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S. Mrówka introduced a topological space ψ whose underlying set is the natural numbers together with an infinite maximal almost disjoint family(MADF) of infinite subsets of natural numbers. A. Dow and J. Vaughan proved a number of results for similar $\psi(\kappa, \mathcal{M})$ spaces based on any cardinal κ together with a MADF of countably infinite subsets of κ. They proved new results, including new results for the case $\kappa = \omega$. In this paper, we will review some properties of the spaces $\psi(\kappa, \mathcal{M})$ for any cardinal κ. We will then extend some of the results of Dow and Vaughan for $\kappa = \omega$ to the $\kappa = \omega_1$ case. Our goal was to show that the cardinal inequality $\mathfrak{a} < \mathfrak{c}$, where $\mathfrak{a}$ is the smallest cardinality of a MADF on ω, is equivalent to the condition that there exists a MADF $\mathcal{M}$ of infinite subsets of ω_1 such that $\mathcal{M}$ has cardinality $\mathfrak{c}$ and a continuous function $f : \psi(\omega_1, \mathcal{M}) \to [0, 1]$ such that for every $r \in [0, 1]$, $|f^{-1}(r)| < \mathfrak{c} = |\mathcal{M}|$. Dow and Vaughan proved that $\mathfrak{a} < \mathfrak{c}$ is equivalent to a similar statement with ω in the place of ω_1, and although we were able to generalize some of the relevant lemmas, at this time we are only able to prove that the existence of such a MADF $\mathcal{M}$ and function f implies that $\mathfrak{a} < \mathfrak{c}$. One important result that we show along the way to our main result is that for any continuous function from $\psi(\kappa, \mathcal{M})$ into the interval $[0, 1]$, there is some $r \in [0, 1]$ such that $|f^{-1}(r) \cap \mathcal{M}|$ is at least $\mathfrak{a}$. Finally, we will provide some generalizations and interpretations of related lemmas in the ω_1 case.

ON $\psi(\kappa, \mathcal{M})$-SPACES WITH $\kappa = \omega_1$

by

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APPROVAL PAGE

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CHAPTER I

ORDINAL AND CARDINAL NUMBERS

Throughout this paper, we will use some basic facts and properties of ordinal and cardinal numbers. Here we will list some properties of ordinal numbers and some basic facts about the cardinality of sets. We will also need some basic facts about functions.

First, we will establish some notation. We let A, B be arbitrary sets and α, β be arbitrary ordinal numbers.

Definition 1 *$A \setminus B$ denotes the set $\{x \in A : x \notin B\}$.*

Next, we will define ordinal numbers and discuss how they are constructed from sets.

Definition 2 *An* ordinal number *α is a set which is well-ordered by $\in$ and is transitive (i.e., if $x \in \alpha$, then $x \subset \alpha$).*

Another way of defining a transitive set (and perhaps more common) is to say that a set α is transitive if $x \in y \in \alpha$ implies $x \in \alpha$. Also, in ZFC, this is equivalent to saying an ordinal is a transitive set, all of whose members are transitive sets.

Definition 3 *If α is an ordinal number, then its* successor *is $\alpha \cup \{\alpha\}$, denoted by $\alpha + 1$.*

The usual well-ordering on the ordinal numbers is given by $\alpha < \beta$ if and only if $\alpha \in \beta$.

Lemma 1 $\alpha + 1$ *is an ordinal.*

Proof: $\alpha + 1 = \alpha \cup \{\alpha\}$. First we will show $\alpha + 1$ is transitive. Suppose $x \in \alpha + 1$. If $x \in \alpha$, then $x \subset \alpha$ because α is an ordinal. Otherwise, $x \in \{\alpha\}$, so $x = \alpha$, and $\alpha \subset \alpha$, so $\alpha \subset \alpha + 1$. So $\alpha + 1$ is transitive. To show $\alpha + 1$ is also well-ordered, we need to show that any subset of $\alpha + 1$ has a smallest element. Suppose $X \subset \alpha+1$. If $X \subset \alpha$, then X has a least element because α is well-ordered. Otherwise $\alpha \in X$. If $X = \{\alpha\}$, then clearly X has a least element. Otherwise, $X \cap \alpha$ is nonempty, and so $(X \cap \alpha) \subset \alpha$ has a least element y because α is well-ordered. Then $y < x$ for all $x \in X \cap \alpha$, and $y < \alpha$ because $y \in \alpha$, so y is the least element of X. Therefore $\alpha + 1$ is well-ordered. Thus $\alpha + 1$ is an ordinal.

Definition 4 *A* limit ordinal *is any ordinal which is not a successor ordinal.*

Each natural number is an ordinal (viewing a natural number n as $\{i \in \mathbb{N} : i < n\}$). In this point of view, $0 = \emptyset$, which is certainly transitive. If X is a set of ordinals, then $\bigcup X$ is also an ordinal, which is called the supremum of X ([6], section 6.2). $\bigcup X$ is the least ordinal which is greater than or equal to all elements of X.

Definition 5 $\omega = \sup \{n : n \in \mathbb{N}\}$.

This means that ω is the first limit ordinal, because for any ordinal $\alpha < \omega$, $\alpha+1 < \omega$, so ω is not a successor ordinal. These definitions are based on the discussion of ordinal numbers in chapter 6 of [6].

Next we will review some facts about cardinality and its relation to ordinal numbers, particularly initial ordinal numbers.

Definition 6 *Two sets A and B* have the same cardinality *if there is a one-to-one, onto function $f : A \to B$ with domain A and range B. The notation for this is $|A| = |B|$.*

Definition 7 *An ordinal α is called an* initial ordinal *if α does not have the same cardinality as β for any $\beta < \alpha$.*

Definition 8 *The* cardinal number *of a set X is the unique initial ordinal with the same cardinality as X.*

We will use both cardinals and ordinals in this paper. We will generally use ordinals when we are concerned about the well-ordering on the number, and cardinals when we are not concerned with the ordering. Every ordinal has the same cardinality as some initial ordinal.

Definition 9 *A set X is* countable *if X has the same cardinality as ω or X is finite. A set X is* uncountable *if X is not countable.*

Therefore ω is the first infinite ordinal, because any smaller ordinal is finite.

Definition 10 *$\omega_0 = \omega$, and $\omega_{\alpha+1}$ is the least ordinal which does not have the same cardinality as any subset of ω_α. If α is a limit ordinal, then $\omega_\alpha = \sup\{\omega_\beta : \beta < \alpha\}$.*

It is a fact that each ω_α is an initial ordinal and ω_1 denotes the first uncountable ordinal. [6]

Definition 11 $\mathfrak{c}$ *denotes the cardinality of the real line.*

CHAPTER II

TOPOLOGY REVIEW

We will also need a number of concepts from topology for the proofs in this paper. To facilitate this, we will review some basic definitions and facts here (See [2]).

Definition 12 *A* topology *on a set X is a set $\mathcal{T} \subset P(X)$ such that:*

1. $X, \emptyset \in \mathcal{T}$
2. *If* $T_1, \cdots, T_n \in \mathcal{T}$*, then* $\bigcap_{i=1}^{n} T_i \in \mathcal{T}$ *for any* $n \in \omega$.
3. *If* $\mathcal{T}' \subset \mathcal{T}$*, then* $\bigcup \mathcal{T}' \in \mathcal{T}$.

Definition 13 *A* topological space *is a pair $(X, \mathcal{T})$, where X is any set and $\mathcal{T}$ is a topology on X.*

The following definitions refer to a topological space $(X, \mathcal{T})$.

Definition 14 *Given a set $Y \subset X$ and a point $x \in Y$, then x is an* interior point *of Y if there exists a set $T \in \mathcal{T}$ such that $x \in T$ and $T \subset Y$.*

Definition 15 *A set $Y \subset X$ is said to be* open *if $Y \in \mathcal{T}$.*

Lemma 2 *[2] $Y \subset X$ is open if and only if every $x \in Y$ is an interior point of Y.*

Definition 16 *For a topological space X, a set $Y \subset X$ such that x is an interior point of Y is called a* neighborhood *of x.*

Definition 17 *Given a set $Y \subset X$ and a point $x \in X$, then x is said to be a* limit point *of Y if every open set containing x contains some $y \in Y$ such that $y \neq x$.*

Definition 18 *A set Y is* closed *if every limit point of Y is a member of Y.*

Definition 19 *A set $Y \subset X$ is* dense in X *if every point of X is either in Y or is a limit point of Y.*

Lemma 3 *The complement of an open set is a closed set.*

Proof: Suppose A is an open set, and consider A^c, which denotes the complement of A. If A^c is not closed, then there is some point x that is a limit point of A^c such that $x \notin A^c$. Therefore $x \in A$. Since x is a limit point of A^c, every open set containing x must contain some point of A^c. But since $x \in A$ and A is open, there is an open set that contains x but no points of A^c, so this is a contradiction. Therefore A^c is closed.

Definition 20 *A* base for a topological space X *is a family $\mathcal{B} \subset \mathcal{T}$ such that every nonempty open subset of X can be written as $\bigcup_\alpha B_\alpha$, where each $B_\alpha \in \mathcal{B}$.*

Definition 21 *A topological space X is said to be* second countable *if X has a countable base.*

Definition 22 *A family $\mathcal{B}$ of neighborhoods of a point x is said to be a* base for the point x *if for every open set O containing x there exists some $B \in \mathcal{B}$ such that $x \in B \subset O$.*

Definition 23 *A topological space X is said to be* first countable *if every point has a countable base.*

Definition 24 *A topological space X is* zero-dimensional *if it has a base of open-and-closed sets.*

The preceding definitions dealing with bases for topological spaces are based on definitions found in [2].

Definition 25 *A set Y is* compact *if every open cover (a collection $\mathcal{Z}$ of open sets such that $Y \subset \bigcup \mathcal{Z}$) has a finite subcover. In other words, there exist $Z_1, \cdots, Z_n \in \mathcal{Z}$ such that $Y \subset \bigcup_{i=1}^{n} Z_i$.*

One fact about compact sets that we will use pertains specifically to compact sets of real numbers. In the real numbers, a compact set is closed and bounded.

Definition 26 *A topological space $(X, \mathcal{T})$ is a* Hausdorff *space if for any $x, y \in X$, $x \neq y$, there exists open sets $T_1, T_2 \in \mathcal{T}$ such that $x \in T_1$, $y \in T_2$, and $T_1 \cap T_2 = \emptyset$.*

Definition 27 *A set $Y \subset X$ is* discrete *if every point $x \in Y$ has a neighborhood that contains no other points of Y.*

Definition 28 *A set X is* locally compact *if every point of X has a compact neighborhood.*

Now we turn to facts and definitions about functions. In particular, we will make use of properties of continuous functions. Since we will be working with a topological space for most of the paper, we will define a continuous function based on the inverse image of an open set in the range.

Definition 29 *A function $f : X \to Y$ is* continuous *if for every open set $V \subset Y$, $f^{-1}(V)$ is open in X.*

Definition 30 *A topological space X is* pseudocompact *if every continuous function $f : X \to \mathbb{R}$ is bounded.*

Lemma 4 *[2] If $f : X \to Y$ is a continuous function, and if $A \subset X$ is compact, then $f(A)$ is compact.*

Proof: Suppose $\{U_\alpha\}$ is an open cover of $f(A)$. Since f is continuous, $f^{-1}(U_\alpha)$ is open for all α. It covers A, because if $x \in A$, then $f(x) \in f(A)$, so $f(x) \in U_\alpha$ for some α. Therefore $x \in f^{-1}(U_\alpha)$. So $\{f^{-1}(U_\alpha)\}$ is an open cover of A. Since A is compact, it has a finite subcover $\{f^{-1}(U_1), f^{-1}(U_2), \cdots, f^{-1}(U_n)\}$. Then we claim $\{U_1, U_2, \cdots, U_n\}$ is an open cover of $f(A)$. Certainly each of the U_i are open by assumption. Suppose $y \in f(A)$. Then there is some $x \in A$ such that $f(x) = y$. Since $x \in A$, there is some $f^{-1}(U_i)$ such that $x \in f^{-1}(U_i)$ for $i \leq n$ because $\{f^{-1}(U_1), f^{-1}(U_2), \cdots, f^{-1}(U_n)\}$ is an open cover of A. Therefore, $y = f(x) \in U_i$ because $x \in f^{-1}(U_i)$. Therefore $\{U_1, U_2, \cdots, U_n\}$ is a finite open subcover of $f(A)$. So $f(A)$ is compact.

Definition 31 *A sequence* $\{x_n : n \in \omega\} \subset X$ *is said to* converge to r *for* $r \in X$ *if for every neighborhood* V *of* r *there exists some* $k \in \omega$ *such that* V *contains* x_n *for* $n \geq k$.

Definition 32 *A countable set* $Y \subset X$ *is said to* converge to r *for* $r \in X$ *if every neighborhood of* r *contains all but finitely many elements of* Y.

Lemma 5 *Suppose* X, Y *are topological spaces and* $\{x_n\}$ *is a sequence converging to* x *in* X. *Let* $f : X \to Y$ *be a continuous function. Then the sequence* $\{f(x_n)\}$ *converges to* $f(x)$.

Proof: Suppose U is an open set containing $f(x)$. Then $f^{-1}(U)$ is also open because f is continuous. Also note that $x \in f^{-1}(U)$. So $f^{-1}(U)$ is an open set containing x. Therefore, since $\{x_n\}$ converges to x, $x_n \in f^{-1}(U)$ for $n \geq k$ for some natural number k. But this means that $f(x_n) \in U$ for $n \geq k$. Therefore U contains all but finitely many $f(x_n)$, so $\{f(x_n)\}$ converges to $f(x)$.

CHAPTER III

INTRODUCTION TO ψ-SPACES

The original construction of a ψ-space originated with three papers by S. Mrówka [3], [4], [5]. Although earlier authors, including P. Alexandroff and P. Urysohn in a 1925 paper, consider almost disjoint families of infinite sets, and put a similar topology on their space, Mrówka's construction added the maximality condition. This construction was based on infinite subsets of $\mathbb{N}$, and because of the maximality condition, it gave the ψ-space some special properties, including being pseudocompact. We recall his construction. Let $\mathcal{R}$ denote the family of all infinite subsets of $\mathbb{N}$. Write $\mathbb{N}$ as the union of infinitely many disjoint infinite subsets of $\mathbb{N}$, say $\mathbb{N} = N_1 \cup N_2 \cup N_3 \cup N_4 \cdots$. Put the family $\mathcal{R} \setminus \{N_1, N_2, N_3, \cdots\}$ in a transfinite sequence $N_\omega, N_{\omega+1}, \cdots, N_\alpha, \cdots$ Then define a family of sets $\mathcal{R}_1$ by transfinite induction as follows: $N_1 \in \mathcal{R}_1$, and for $\alpha > 1$, $N_\alpha \in \mathcal{R}_1$ if and only if for every $N_\beta \in \mathcal{R}_1$, with $\beta < \alpha$, $N_\alpha \cap N_\beta$ is finite. Finally, let $\psi = \mathbb{N} \cup \mathcal{R}_1$. This is the historical version of space we wish to work with. The neighborhoods in ψ are defined to be as follows: If $x \in \mathbb{N}$, then $O(x) = \{x\}$ is a neighborhood of x. If $x \in \mathcal{R}_1$, then $O(x) = \{x\} \cup (x \setminus S)$ is a neighborhood, where S is any finite subset of x. Now we define a general kind of ψ-space, by replacing $\mathbb{N}$ (or ω) by an arbitrary cardinal $\kappa \geq \omega$. This generalization was introduced by Dow and Vaughan [1].

Definition 33 *An* almost disjoint family *(ADF) is an infinite collection of infinite sets such that the intersection of any two is finite.*

Definition 34 $[X]^\omega$ *denotes the set of all infinite countable subsets of the set* X*.*

Definition 35 *$\mathcal{M}$ is an* ADF on a set X *means* $\mathcal{M} \subset [X]^{\omega}$ *and* $\mathcal{M}$ *is an ADF.*

Definition 36 *Let X be an infinite set. A* maximal almost disjoint family *(MADF) $\mathcal{M}$ on X is an ADF on X that is not properly contained in any other ADF on X.*

Note than an equivalent maximality condition is that for all sets $Y \in [X]^{\omega}$, there is some $M \in \mathcal{M}$ such that $M \cap Y$ is infinite. This is the condition we will generally use when proving that a family is a MADF. If this condition holds, then $\mathcal{M}$ cannot be properly contained in any other ADF $\mathcal{A}$. So there is some $M \in \mathcal{M}$ such that $M \cap Y$ is infinite, so $\mathcal{M} \cup \{Y\}$ is not almost disjoint. On the other hand, if $\mathcal{M}$ is a MADF, then for any $Y \notin \mathcal{M}$, $\mathcal{M} \cup \{Y\}$ is not an ADF, so there must be some $M \in \mathcal{M}$ such that $M \cap Y$ is infinite.

Lemma 6 *$\mathcal{R}_1$ is a MADF on $\mathbb{N}$.*

Proof: $\mathcal{R}_1$ is almost disjoint by construction. Suppose it is not maximal. Then there is some subset $N_\alpha \in \mathcal{R} \setminus \mathcal{R}_1$ such that $N_\alpha \cap N_\beta$ is finite for all $N_\beta \in \mathcal{R}_1$. But this means that we did not include N_α in our inductive construction of $\mathcal{R}_1$, which could only happen if $N_\alpha \cap N_\beta$ was infinite for some $N_\beta \in \mathcal{R}_1$, with $\beta < \alpha$, a contradiction. So $\mathcal{R}_1$ is a MADF.

Definition 37 *For a MADF $\mathcal{M} \subset [\kappa]^{\omega}$, let $\psi(\kappa, \mathcal{M})$ denote the topological space on the set $\kappa \cup \mathcal{M}$ where basic neighborhoods have the form $\{x\}$ for $x \in \kappa$ and $\{M\} \cup (M \setminus S)$ for $M \in \mathcal{M}$, where S is any finite subset of M. When κ and $\mathcal{M}$ are understood, we will write simply ψ. $O(x)$ will denote a basic neighborhood of x in ψ. Thus $O(x) = \{x\}$ if $x \in \kappa$ and $O(x) = \{M\} \cup M \setminus S$ if $x = M \in \mathcal{M}$, where S is finite.*

Definition 38 *The cardinal number $\mathfrak{a}$ denotes the smallest cardinality of a MADF on ω.*

We will review some basic facts about the topology of $\psi = \psi(\kappa, \mathcal{M})$. First, in this thesis, we will always assume the almost disjoint family $\mathcal{M}$ is maximal. However, other authors have considered similar spaces $\psi(\kappa, \mathcal{A})$, with the same neighborhoods described above, where $\mathcal{A}$ is an almost disjoint family but not maximal. Next, we note that the neighborhoods defined above are a neighborhood base for the ψ-space. We will define a system of neighborhoods of x for each $x \in \psi$. Let $B(x) = \{\{x\}\}$ if $x \in \kappa$, and let $B(M) = \{\{M\} \cup (M \setminus S) : S \subset M$ and S is finite$\}$. Let $\mathcal{T}$ be the family of all subsets of ψ that are unions of subfamilies of $\bigcup_{x\in\psi} B(x)$.

Lemma 7 *$\psi(\kappa, \mathcal{M})$ is a topological space with the topology $\mathcal{T}$.*

Proof: By [2], 1.2.3, $\mathcal{T}$ is a topology if the collection $\{B(x)\}_{x\in\psi}$ satisfies three conditions.

1. For every $x \in \psi$, $B(x) \neq \emptyset$ and for every $U \in B(x)$, $x \in U$. Clearly none of our $B(x)$ as defined above are empty because each has at least 1 element, and every set in $B(x)$ contains x.

2. If $x \in U \in B(y)$, then there exists a $V \in B(x)$ such that $V \subset U$. Suppose $x \in U \in B(y)$. If $y \in \kappa$, then $U = \{y\}$. Since $x \in U$, $x = y$, so $\{x\} \in B(x)$ is the desired V. Otherwise $y \in \mathcal{M}$, so say $y = M$. Then $U = \{M\} \cup (M \setminus S)$ for some finite $S \subset M$. Suppose $x \in U$. If $x \in \kappa$, then $\{x\} \in B(x)$ and $\{x\} \subset U$. If $x \in \mathcal{M}$, then we must have $x = M$, because U contains no other points of $\mathcal{M}$. Then $V = U \subset U$, and $V \in B(M)$. So the condition is satisfied.

3. For any $U_1, U_2 \in B(x)$, there exists a $U \in B(x)$ such that $U \subset U_1 \cap U_2$. Suppose $U_1, U_2 \in B(x)$. If $x \in \kappa$, then $U_1 = U_2 = \{x\}$. So $U = \{x\} \subset U_1 \cap U_2 = \{x\}$, and $U \in B(x)$. Otherwise, $x \in \mathcal{M}$ so $x = M$. Say $U_1 = \{M\} \cup (M \setminus S_1)$ and $U_2 = \{M\} \cup (M \setminus S_2)$. Then $U = \{M\} \cup (M \setminus (S_1 \cup S_2)) \subset U_1 \cap U_2$, and $U \in B(M)$. Therefore the third condition is satisfied.

Therefore $\mathcal{T}$ is a topology, and ψ is a topological space.

Lemma 8 *Every $x \in \kappa$ is an isolated point of $\psi(\kappa, \mathcal{M})$.*

Proof: In order for a point x to be isolated, there must be a neighborhood of x that contains no points of ψ other than x itself. For any $x \in \kappa$, $\{x\}$ is such a neighborhood, so every $x \in \kappa$ is isolated.

Lemma 9 *κ is open and dense in ψ.*

Proof: Since a union of open sets is open, and $\kappa = \bigcup_{x \in \kappa} \{x\}$, where each $\{x\}$ is open by definition, κ is open in ψ. To show κ is dense, we will show that any $x \in \psi \setminus \kappa$ is a limit point of κ. Since $x \in \psi \setminus \kappa$, $x = M$ for some $M \in \mathcal{M}$. Then every neighborhood of x has the form $O(x) = \{M\} \cup (M \setminus S)$, where S is a finite subset of M. Since every $O(x)$ contains infinitely many points of κ, x is a limit point of κ. Therefore κ is dense in ψ.

Lemma 10 *ψ is a Hausdorff space.*

Proof: Let $x, y \in \psi$ with $x \neq y$. There are three cases.

1. $x \in \kappa$ and $y \in \kappa$. Then $\{x\}$ and $\{y\}$ are open sets in ψ, with $\{x\} \cap \{y\} = \emptyset$.

2. $x \in \kappa$ and $y \in \mathcal{M}$. Then $\{y\} \cup y \setminus \{x\}$ and $\{x\}$ are open sets in ψ, and $(\{y\} \cup y \setminus \{x\}) \cap \{x\} = \emptyset$.

3. $x \in \mathcal{M}$ and $y \in \mathcal{M}$. Since $\mathcal{M}$ is a MADF, $x \cap y$ is finite. So $V = \{y\} \cup y \setminus (x \cap y)$ and $W = \{x\} \cup x \setminus (x \cap y)$ are open sets in ψ, and $W \cap V = \emptyset$.

So ψ is a Hausdorff space.

Lemma 11 *$\mathcal{M} = \psi \setminus \kappa$ is closed and discrete in ψ.*

Proof: Since $\mathcal{M}$ is the complement of κ in ψ, and the complement of an open set is closed, then $\mathcal{M}$ is closed. To show that it is discrete, let $M \in \mathcal{M}$. We need to show that there is a neighborhood of M that contains no points of $\mathcal{M}$ other than itself. Consider $O(M) = \{M\} \cup M$. $O(M)$ consists of the point M together with members of κ. $O(M)$ cannot contain any other element of $\mathcal{M}$, because elements of $\mathcal{M}$ are infinite subsets of κ, and the only subset of κ which is an element of $O(M)$ is M. Therefore $\mathcal{M}$ is discrete.

Lemma 12 *For any $x \in \psi$, $O(x)$ is both closed and open. Therefore ψ is zero-dimensional.*

Proof:

1. If $x \in \kappa$, then $O(x)$ is closed because it is finite, and a finite set cannot have any limit points in a Hausdorff topology. $O(x)$ is open by definition.

2. If $x \in \mathcal{M}$, then $x = M$, where $M \in \mathcal{M}$, and $O(M) = \{M\} \cup M \setminus S$. $O(M)$ is open by definition. To show $O(M)$ is closed, suppose y is a limit point of $O(M)$. Since every point in κ is isolated, the only possible limit points are elements of $\mathcal{M}$. Suppose some $M_1 \neq M$ is a limit point of $O(M)$. So every neighborhood of M_1 must contain some element of $O(M)$. However, $\{M_1\} \cup M_1 \setminus (M_1 \cap M)$ is a neighborhood of M_1 because $M_1 \cap M$ is finite, and it contains no member of $O(M)$, so M_1 cannot be a limit point. Therefore $O(M)$ contains all its limit points, so $O(M)$ is closed.

Therefore for any $x \in \psi$, the basic neighborhoods of x are both closed and open, so ψ is zero-dimensional.

Lemma 13 *For all $x \in \psi$, $O(x)$ is compact. Therefore ψ is locally compact.*

Proof:

1. If $x \in \kappa$, then $O(x) = \{x\}$, which is finite and therefore compact.

2. The set $O(M) = \{M\} \cup M$, for $M \in \mathcal{M}$, is compact because the countable set M converges to the point M in the ψ-space topology, where M is the only possible limit point (Lemma 12), and a convergent sequence together with its limit points must be compact.

Therefore for all $x \in \psi$, $O(x)$ is compact. Since basic neighborhoods of points in ψ are either finite ($\{x\}$ if $x \in \kappa$) or countable ($\{M\} \cup M \setminus S$ if $M \in \mathcal{M}$), every point has a compact, countable neighborhood. In particular ψ is locally compact.

Lemma 14 *If a function $f : X \to \mathbb{R}$ is continuous and $D \subset X$ is dense in X, then f is bounded on X if and only if f is bounded on D.*

Proof: Clearly if f is bounded on X, then f is bounded on D because $D \subset X$. Now suppose f is bounded on D. Then there is some $r \in \mathbb{R}$ such that $|f(y)| \leq r$ for all $y \in D$. By way of contradiction, suppose f is unbounded on X. Then there exists some $x \in X \setminus D$ such that $|f(x)| > r$. Let us say $f(x) > r$. So there exists some $\epsilon > 0$ such that $r \notin V = (f(x) - \epsilon, f(x) + \epsilon)$. But V is open and f is continuous, so $f^{-1}(V)$ is open in X and contains x. But either $x \in D$ or x is a limit point of D. So there is some $y \in D$ such that $y \in f^{-1}(V)$. But then $f(y) \in V$, and therefore $f(y) > r$, a contradiction. Thus f must be bounded on X.

Theorem 1 *ψ is pseudocompact.*

Proof: By our definition of $\psi(\kappa, \mathcal{M})$, $\mathcal{M}$ is maximal. Suppose $f : \psi \to \mathbb{R}$ is continuous. Let $M_1 \in \mathcal{M}$. By Lemma 13, $O(M_1)$ is compact. Since $M_1 \cup O(M_1)$ has only finitely many more points than $O(M_1)$, $M_1 \cup O(M_1)$ is compact as well. We know that the continuous image of a compact set is compact from Lemma 4 and Theorem 3.1.10 of [2]. Since $f(M_1 \cup O(M_1))$ is a compact subset of $\mathbb{R}$, it is bounded. So f is bounded on the set M_1 for all $M_1 \in \mathcal{M}$. By way of contradiction, suppose f is not bounded on κ. Then we can inductively choose a sequence of elements of κ as follows: Choose x_1 arbitrarily. For $k > 1$, choose x_k such that $|f(x_k)| > |f(x_{k-1})| \cdots > |f(x_1)|$ and $|f(x_k)| \geq k$. This is always possible because we assumed f is unbounded on κ. Let $X = \{x_n : n < \omega\}$. Then there is some $M_2 \in \mathcal{M}$ such that $X \cap M_2$ is infinite. Certainly f is unbounded on X by construction, hence f is unbounded on M_2. But above, we showed that f is bounded on all $M \in \mathcal{M}$. This is a contradiction, so we must have that f is bounded on κ. Since κ is dense in ψ by Lemma 9, by the previous lemma f is bounded on ψ. Therefore ψ is pseudocompact.

We will now show that the maximality of the ADF is necessary in order to have the pseudocompact property. Recall that other authors described a ψ-space where the ADF was not maximal. These spaces are not pseudocompact.

Theorem 2 *If $\psi(\kappa, \mathcal{M})$ is a topological space where $\mathcal{M}$ is an ADF but not maximal, with the same neighborhoods as described in our definition of a ψ-space (definition 37), then ψ is not pseudocompact.*

Proof: Suppose $\mathcal{M}$ is not maximal. We will construct a continuous function on ψ which is not bounded. There exists a set $X \subset [\kappa]^\omega$ ($X \notin \mathcal{M}$) such that $X \cap M$ is finite for all $M \in \mathcal{M}$. Hence X contains all its limit points: Each $y \in \kappa$ is isolated, and so cannot be a limit point of X. If some $M \in \mathcal{M}$ is a limit point of X, then every neighborhood of M contains all but finitely many points of X. In particular, then since $M \cap X$ is finite, $O(M) = \{M\} \cup (M \setminus (M \cap X))$ contains all but finitely many points of X, but it contains no points of X. Therefore M is not a limit point, so X is closed. Say $X = \{x_1, x_2, \cdots\}$. Define $f : \psi \to \mathbb{R}$ by

$$f(x) = \begin{cases} i \text{ if } x = x_i \text{ for some } i \geq 1 \\ 0 \text{ if } \qquad x \notin X \end{cases}$$

First we will show that f is continuous. For $M \in \mathcal{M}$, $f(M) = 0$, so let U be a neighborhood of 0. Choose $\epsilon > 0$ such that $\epsilon < 1$ and such that the interval $(-\epsilon, \epsilon) \subset U$. Then $f^{-1}(-\epsilon, \epsilon) = \psi \backslash X$. Since X is closed, this means that $f^{-1}(-\epsilon, \epsilon)$ is open. Then $f^{-1}(U) = f^{-1}(-\epsilon, \epsilon) \cup X'$ for some $X' \subset X$. Since each point $x_i \in X$ is open, X' is open as the union of open sets. Thus $f^{-1}(U)$ is open. If U is an open set in $\mathbb{R}$ not containing 0, then $f^{-1}(U) = X'$ for some $X' \subset X$, which is open. Therefore f is continuous, but f is clearly unbounded by construction. So ψ is not pseudocompact.

Lemma 15 *If $\mathcal{M}$ is a MADF, then $\mathcal{M}$ is uncountable.*

Proof: Suppose $\mathcal{M}$ is countable. Then we can label the elements of $\mathcal{M}$ as $N_1, N_2, \cdots$. Let x_1 be the smallest element of N_1. (We can do this because κ is well-ordered.) Then by induction, choose x_k such that $x_k > x_{k-1}$ and $x_k \notin N_l$ for $l < k$. This is possible because for each $l < k$, $N_l \cap N_k$ is finite. Let $X = \{x_k : k \in \omega\}$. Then for all $i \in \omega$, $X \cap N_i$ is finite because it cannot contain x_k for $k > i$, so it contains at most i elements of X. Therefore there exists some $M \in \mathcal{M}$ such that $M \cap X$ is infinite, but M cannot be any of the N_i because $X \cap N_i$ is finite for all natural numbers i. This is a contradiction, so $\mathcal{M}$ is uncountable.

Lemma 16 *ψ is not compact.*

Proof: Consider the open cover given by $\{A \cup \{A\} : A \in \mathcal{M}\} \cup \kappa$. This cover is uncountable, by the previous lemma. So there is no finite subcover, nor even a countable subcover, because no element of this cover contains two or more elements of $\mathcal{M}$. Then ψ is not compact.

Lemma 17 *ψ is first countable but not second countable.*

Proof:

1. ψ is first countable: Let $x \in \psi$, and let V be a neighborhood of x. If $x \in \kappa$, then $O(x) = \{x\}$ is a base for the point x. It is finite and therefore countable. Now suppose $x = M$ for some $M \in [\kappa]^\omega$. We described a base for the point M in Definition 37 to be $B(M) = \{\{M\} \cup (M \setminus S) : S \subset M$ and S is finite$\}$. So every $O(M)$ has the form $O(M) = \{M\} \cup M \setminus S$, where S is a finite subset of M. Since M is countable, there are countably many such finite sets S. Therefore, $B(M)$ is countable, so $B(M)$ is a countable base for the point M. So ψ is first countable.

2. ψ is not second countable: If ψ had a countable base, then every open cover of ψ has a countable subcover by [2], Theorem 3.8.1. However, we showed in Lemma 16 that there is an open cover that does not have a countable subcover, so ψ has no countable base and is not second countable.

CHAPTER IV

ψ-SPACES WITH $\kappa = \omega_1$

In this section we will present our main results. Our goal was to generalize Vaughan and Dow's Theorem 11.1 [1] for a MADF on ω_1 instead of on ω. In order to do this, we would need to generalize several related lemmas. First, Dow and Vaughan proved the following theorem:

Theorem 3 *[1] The following are equivalent:*

1. $\mathfrak{a} < \mathfrak{c}$
2. *There exists a MADF* $\mathcal{M} \subset [\omega]^\omega$, $|\mathcal{M}| = \mathfrak{c}$ *and there exists a continuous function* $f : \psi \to [0,1]$ *such that for every* $r \in [0,1]$, $|f^{-1}(r)| < \mathfrak{c} = |\mathcal{M}|$.

Our aim was to consider a third condition:

Condition (*): *There exists a MADF* $\mathcal{M} \subset [\omega_1]^\omega$, $|\mathcal{M}| = \mathfrak{c}$ *and there exists a continuous function* $f : \psi \to [0,1]$ *such that for every* $r \in [0,1]$, $|f^{-1}(r)| < \mathfrak{c} = |\mathcal{M}|$.

The problem we consider is whether Condition (*) is equivalent to (1). We can generalize one key lemma from the relevant section of [1], and this is the first of our main results. However, we are not currently able to generalize the next lemma in Dow and Vaughan's work leading to the construction of a MADF $\mathcal{M}$ and function f satisfying Condition (*). Therefore we will only prove that Condition (*) implies (1), and this is our second main result. Whether (1) implies (*) is still an open problem. Our main results are as follows:

Theorem 4 *Let $\mathcal{P} = \{P_i : i \in \omega_1\} \subset [\omega_1]^\omega$ be an uncountable family of pairwise disjoint infinite sets. Then there exists a MADF $\mathcal{M} \subset [\omega_1]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$.*

Theorem 5 *If there exists a MADF $\mathcal{M} \subset [\omega_1]^\omega$, $|\mathcal{M}| = \mathfrak{c}$ and there exists a continuous function $f : \psi \to [0,1]$ such that for every $r \in [0,1]$, $|f^{-1}(r)| < \mathfrak{c} = |\mathcal{M}|$, then $\mathfrak{a} < \mathfrak{c}$.*

We will first give a few preliminary lemmas, one dealing with the smallest cardinality of a MADF on ω_1, and the other dealing with the inverse continuous image of a ψ-space. Recall that $\mathfrak{a}$ denotes the smallest cardinality of a MADF on the countable set ω. A natural question, then, is: What is the smallest cardinality of a MADF on ω_1? We show that the answer to this question is exactly $\mathfrak{a}$.

Lemma 18 *For every MADF $\mathcal{M} \subset [\omega_1]^\omega$, $\mathfrak{a} \leq |\mathcal{M}|$.*

Proof: The proof will be by contradiction. Suppose $\mathcal{M}$ is a MADF on ω_1 of cardinality $\kappa < \mathfrak{a}$. Let $\mathcal{N} = \{N_i : i \in \omega\} \subset \mathcal{M}$, where the N_i are distinct. Let $X = \bigcup\{N_i : i \in \omega\}$. So X is a countable subset of ω_1. Then define $\mathcal{M}_X = \{M \cap X : M \in \mathcal{M} \text{ and } |M \cap X| = |\omega|\}$. First, we need to show that $\mathcal{M}_X$ is infinite. In particular, each $N_i \in \mathcal{M}_X$, so $\mathcal{M}_X$ must be infinite. We claim $\mathcal{M}_X$ is a MADF on X. Suppose $A, B \in \mathcal{M}_X$. Then $A = M_1 \cap X$ and $B = M_2 \cap X$ for some $M_1, M_2 \in \mathcal{M}$. Since $A \subset M_1$ and $B \subset M_2$ and $M_1 \cap M_2$ is finite, we must have that $A \cap B$ is finite as well, so $\mathcal{M}_X$ is almost disjoint. To show $\mathcal{M}_X$ is maximal, suppose $A \in [X]^\omega$. Then $A \in [\omega_1]^\omega$ as well, so there is some $M \in \mathcal{M}$ such that $M \cap A$ is infinite. Since $A \subset X$, this means that $M \cap X$ is infinite as well, so $M \cap X \in \mathcal{M}_X$. Therefore $\mathcal{M}_X$ is a MADF on X.

Let $\tilde{\mathcal{M}} = \{M \in \mathcal{M} : M \cap X \in \mathcal{M}_X\}$, and define $f : \tilde{\mathcal{M}} \to \mathcal{M}_X$ by $f(M) = M \cap X$. Then f is onto $\mathcal{M}_X$ because if $M \cap X \in \mathcal{M}_X$, then $M \in \tilde{\mathcal{M}}$, so $f(M) =$

$M \cap X$. Therefore, $\left|\tilde{\mathcal{M}}\right| \geq |\mathcal{M}_X|$. Thus, we have that $|\mathcal{M}| \geq \left|\tilde{\mathcal{M}}\right| \geq |\mathcal{M}_X| \geq \mathfrak{a}$. So $\mathfrak{a} \leq |\mathcal{M}|$ as desired.

Lemma 19 *Given a ψ-space and a continuous function $f : \psi(\kappa, \mathcal{M}) \to [0,1]$, then there is some $r \in [0,1]$ such that $f^{-1}(r) \cap \mathcal{M}$ is infinite.*

Proof: The proof is by contradiction. Suppose for any $r \in [0,1]$, $f^{-1}(r) \cap \mathcal{M}$ is finite. For $n \geq 0$, we will inductively construct V_n, M_n, and α_n such that:

1. $V_n = f^{-1}[a_n, b_n] \cap \mathcal{M}$ for some $a_n, b_n \in [0,1]$ and V_n is infinite,
2. $a_n < b_n$ and $b_n - a_n \leq \frac{1}{2^n}$,
3. $M_n \in f^{-1}(a_n, b_n) \cap \mathcal{M}$,
4. $\alpha_n \in M_n$ and $f(\alpha_n) \in (a_n, b_n)$.

First we will construct a_n, b_n and the sets V_n. Let $b_0 = 0, a_0 = 1$, and $V_0 = f^{-1}[0,1] \cap \mathcal{M} = \mathcal{M}$. Note that V_0 is infinite because $\mathcal{M}$ is infinite. Now assume we have created $V_i = f^{-1}[a_i, b_i] \cap \mathcal{M}$ for $i \leq n$, where V_i is infinite. Consider the sets $f^{-1}\left[a_n, \frac{a_n+b_n}{2}\right] \cap \mathcal{M}$ and $f^{-1}\left[\frac{a_n+b_n}{2}, b_n\right] \cap \mathcal{M}$. Since the union of these two sets is V_n and V_n is infinite, at least one of the two sets must be infinite as well. Suppose $f^{-1}\left[a_n, \frac{a_n+b_n}{2}\right] \cap \mathcal{M}$ is infinite. Then define $a_{n+1} = a_n$ and $b_{n+1} = \frac{a_n+b_n}{2}$, and $V_{n+1} = f^{-1}[a_{n+1}, b_{n+1}] \cap \mathcal{M}$. If $f^{-1}\left[\frac{a_n+b_n}{2}, b_n\right] \cap \mathcal{M}$ is infinite, then $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = b_n$, and $V_{n+1} = f^{-1}[a_{n+1}, b_{n+1}] \cap \mathcal{M}$. Therefore V_{n+1} is infinite as well.

Since each $[a_n, b_n]$ is a closed, compact set in $\mathbb{R}$ and $[a_{n+1}, b_{n+1}] \subset [a_n, b_n]$ for all n, then $\bigcap_{n\in\omega}[a_n, b_n] \neq \emptyset$, and by 2), $\bigcap_{n\in\omega}[a_n, b_n] = \{r\}$ for some $r \in [0,1]$. Since $f^{-1}(r) \cap \mathcal{M}$ is finite, then suppose we have $\mathcal{N} = \{N_1, N_2, \ldots, N_k\} \subset \mathcal{M}$ such that $f(N_i) = r$ and these are all such sets in $\mathcal{M}$.

Now we will construct the M_n and α_n. We will need the following fact: If $M \in \mathcal{M}$, then since the set M converges to the point M in the ψ-space, the set $\{f(\alpha) : \alpha \in M\}$ converges to $f(M)$ in $[0,1]$ for $\alpha \in M$ because f is continuous. Since $f^{-1}(0) \cap \mathcal{M}$ and $f^{-1}(1) \cap \mathcal{M}$ are finite, and V_1 is infinite, then there is some $M_1 \in (f^{-1}(0,1) \cap \mathcal{M}) \setminus \mathcal{N}$ because $f^{-1}(0,1) \cap \mathcal{M}$ is infinite and $\mathcal{N}$ is finite. Then since $f(M_1) \in (0,1)$ which is open, there is some neighborhood U of $f(M_1)$ such that $U \subset (0,1)$. Since f is continuous, there is $\alpha_1 \in M_1 \setminus \bigcup \mathcal{N}$ such that $f(\alpha_1) \in U \subset (0,1)$. (Since $M_1 \cap N_i$ is finite for $i \leq k$, $M_1 \setminus \bigcup \mathcal{N} \neq \emptyset$.)

Now assume we have constructed M_i and α_i for $i \leq n$. Since $f^{-1}(a_{n+1}) \cap \mathcal{M}$ and $f^{-1}(b_{n+1}) \cap \mathcal{M}$ are finite, then $(f^{-1}(a_{n+1}, b_{n+1}) \cap \mathcal{M}) \subset V_{n+1}$ is infinite. Therefore we can choose $M_{n+1} \in (f^{-1}(a_{n+1}, b_{n+1}) \cap \mathcal{M}) \setminus (\{M_i : i \leq n\} \cup \mathcal{N})$ because $\mathcal{N}$ and $\{M_i : i \leq n\}$ are finite. Then $f(M_{n+1}) \in (a_{n+1}, b_{n+1})$, so there is a neighborhood U of $f(M_{n+1})$ such that $U \subset (a_{n+1}, b_{n+1})$. Since f is continuous, then there is some $\alpha_{n+1} \in M_{n+1} \setminus (\bigcup \{M_i : i \leq n\} \cup \{N_1, N_2, \cdots, N_k\})$ such that $f(\alpha_{n+1}) \in U \subset (a_{n+1}, b_{n+1})$. This completes the construction of M_n and α_n for $n \geq 0$. Let $A = \{\alpha_i : i \in \omega\}$.

Let $(r - \epsilon, r + \epsilon)$ be a neighborhood of r. We claim that there is some $n \in \omega$ such that $[a_n, b_n] \subset (r - \epsilon, r + \epsilon)$. The proof is by contradiction. Suppose there is no n such that $[a_n, b_n] \subset (r - \epsilon, r + \epsilon)$. Then for each n, either $a_n \leq r - \epsilon < r$ or $r < r + \epsilon \leq b_n$. Consider $[a_i, b_i]$, and suppose $[a_i, b_i] \not\subset (r - \epsilon, r + \epsilon)$. If $a_i \leq r - \epsilon < r$, then there must be some $k > i$ such that $r - \epsilon < a_k < r$ because otherwise, we would have $a_n \leq r - \epsilon < r \leq b_n$ for all n, and therefore $r - \epsilon \in \bigcap [a_n, b_n] = \{r\}$, which is a contradiction. Similarly if $r < r + \epsilon \leq b_i$, there must be some $l > i$ such that $r < b_l < r + \epsilon$ because otherwise $r + \epsilon \in \bigcap [a_n, b_n] = \{r\}$, a contradiction. Let $j = \max\{k, l\}$. Then $[a_j, b_j] \subset (r - \epsilon, r + \epsilon)$, so we have the desired result. Therefore, $f(M_j) \in (r - \epsilon, r + \epsilon)$, and by our choice of α_j, $f(\alpha_j) \in (a_j, b_j) \subset (r - \epsilon, r + \epsilon)$.

Therefore the values $f(\alpha_n)$ converge to r.

Since A is an infinite subset of κ, there is some $M \in \mathcal{M}$ such that $M \cap A$ is infinite. Since each α_n was chosen to be not in any of $N_1, N_2, \cdots, N_k$, then $M \notin \{N_1, N_2, \cdots, N_k\}$, so $f(M) \neq r$ by assumption. However, in $\psi(\kappa, \mathcal{M})$, the set $M \cap A$ converges to the point M, and since f is continuous, this means that $\{f(\alpha_n) : \alpha_n \in M \cap A\}$ must converge to $f(M)$. But $f(\alpha_n)$ converges to r by our choice of α_n, so $f(M) = r$, a contradiction. Therefore $f^{-1}(r) \cap \mathcal{M}$ must be infinite.

Next, we will prove our first main theorem, which is an ω_1 analog of Dow and Vaughan's Lemma 11.2. [1] This is a key lemma in both Dow and Vaughan's work and our attempt to prove that $\mathfrak{a} < \mathfrak{c}$ implies the existence of a MADF $\mathcal{M} \subset [\omega_1]^\omega$ and continuous function f satisfying Condition (*). In order to do this, we make use of one of the lemmas from Dow and Vaughan's work, so we state that here for convenience. We will then prove some generalizations of this lemma, followed by our first main theorem.

Lemma 20 *For every countable family $\mathcal{P} = \{P_i : i \in \omega\} \subset [\omega]^\omega$ of pairwise disjoint infinite sets, there exists a MADF $\mathcal{M} \subset [\omega]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$. [1]*

We now will give proofs for some related generalizations of Dow and Vaughan's Lemma 20, which we will use in our first main result. The proofs follow much the same procedure as Dow and Vaughan's work for the ω case. We will show that Lemma 20 is also true for a countable family of countably infinite sets, where the MADF is created on the union of the sets in the family. The lemma also holds for a countable family of countably infinite almost disjoint subsets instead of pairwise disjoint sets.

Lemma 21 *Let $\mathcal{P} = \{P_i : i \in \omega\}$ be a countable family of pairwise disjoint countably infinite sets. Then there exists a MADF $\mathcal{M} \subset [\bigcup \mathcal{P}]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and*

$|\mathcal{M}| = \mathfrak{a}$.

Proof: Since $\mathcal{P}$ is a countably infinite family of countably infinite sets, $|\bigcup \mathcal{P}| = |\omega|$. Let $X = \bigcup \mathcal{P}$, and $f : \omega \to X$ be a bijection. Let $Q_i = f^{-1}(P_i)$, and define $\mathcal{Q} = \{Q_i : i \in \omega\}$. Since f is a bijection and each P_i is countably infinite, each Q_i is countably infinite as well. Also, $\mathcal{Q}$ is pairwise disjoint: For a contradiction, suppose $x \in Q_i$ and $x \in Q_j$ for $i \neq j$. Then $f(x) \in P_i$ and $f(x) \in P_j$, which contradicts that $\mathcal{P}$ is pairwise disjoint. So $Q_i \cap Q_j = \emptyset$, so $\mathcal{Q}$ is a pairwise disjoint countably infinite family of countably infinite subsets of ω. Therefore $\mathcal{Q}$ satisfies the hypotheses of Lemma 20, so there is a MADF $\mathcal{N} \subset [\omega]^\omega$ such that $\mathcal{Q} \subset \mathcal{N}$ and $|\mathcal{N}| = \mathfrak{a}$. Let $\mathcal{M} = \{f(N) : N \in \mathcal{N}\}$.

We claim $\mathcal{M}$ is a MADF on X. First we will show that it is an ADF. Suppose $M_1, M_2 \in \mathcal{M}$, where $M_1 = f(N_1)$ and $M_2 = f(N_2)$ for some $N_1, N_2 \in \mathcal{N}$. Since f is a bijection, $f(N_1 \cap N_2) = f(N_1) \cap f(N_2) = M_1 \cap M_2$. Since $N_1 \cap N_2$ is finite (because $\mathcal{N}$ is an ADF on ω), $M_1 \cap M_2$ is finite as well, so $\mathcal{M}$ is an ADF. To show $\mathcal{M}$ is maximal, suppose $C \in [X]^\omega$. Let $C' = f^{-1}(C)$. Then $C' \in [\omega]^\omega$, so there is some $N \in \mathcal{N}$ such that $N \cap C'$ is infinite. Again, because f is a bijection, $f(N \cap C') = f(N) \cap f(C') = f(N) \cap C$. Since $N \cap C'$ is infinite, $f(N) \cap C$ is infinite as well, and since $f(N) \in \mathcal{M}$, $\mathcal{M}$ is maximal. So we have shown that $\mathcal{M}$ is a MADF on X. $\mathcal{P} \subset \mathcal{M}$ because for each $i \in \omega$, $f(Q_i) = P_i \in \mathcal{M}$ because $Q_i \in \mathcal{N}$. Also, since $|\mathcal{N}| = \mathfrak{a}$ and f is a bijection, $|\mathcal{M}| = \mathfrak{a}$ as well. Therefore $\mathcal{M}$ is the desired MADF.

Corollary 1 *Let $\mathcal{P} = \{P_i : i \in \omega\} \subset [\kappa]^\omega$ be a countable family of pairwise disjoint infinite sets, where $\kappa < \omega_1$. Then there exists a MADF $\mathcal{M} \subset [\kappa]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$.*

Proof: If $\bigcup \mathcal{P} = \kappa$, we simply apply Lemma 21. If $\bigcup \mathcal{P} \setminus \kappa$ is countable and infinite, let $N = \bigcup \mathcal{P} \setminus \kappa$, and define $\mathcal{P}' = \mathcal{P} \cup \{N\}$. $\mathcal{P}'$ is still a countably infinite family of countably infinite sets, and is still pairwise disjoint, so we apply Lemma 21. If $\bigcup \mathcal{P} \setminus \kappa$ is finite, then Lemma 21 will give us a MADF $\mathcal{M}$ on $\bigcup \mathcal{P} \setminus \kappa$. $\mathcal{M}$ still consists of countably infinite subsets of κ, and is an ADF on κ. It is also maximal because if $C \in [\kappa]^\omega$, then $C \cap (\bigcup \mathcal{P})$ must be infinite, so there is some $M \in \mathcal{M}$ such that $M \cap (C \cap \bigcup \mathcal{P})$ is infinite, and therefore $M \cap C$ is infinite. So $\mathcal{M}$ is a MADF on κ satisfying all the conditions.

Lemma 22 *Let $\mathcal{P} = \{P_i : i \in \omega\}$ be a countable family of countably infinite almost disjoint sets. Then there exists a MADF $\mathcal{M} \subset [\bigcup \mathcal{P}]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$.*

Proof: For each P_i, we can find a finite set B_i such that $\mathcal{N} = \{P_i \setminus B_i : i \in \omega\}$ is pairwise disjoint. To do this, let $B_0 = \emptyset$. Then for $i > 0$, let $B_i = \bigcup_{j<i}(P_i \cap P_j)$. B_i is finite because each $P_i \cap P_j$ is finite for the finitely many $j < i$. By Lemma 21, we can find a MADF $\mathcal{A}$ on $\bigcup \mathcal{P}$ of cardinality $\mathfrak{a}$ containing $\mathcal{N}$. Then define $\mathcal{M} = (\mathcal{A} \setminus \mathcal{N}) \cup \mathcal{P}$. Clearly $\mathcal{M}$ has cardinality $\mathfrak{a}$ as well. We claim $\mathcal{M}$ is a MADF. Suppose $X, Y \in \mathcal{M}$. If both $X, Y \in \mathcal{P}$ or both $X, Y \in \mathcal{A} \setminus \mathcal{N}$ then they must be almost disjoint because both $\mathcal{P}$ and $\mathcal{A}$ are almost disjoint. Otherwise, suppose $X \in \mathcal{A} \setminus \mathcal{N}$ and $Y \in \mathcal{P}$. Then $Y = P_i$ for some $i < \omega$. Then X and $Y \setminus B_i$ are in $\mathcal{A}$, so $X \cap (Y \setminus B_i)$ is finite. Therefore, since B_i is finite, $X \cap Y$ is finite as well, so $\mathcal{M}$ is almost disjoint. For maximal, suppose $X \in [\bigcup \mathcal{P}]^\omega$. We know there is some $A \in \mathcal{A}$ such that $A \cap X$ is infinite. If $A \notin \mathcal{N}$, then $A \in \mathcal{M}$. Otherwise, $A \in \mathcal{N}$, so $A = P_i \setminus B_i$ for some $i < \omega$. Since $A \cap X$ is infinite, then $(A \cup B_i) \cap X = P_i \cap X$ is infinite as well, and $A \cup B_i = P_i \in \mathcal{M}$. Therefore, $\mathcal{M}$ is maximal, so $\mathcal{M}$ is a MADF of cardinality $\mathfrak{a}$ on $\bigcup \mathcal{P}$ containing $\mathcal{P}$.

Corollary 2 *Let $\mathcal{P} = \{P_i : i \in \omega\} \subset [\kappa]^\omega$ be a countable family of almost disjoint infinite sets, where $\kappa < \omega_1$. Then there exists a MADF $\mathcal{M} \subset [\kappa]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$.*

Proof: As in Corollary 1, if $\bigcup \mathcal{P} = \kappa$, we apply Lemma 22 to obtain the desired MADF. If $\bigcup \mathcal{P} \setminus \kappa$ is countable and infinite, let $N = \bigcup \mathcal{P} \setminus \kappa$, and define $\mathcal{P}' = \mathcal{P} \cup \{N\}$. $\mathcal{P}'$ is still a countably infinite family of countably infinite sets, and is still pairwise disjoint, so we apply Lemma 22 and obtain the desired MADF. If $\bigcup \mathcal{P} \setminus \kappa$ is finite, then Lemma 22 will give us a MADF $\mathcal{M}$ on $\bigcup \mathcal{P} \setminus \kappa$. As we showed in Corollary 1, this $\mathcal{M}$ is also a MADF on κ satisfying the desired conditions.

Now we will prove an ω_1 analogue of Dow and Vaughan's Lemma 20. We will restate Theorem 4 here for convenience.

Theorem 4: *Let $\mathcal{P} = \{P_i : i \in \omega_1\} \subset [\omega_1]^\omega$ be an uncountable family of pairwise disjoint infinite sets. Then there exists a MADF $\mathcal{M} \subset [\omega_1]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$.*

Proof: We will construct $\mathcal{M}$ by transfinite recursion. For each $\alpha < \omega_1$ we will construct $X_\alpha \subset \omega_1$ and a MADF $\mathcal{M}_\alpha$ such that:

1. $X_\alpha = \bigcup \{P_i : i < \alpha \cdot \omega\}$
2. $\mathcal{M}_\alpha$ is a MADF on X_α,
3. $\alpha < \beta$ implies $\mathcal{M}_\alpha \subset \mathcal{M}_\beta$
4. $|\mathcal{M}_\alpha| = \mathfrak{a}$
5. $\{P_i : i < \alpha \cdot \omega\} \subset \mathcal{M}_\alpha$.

We may assume that $\bigcup \mathcal{P} = \omega_1$. If $\bigcup \mathcal{P} \neq \omega_1$, we proceed as follows: If $\omega_1 \setminus \bigcup \mathcal{P}$ is countable, then let $N_0 = \omega_1 \setminus \bigcup \mathcal{P}$, and redefine $\mathcal{P}' = \{P_i : i \in \omega_1\} \cup \{N_0\}$.

$\mathcal{P}'$ is still pairwise disjoint, with $\mathcal{P} \subset \mathcal{P}'$, and $|\mathcal{P}'| = |\omega_1|$. Otherwise, if $\omega_1 \setminus \bigcup \mathcal{P}$ is uncountable (note that $|\omega_1 \setminus \bigcup \mathcal{P}| \leq |\omega_1|$), then order the elements x_α of $\omega_1 \setminus \bigcup \mathcal{P}$. Define $N_0 = \{x_\alpha : \alpha \in \omega\}$, and for $\beta \geq 1$, define $N_\beta = \{x_\alpha : \beta\omega \leq \alpha < (\beta+1)\omega\}$. Note that there cannot be more than ω_1 of these sets, so we can redefine $\mathcal{P}' = \{P_i : i \in \omega_1\} \cup \{N_i : i \in \omega_1\}$, and still have $|\mathcal{P}'| = |\omega_1|$, and $\mathcal{P} \subset \mathcal{P}'$. Finally, if $\omega_1 \setminus \bigcup \mathcal{P}$ is finite, we note that a MADF on $\bigcup \mathcal{P}$ will automatically be a MADF with the same properties on ω_1. So we will work with $\mathcal{P}$ assuming $\bigcup \mathcal{P} = \omega_1$.

Let $X_1 = \bigcup \{P_i : i < \omega\}$. By Lemma 21, we can expand $\{P_i : i < \omega\}$ to a MADF $\mathcal{M}_1$ on X_1 with $|\mathcal{M}_1| = \mathfrak{a}$. Now assume X_α and $\mathcal{M}_\alpha$ are created for all $\alpha < \gamma$, $\gamma \in \omega_1$. If $\gamma = \alpha + 1$, then we have X_α and $\mathcal{M}_\alpha$, where $X_\alpha = \bigcup \{P_i : i < \alpha\omega\}$. Let $Y = \bigcup \{P_i : \alpha\omega \leq i < (\alpha+1)\omega\}$ and define $X_{\alpha+1} = X_\alpha \cup Y$. Again by Lemma 21, we can expand $\{P_i : \alpha\omega \leq i < (\alpha+1)\omega\}$ into a MADF $\mathcal{N}$ on Y with $|\mathcal{N}| = \mathfrak{a}$. Let $\mathcal{M}_{\alpha+1} = \mathcal{M}_\alpha \cup \mathcal{N}$. Therefore we have that $\mathcal{M}_\alpha \subset \mathcal{M}_{\alpha+1}$, and $\{P_i : i < (\alpha+1)\,\omega\} \subset \mathcal{M}_{\alpha+1}$ because $\{P_i : i < \alpha\omega\} \subset \mathcal{M}_\alpha$ and $\{P_i : \alpha\omega \leq i < (\alpha+1)\,\omega\} \subset \mathcal{N}$. Also, $|\mathcal{M}_{\alpha+1}| = \mathfrak{a} + \mathfrak{a} = \mathfrak{a}$.

We claim $\mathcal{M}_{\alpha+1}$ is a MADF on $X_{\alpha+1}$. To show that $\mathcal{M}_{\alpha+1}$ is almost disjoint, suppose $M_1, M_2 \in \mathcal{M}_{\alpha+1}$. If $M_1, M_2 \in \mathcal{M}_\alpha$ or $M_1, M_2 \in \mathcal{N}$, then we know $M_1 \cap M_2$ is finite because both $\mathcal{M}_\alpha$ and $\mathcal{N}$ are MADF. Otherwise suppose $M_1 \in \mathcal{M}_\alpha$ and $M_2 \in \mathcal{N}$. Then $M_1 \in [X_\alpha]^\omega$ and $M_2 \in [Y]^\omega$. In particular, $M_1 \subset X_\alpha$ and $M_2 \subset Y$. Since $X_\alpha \cap Y = \emptyset$, $M_1 \cap M_2 = \emptyset$, so their intersection is clearly finite. So $\mathcal{M}_{\alpha+1}$ is almost disjoint. To prove $\mathcal{M}_{\alpha+1}$ is maximal, let $A \in [X_{\alpha+1}]^\omega$. Let $B = A \cap X_\alpha$. If B is infinite, then there is some $M_1 \in \mathcal{M}_\alpha$ such that $M_1 \cap B$ is infinite because $\mathcal{M}_\alpha$ is a MADF on X_α. Therefore, $M_1 \cap A$ is infinite also because $B \subset A$. Otherwise, B is finite, so $C = A \cap Y$ must be infinite. By the same argument, there is some $M_2 \in \mathcal{N}$ such that $M_2 \cap C$ is infinite, so $M_2 \cap A$ is infinite as well. Therefore $\mathcal{M}_{\alpha+1}$ is a MADF on $X_{\alpha+1}$.

Now assume γ is a limit ordinal, with $\gamma < \omega_1$. Pick $\{\alpha_i : i \in \omega\}$ such that $\alpha_i < \alpha_{i+1}$ for $i \in \omega$ and $\gamma = \sup\{\alpha_n : n \in \omega\}$. Let $X_\gamma = \bigcup\{X_{\alpha_i} : i \in \omega\}$. Each X_{α_i} is made by taking the union of countably many P_i, and therefore each X_{α_i} is countable. Also, $\mathcal{M}_{\alpha_i}$ is the (already created) MADF on X_{α_i} with $|\mathcal{M}_{\alpha_i}| = \mathfrak{a}$. Put $\mathcal{A} = \left\{X_{\alpha_{i+1}} \setminus X_{\alpha_i} : i \in \omega\right\} \cup \{X_{\alpha_1}\}$. Denote $X_{\alpha_{i+1}} \setminus X_{\alpha_i}$ by A_{i+1}. Then $\mathcal{A}$ is a pairwise disjoint, countable family of countable subsets of X_γ, so by Lemma 21, we can extend $\mathcal{A}$ to a MADF $\mathcal{N}$ on X_γ where $|\mathcal{N}| = \mathfrak{a}$. Let $\mathcal{B} = \mathcal{N} \setminus \mathcal{A}$. Note that $|\mathcal{B}| = \mathfrak{a}$ because $|\mathcal{N}| = \mathfrak{a}$ and $\mathcal{A}$ is countable. Define $\mathcal{M}_\gamma = \bigcup\{\mathcal{M}_{\alpha_i} : i \in \omega\} \cup \mathcal{B}$.

We claim $\mathcal{M}_\gamma$ is a MADF on X_γ. Suppose $C, D \in M_\gamma$. If both $C, D \in \mathcal{B}$, then they are almost disjoint because $\mathcal{B} \subset \mathcal{N}$, which is a MADF. If $C, D \in \bigcup\{\mathcal{M}_{\alpha_i} : i \in \omega\}$, then $C \in \mathcal{M}_{\alpha_i}$ and $D \in \mathcal{M}_{\alpha_j}$ for some $i, j < \omega$. Let $\alpha = \max\{\alpha_i, \alpha_j\}$. Then $\mathcal{M}_{\alpha_i} \subset \mathcal{M}_\alpha$ and $\mathcal{M}_{\alpha_j} \subset \mathcal{M}_\alpha$. Therefore, $C, D \in \mathcal{M}_\alpha$, so C and D are almost disjoint. Otherwise, say $C \in \mathcal{B}$ and $D \in \mathcal{M}_{\alpha_i}$ for some $i \in \omega$. Then $D \subset X_{\alpha_i} = \bigcup_{j \leq i} A_j$. Since $C \in \mathcal{N}$ and $A_i \in \mathcal{A} \subset \mathcal{N}$, we must have that $C \cap A_i$ is finite for all $i \in \omega$ because $\mathcal{N}$ is a MADF. But because $D \subset X_{\alpha_i}$, this means that $C \cap D$ must be finite, so $\mathcal{M}_\gamma$ is almost disjoint. To show $\mathcal{M}_\gamma$ is maximal, suppose $D \in [X_\gamma]^\omega$. If $D \cap X_{\alpha_i}$ is infinite for any $i \in \omega$, then there is some $M \in \mathcal{M}_{\alpha_i}$ such that $D \cap M$ is infinite. Otherwise, $D \cap X_{\alpha_i}$ is finite for all $i \in \omega$. Then there is some $N \in \mathcal{N}$ such that $D \cap N$ is infinite because $\mathcal{N}$ is a MADF. Since $D \cap X_{\alpha_i}$ is finite for all $i \in \omega$, $N \notin \mathcal{A}$. Therefore $N \in \mathcal{B}$, so $N \in \mathcal{M}_\gamma$. Therefore $\mathcal{M}_\gamma$ is a MADF on X_γ. Clearly if $\alpha < \gamma$, then $\mathcal{M}_\alpha \subset \mathcal{M}_\gamma$. Also, $\{P_i : i < \gamma\omega\} \subset \mathcal{M}_\gamma$ because if $j < \gamma\omega$, we must have that $j < \alpha_i \cdot \omega$ for some $i \in \omega$ since $\gamma = \sup\{\alpha_n : n \in \omega\}$. Therefore $P_j \in \mathcal{M}_{\alpha_i}$ for some $i \in \omega$, so $P_j \in \mathcal{M}_\gamma$. Finally, $|\mathcal{M}_\gamma| = |\omega|\,\mathfrak{a} + \mathfrak{a} = \mathfrak{a}$.

Define $\mathcal{M} = \bigcup_{\alpha \in \omega_1} \mathcal{M}_\alpha$. Certainly $\mathcal{P} \subset \mathcal{M}$, and $|\mathcal{M}| = |\omega_1|\,\mathfrak{a} = \mathfrak{a}$. To show $\mathcal{M}$ is almost disjoint, suppose $C, D \in \bigcup\{\mathcal{M}_\alpha : \alpha < \omega_1\}$. Then $C \in \mathcal{M}_{\alpha_1}$ and $D \in \mathcal{M}_{\alpha_2}$ for some $\alpha_1, \alpha_2 < \omega_1$. Let $\alpha = \max\{\alpha_1, \alpha_2\}$. Then $\mathcal{M}_{\alpha_1} \subset \mathcal{M}_\alpha$ and

$\mathcal{M}_{\alpha_2} \subset \mathcal{M}_\alpha$. Therefore, $C, D \in \mathcal{M}_\alpha$, so C and D are almost disjoint. Suppose $C \in [\omega_1]^\omega$. Let $\alpha = \sup\{i : \text{There exists } x \in C \text{ such that } x \in P_i\}$. Then α is a countable ordinal by [6], Lemma 2.9, because it is a countable union of countable sets. We claim $C \subset X_\alpha$. If $x \in C$, then certainly $x \in P_i$ for some $i < \alpha$. Since $\alpha < \alpha \cdot \omega$, and $X_\alpha = \bigcup\{P_i : i < \alpha \cdot \omega\}$, then $x \in X_\alpha$, so $C \subset X_\alpha$. Therefore there is some $M \in \mathcal{M}_\alpha \subset \mathcal{M}$ such that $M \cap C$ is infinite. Therefore there is a MADF $\mathcal{M}$ on ω_1 such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$.

Corollary 3 *Let $\mathcal{P} = \{P_i : i \in \omega\} \subset [\omega_1]^\omega$ be a countable family of pairwise disjoint infinite sets. Then there exists a MADF $\mathcal{M} \subset [\omega_1]^\omega$ such that $\mathcal{P} \subset \mathcal{M}$ and $|\mathcal{M}| = \mathfrak{a}$.*

Proof: To begin, note that $|\bigcup \mathcal{P}| = |\omega|$, so $|\omega_1 \setminus \bigcup \mathcal{P}| = |\omega_1|$. Put the elements of $\omega_1 \setminus \bigcup \mathcal{P}$ into ω_1 disjoint countably infinite sets. Let $\mathcal{N} = \{N_i : i \in \omega_1\}$, where each $N_i \subset \omega_1 \setminus \bigcup \mathcal{P}$ and is countable, and $\mathcal{N}$ is pairwise disjoint. Let $\mathcal{P}' = \mathcal{P} \cup \mathcal{N}$. Then $\mathcal{P}'$ satisfies the hypotheses of Theorem 4, so it can be extended to a MADF $\mathcal{M}$ of cardinality $\mathfrak{a}$ containing $\mathcal{P}'$. Since $\mathcal{P} \subset \mathcal{P}'$, $\mathcal{M}$ also contains $\mathcal{P}$.

Next, we will prove our second main result, beginning with a key preliminary lemma.

Lemma 23 *If $f : \psi(\kappa, \mathcal{M}) \to [0, 1]$ is a continuous function, then there exists $r \in [0, 1]$ such that $|f^{-1}(r) \cap \mathcal{M}| \geq \mathfrak{a}$.*

Proof: In lemma 19, we have already shown that there exists some $r \in [0, 1]$ such that $f^{-1}(r) \cap \mathcal{M}$ is infinite. For such an r, we can pick distinct $M_i \in \mathcal{M}$ for $i \in \omega$ such that $f(M_i) = r$. Since f is a continuous function and the countable set M_i converges to the point M_i in the ψ-space topology, the set $\{f(x) : x \in M_i\}$ converges to $f(M_i) = r$. Therefore, for each i, all but finitely many elements of M_i are such that $f(x)$ is in the interval $\left(r - \frac{1}{i}, r + \frac{1}{i}\right)$. Define $M_i' = M_i \setminus S_i$ where

$S_i = \left\{x \in M_i : f(x) \notin \left(r - \frac{1}{i}, r + \frac{1}{i}\right)\right\}$. In other words, if $x \in M_i'$, then $|f(x) - r| < \frac{1}{i}$. Note that S_i is finite for all i. Then let $X = \bigcup M_i'$. Since X is a countable union of countable sets, X is countable. Define $\mathcal{M}_X = \{M \cap X : M \in \mathcal{M} \text{ and } |M \cap X| = \omega\}$.

We claim $\mathcal{M}_X$ is a MADF on X. First, note that $\mathcal{M}_X$ is infinite because for each i, $(M_i \cap X) \in \mathcal{M}_X$ because $(M_i \cap X)$ is infinite. Suppose $A, B \in \mathcal{M}_X$. Then $A = M_1 \cap X$ and $B = M_2 \cap X$ for some $M_1, M_2 \in \mathcal{M}$. Since $A \subset M_1$ and $B \subset M_2$ and $M_1 \cap M_2$ is finite, we must have that $A \cap B$ is finite as well, so $\mathcal{M}_X$ is an almost disjoint family. To show $\mathcal{M}_X$ is maximal, suppose $A \in [X]^\omega$. Then $A \in [\kappa]^\omega$ as well, so there is some $M \in \mathcal{M}$ such that $M \cap A$ is infinite. Since $A \subset X$, this means that $M \cap X$ is infinite as well, so $M \in \mathcal{M}_X$. Therefore $\mathcal{M}_X$ is a MADF on X.

Next, we claim that for each $M \in \mathcal{M}$ such that $M \cap X \in \mathcal{M}_X$, $f(M) = r$. Suppose we have $M \in \mathcal{M}$ such that $M \cap X \in \mathcal{M}_X$. If $M = M_i$ for some $i \in \omega$, then $f(M_i) = r$ by our choice of M_i. Otherwise, $M \notin \{M_i : i \in \omega\}$, but $M \cap X$ is infinite, so M intersects infinitely many M_i' in a finite set. Fix i, so $\left(r - \frac{1}{i}, r + \frac{1}{i}\right)$ is a neighborhood of r. For $k \geq i$, if $x \in M_k'$, then $f(x) \in \left(r - \frac{1}{k}, r + \frac{1}{k}\right) \subset \left(r - \frac{1}{i}, r + \frac{1}{i}\right)$. But there are only finitely many $x \in M \cap X$ such that $x \in M_k$ for $k < i$, so therefore all but finitely many $x \in M \cap X$ are such that $f(x) \in \left(r - \frac{1}{i}, r + \frac{1}{i}\right)$. Thus for each i, $\left(r - \frac{1}{i}, r + \frac{1}{i}\right)$ contains $f(x)$ for all but finitely many $x \in M \cap X$, so $\{f(x) : x \in M \cap X\}$ converges to r. But since $M \cap X$ is infinite, we have an infinite subsequence $\{x : x \in M \cap X\}$ of M such that $\{f(x) : x \in M \cap X\}$ converges to r. Since this subsequence converges to M in $\psi(\kappa, \mathcal{M})$, and the set M converges to the point M, and f is continuous, we must have that $f(M) = r$.

Since $\mathcal{M}_X$ is a MADF on X, $\mathfrak{a} \leq |\mathcal{M}_X|$. Let $g : \mathcal{M}_X \to f^{-1}(r) \cap \mathcal{M}$ be given by $g(M \cap X) = M$. First we will show g is a function. Suppose $A \in \mathcal{M}_X$ and $g(A) = M_1$ and $g(A) = M_2$ for distinct $M_1, M_2 \in \mathcal{M}$. Then $A = M_1 \cap X = M_2 \cap X$. But then $A \subset M_1 \cap M_2$, and since A is infinite, this means that $M_1 \cap M_2$ is infinite.

Since $\mathcal{M}$ is a MADF, this implies that $M_1 = M_2$, and thus g is a function. Next we will show g is one to one. Suppose $g(N_1) = g(N_2)$ for $N_1, N_2 \in \mathcal{M}_X$. Then $N_1 = M_1 \cap X$ and $N_2 = M_2 \cap X$ for $M_1, M_2 \in \mathcal{M}$. Then $g(N_1) = M_1$ and $g(N_2) = M_2$, so $M_1 = M_2$. But then $M_1 \cap X = M_2 \cap X$, so $N_1 = N_2$, and g is one to one. Therefore we have $\mathfrak{a} \leq |\mathcal{M}_X| \leq |f^{-1}(r) \cap \mathcal{M}|$, so $\mathfrak{a} \leq |f^{-1}(r) \cap \mathcal{M}|$.

Proof of theorem 5 (i.e., (*) implies (1)): Suppose there exists a MADF $\mathcal{M} \subset [\omega_1]^\omega$, $|\mathcal{M}| = \mathfrak{c}$ and there exists a continuous function $f : \psi \to [0,1]$ such that for every $r \in [0,1]$, $|f^{-1}(r)| < \mathfrak{c} = |\mathcal{M}|$. We have shown in Lemma 23 that there is some $r \in [0,1]$ such that $\mathfrak{a} \leq |f^{-1}(r) \cap \mathcal{M}|$. But since $|f^{-1}(r)| < \mathfrak{c}$ for all $r \in [0,1]$, we have $\mathfrak{a} \leq |f^{-1}(r) \cap \mathcal{M}| <$, so $\mathfrak{a} < \mathfrak{c}$.

As a side note, Lemma 23 allows us to show that condition (*) implies $\mathfrak{a} < \mathfrak{c}$ for $\psi(\kappa, \mathcal{M})$ for any infinite ordinal κ. However, we cannot yet show that $\mathfrak{a} < \mathfrak{c}$ implies condition (*) for any cardinal beyond ω.

Whether $\mathfrak{a} < \mathfrak{c}$ implies that there exists a MADF $\mathcal{M} \subset [\omega_1]^\omega$ satisfying the desired conditions is still an open question. We have been able to prove all the relevant lemmas except the following, which used in Dow and Vaughan's construction of such a MADF and continuous function f.

Conjecture 1 *Let Q be a subset of $[0,1]$ such that Q contains the rational numbers in $[0,1]$ and $|Q| = \omega_1$. Then for each $r \in [0,1]$, there is an ADF $\mathcal{M}_r \subset [Q]^\omega$ such that $|\mathcal{M}_r| = \mathfrak{a}$, every $M \in \mathcal{M}_r$ converges to r, and and $\mathcal{M}_r$ satisfies the following maximality condition: for every $C \in [Q]^\omega$, if C converges to r then there exists $M \in \mathcal{M}_r$ such that $|M \cap C| = \omega$.*

If this conjecture could be proven, then we could use the $\mathcal{M}_r$ to create a MADF on ω_1 and continuous function f satisfying condition (*), thus giving the equivalence of condition 1 and condition (*). For now, it remains an open problem.

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