SHER, LAUREN DANIELLE, M.A. Asymptotic Dimension and Asymptotic Property C. (2011)
Directed by Gregory Bell. 32 pp.

This thesis will be concerned with the study of some "large-scale" properties of metric spaces. This area evolved from the study of geometric group theory.

Chapter 1 lays out some of the fundamental notions of geometric group theory including information about word metrics, Cayley graphs, quasi-isometries, and ends of groups and graphs.

Chapter 2 introduces the idea of "large-scale" or "asymptotic" properties of metric spaces along the lines proposed by Gromov in [Gro93]. After looking at some elementary asymptotic versions of common topological notions, such as connectedness, we focus on asymptotic dimension, the large-scale analog of ordinary covering dimension.

In the final chapter, we focus on Dranishnikov's asymptotic version of Haver's property C; see [Dra00]. We provide some basic results on metric spaces with asymptotic property C, studying subspaces and unions. We also prove a result involving the product of metric spaces with asymptotic property C and exhibit a metric space with asymptotic property C and infinite asymptotic dimension. In addition, we study the relationships between asymptotic property C and some of our previously introduced concepts such as quasi-isometries and asymptotic dimension.

ASYMPTOTIC DIMENSION AND ASYMPTOTIC PROPERTY C

by

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A Thesis Submitted to
the Faculty of the Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
2011

Approved by

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To my Grampa who helped me achieve all of my educational goals and to my Grandma who provided all the emotional support in the world.

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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ACKNOWLEDGEMENTS

I would like to express my gratitude to my advisor Dr. Bell for all of his help and to Dr. Duvall and Dr. Yasaki, members of my committee.

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CHAPTER I

GEOMETRIC GROUP THEORY

A group G is *finitely generated*, or of *finite type*, if there exists a finite subset $S \subset G$ such that for any $g \in G$, there exists a finite sequence $s_1, s_2, ..., s_n$ of elements in $S \cup S^{-1}$ with $g = s_1 s_2 ... s_n$. The, not necessarily unique, product $s_1 s_2 ... s_n$ is called an *S-word* representing g. For instance, it can easily be shown that every finite group is finitely generated by taking the generating set S to be the group itself. Similarly, $(\mathbb{Z}, +)$ is finitely generated as one can take $S = \{1\}$. On the other hand, $(\mathbb{Q},+)$ is not finitely generated. This is because if we let S be any finite subset of $\mathbb{Q}$, the only elements generated by $S \cup S^{-1}$ are rational numbers whose denominators divide the least common denominator of elements in S. Hence, S does not generate all of $\mathbb{Q}$.

A *metric space* is defined as a pair (X, d) where X is a set and d is a metric on X. Specifically, d is a function $d : X \times X \to \mathbb{R}$ such that

(i) $d(x, y) \geq 0$ for all $x, y \in X$; equality holds if and only if $x = y$.

(ii) $d(x, y) = d(y, x)$ for all $x, y \in X$.

(iii) $d(x, y) + d(y, z) \geq d(x, z)$ for all $x, y, z \in X$.

Hence, to transform a group G into a metric space, we must find a suitable metric on G. The most useful metric for our purposes is known as the *word metric*. We can define this as follows: Let G be a finitely generated group and let S be a finite set of generators. Let $\|g\|_S$, or just $\|g\|$ when the choice of S is clear, be the minimal length of any S-word representing an element $g \in G$ with the added condition that

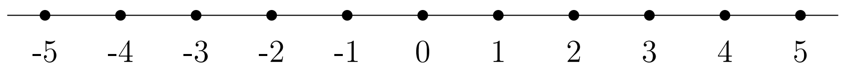

Figure 1. The Cayley graph of $\mathbb{Z}$ with generating set $S = \{1\}$.

$\|e\| = 0$ where e is the identity, represented by the empty word. The *left-invariant word metric*, d_S on G is defined by $d_S(g_1, g_2) = \|g_1^{-1}g_2\|$. The importance of this metric is that it is determined by the group structure on G.

Since $\|g\|$ is an integer for each $g \in G$, the open neighborhood, $N_{\frac{1}{2}}(g) = \{g\}$ so every subset of G is open, i.e. the word metric induces the discrete topology on G. It seems unintuitive that we will obtain any geometric information from this, since the discrete metric gives no more information than we get from ordinary set theory. For instance, any two finitely generated groups with the same cardinality under the discrete topology would be topologically equivalent. We will use the construction of Cayley graphs below to interpret the group G geometrically.

For any finitely generated group G and associated finite generating set S, one can construct a graph known as a Cayley Graph. Let G be a group with finite generating set S. The *Cayley graph* $\mathrm{Cay}(G, S)$ is the graph whose vertex set is G, one vertex for each element in G, and such that two vertices $g_1, g_2 \in G$ are incident with an edge y if and only if $g_1^{-1}g_2 \in S \cup S^{-1}$.

For example, Figure 1 shows the Cayley graph of $(\mathbb{Z}, S)$ where S is the generating set $\{1\}$.

A natural question to ask is whether or not a group can have non-isomorphic Cayley Graphs. The answer is yes. For example, the Cayley graph of $\mathbb{Z}$ with the generating set $S = \{1\}$ is shown in Figure 1 and is clearly not isomorphic to the

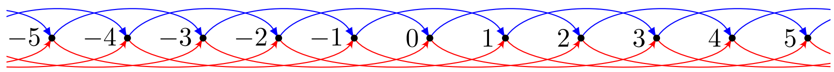

Figure 2. The Cayley graph of $\mathbb{Z}$ with generating set $S = \{2, 3\}$.

Cayley graph shown in Figure 2. (In Figure 2 we have vertices of degree four while in Figure 1 all vertices have degree two.)

At this point, one might wonder whether Cayley graphs are unique to specific groups. This is not the case; i.e., non-isomorphic groups can have isomorphic Cayley graphs. Note that $\mathbb{Z}/4\mathbb{Z} \ncong V_4$ where V_4 is the Klein 4-group. On the other hand, when taking each respective generating set to be the group itself, the Cayley graphs obtained are both the complete graph on four vertices and thus are isomorphic.

Thus, we see that Cayley graphs depend not only on their group, but also on a choice of finite generating set. However, after we define a metric on Cayley graphs, we shall see that there is a relationship between the Cayley graphs that can be described using the notion of quasi-isometry to be introduced shortly. In particular, the graphs of Figures 1 and 2 "look the same when viewed from far away," an idea that will be made more precise by noting that these graphs are quasi-isometric. As we shall see, a finitely generated group's word metric is unique up to quasi-isometries, which is why a specific generating set need not always be mentioned.

Suppose (X, d) and (X', d') are metric spaces with distance functions d and d' respectively. A mapping $\psi : X \to X'$ such that $d'(\psi(x), \psi(y)) = d(x, y)$ for all $x, y \in X$ is an *isometric embedding*. If ψ is also surjective, ψ is an *isometry* and X and X' are said to be *isometric*. As one might expect, an isometry must be a topological equivalence. Isometries are, in fact, stronger than topological equivalences in that they preserve distances. Therefore isometries are useful in the study of geometric structures and their preservation by maps. However, isometries are too strong to be of use in our study of geometric group theory. In our study, we will focus on the weaker notion of quasi-isometry introduced in the following paragraph.

Definition I.1. Let (X, d) and (X', d') be two metric spaces. A function $\psi : X \to X'$ is a *quasi-isometric embedding* if there exist constants $\lambda \geq 1$, $C \geq 0$, such that

$$\frac{1}{\lambda}d(x,y) - C \leq d'(\psi(x), \psi(y)) \leq \lambda d(x,y) + C$$

for all $x, y \in X$. We say that ψ is a *quasi-isometry* if there exists $D \geq 0$ such that for all $x \in X'$ there exists $y \in \psi(X)$ such that $d'(x,y) \leq D$. If such a ψ exists, X and X' are said to be *quasi-isometric*.

Note that a quasi-isometric embedding need not be continuous. Observe that if X and Y are compact metric spaces, and $f : X \to Y$ is any function, then f is a quasi-isometric embedding.

For example, the function $f : \mathbb{R} \to \mathbb{Z}$ given by $f(x) = \lfloor x \rfloor$ is a quasi-isometry. But, the function $\phi : \mathbb{R} \to \mathbb{R}$ given by $\phi(x) = x^2$ is not.

We are now in a position to assign a metric to $\mathrm{Cay}(G, S)$ where G is a group with finite generating set S. Construct this metric as follows: For each edge y of $\mathrm{Cay}(G, S)$ we assign a measure of length so that y is isometric to [0,1]. If $x_1, x_2 \in \mathrm{Cay}(G, S)$ and P is a path from x_1 to x_2, we may use the measure of length on the edges of the graph to define $l(P)$, the length of P. Let

$$d(x, y) = \min\{l(P) \mid P \text{ is a path from } x_1 \text{ to } x_2\}.$$

It can be shown that d is a metric on $\mathrm{Cay}(G, S)$. Throughout this paper, when dealing with topological notions, we shall always assume that $\mathrm{Cay}(G, S)$ is endowed with this metric.

Consider the following examples:

(i) Let X be a metric space of finite diameter. Then X is quasi-isometric to a point; we simply take ψ to be the constant function, C to be 1, and λ to be $B + 1$, where B is the diameter of X.

(ii) Let G be a group, and let S be a finite generating set for G. Then the natural inclusion from G to $\mathrm{Cay}(G, S)$ is a quasi-isometry (see I.3). This is evident from the fact that the natural inclusion preserves distances along with the fact that each point of $\mathrm{Cay}(G, S)$ is no more than a distance of 1 from some vertex of $\mathrm{Cay}(G, S)$.

We now prove some results which will lead to Theorem I.5 stating that any two word metrics on a group G yield quasi-isometric spaces.

Theorem I.2 ([Mei08, Corollary 11.3])**.** *If G is a group with two finite generating sets S and T then the identity on G is a quasi-isometry from (G, d_S) to (G, d_T).*

Proof. If G is trivial, we are done. So, suppose G is not trivial. Let $S = \{s_1, s_2, ..., s_n\}$ and $T = \{t_1, t_2, ..., t_m\}$. Let $\lambda = \max\{\|t_i\|_S, \|s_i\|_T\}$. Note that λ is finite since

$|S|, |T| < \infty$ and $\lambda \geq 1$, since $G \neq \{e\}$. Let $g^{-1}h \in G$ such that

$$g^{-1}h = s_{i_1}s_{i_2}\cdots s_{i_k} = t_{j_1}t_{j_2}\cdots t_{j_l},$$

where $s_{i_1}s_{i_2}\cdots s_{i_k}$ and $t_{j_1}t_{j_2}\cdots t_{j_l}$ are reduced words in S and T respectively, $\|g^{-1}h\|_S = k$, and $\|g^{-1}h\|_T = l$. Now,

$$\begin{aligned}
d_T(g,h) &= \|g^{-1}h\|_T \\
&= \|s_{i_1}\cdots s_{i_k}\|_T \\
&\leq \|s_{i_1}\|_T + \|s_{i_2}\|_T + ... + \|s_{i_k}\|_T \\
&\leq k\lambda \\
&= \lambda\|g^{-1}h\|_S \\
&= \lambda d_S(g,h).
\end{aligned}$$

And,

$$\begin{aligned}
d_S(g,h) &= \|g^{-1}h\|_S \\
&= \|t_{j_1}\cdots t_{j_l}\|_S \\
&\leq \|t_{j_1}\|_S + \|t_{j_2}\|_S + ... + \|t_{j_l}\|_S \\
&\leq l\lambda \\
&= \lambda\|g^{-1}h\|_T \\
&= \lambda d_T(g,h).
\end{aligned}$$

Thus we have $\frac{1}{\lambda}d_T(g,h) \leq d_S(g,h) \leq \lambda d_T(g,h)$ as desired. $\square$

The following theorem can be found in [dlH00] without proof.

Theorem I.3 ([dlH00, 21. Examples (ii)])**.** *If G is a group with finite generating set S, then there is a quasi-isometry from (G, d_S) to* $\mathrm{Cay}(G, S)$*.*

Proof. Let $i : (G, d_S) \to \mathrm{Cay}(G, S)$ be the inclusion function. This is an isometric embedding, and thus a quasi-isometric embedding, because the distance between any two elements as measured in G equals the distance between the two elements as measured in the Cayley graph for if $s_1 s_2 ... s_k$ is a shortest s-word representing $x^{-1}y$ then $x, xs_1, xs_1s_2, ..., xs_1s_2...s_k = y$ is a unique path in the Cayley graph from x to y. $\square$

Theorem I.4 ([Mei08, Lemma 11.38])**.** *Let X and Y be metric spaces. If there is a quasi-isometry from X to Y then there is a quasi-isometry from Y to X.*

We proceed by giving a brief outline of the proof. The entire proof may be found in [Mei08]. Let X and Y be metric spaces. Let $\psi : X \to Y$ be a quasi-isometry with associated constant D, so that each point of Y is within D of some point of $\psi(x)$. Let $\phi : Y \to X$ be a function such that $\phi(y) = x$ where $x \in X$ is chosen, using the axiom of choice, so that $d_Y(\psi(x), y) \leq D$. Then ϕ is a quasi-isometry from Y to X.

We also note that there exist constants α and β such that:

(i) for all $x \in X$, $d_X(x, \phi\psi(x)) \leq \alpha$ and

(ii) for all $y \in Y$, $d_Y(y, \psi\phi(y)) \leq \beta$.

To verify (i), we use the fact that ψ is a quasi-isometric embedding. Let $\lambda \geq 1$ be chosen so that for all $x, x^{'} \in X$, $\frac{1}{\lambda} d_X(x, x^{'}) - \lambda \leq d_Y(\psi(x), \psi(x^{'}))$, and let $\alpha = \lambda(\lambda + D)$. Taking $x \in X$ and $x^{'} = \phi(\psi(x))$, we see from the definition of ϕ

that $d_Y(\psi(x^{'}), \psi(x)) \leq D$. So, for each $x \in X$, $\frac{1}{\lambda}d_X(x, \phi\psi(x)) - \lambda \leq D$ and so $d_X(x, \phi\psi(x)) \leq \lambda(\lambda + D) = \alpha$.

Part (ii) immediately follows from the definition of ψ by letting $\beta = D$.

Suppose X and Y are metric spaces and there exists a quasi-isometric embedding $\psi : X \to Y$. Then we will write $X \sim_Q Y$.

Theorem I.5 ([Mei08, Proposition 11.39])**.** *$\sim_Q$ is an equivalence relation.*

Proof. Reflexivity follows using the identity function. The fact that $\sim_Q$ is symmetric follows from Theorem I.4.

It remains to prove transitivity. Let $\psi_1 : X \to Y$ and $\psi_2 : Y \to Z$ be quasi-isometries. First, we wish to show that $\psi_2 \circ \psi_1 : X \to Z$ is a quasi-isometric embedding. By replacing constants by larger values if necessary, there exists $\lambda > 1$ such that for all $x, x' \in X$,

$$\frac{1}{\lambda}d_X(x, x') - \lambda \leq d_Y(\psi_1(x), \psi_1(x')) \leq \lambda d_X(x, x') + \lambda.$$

And, for all $y, y' \in Y$,

$$\frac{1}{\lambda}d_Y(y, y') - \lambda \leq d_Z(\psi_2(y), \psi_2(y')) \leq \lambda d_Y(y, y') + \lambda.$$

Let $\beta = \lambda^2 + \lambda$. Now by substituting y for $\psi_1(x)$ and y' for $\psi_1(x')$, we get

$$\frac{1}{\lambda}d_Y(\psi_1(x), \psi_1(x')) - \lambda \leq d_Z(\psi_2\psi_1(x), \psi_2\psi_1(x')) \leq \lambda d_Y(\psi_1(x), \psi_1(x')) + \lambda.$$

So, we have that

$$\frac{1}{\lambda}[\frac{1}{\lambda}d_X(x,x') - \lambda] - \lambda \le d_Z(\psi_2\psi_1(x), \psi_2\psi_1(x')) \le \lambda[\lambda d_X(x,x') + \lambda] + \lambda.$$

Then,

$$\frac{1}{\beta}d_X(x,x') - \beta \le d_Z(\psi_2\psi_1(x), \psi_2\psi_1(x')) \le \beta d_X(x,x') + \beta.$$

Thus, $\psi_2 \circ \psi_1$ is a quasi-isometric embedding.

Now, we must show that in fact $\psi_2 \circ \psi_1$ is a quasi-isometry by showing there exists $D > 0$ such that for all $z \in Z$ there exists $x \in X$ such that $d_Z(\psi_2\psi_1(x), z) < D$. Since ψ_1 is a quasi-isometry, there exists D_X such that for all $y \in Y$ there exists $x \in X$ such that $d_Y(\psi_1(x), y) < D_X$. Also, since ψ_2 is a quasi-isometry, there exists D_Y such that for all $z \in Z$ there exists $y \in Y$ such that $d_Z(\psi_2(y), z) < D_Y$. Let $D = D_Y + \lambda D_X + \lambda$, and let $z \in Z$. Now, there exists $y \in Y$ such that $d_Z(\psi_2(y), z) < D_Y$, and there exists $x \in X$ such that $d_Y(\psi_1(x), y) < D_X$. Now,

$$\begin{aligned} d_Z(z, \psi_2\psi_1(x)) &\le d_Z(z, \psi_2(y)) + d_Z(\psi_2(y), \psi_2(\psi_1(x)) \\ &\le d_Z(z, \psi_2(y)) + \lambda d_Y(y, \psi_1(x)) + \lambda \\ &< D_Y + \lambda D_X + \lambda \\ &= D. \end{aligned}$$

Thus, $\psi_2 \circ \psi_1$ is a quasi-isometry and therefore $X \sim_Q Z$. □

We are now in a position to prove the fact, mentioned earlier, that the Cayley graph of the group G does not depend, up to quasi-isometry, on the choice of generating set for G.

Theorem I.6. *Let S and T be finite generating sets for G, then* $\mathrm{Cay}(G,S) = \mathrm{Cay}(G,T)$.

Proof. $\mathrm{Cay}(G,S) \sim_Q (G,d_S) \sim_Q (G,d_T) \sim_Q \mathrm{Cay}(G,T)$. $\square$

Several definitions will be needed before stating the Švarc-Milnor Lemma which relates groups and metric spaces.

Definition I.7. A (left) *group action* of a group G on a set A is a map from $G \times A$ to A, written $(g,a) \to g \cdot a$ for all $g \in G$ and $a \in A$, such that for all $g_1, g_2 \in G$ and $a \in A$, $g_1 \cdot (g_2 \cdot a) = (g_1 g_2) \cdot a$ and $e \cdot a = a$.

The *trivial action* from $G \times X \to X$ is defined by $g \cdot x = x$ for all $g \in G$ and $x \in X$ and clearly satisfies the two conditions of a group action. For an example of a nontrivial action, let $G = \{z \in \mathbb{C} \mid |z| = 1\}$. Then, G acts on itself by ordinary multiplication.

Let G be a group acting by homeomorphisms on a locally compact space X; such a group action is *proper* if for every compact subspace K of X, the set $\{g \in G \mid gK \cap K \neq \emptyset\}$ is finite.

If X is a metric space and G is a group, an *action by isometries* is an action such that $d(x,y) = d(g \cdot x, g \cdot y)$ for all $g \in G$.

Let (X,d) be a metric space. If $p,q \in X$ and $p \neq q$, an *arc* from p to q is the image of a topological embedding $\alpha : [a,b] \to X$, where $[a,b]$ is a closed interval, $\alpha(a) = p$ and $\alpha(b) = q$. If such an α is an isometric embedding, the associated arc from p to q is said to be a *geodesic segment*.

Definition I.8. A metric space X is *geodesic* if any two points in X can be joined by at least one geodesic segment.

Definition I.9. A metric space X is said to be *proper* if its closed balls of finite radius are compact.

For instance, $\mathbb{R}$ is a proper metric space, but $\mathbb{Q}$ is not.

Let G be a finitely generated group and let C be a Cayley graph of G. Using the above terminology, it can be seen that C is a proper geodesic space and that G acts on C by isometries.

Suppose a group G acts on a metric space X. If $x \in X$, the *orbit* of x under this action is the set $H_x = \{gx \mid g \in G\}$. The sets $\{H_x \mid x \in X\}$ form a partition of X. This partition determines an equivalence relation on X, say $\sim$, and we denote the quotient space $\sim \backslash X$ by $G \backslash X$.

Lemma I.10 (Švarc-Milnor)**.** *Let X be a metric space which is geodesic and proper, let G be a group, and let $G \times X \to X$ be an action by isometries (say from the left) so $d(x, y) = d(g \cdot x, g \cdot y)$ for all $x, y \in X$ and $g \in G$. Assume that the action is proper and that the quotient $G \backslash X$ is compact. Then the group G is finitely generated and quasi-isometric to X. More precisely, for any $x_0 \in X$, the mapping $G \to X$ given by $g \mapsto g \cdot x_0$ is a quasi-isometry.*

Lemma I.10 and its proof can be found in [Mei08].

For example, observe that the torus T^2 has fundamental group $\mathbb{Z} \times \mathbb{Z}$. The universal cover of T^2 is $\mathbb{R}^2$. Note that $\mathbb{Z} \times \mathbb{Z}$ acts properly on $\mathbb{R}^2$ by isometries (translations) in a way that commutes with the covering map. So, $(\mathbb{Z} \times \mathbb{Z}) \backslash \mathbb{R}^2 \cong T^2$. Therefore, $(\mathbb{Z} \times \mathbb{Z}) \backslash \mathbb{R}^2$ is compact. Now, by I.10, $\mathbb{Z} \times \mathbb{Z}$ is quasi-isometric to $\mathbb{R}^2$.

We conclude this chapter with a discussion of the *ends of graphs* and the *ends of groups*. The study of ends was one of the earliest geometric group theory topics studied.

Let Γ be a connected, locally finite graph, let $B(n)$ be the closed ball of radius n in Γ based at some fixed vertex $v \in V(\Gamma)$, and let $\|\Gamma - B(n)\|$ be the number of connected, unbounded components in the complement of $B(n)$. Note that if $m, n \in \mathbb{Z}$ such that $m < n$ then $\|\Gamma - B(m)\| < \|\Gamma - B(n)\|$. This comes from the fact that if you have an unbounded, connected component of $\Gamma - B(m)$, then after removing $B(n)$, the said component remains connected or yields additional unbounded components. Thus, every unbounded, connected component of $\Gamma - B(m)$ contributes at least one unbounded, connected component to $\Gamma - B(n)$.

Definition I.11. The *number of ends* of Γ is $e(\Gamma) = \lim_{n\to\infty} \|\Gamma - B(n)\|$.

This limit exists in the extended real number system due to the fact that $\{\|\Gamma - B(n)\|\}$ is a non-decreasing sequence of integers. Note that there is no requirement that the sequence must converge to a number and thus the limit may be infinity.

As an example, the graphs shown in Figures 1 and 2 have two ends. Also, the graph Γ consisting of the x and y-axes in $\mathbb{R}^2$, with vertices being points with integral coordinates, has four ends.

The following Lemma and Theorem along with their proofs appear in [Mei08]. These results connect the study of ends to the previous topic of Cayley graphs.

Lemma I.12 ([Mei08, Lemma 11.22])**.** *Let S and T be finite generating sets for the group G. If $B_S(n)$ and $B_T(n)$ are balls of radius n in* $\mathrm{Cay}(G,S)$ *and* $\mathrm{Cay}(G,T)$ *respectively, then there exists $\mu \geq 1$ such that if g and h are vertices of* $\mathrm{Cay}(G,S)$

that can be joined by an edge path outside of $B_S(\mu n + \mu)$ then $g, h \in \mathrm{Cay}(G,T)$ are outside $B_T(n)$ and can be joined by a path that stays outside of $B_T(n)$.

Theorem I.13 ([Mei08, Theorem 11.23])**.** *Let S and T be finite generating sets for the group G. Then $e(\mathrm{Cay}(G,S)) = e(\mathrm{Cay}(G,T))$.*

Proof. By Lemma I.12, if two vertices in $\mathrm{Cay}(G,S)$ can be connected in the complement of $B_S(\mu n + \mu)$ then they can be connected in the complement of $B_T(n)$. Hence,

$$\|\mathrm{Cay}(G,S) - B_S(\mu n + \mu)\| \geq \|\mathrm{Cay}(G,T) - B_T(n)\|.$$

So,

$$\lim_{n\to\infty} \|\mathrm{Cay}(G,S) - B_S(n)\| \geq \lim_{n\to\infty} \|\mathrm{Cay}(G,T) - B_T(n)\|,$$

and thus $e(\mathrm{Cay}(G,S)) \geq e(\mathrm{Cay}(G,T))$. Similarly, $e(\mathrm{Cay}(G,S)) \leq e(\mathrm{Cay}(G,T))$. Therefore, we conclude $e(\mathrm{Cay}(G,S)) = e(\mathrm{Cay}(G,T))$. □

Definition I.14. Let G be a finitely generated group. The *number of ends* of G is the number of ends of any of its Cayley graphs and is denoted by $e(G)$.

By Theorem I.13, $e(G)$ is well-defined.

It should be noted that a finitely generated group G has zero ends if and only if G is finite [Mei08]. Also, by looking at the Cayley graph of $\mathbb{Z}$ shown in either Figure 1 or 2, one can see that the graph has two ends. Thus, $e(\mathbb{Z}) = 2$.

One may wonder if there are any limitations on the number of ends a group may have. This question was answered by Freudenthal and Hopf sometime in the 1930's [Mei08].

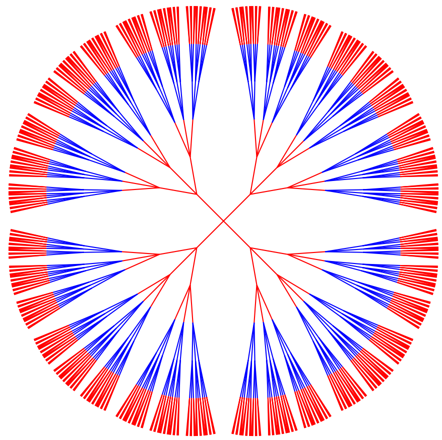

Figure 3. The Cayley graph of $\mathbb{F}^n$.

Theorem I.15 (Freudenthal-Hopf)**.** *Every finitely generated group has either zero, one, two, or infinitely many ends.*

Stallings went on to further classify which groups can have zero, one, two, or infinitely many ends. A portion of this result states that a finitely generated group has two ends if and only if it has an infinite cyclic subgroup of finite index. The remainder of the theorem is more technical and may be found in [Sta68].

Examples:

(i) $\mathbb{Z}^2$ has one end.

(ii) $\mathrm{Cay}(\mathbb{F}^n, \{a_1, a_2, ..., a_n\})$ with $n \geq 2$ has infinitely many ends.

CHAPTER II

ASYMPTOTIC DIMENSION

Gromov was the first mathematician to define asymptotic dimension, as a part of his program of studying the "large-scale properties" of spaces; see [Gro93]. The idea behind asymptotic topology is to gain information by considering small-scale information, like local connectedness, to be irrelevant and to look only at the large-scale properties of the space. For example, even though $\mathbb{R}^2$ and the integer lattice in $\mathbb{R}^2$ are not topologically equivalent because $\mathbb{R}^2$ is connected and the integer lattice is not, they are considered to be the same when looking at the spaces asymptotically. One can see this by envisioning getting farther and farther away from the integer lattice. As we do so, the "gaps" in the integer lattice appear to go away until there are none, as is the case with $\mathbb{R}^2$. Gromov describes this saying points "coalesce into a connected continuous solid unity which occupies the visual horizon without any gaps or holes and fills our geometer's heart with joy" [Gro93]. It may seem that a lot of information about the spaces may be lost, but Gromov points out that this idea can be beneficial because it allows for the "analysis of infinity" and that it is possible "the most essential invariants of an infinite group are large-scale invariants."

Before defining asymptotic dimension, we give some definitions that help illustrate how one can look at metric spaces asymptotically, or in the large-scale.

One of the simplest of Gromov's ideas is the following way of looking asymptotically at the usual notion of connectedness; see [Gro93].

Definition II.1. A metric space X is called *long-range connected* (or *large-scale con-*

nected or *asymptotically connected*), if there exists a constant $c > 0$ such that every two points $x, y \in X$ can be joined by a finite chain of points $x = x_0, x_1, ..., x_n = y$ such that $d(x_i, x_{i-1}) \le c$, $i = 1, ..., n$.

It is easy to see, for example, that any bounded metric space is long-range connected, as is $\mathbb{Z}$ with the usual metric. With the usual metric, $\{10^n \mid n = 0, 1, ...\} \subset \mathbb{Z}$ is not long-range connected. Therefore, long-range connectivity is not a topological property.

Definition II.2. Let X be a metric space and $c > 0$ be a real number. A *c-thickening* of (X, d_X) is a metric space (Y, d_Y) such that

(i) $X \subset Y$,

(ii) d_Y when restricted to X is exactly d_X, and

(iii) for each $y \in Y$, $d_Y(y, X) = \inf_{x \in X} d_Y(y, x) \le c$.

As an example, if G is a finitely generated group, the Cayley graph of G is a $\frac{1}{2}$-thickening of G because each point in the Cayley graph is within distance $\frac{1}{2}$ of an element of G.

The following definition provides a way of looking asymptotically at the familiar topological concepts of contractibility and simple connectivity. Recall that a space X is said to be *simply connected* if X is path-connected and $\pi_1(X, x_0)$ is the trivial group for some $x_0 \in X$. Our definition is slightly different from that given by Gromov in [Gro93].

Definition II.3. The metric space X is *large-scale contractible* (respectively *large-scale simply connected*) if for every $\epsilon > 0$ there exists a ϵ-thickening Y of X such that

Y is contractible (respectively simply connected).

We now give a simple example of a space that is not simply connected, and hence not contractible, but is large-scale contractible. Our example is

$$X = \{(x, y, z) \in \mathbb{R}^3 \mid x \geq 1, y^2 + z^2 = 1/x\}$$

where X has the usual euclidean metric inherited from $\mathbb{R}^3$ and is a simplified version of example $1.D_1'$ in [Gro93]. This space is, intuitively, like an infinite "trumpet" that tapers down with the x-axis as an asymptote. So, when finding an ϵ-thickening, one can go "far enough" down the trumpet and plug up everything to the right of that to form a contractible space.

It should be emphasized that these large-scale properties are not topological properties. Rather, they are dependent on a metric and are therefore appropriate for the study of the geometry of metric spaces. To see this, the space X above is topologically equivalent to $\mathbb{R}^2 - N_1(0)$. On the other hand, with the normal euclidean metric, $\mathbb{R}^2 - N_1(0)$ is not large-scale contractible.

According to the classical definition of dimension, the (covering) dimension of a metric space X does not exceed $n \in \mathbb{Z}^+ \cup \{0\}$, written $\dim X \leq n$, if and only if for every finite open cover $\mathcal{U}$ of X there exists an open cover $\mathcal{V}$ refining $\mathcal{U}$ with multiplicity at most $n+1$ (i.e. each element of $\mathcal{V}$ lies in some element of $\mathcal{U}$ and no more than $n+1$ elements of $\mathcal{V}$ have a point in common). We say that $\dim X = n$ if $\dim X \leq n$ but $\dim X \leq n+1$ is false. We write $\dim X = \infty$ if $\dim X \leq n$ is false for all $n \in \mathbb{Z}^+ \cup \{0\}$.

Examples:

(i) Any discrete space has dimension 0.

(ii) $\dim \mathbb{R}^n = n$ for all $n \in \mathbb{Z}^+ \cup \{0\}$ [BD11].

(iii) Let $I^\infty = \prod_{i=1}^{\infty} I_i$, where $I_i = [0,1]$ for all $i \in \mathbb{Z}^+$. I^∞ is known as the Hilbert Cube and $\dim I^\infty = \infty$.

Asymptotic dimension is the large-scale version of ordinary covering dimension and is defined in a way similar to the classical definition above. Many asymptotic properties have been discovered by determining the relationship between dimension and asymptotic dimension and by comparing results that have been obtained in the study of each type of dimension.

Definition II.4. Let X be a metric space. We say that the *asymptotic dimension* of X does not exceed $n \in \mathbb{Z}^+ \cup \{0\}$, written $\operatorname{asdim} X \leq n$, provided for every uniformly bounded open cover $\mathcal{V}$ of X there is a uniformly bounded open cover $\mathcal{U}$ of X of multiplicity $\leq n+1$ so that $\mathcal{V}$ refines $\mathcal{U}$. We say that $\operatorname{asdim} X = n$ if $\operatorname{asdim} X \leq n$ but $\operatorname{asdim} X \leq n+1$ is false. We write $\operatorname{asdim} X = \infty$ if $\operatorname{asdim} X \leq n$ is false for all $n \in \mathbb{Z}^+ \cup \{0\}$.

We note the obvious fact that if X is any bounded metric space, then $\operatorname{asdim} X = 0$; we simply take $\mathcal{U}$ to be $\{X\}$.

One may be tempted to conclude that every space X with asymptotic dimension 0 is bounded. However, that is not the case. For example, let $X = \{10^n \mid n \in \mathbb{Z}^+\}$. Let $\mathcal{V}$ be a uniformly bounded cover of X. Since $\mathcal{V}$ is uniformly bounded, each element of $\mathcal{V}$ is finite and all but finitely many members of $\mathcal{V}$ are singletons. We let $\mathcal{U}$ consist of those members of $\mathcal{V}$ that are singletons along with the union of the non-singleton members of $\mathcal{V}$. Then $\mathcal{U}$ is uniformly bounded, $\mathcal{V}$ is a refinement of $\mathcal{U}$, and the multiplicity of $\mathcal{U}$ is 1. Hence, $\operatorname{asdim} X = 0$, but X is unbounded.

Before stating the next result, several definitions will be needed. Let X be a metric space and let $r < \infty$. A family $\mathcal{U}$ of subsets of X is said to be *r-disjoint* if

$$d(U, U') = \inf\{d(x, x') \mid x \in U, x' \in U'\} > r$$

for every $U \neq U' \in \mathcal{U}$. The *d-multiplicity* of a family $\mathcal{U}$ of subsets of X is the largest n such that there exists $x \in X$ so that $B_d(x)$ intersects n of the sets from $\mathcal{U}$. The *Lebesgue number* of a cover $\mathcal{U}$ of X is the largest number λ so that if $A \subset X$ and $\operatorname{diam}(A) \leq \lambda$ then there exists $U \in \mathcal{U}$ such that $A \subset U$. A *uniform complex* is a simplicial complex considered to be a subset of l_2 with each vertex at some basis element, with the l_2 metric. A *uniformly cobounded map to a uniform complex* is one in which the diameter of each inverse image of each simplex is uniformly bounded.

There are several equivalent definitions of asymptotic dimension.

Theorem II.5. *Let X be a metric space. Then, the following are equivalent.*

(1) $\operatorname{asdim} X \leq n$*;*

(2) *For every $r < \infty$ there exist uniformly bounded, r-disjoint families $\mathcal{U}^0, \mathcal{U}^1, ..., \mathcal{U}^n$ of subsets of X such that $\cup_i \mathcal{U}^i$ is a cover of X;*

(3) *For every $d < \infty$ there exists a uniformly bounded cover $\mathcal{V}$ of X with d-multiplicity $\leq n + 1$;*

(4) *For every* $\lambda < \infty$ *there is a uniformly bounded cover* $\mathcal{W}$ *of* X *with Lebesgue number* $> \lambda$ *and multiplicity* $\leq n+1$*;*

(5) *For every* $\epsilon > 0$ *there is a uniformly cobounded,* ϵ*-Lipschitz map* $\phi : X \to K$ *to a uniform simplicial complex of dimension* n*.*

The proofs of these equivalences may be found in [BD11].

In many cases, it may be more helpful to use one of these properties as opposed to directly applying the definition to determine the asymptotic dimension of a metric space. For example, (5) is useful in putting a bound on the asymptotic dimension of a product of two metric spaces that have finite asymptotic dimension.

The following two theorems can be found in [BD11]. We provide our own proofs.

Theorem II.6 ([BD11, Proposition 5])**.** *Let* X *and* Y *be metric spaces where* $Y \subset X$ *with the metric inherited from* X*. Then* $\operatorname{asdim} Y \leq \operatorname{asdim} X$*.*

Proof. If $\operatorname{asdim} X = \infty$, we are done, so suppose $\operatorname{asdim} X = n$. Let $r < \infty$. Since $\operatorname{asdim} X = n$, there exist uniformly bounded r-disjoint families $\mathcal{U}^0, \mathcal{U}^1, ...\mathcal{U}^n$ of subsets of X such that $\bigcup_{i=0}^n \mathcal{U}^i$ covers X. Let $\mathcal{V}^i = \{U \cap Y \mid U \in \mathcal{U}^i\}$. Then, $\bigcup_{i=0}^n \mathcal{V}^i$ covers Y. And, the families $\mathcal{V}^0, \mathcal{V}^1, ...\mathcal{V}^n$ are uniformly bounded and r-disjoint since the families $\mathcal{U}^0, \mathcal{U}^1, ...\mathcal{U}^n$ are. Therefore, $\operatorname{asdim} Y \leq n$ and hence $\operatorname{asdim} Y \leq \operatorname{asdim} X$. □

It is important to note that asymptotic dimension, like our other large-scale properties, is not a topological invariant. For example, $\mathbb{R}$ and (0,1) with their usual metrics are topologically equivalent but do not have the same asymptotic dimension; in particular, $\operatorname{asdim} \mathbb{R} = 1$ while $\operatorname{asdim}(0,1) = 0$. However, it is the case that if X and Y are metric spaces that are quasi-isometric, then X and Y do have the same asymptotic dimension.

Theorem II.7 ([BD11, Proposition 2])**.** *If X and Y are metric spaces which are quasi-isometric then* $\operatorname{asdim} X = \operatorname{asdim} Y$.

Proof. We begin by showing that $\operatorname{asdim} Y \leq \operatorname{asdim} X$. If $\operatorname{asdim} X = \infty$, we are done. So, suppose $\operatorname{asdim} X = n < \infty$.

Let $r < \infty$. We wish to show there exist uniformly bounded, r-disjoint families $\mathcal{V}^0, \mathcal{V}^1, ...\mathcal{V}^n$ of subsets of Y such that $\bigcup_{i=1}^n \mathcal{V}^i$ covers Y.

First, since X and Y are quasi-isometric, there exists constants λ, C, and D and a function $\phi : X \to Y$ such that for all $x, y \in X$,

$$\frac{1}{\lambda} d_X(x, y) - C \leq d_Y(\phi(x), \phi(y)) \leq \lambda d_X(x, y) + C$$

and for all $y \in Y$ there exists $x \in X$ such that $d_Y(y, \phi(x)) < D$. Let $R > \lambda(C + 2D + r)$. Then, since $\operatorname{asdim} X = n$, there exist uniformly bounded, R-disjoint families $\mathcal{U}^0, \mathcal{U}^1, ..., \mathcal{U}^n$ of subsets of X such that $\bigcup_{i=1}^n \mathcal{U}^i$ covers X.

For $i = 0, 1, ..., n$, let $\mathcal{V}^i = \{N_D\phi(U) \mid U \in \mathcal{U}^i\}$. Let $y \in Y$. Then there exists $x \in X$ such that $d_Y(y, \phi(x)) < D$. Since $\bigcup_{i=1}^n \mathcal{U}^i$ covers X, there exists $U \in \bigcup_{i=1}^n \mathcal{U}^i$ such that $x \in U$. Then $y \in N_D\phi(U)$, showing that $\bigcup_{i=1}^n \mathcal{V}^i$ covers Y.

Now, we must show that the families $\mathcal{V}^0, \mathcal{V}^1, ..., \mathcal{V}^n$ are uniformly bounded. Since the families $\mathcal{U}^0, \mathcal{U}^1, ..., \mathcal{U}^n$ are uniformly bounded, there exists B such that $\operatorname{diam} U < B$ for all $U \in \bigcup_{i=0}^n \mathcal{U}^i$. Let $p, q \in N_D(\phi(U)), U \in \mathcal{U}^i$. Then there exists $x, x' \in U$ such that $d_Y(p, \phi(x)) < D$ and $d_Y(q, \phi(x')) < D$. So,

$$\begin{aligned} d_Y(p, q) &\leq d_Y(\phi(x), \phi(x')) + 2D \\ &\leq \lambda d_X(x, x') + C + 2D \\ &< \lambda B + C + 2D. \end{aligned}$$

Thus, $\mathcal{V}^0, \mathcal{V}^1, ... \mathcal{V}^n$ are uniformly bounded.

It remains to show that the families $\mathcal{V}^0, \mathcal{V}^1, ..., \mathcal{V}^n$ are r-disjoint. Suppose $V_1, V_2 \in \mathcal{V}^i$, such that $V_1 \neq V_2$. Then there exists $U_1, U_2 \in \mathcal{U}^i$ such that $V_1 = N_D(\phi(U_1)), V_2 = N_D(\phi(U_2))$. Let $p \in V_1, q \in V_2$. Then there exists $x \in U_1$ such that $d_Y(p, \phi(x)) < D$ and $y \in U_2$ such that $d_Y(q, \phi(y)) < D$. Then $d_Y(\phi(x), \phi(y)) < d_Y(p, q) + 2D$. So,

$$\begin{aligned} d_Y(p, q) &> d_Y(\phi(x), \phi(y)) - 2D \\ &\geq \frac{1}{\lambda} d_X(x, y) - C - 2D \\ &\geq \frac{1}{\lambda}(R) - C - 2D \\ &> \frac{1}{\lambda}[\lambda(C + 2D + r)] - C - 2D \\ &= r. \end{aligned}$$

Thus, $\operatorname{asdim} Y \leq \operatorname{asdim} X$. A similar argument will show that $\operatorname{asdim} X \leq \operatorname{asdim} Y$, and hence $\operatorname{asdim} X = \operatorname{asdim} Y$. □

For example, $\mathbb{R}^n$ is quasi-isometric to the set of integer lattice points in $\mathbb{R}^n$. The inclusion map of the lattice points into $\mathbb{R}^n$ is an isometry, and every point of $\mathbb{R}^n$ is within distance $\frac{\sqrt{n}}{2}$ of a lattice point. Therefore, by II.7, the asymptotic dimension of the integer lattice points is equal to the asymptotic dimension of $\mathbb{R}^n$, which is n [BD11]. This may seem surprising since in ordinary dimension theory the set of integer lattice points has dimension 0. It can also be seen that the space $B_n = (n\mathbb{Z})^n \subset \mathbb{R}^n$ with the metric inherited from $\mathbb{R}^n$ is quasi-isometric to $\mathbb{R}^n$ and thus has asymptotic dimension n.

The following result is a consequence of II.7 and Theorem I.2 where if G is a group with finite generating set S, then $\operatorname{asdim} G = \operatorname{asdim} \operatorname{Cay}(G, S)$.

Theorem II.8 ([BD11, Corollary 3]). *Let G be a finitely generated group. Then the asymptotic dimension of G is independent of the choice of a finite generating set for G.*

Theorem II.9. *Suppose a metric space X is unbounded and large-scale connected. Then* $\operatorname{asdim} X \geq 1$.

Proof. Since X is large-scale connected, there exists $d > 0$ such that every two points of X can be connected by a finite d-chain. If $\operatorname{asdim} X = 0$, there exists a uniformly bounded cover $\mathcal{U}'$ of X which is d-disjoint. Let $U \in \mathcal{U}'$, $x \in U$ and $y \in X$. Since there is a d-chain from x to y, $y \in U$. It follows that $U = X$ and since $\mathcal{U}'$ is uniformly bounded, X is bounded, a contradiction. Therefore, $\operatorname{asdim} X \geq 1$. $\square$

For example, let $X = \{10^n \mid n = 0, 1, 2, , , \} \in \mathbb{Z}$. Then X is unbounded, discrete, and $\operatorname{asdim} X = 0$. So, X is not large-scale connected, as we showed directly earlier.

Another important result is the Hurewicz theorem which can be formulated using the notion of either classical dimension or asymptotic dimension; see [BD06].

The following is the Hurewicz Theorem with regards to classical dimension.

Theorem II.10 ([BD06]). *Let X and Y be compact metric spaces and $f : X \to Y$ be a continuous map. Suppose that there is some n so that for every $y \in Y$, $\dim f^{-1}(y) \leq n$. Then $\dim X \leq \dim Y + n$.*

A family $\{F_\alpha\}$ of subsets of a metric space X is said to satisfy $\operatorname{asdim} \mathbb{F}_\alpha \leq n$ *uniformly* if for all $r < \infty$ there exists a constant B so that for every α there exist r-disjoint, B-bounded families $\mathcal{U}_\alpha^0, \mathcal{U}_\alpha^1, ..., \mathcal{U}_\alpha^n$ of subsets of $\mathbb{F}_\alpha$ such that $\bigcup_{i=0}^n \mathcal{U}_\alpha^i$ covers $\mathbb{F}_\alpha$.

Now we state the Hurewicz Theorem with regards to asymptotic dimension.

Theorem II.11 ([BD06, Theorem 1]). *Let $f : X \to Y$ be a Lipschitz map of a geodesic metric space to a metric space. Suppose that for every $R > 0$, $\{f^{-1}(B_R(y))\}_{y \in Y}$ satisfies the inequality $\operatorname{asdim} \leq n$ uniformly. Then $\operatorname{asdim} X \leq \operatorname{asdim} Y + n$.*

The benefit of the Hurewicz theorem is that it allows us to estimate the asymptotic dimension of a product of two metric spaces. The following corollaries can be found in [BD06].

Corollary II.12 ([BD11, Corollary 14]). *Let X and Y be metric spaces. Then $\operatorname{asdim} X \times Y \leq \operatorname{asdim} X + \operatorname{asdim} Y$.*

For example, $\operatorname{asdim} \mathbb{R}^n \leq n$.

Corollary II.13 ([BD06, Theorem 7]). *Let $\phi : G \to H$ be a surjective homomorphism of a finitely generated group with kernel K. Suppose that $\operatorname{asdim} H = n$ and $\operatorname{asdim} K = k$. Then, $\operatorname{asdim} G \leq n + k$.*

CHAPTER III

ASYMPTOTIC PROPERTY C

In this chapter, we will concentrate on the notion of asymptotic property C. Asymptotic property C was first defined by Dranishnikov, see [Dra00], and is an asymptotic analog of property C which was first defined by Haver [DF74].

According to Haver [DF74], a metric space (X, d) is said to have *property C* if for each sequence of positive numbers $\{\epsilon_i\}_{i=1}^{\infty}$, there exists a sequence of collections of open sets $\mathcal{U}^1$,$\mathcal{U}^2$, ... such that

(i) if $U_i \in \mathcal{U}^i$, then diam$U_i < \epsilon_i$

(ii) if $U_i, U_{i'} \in \mathcal{U}^i$ and $U_i \neq U_{i'}$, then $U_i \cap U_{i'} = \emptyset$

(iii) $\mathcal{U} = \bigcup_{i=1}^{\infty} \mathcal{U}^i$ is a cover of X.

Asymptotic property C was first defined by A.N. Dranishnikov; see [Dra00]. This is an asymptotic analog of property C.

Definition III.1. A metric space X has *asymptotic property C* if for any number sequence $R_1 \leq R_2 \leq R_3 \leq ...$ there is a finite sequence of uniformly bounded families of open subsets $\{\mathcal{U}^i\}_{i=1}^{k}$ such that the union $\bigcup_{i=1}^{k} \mathcal{U}^i$ is a covering of X and the family $\mathcal{U}^i$ is R_i-disjoint.

Our first proposition shows that the condition that the families $\mathcal{U}^i$ consist of open sets is superfluous.

Theorem III.2. *A metric space X has asymptotic property C if and only if for any number sequence $R_1 \leq R_2 \leq R_3 \leq ...$ there is a finite sequence of uniformly bounded* *families of sets* $\{\mathcal{U}^i\}_{i=1}^k$ *such that the union* $\bigcup_{i=1}^k \mathcal{U}^i$ *is a covering of* X *and the family* $\mathcal{U}^i$ *is* R_i*-disjoint.*

Proof. First suppose X has asymptotic property C. Then for any number sequence $R_1 \leq R_2 \leq R_3 \leq \ldots$ there is a finite sequence of uniformly bounded families of open subsets, and therefore sets, $\{\mathcal{U}^i\}_{i=1}^k$ such that the union $\bigcup_{i=1}^k \mathcal{U}^i$ is a covering of X and the family $\mathcal{U}^i$ is R_i-disjoint.

For the converse, let $S_1 \leq S_2 \leq S_3 \leq \ldots$ be a number sequence. Let $\epsilon > 0$ and let $R_i = S_i + \epsilon$. Then there exists a finite sequence of uniformly bounded families of sets $\{\mathcal{U}^i\}_{i=1}^k$ whose union covers X and such that each $\mathcal{U}^i$ is R_i-disjoint. Let $\mathcal{V}^i = \{N_{\frac{\epsilon}{3}}(U) \mid U \in \mathcal{U}^i\}$. Then $\mathcal{V}^i$ is a collection of open sets and $\bigcup_{i=1}^n \mathcal{V}^i$ covers X. Since $\mathcal{U}^i$ is uniformly bounded, so is $\mathcal{V}^i$. Now suppose V_1 and V_2 are distinct elements of $\mathcal{V}^i$. Then there exist $U_1, U_2 \in \mathcal{U}^i$ such that $V_1 = N_{\frac{\epsilon}{3}}(U_1)$ and $V_2 = N_{\frac{\epsilon}{3}}(U_2)$. So, $d(V_1, V_2) \geq R_i - \frac{2\epsilon}{3} > S_i$. Thus, the family $\mathcal{V}^i$ is S_i-disjoint and X has asymptotic property C. □

We now relate asymptotic property C to the earlier introduced concepts of quasi-isometry and asymptotic dimension.

Theorem III.3. *Let* X *and* Y *be quasi-isometric spaces. If* X *has asymptotic property C then* Y *has asymptotic property C.*

Proof. Suppose X and Y are quasi-isometric spaces and that X has asymptotic property C. Then there exist constants λ, C, and D and a function $\phi : X \to Y$ such that for all $x, y \in X$,

$$\frac{1}{\lambda} d_X(x, y) - C \leq d_Y(\phi(x), \phi(y)) \leq \lambda d_X(x, y) + C$$

and for all $y \in Y$, there exists $x \in X$ such that $d_Y(y, \phi(x)) \leq D$.

Let $R_1 \leq R_2 \leq R_3 \leq ...$ be a number sequence. Let $S_1 \leq S_2 \leq S_3 \leq ...$ be a number sequence such that $S_i > \lambda(C + 2D + R_i)$. Since X has asymptotic property C, there exist uniformly bounded families $\mathcal{U}^i$, $i = 1, 2, ..., k$, such that $\bigcup_i^k \mathcal{U}^i$ covers X and $\mathcal{U}^i$ is S_i-disjoint for $i = 1, 2, ..., k$. For $i = 0, 1, ..., n$, let $\mathcal{V}^i = \{N_D\phi(U) \mid U \in \mathcal{U}^i\}$. Let $y \in Y$. Then there exists $x \in X$ such that $d_Y(y, \phi(x)) < D$. Since $\bigcup_{i=1}^n \mathcal{U}^i$ covers X, there exists $U \in \bigcup_{i=1}^n \mathcal{U}^i$ such that $x \in U$. Then $y \in N_D\phi(U)$, showing that $\bigcup_{i=1}^n \mathcal{V}^i$ covers Y.

To show the families $\mathcal{V}^0, \mathcal{V}^1, ..., \mathcal{V}^n$ are uniformly bounded, let $p, q \in V_i$. As we have seen in the proof of II.7, $d_Y(p, q) < \lambda B + C + 2D$. Therefore, $\mathcal{V}^0, \mathcal{V}^1, ...\mathcal{V}^n$ are uniformly bounded.

It remains to show that the families $\mathcal{V}^0, \mathcal{V}^1, ..., \mathcal{V}^n$ are R_i-disjoint. Suppose $V_1, V_2 \in \mathcal{V}^i$, such that $V_1 \neq V_2$. Then there exists $U_1, U_2 \in \mathcal{U}^i$ such that $V_1 = N_D(\phi(U_1)), V_2 = N_D(\phi(U_2))$. Let $p \in V_1, q \in V_2$. Then there exists $x \in U_1$ such that $d_Y(p, \phi(x)) < D$ and $x' \in U_2$ such that $d_Y(q, \phi(x')) < D$. Then $d_Y(\phi(x), \phi(x')) < d_Y(p, q) + 2D$. So,

$$\begin{aligned}
d_Y(p, q) &> d_Y(\phi(x), \phi(x')) - 2D \\
&\geq \frac{1}{\lambda} d_X(x, x') - C - 2D \\
&\geq \frac{1}{\lambda}(S_i) - C - 2D \\
&> \frac{1}{\lambda}[\lambda(C + 2D + R_i)] - C - 2D \\
&= R_i.
\end{aligned}$$

Therefore, Y has asymptotic property C. $\square$

We now show that asymptotic property C is implied by finite asymptotic dimension.

Theorem III.4. *If X has finite asymptotic dimension, then X has asymptotic property C.*

Proof. Suppose X is a metric space and $\operatorname{asdim} X = n < \infty$. Let $R_1 \leq R_2 \leq R_3 \leq \ldots$ be a number sequence. Let $r = R_{n+1}$. Then there exist uniformly bounded r-disjoint families $\mathcal{U}^1, \mathcal{U}^2, \ldots, \mathcal{U}^{n+1}$ of subsets of X whose union covers X. Then, since $r \geq R_i$ for all $i = 1, \ldots, n+1$, $\mathcal{U}^i$ is R_i-disjoint and therefore X has asymptotic property C. $\square$

One might wonder if there are any spaces having infinite asymptotic dimension and asymptotic property C. Our next example shows that the answer to this question is yes. We remark that this example is essentially due to Radul [Rad10]. Radul's work seems to imply that the space has asymptotic property C; however, instead of following the more involved program of Radul's paper, we prove this directly.

Example: Let $B_n = (n\mathbb{Z})^n \subset \mathbb{R}^n$ and let $X = \bigcup_{n=1}^{\infty} B_n$.

First we shall define a metric d on X. Let $p, q \in X$. If $p, q \in B_i$ for some $i \in \mathbb{Z}^+$, let $d(p,q) = d_{R^i}(p,q)$ where $\mathbb{R}^i$ is the usual Euclidean metric on $\mathbb{R}^i$. If not, let $\{p,q\} = \{x,y\}$ where $x \in B_i, y \in B_j$, and $i < j$. Define $d(p,q)$ to be $(\sum_{n=i+1}^{j} n) + d_{\mathbb{R}^j}(x', q)$, where x' is the image of x under the natural inclusion of $\mathbb{R}^i$ into $\mathbb{R}^j$. Then d is a metric, the restriction of d to B_n is the usual euclidean metric for $n = 1, 2, \ldots$, and for $p \in B_n$ and $q \in B_m$ with $p \neq q$, $d(p,q) \geq \max(n,m)$.

Now we wish to show that X has infinite asymptotic dimension. Suppose $\operatorname{asdim} X \leq n$. Then, since $B_{n+1} \subset X$, $\operatorname{asdim} B_{n+1} \leq n$ by II.6. But, $B_{n+1} \sim_Q \mathbb{R}^{n+1}$ and so asdim $B_{n+1} = n+1$, a contradiction. Therefore, X has infinite asymptotic dimension.

It remains to show that X has asymptotic property C. Let $R_1 \leq R_2 \leq R_3 \leq ...$ be a number sequence. Let n_0 be an integer such that n and $n_0 > 1$. Let $r = R_{In_0+1}$. Let $Y = \bigcup_{i \leq n_0} B_i$. By appending the appropriate number of zeros to the elements of Y, we see that there is a natural projection of Y into $\mathbb{R}^{n_0}$. This is a quasi-isometry and therefore asdim $Y = n_0$. Since asdim $Y = n_0$, there exist uniformly bounded r-disjoint families $\mathcal{U}^1$,$\mathcal{U}^2$, ...,$\mathcal{U}^{n_0+1}$ of subsets of Y which cover Y. Then $\mathcal{U}^i$ is R_i-disjoint for $i = 1, 2, ..., n_0 + 1$. Let $\mathcal{V}^1 = \mathcal{U}^1 \bigcup \{\{x\} \mid x \in X - Y\}$. Then $\mathcal{V}^1$ is uniformly bounded and R_1-disjoint. Letting $\mathcal{V}^i = \mathcal{U}^i$ for $i = 2, 3, ...n_0 + 1$, we see that the families $\mathcal{V}^1, \mathcal{V}^2, ..., \mathcal{V}^{n_0+1}$ are uniformly bounded, $\mathcal{V}^i$ is R_i-disjoint for $i = 1, 2, ..., n_0 + 1$ and $\bigcup_{i=1}^{n_0+1} \mathcal{V}^i$ covers X.

Thus, X has infinite asymptotic dimension and asymptotic property C.

We finish this chapter by considering subspaces, unions, and products.

Theorem III.5. *If X is a metric space having asymptotic property C and $Y \subset X$ is equipped with the metric inherited from X, then Y has asymptotic property C.*

Since the proof of this exactly follows the proof of II.6, we omit it here.

Theorem III.6. *Let $Z = X \cup Y$, where X and Y are metric spaces with asymptotic property C. Then Z has asymptotic property C.*

Proof. Let $R_1 \leq R_2 \leq R_3 \leq ...$ be a number sequence. Since X has asymptotic property C, there exists a finite sequence of uniformly bounded families of sets $\{\mathcal{U}^i\}_{i=1}^{k}$ such that the union $\bigcup_{i=1}^{k} \mathcal{U}^i$ is a covering of X and the family $\mathcal{U}^i$ is R_i-disjoint. Now consider $R_{k+1} \leq R_{k+2} \leq R_{k+3} \leq ....$ Since Y has asymptotic property C, there exists a finite sequence of uniformly bounded families of sets $\{\mathcal{V}^i\}_{i=1}^n$ such that $\bigcup_{i=1}^n \mathcal{V}^i$ is a covering of Y and the family $\mathcal{V}^i$ is R_{k+i}-disjoint. Let

$$\mathcal{W}^i = \begin{cases} \mathcal{U}^i & \text{if } i \leq k \\ \mathcal{V}^{i-k} & \text{if } k+1 \leq i \leq k+n. \end{cases}$$

Now, $\bigcup_{i=1}^{k+n} \mathcal{W}^i$ covers X and each $\mathcal{W}^i$ is uniformly bounded since each $\mathcal{U}^j$ and $\mathcal{V}^l$ is uniformly bounded. In addition, $\mathcal{W}^i$ is R_i-disjoint since if $i \leq k$, $\mathcal{U}^i$ is R_i-disjoint and if $k+1 \leq i$, $\mathcal{V}^{i-k}$ is $R_{k+(i-k)} = R_i$-disjoint. □

The following is immediate using Theorem III.6 and induction.

Corollary III.7. *If $X = \bigcup_{i=1}^n X_i$, where X_i has asymptotic property C for $i = 1, 2, ..., n$, then X has asymptotic property C.*

We note that Corollary III.7 does not hold for countable unions. In [Rad10], Radul defines a metric d_∞ on the set $L_\infty = \bigcup_{i=1}^\infty \mathbb{Z}^i$ in such a way that the restriction of d_∞ to $\mathbb{Z}^i$ is the sup-metric on $\mathbb{Z}^i$ and L_∞ does not have asymptotic property C. But, $\mathbb{Z}^i$, with the sup-metric, has asymptotic dimension i, and therefore satisfies asymptotic property C.

The following Theorem is a special case of showing that a product of metric spaces of finite asymptotic dimension has asymptotic property C. The more general question of whether or not the product of two metric spaces with asymptotic property C has asymptotic property C is still open. In the next theorem, we will endow $X \times Y$ with the ∞-product metric so $d((x_1, y_1), (x_2, y_2)) = \sup(d(x_1, x_2), d(y_1, y_2))$.

Theorem III.8. *If X and Y are metric spaces such that X has asymptotic property*

C and Y has finite asymptotic dimension, then $X \times Y$ has asymptotic property C.

Proof. Since Y has finite asymptotic dimension, $\operatorname{asdim} Y = n$ for some $n \in \mathbb{Z}^+$. Let $R_1 \leq R_2 \leq R_3 \leq ...$ be a number sequence. Now look at the number sequence $R_{n+1} \leq R_{2(n+1)} \leq R_{3(n+1)} \leq ....$ Since X has asymptotic property C, there exist collections of subsets of X, $\mathcal{U}^1, \mathcal{U}^2, ..., \mathcal{U}^k$, such that each $\mathcal{U}^i$ is uniformly bounded and $R_{i(n+1)}$-disjoint for $i = 1, 2, ..., k$ and such that $\bigcup_{i=1}^{k} \mathcal{U}^i$ covers X. Now let $R = R_{k(n+1)}$. Since $\operatorname{asdim} Y = n$, there exist uniformly bounded collections of subsets of Y, $\mathcal{V}^1, \mathcal{V}^2, ..., \mathcal{V}^{n+1}$, that cover Y and each $\mathcal{V}^i$ is R-disjoint.

We enumerate the $k(n+1)$ collections of sets, $\mathcal{U}^i \times \mathcal{V}^j$, as $\mathcal{W}^1, \mathcal{W}^2, ..., \mathcal{W}^{k(n+1)}$, where $\mathcal{W}^{j(n+1)+i} = \mathcal{U}^{j+1} \times \mathcal{V}^i$ for $j = 0, 1, ..., k-1$ and $i = 1, 2, ..., n+1$. Note that $\bigcup_{i=1}^{k(n+1)} \mathcal{W}^i$ covers $X \times Y$ and each $\mathcal{W}^i$ is uniformly bounded since $\bigcup_{i=1}^{k} \mathcal{U}^i$ covers X, $\bigcup_{i=1}^{n+1} \mathcal{V}^i$ covers Y and each $\mathcal{U}^i$ and $\mathcal{V}^j$ is uniformly bounded. It remains to show that each $\mathcal{W}^i$ is R_i-disjoint. So, let $U_1 \times V_1, U_2 \times V_2$ be distinct elements of $\mathcal{W}^i$. Note that since $\mathcal{U}^i$ is $R_{i(n+1)}$-disjoint and $\mathcal{V}^j$ is R-disjoint, $\mathcal{U}^i \times \mathcal{V}^j$ is $\min(R_{i(n+1)}, R)$-disjoint. By our choice of R, $\min(R_{i(n+1)}, R) \geq R_{i(n+1)} \geq R_i$. Therefore, $d((U_1 \times V_1), (U_2 \times V_2)) \geq R_i$ and therefore, $\mathcal{W}^i$ is R_i-disjoint. Thus, $X \times Y$ has asymptotic property C. $\square$

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