τ, according to Claim 5, the inclusion i_α^{α+1}: X_α → X_{α+1} is isomorphic to the inclusion X_α ⊆ X_α \bigoplus X_{α+1}/X_α. In this situation the straightforward transfinite induction shows that X is isomorphic to the locally convex direct sum X_0 \bigoplus \bigoplus\{X_{α+1}/X_α: α < τ\}\).

By construction, the set T_0 is countable and X_0 is a complemented subspace of \(\bigoplus B_t: t ∈ T_0\). Note also that for each α < τ the set T_{α+1} − T_α = A_{α+1} is countable and X_{α+1}/X_α is a complemented subspace of \(\bigoplus B_t: t ∈ A_{α+1}\). This completes the proof of Theorem 2.2.

The following statement, as was noted in the Introduction, provides a complete description of complemented subspaces of locally convex direct sums of uncountable collections of \(ℓ_1(Γ)\)-spaces.

**Corollary 2.3.** Let X be a complemented subspace of \(\bigoplus \{ℓ_1(Γ_t): t ∈ T\}\). Then X is isomorphic to \(\bigoplus \{ℓ_1(Λ_t): i ∈ I\}\).

**Proof.** For countable T results follows from [3] and [2]. Let now T is uncountable and X be a complemented subspace of a locally convex direct sum \(\bigoplus \{ℓ_1(Γ_t): t ∈ T\}\). By Theorem 2.2, X is isomorphic to a locally convex direct sum \(\bigoplus \{F_j: j ∈ J\}\), where F_j is a complemented subspace of the countable sum \(\bigoplus \{ℓ_1(Γ_t): t ∈ T_j\}\) where |T_j| = ω for each j ∈ J. According to [2], \(F_j = \bigoplus \{ℓ_1(Λ_t): t ∈ T_j\} \)
for each $j \in J$. Consequently, $X$ is isomorphic to the locally convex direct sum
\[ \bigoplus \bigoplus \{ \ell_1(\Lambda_t): t \in T_j \}: j \in J \} = \bigoplus \{ \ell_1(\Lambda_t): t \in \cup\{T_j: j \in J\} \} \] as required. 

References