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Recent curricular developments suggest that students at all levels need to be statistically literate and able to efficiently and accurately make probabilistic decisions. Furthermore, statistical literacy is a requirement to being a well-informed citizen of society. Research also recognizes that the ability to reason probabilistically is supported and enabled by other forms of mathematical reasoning and concepts. One of these supporting concepts is sample space, the enumeration of all outcomes of a probability experiment. As a concept, sample space supports the construction of probability distributions, which in turn enables statistical inference, a form of probabilistic reasoning.

This mixed methods study investigated how undergraduate pre-service elementary teachers construct and generalize their understanding about sample space. One hundred fifty students participated in a series of three tasks designed to investigate the ways in which they enumerate sample space and the associations between their enumeration strategies and their generalization rules. A subset of eight participants engaged in follow-up interviews designed to explore their understandings of sample space enumeration and generalization. Findings from the study suggest that there was growth across tasks in the sophistication of the enumeration strategies used and that participants attempted to find explicit and formalized generalizations. However, in spite of this growth in the sophistication of enumeration, there was little association between the enumeration strategies participants used and the generalizations that they constructed. Students

compartmentalized their understanding of generalization rules, often looking for a numeric formula that had little do to with their enumerated solutions.

CONSTRUCTING SAMPLE SPACE WITH
COMBINATORIAL REASONING:
A MIXED METHODS STUDY

by

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For my loving wife Chrissy who has supported me through this long process as well my parents who have both been inspirational, supportive, and acted as role models and to my other family and friends who have encouraged me throughout

APPROVAL PAGE

This dissertation has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I

THE NEED FOR STATISTICS EDUCATION

Recommendations by National Council of Teachers of Mathematics (1989, 2000) indicate the importance of all students developing an awareness of probabilistic concepts and their applications within mathematics. Until the publication of the *Curriculum and Evaluation Standards for School Mathematics* (NCTM, 1989), the position of probability within the curriculum was at best uncertain (Watson & Kelly, 2007). With the introduction of this landmark document, probability gained its own standards for grades 5-8 and 9-12 and a shared standard with statistics for grades K-4. These standards suggest that students learn to conduct probabilistic experiments and develop theoretical ideas about concepts such as sample space and distribution. This emphasis continued with the publication of the *Principles and Standards for School Mathematics* (2000), where NCTM recommends that the probability and statistics standards from its earlier document be strengthened and that concepts and procedures become increasingly sophisticated across grades levels. As a result, by the end of high school, students will possess a sound understanding of elementary statistics. However, this emphasis on and interest in probabilistic and statistical understanding is not limited to elementary, middle-grades, and secondary educational levels. Colleges and universities are issuing their own calls for probabilistic and statistical knowledge for undergraduate students (CBMS, 2010).

This collective call for probabilistic knowledge at all levels of the educational enterprise exists for a variety of reasons. One of these, as suggested by Franklin et al. (2007) in their *Guidelines for Assessment and Instruction in Statistics Education Report* [GAISE], is that an outcome of a well-rounded education in mathematics and statistics is that every high-school graduate be able to employ sound statistical and probabilistic reasoning. This supports the requirements of citizenship, employment, and their family. Probability and statistics provide powerful links between mathematics and other educational areas where mathematical concepts can be utilized to reason about real world situations as students study subjects such as science, geography, and history (Council of Chief State School Officers, 2010; Franklin et al., 2007; Konold & Higgins, 2003).

However, this increased emphasis on statistical and probabilistic proficiency stands in stark contrast to the placement of statistics and probability in recent curricular developments. The placement of probability and statistics is troubling for several reasons. First, it seems to ignore the fact that students are capable of sophisticated probabilistic thought (Fischbein, 1975; Jones, Langrall, Thornton, & Mogill, 1997, 1999; Shaughnessy, 2003). Second, it disregards much of the reasoning behind the push for probabilistic and statistical literacy. Students, as well as and people in general, live in a complex and technological world that requires probabilistic thinking and understanding (Franklin et al., 2007; Shaughnessy, 2006). By ignoring or disregarding the experiences that students bring to school, educators lose a potential source of interesting and motivating tasks and problems (Donovan & Bransford, 2005). Even though some researchers (Jones et al., 1999; Nilsson, 2007, 2009; Pratt, 2000; Shaughnessy &

Ciancetta, 2002; Watson & Kelly, 2007) have conducted research in the area of probabilistic and statistical learning, much still remains to be discovered about this topic (Jones, Langrall, & Mooney, 2007; Shaughnessy, 2003, 2007).

It is essential that everyone has access to knowledge that will enable them to make reasoned decisions when presented on a daily basis with information from newspapers, medical and consumer reports, and a plethora of other informational sources (Franklin et al., 2007; Shaughnessy, 2003). Shaughnessy (2003) called this statistical literacy and argued that most graduating secondary students are not knowledgeable enough in these areas to be considered adept in decision-making that involve data and chance. Davis and Hersh (1991) made an argument for a worldview and an epistemology based almost wholly on uncertainty. They suggested that bottom line probabilities, or hunches, which students possess about the likelihood of events, will always be present. This observation resonates with the findings of other researchers who have documented probabilistic difficulties in mathematics education and cognitive psychology. Examples of these are the representativeness heuristic, availability heuristic, (Konold, Pollatsek, Well, Lohmeier, & Lipson, 1993; Shaughnessy, 1992; Tversky & Kahneman, 1973), and the outcome approach (Konold, 1989, 1991). Each of the probabilistic approaches, mentioned above, is subjective and is based upon the hunches students possess. Tversky and Kahneman's (1973) availability heuristic dealt with obtaining probability based on the ease with which students were able to bring a particular instance of an event to mind. The outcome approach is one where each trial of a random event is thought of as an individual phenomenon as opposed to repeated trials of the same process. However, teachers can use these misconceptions to encourage students to investigate their intuitions both formally and informally (Shaughnessy, 2003).

Sample Space: An Important Issue

Students have various ways of investigating data in an informal manner. Petrosino, Lehrer, and Schauble (2003) found that fourth grade students were able to construct sophisticated comparison techniques for distributions that do not involve formal inference and when encouraged to use their intuitions, they could make sense out of data. To enable students to engage in statistical inference, they first needed to understand the concept of distribution, defined as associating a probability with all the possible values of a random variable (Petrosino et al., 2003). However, distribution is a complex concept that required coordinating probability and more deterministic topics such as sample space (Jones et al., 2007).

Piaget and Inhelder (1951) suggested that the probabilistic thought required for understanding distribution is necessarily preceded by other concepts such as combinatorics. Thus, construction and understanding of sample space is a precursor to probabilistic thinking, and investigating the process of sample space construction is a starting point for understanding the probabilistic thinking that will follow it. Sample space here is defined as the enumeration of all possible outcomes within a probability experiment, or expressed another way, it is all values that a random variable can assume. For example, in a simple random experiment, such as tossing a six-sided die, the sample space is: 1, 2, 3, 4, 5, and 6, which are all possible values of rolling a six-sided die. In a compound event, for example tossing a coin twice, the resulting sample space would be

{(H, H), (H, T), (T, T), and (T, H)}. After the construction of sample space, probabilistic connections are made between the values of the sample space and their associated probabilities, which together form a probability distribution and enable probabilistic decision-making (Jones et al., 1999). However, many researchers have demonstrated that students do not consider the sample space when making probabilistic judgments (Ayres & Way, 2000; Fischbein & Schnarch, 1997; Shaughnessy & Ciancetta, 2002). Therefore, an understanding of how students construct sample space may serve as a starting point to better help them gain proficiency with probabilistic reasoning. This understanding of sample space construction will also assist researchers and will give them a place to start in understanding how students gain an understanding of probability. Sample space construction is the first step in making probabilistic decisions since it is required for an understanding of distribution but is not necessarily a probabilistic concept.

Determining the sample space requires systematically and exhaustively generating all possible outcomes of a probability experiment or random variable. However, this process of constructing sample space requires other understandings such as how to systematically enumerate (Piaget & Inhelder, 1951), and it is known that enumeration for even a simple random experiment is a nontrivial task for students (Jones et al., 1999). Jones et al. (1999), in their investigation of students’ ability to identify all possible outcomes of a probability experiment, found that students either stated a single outcome as the answer or eliminated outcomes that had occurred on prior trials. Enumeration of the outcomes of compound events is a more complex undertaking for students than enumeration of simple events (Jones et al., 2007). Much of this complexity in constructing the sample space stems from the dearth of experience students have with combinatorial reasoning (Batanero, Navarro-Pelayo, & Godino, 1997; Fischbein & Grossman, 1997), which is a topic not well represented in the curriculum of school mathematics (English, 2005). Thus, exploring how combinatorial reasoning is used in sample construction will aid in understanding how students begin to construct probabilistic reasoning.

Constructing Sample Space with Combinatorics

Combinatorics is a branch of mathematics concerned with the study of finite or countable discrete structures. It plays a part in sample space construction because it is an approach students can utilize for enumeration of the outcomes of a probabilistic situation (Jones et al., 2007). Piaget and Inhelder (1951/1975) first studied the development of combinatorial reasoning. They suggested that students follow a developmental sequence in their combinatorial thinking, dividing combinatorial reasoning into combinations, permutations, and arrangements. They defined a combination is a way of selecting several things out of a larger group where order does not matter and repetition is not allowed. According to Piaget and Inhelder (1951/1975), a permutation was a way of selecting several things from a group where order does matter and repetition is not allowed whereas arrangements are simply combinations and permutations where repetition is allowed. An example of a combination is selecting two fruits from a total of three fruits, such as an apple, orange, and banana. How many different combinations of two fruits can be selected? Three combinations of two can be drawn from this set if repetition is not allowed: an apple and a banana, an apple and an orange, and a banana and an orange. However if repetition were allowed, which Piaget and Inhelder referred to as an arrangement, the original combinations could each produce one additional combination simply by reversing which fruit was chosen first. For example, the apple and banana combination would produce an apple and a banana or a banana and an apple. See Table 1 for a graphical representation of combinations.

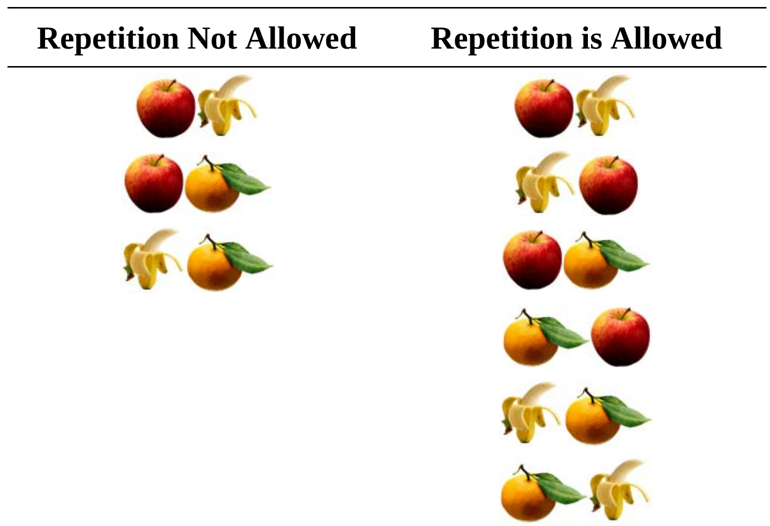


Figure 1. Combinations

Unlike combinations, permutations take order into account where repetition is not allowed. For example, in the selection of a two-digit number where repetition of numbers is allowed, 10 digits can occupy both the tens place and the ones place. Thus, 100 possible two-digit numbers are possible. On the other hand, when repetition is not allowed there would be 10 digits for the tens place but only 9 for the ones place since one digit has already been selected for use in the tens place. Thus, only 10*9 or 90 two-digit numbers are possible if repetition is not allowed. Piaget and Inhelder called a situation an arrangement.

Piaget and Inhelder (1951/1975) found that permutations were more difficult for students than combinations. That is, students found combinations less difficult than permutations. However, Fischbein (1975) established that systematic instruction could assist students in understanding permutations. Researchers have also shown that students are able to generate creative strategies for combinatorics problems (English, 1991, 1993; Maher, Sran, & Yankelewitz, 2010a, 2010b; Maher & Yankelewitz, 2010; Tarlow, 2010; Tarlow & Uptegrove, 2010).

Sample Space Construction: A Deterministic Endeavor

Piaget and Inhelder (1951) argued that combinatorial reasoning preceded probabilistic reasoning, while Fischbein (1975) raised an important objection to, and caveat surrounding, probabilistic thinking. He suggested that students were capable of grasping the notion of chance at stages earlier in the developmental cycle than suggested by Piaget and Inhelder. However, this does not necessarily mean students are capable of formal statements and calculations of odds or probabilities. Instead, with systematic instruction and the use of organizational techniques, such as tree diagrams, students are capable of intuitive estimations of combinations and permutations (Fischbein, Pampu, & Manzat, 1970). This means that students are capable of the construction of sample space since it often makes use of combinatorial knowledge of combinations and permutations (Jones et al., 2007).

At a practical level, instruction in probability can often be limited by the lack of more deterministic skills and reasoning abilities. Reasoning that is deterministic occurs when there is no inherent uncertainty in a context, situation, or problem (Stohl, 2005).

Probability by its very definition deals with uncertainty. Ritson (1998) argued that the understanding of fractions, a deterministic concept, also contributes to probabilistic reasoning. However, if there is a lack of familiarity or understanding of fractions, then this lack tended to limit probabilistic reasoning. Understanding fractions does not involve uncertainty and is therefore deterministic. As argued in preceding sections, the ability to make informed probabilistic judgments is impossible without the ability to construct and manipulate sample space, which is an understanding that involves the use of combinatorics. Combinatorics, like fractions, is a deterministic understanding that does not involve uncertainty (Piaget & Inhelder, 1951). This study, similar to that of Ritson (1998), argues that a naïve approach to combinatorics and construction of sample space limits probabilistic understanding and probabilistic decision-making.

Research Problem and Questions

This study occupies a unique place within the body of literature since it investigated the precursors that support probabilistic reasoning by examining how undergraduates use combinatorial reasoning to construct sample space (Jones et al., 2007, 1997, 1999; Shaughnessy, 2003). Combinatorial reasoning precedes probabilistic reasoning, as argued earlier. However, probabilistic reasoning is difficult for elementary, middle-grade, and secondary students to develop (Shaughnessy, 2007). Even for teachers probabilistic reasoning continues to be an issue (Jones et al., 2007). Thus, one place to begin investigating this difficulty is with undergraduate students who are future teachers and investigate how they construct sample space with combinatorial reasoning. This study was concerned with the following research questions:

1. What combinatorial strategies do undergraduate students use to construct the sample space?
2. When undergraduate students enumerate the sample space, what generalizations do they generate?
3. What are the associations between the generalization rules and enumeration strategies?

CHAPTER II

REVIEW OF LITERATURE AND FRAMEWORKS

It is important for students to understand the concept of sample space (Jones, Langrall, & Mooney, 2007; Shaughnessy, 2003), and more specifically, how to generate it since sample space as a concept is linked to the statistical concept of distribution (Ayres & Way, 2000; Jones, Langrall, Thornton, & Mogill, 1999; Jones et al., 2007; Shaughnessy, 2003). However, because sample space is not a probabilistic concept but is none the less required for probabilistic decision making, it is necessarily preceded by deterministic understandings such as knowing how to systematically enumerate possible values of a random variable. This amounts to the construction of sample space (Franklin et al., 2007; NCTM, 2000). The concept of sample space is a complex and nuanced topic for students to understand fully. To capture this complexity and nuance, this chapter will be divided into several sections with each describing a different facet of the understanding of sample space.

To review the literature on sample space, this chapter will utilize the various understandings required, as outlined by Horvath and Lehrer (1998) (see Figure 2). Horvath and Lehrer (1998) noted that the understanding and use of sample space required that students: (1) Recognize different ways of obtaining an outcome; (2) Know how to map probabilities onto the distribution of outcomes; (3) Enumerate all possible outcomes systematically and completely. These three aspects of understanding sample space will serve to organize the first portion of this chapter. For a full understanding of sample space construction, all three aspects will be discussed. Finally, the chapter concludes with a discussion of the frameworks that will be used to investigate how students systematically enumerate sample space outcomes, and how they generalize their combinatorial solutions for those systematic enumerations.

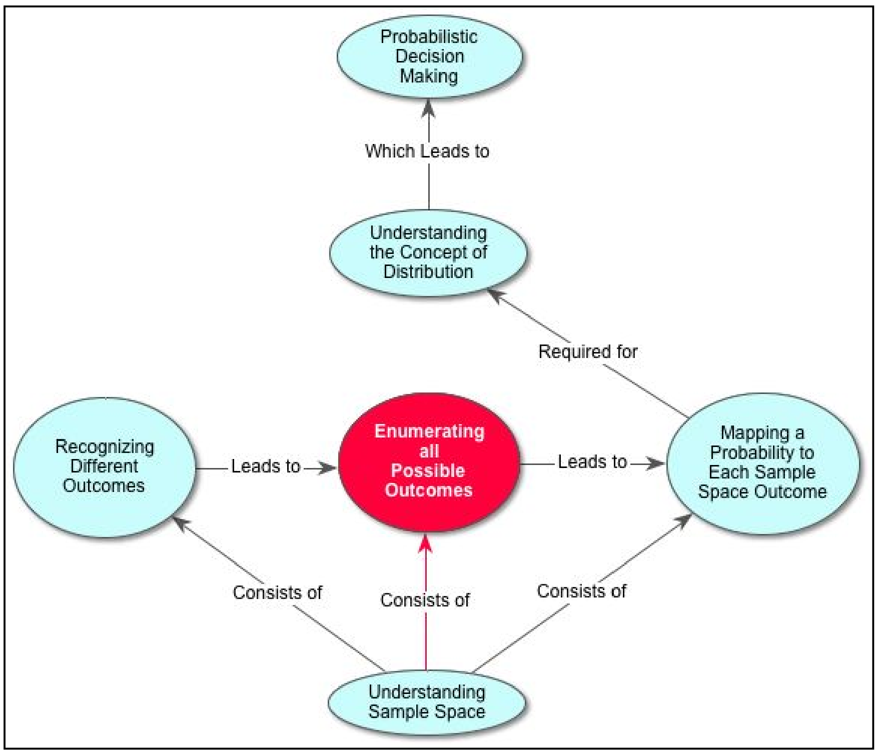


Figure 2. Development in the Understanding of Sample Space

Recognizing Different Outcomes

The concept of sample space is fundamental to the process of mathematically understanding random events and phenomena (Jones et al., 2007). It serves as a scheme for describing the outcomes of a random event, which are characterized by having multiple outcomes, and the sample space is defined as the set of all possible outcomes. Shaughnessy (2003) called attention to the fact that students have trouble with the concept of sample space and recognized that it is valuable for students to engage in constructing the sample space. Furthermore, he indicated that students were generally unsophisticated in their understanding of this concept based on the National Assessment of Educational Progress [NAEP] data, but he did not explain why they had difficulty with it. Horvath and Lehrer (1998) contended that the first step in students' understanding sample space is the recognition of what a possible outcome is and how it is different from or similar to other possible outcomes.

However, identifying all possible outcomes from even a simple random experiment, such as tossing a fair six-sided die once, is not an easily accomplished task for students (Jones et al., 1999; Nilsson, 2007; Watson & Kelly, 2007). Jones et al. (1999) examined 37 eight to year old children because they were interested in examining how these students were able to construct and identify sample space with no prior instruction. They found that 15 of the 37 students were not able to identify and label all the outcomes of a random event. These 15 students gave one of two explanations. Some, for instance, stated that the outcome of rolling a die might be just '2' instead of all numbers 1, 2, 3, 4, 5, or 6, or once a '2' was rolled they thought it would not be rolled again. Others, however, eliminated some outcomes because they had occurred on previous trials. For example, when students rolled a six-sided die, they excluded the value of 4 in the sample space if they had seen a 4 on a previous trial. The researchers referred to this as the *sample space misconception* and concluded that students in their study seemed to approach the construction of sample space in a predictive and deterministic manner, inferring certainty to situations where uncertainty existed. Furthermore, they indicated this deterministic thinking persisted even after students were exposed to extensive experiences with varied random generators. This suggests that although sample space construction is required for probabilistic thought, it is not itself probabilistic in nature.

Other researchers (e. g. Borovcnik & Bentz, 1991; Jones et al., 1997, 1999, 2007) have indicated that sample space is an important topic but have given little attention to explaining the mechanisms that students use in its construction (Jones et al., 2007). Sample space construction requires the systematic enumeration of all possible outcomes of a probability event and the mapping of a probability to each of those elements. However, this enumeration does not involve probabilistic reasoning but instead requires students to reason in a more deterministic manner. Piaget and Inhelder (1951) argued that certain types of reasoning preceded the ability to reason in a probabilistic manner.

Mapping the Sample Space to the Distribution of Outcomes

Another facet of understanding sample space construction and how it is used pertains to how probabilities are mapped to each element of the sample space. Mapping a probability value to a sample space element is one step that is required for the construction of an understanding of distribution (see Figure 2). It is well known that many students do not take the sample space into account when they examine outcome frequencies and determine probabilities (Ayres & Way, 2000; Fischbein & Schnarch, 1997; Shaughnessy & Ciancetta, 2002). Yet, examining outcome frequencies and assigning probabilities are both important actions when defining a probability distribution, which itself is important in probabilistic decision-making (Jones et al., 2007).

The work of Fischbein, Nello, and Marino (1991) illustrated this tendency of students not considering the sample space of an experiment when assigning probabilities or making decisions. The researchers examined the ideas and understandings that students possess about compound events involving rolling a six-sided die and flipping a fair coin twice. Students were first asked a context specific question, where they had to compare the likelihood of rolling a 5 with one die and a 6 with the other or rolling a 6 with both dice. Most students in Fischbein et al.'s (1991) study stated that they thought the likelihood of these two compound events were equal. They justified this statement by saying either that the situation involved chance and therefore the events were equally likely or that each toss was independent, meaning that the outcomes were equally likely. The first response demonstrated Konold's (1991) outcome approach where students do not see a single outcome as just one possible outcome that will vary across the sample space if the experiment were conducted numerous times. The second justification, which was more sophisticated, seemed to indicate that students had received some instruction in probability and had been influenced by it in a beneficial way. However, both responses showed that students do not consider the sample space of the experiment when assigning probabilities to outcome events.

Horvath and Lehrer (1998) investigated what effect giving students ‘notational assistance’ had on their understanding of the relationship between sample space and distribution. By notational assistance, the researchers meant some way of helping students see or understand what was going on, specifically using bar graphs. Students in their study were asked to make predictions with 6, 8, and 12 sided dice for most and least favorable outcomes based upon experimentation and data collection. They found that most students did not relate the sample space to their experimental data, especially when they were asked to make predictions based upon the rolls of two dice. However, when ‘notational assistance’ in the form of bar graphs was introduced, students were able to identify some of the relationships between the sample space of the probability experiment in question and their experimental data. For example, when they rolled two 6 sided dice, students were able to recognize that rolling a 3 was more likely than rolling a 2 since each roll was made up of a sum of the numbers on each die and that there are two ways of getting a sum for 3 and only one way of getting a sum for 2. Yet, when their experimentation offered a counter example to this, for instance rolling more 2’s than 3’s in a set number of trials, they changed their minds instead of relying on their knowledge of the sample space. Students’ preference for experimental data was much more pronounced when the context of the problem was changed and this notational assistance was removed. Horvath and Lehrer (1998) found similar results in older students, but when their notational assistance was removed, these older students were much more likely to incorporate sample space into their thinking.

Shaughnessy and Ciancetta (2002) also examined students’ understanding of sample space in an experimental and more general context. In order to obtain information about this, they used the task in Figure 3, drawn from the 1996 NAEP test, and gave it to

652 mathematics students in grades 6-12 from five different schools. They found that: (a) approximately 20% of middle grades students (grades 6-8) answered the question correctly; (b) 43% of students in grade 9, enrolled in an Interactive Mathematics Curricula class, answered the question correctly; (c) only 34% of students in grades 9-12, enrolled in an introductory mathematics class, were able to answer the question successfully; and (d) 90% of students in grade 12, enrolled in an advanced mathematics class, such as AP Calculus or AP Statistics, were able to answer the question correctly.

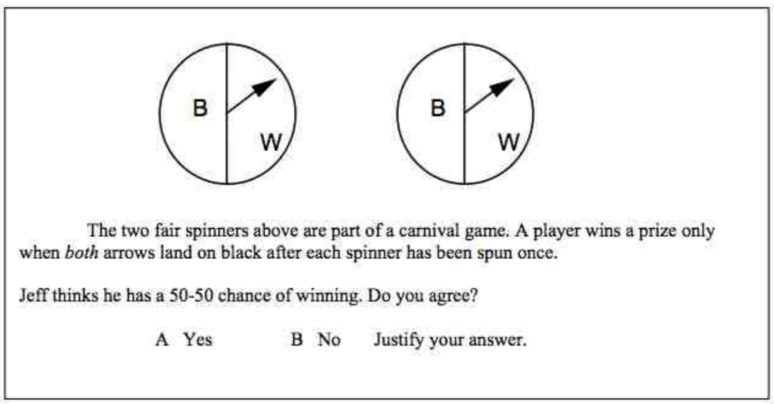


Figure 3. Spinner Task from 1996 NAEP (Shaughnessy & Ciancetta, 2002)

Shaughnessy and Ciancetta (2002) noted that a high percentage of the students they studied did not consider the sample space of the probability experiment when giving their response. To test if playing the game with spinners would change students' predictions, they selected 28 students in grades 8-12 to interview. The interviewees were first asked who they thought the winner would be. They were then asked to play the game 10 times, collect data and afterward rethink their prior predictions. Before playing the game, only 13 of 28 students interviewed responded correctly. However, only five of these students gave a satisfactory explanation of their answer, and only 4 of those 5 listed the sample space while the other used a multiplication principle to explain why the probability of winning was one fourth. After gathering data, 11 students revised their predictions and answered correctly, with 8 of them listing the sample space in their explanation. This study, along with that of Fischbein et al. (1991), illustrated the power that experimentation had in helping students understand and see the need for the construction of sample space.

Other researchers have utilized tasks such as dice games to study sample space (e.g. Nilsson, 2007; Pratt, 2000; Speiser & Walter, 1998; Vidakovic, Berenson, & Brandsma, 1998; Watson & Kelly, 2007). Speiser and Walter (1998) studied how five pre-service teachers informally constructed sample space after playing dice games. The first dice game had player 1 receive a point when rolling a 1, 2, 3, or 4, and player 2 win a point when rolling a 5 or 6. The second game involved rolling two dice where player 1 receives a point for rolling a 2, 3, 4, 10, 11, or 12 and player 2 receives a point for rolling a 5, 6, 7, 8, or 9. The pre-service teachers concluded that the result of a hypothetical game involving dice that were distinguishable was required in order to enable them to determine the sample space for the second dice game. The researchers noted that the sample space of the game involving indistinguishable dice had a sample space that contained only 21 outcomes since a result of (1,2) and (2,1) was considered to be the same, and that the game involving distinguishable dice had a sample space that consisted of 36 outcomes since (1,2) and (2,1) were now recognizably different.

Polaki (2002) illustrated this same tendency in another cultural setting where his teaching experiment involved 4th and 5th grade students in Lesotho, South Africa. He assigned his students to two instructional groups. The first focused on a small set of experimental data drawn from repeatedly playing a game, while the second focused on computer simulation of the same game run over larger sample sizes. Each group was asked to make predications and begin trying to assign probabilities to the sample space of the game. Polaki (2002) found that both groups were better able to connect sample space and probability at the end of his teaching experiment. However, interestingly, no statistical difference existed between the two groups.

Mapping the sample space onto the distribution of outcomes is the last step in understanding sample space (Horvath & Lehrer, 1998). However, the purpose of this study was not to investigate probabilistic reasoning, which this mapping would entail. Mapping the sample space onto the distribution of outcomes involves using the understanding of the construction of sample space to associate a probability with each sample space element. Instead, this study focused on the combinatorial reasoning involved in the systematic enumeration of sample space. Piaget and Inhelder (1951) argued that this combinatorial reasoning ability directly precedes and supports probabilistic understanding.

Systematic Enumeration

Recognizing possible outcomes is a first step in constructing sample space whereas mapping a probability value to each element of the sample space, as discussed in the previous section, is the last step in constructing sample space (Horvath & Lehrer, 1998). What constitutes the heart of this process is a way to enumerate all possible outcomes completely and systematically (see Figure 2). The process of generating all possible outcomes for a task such as tossing two fair dice or a fair coin multiple times is difficult for students (Jones et al., 2007). Part of this difficulty is due to students' underdeveloped combinatorial reasoning (Batanero, Navarro-Pelayo, & Godino, 1997; Fischbein & Grossman, 1997). This section will first discuss how to use combinatorics to frame systematic enumeration, which is then followed by defining combinatorics in the field of mathematics and discussing the historic research of Piaget and Inhelder about combinatorial reasoning. Attention will also be given to how combinatorics has been examined in mathematics education literature up to this point.

Combinatorics as a Way to Frame Systematic Enumeration

Combinatorics is a branch of discrete mathematics that explores the relationships among discrete structures. It includes, but is not limited to, graph theory, coding theory, discrete optimization, and counting techniques. Discrete mathematics is also an important subject in other disciplines such as chemistry, computer science, and physics (Eizenberg & Zaslavsky, 2004; Kavousian, 2008). Combinatorics, as one of the major components of discrete mathematics, has been included in curricula in schools across the nation. It has many applications in both business and industry, but even more importantly it offers unique opportunities for students to learn how to solve creative, challenging, and interesting problems with little prior mathematical instruction (Kavousian, 2008; Lockwood, 2010; Maher & Martino, 1996).

In spite of these positive characteristics, combinatorics is still considered one of the more difficult mathematical topics to teach and learn because disparate solutions to the same problem often may seem equally convincing (Batanero et al., 1997; Eizenberg & Zaslavsky, 2004; English, 1991; Fischbein & Gazit, 1988; Hadar & Hadass, 1981; Tversky & Kahneman, 1973). Researchers suggest that the use of combinatorial problems in mathematics instruction can assist students in the process of learning to reason and generalize (Maher & Martino, 1996; Martino & Maher, 1999). One important aspect of combinatorics is its link to probability, as first postulated by Piaget and Inhelder (1951).

Definition of combinatorics. Combinatorics, sometimes called the science of counting, has two foundational principles, the addition and multiplication principle (Tucker, 1980). The addition principle states that if there are *a* possible outcomes for an event and *b* possible outcomes for another event, then the total possible outcomes are $a + b$ for the events, provided the two events cannot both occur simultaneously (see Table 1).

Table 1

The Addition and Multiplication Principle

Addition Principle		
Number of Outcomes of Event A	Number of Outcomes of Event B	Total Outcomes
a	b	a + b
Multiplication Principle		
Number of ways to Perform Event A	Number of Ways to Perform Event B	Total ways to perform both
a	b	a*b

More formally stated, the sum of the sizes of two disjoint sets is equal to the size of their union. The multiplication principle is a basic counting principle that states if there are a outcomes in event A and b outcomes in event B, then there are $a \cdot b$ total outcomes of both events happening together.

More advanced forms of combinatorics are concerned with the selection and arrangement of elements within sets. Combinatorial theory can further be defined by dividing it into *enumeration*, *existence*, and *structure* problems. Enumeration problems examine how many arrangements are possible out of a given number of objects. For example, counting the number of different passwords of length five that can be made using ten numerals. Problems that investigate combinatorial *existence* search for how many particular arrangements of a certain type exist. For example, using the five cards 2, 3, 4, 5, and 6, how many arrangements of two cards have a 3 and 4 in them? A combinatorial problem that investigates *structure* has the student seek to find the optimum combinatorial solution (Hall, 1998; Kavousian, 2008). For example, in the famous knapsack problem, where a student is given a set of items, each with a weight and a value, they then have to determine the number of each item to include in a collection so that the total weight is less than or equal to a given limit and the total value is as large as possible.

Enumeration was the focus of this study because it is the most mathematically appropriate and commonly used part of combinatorial theory in the early combinatorics curriculum. Thus, obtaining a better understanding of how students think about combinatorial situations, especially those that lead to construction of sample space, will enable educators and mathematics educators to have a better understanding of how students are reasoning about those concepts that support probabilistic thinking. In turn, this may lead to different and potentially more successful curricular developments in the support of instruction on probabilistic decision-making. Thus, this study explored enumeration as a way to frame and organize the construction of sample space.

Piaget and Inhelder's research about combinatorics. In their book *The Origin of the Idea of Chance in Children,* Piaget and Inhelder (1951) first investigated how ideas of probability and chance developed. They stated "the formation of the ideas of chance and probability depend in a very strict manner on the evolution of combinatoric operations" (Piaget & Inhelder, 1951, p. 161). They argued that the idea of chance properly comes into being only when combinatoric operations precede it and are understood by students. They divided the combinatorial reasoning of students into three categories: Combinations, Permutations, and Arrangements.

Combinations were, in the view of Piaget and Inhelder (1951), the simplest and generally the first type of combinatorial operation to emerge and be understood by students. As described previously, a combination is a way of selecting several things out of a larger group where order does not matter and repetitions are not allowed. Piaget and Inhelder (1951) found that reasoning about combinations emerged in three ways. First, students sought to find combinations by simple brute force manipulation that involved trial and error. Second, they might then also have looked for a system that helped them generate combinations but were still relying heavily upon trial and error and other brute force techniques. Finally, students could also have developed a well-organized system for enumeration of possibilities. However, even when students were able to develop such a system, Piaget and Inhelder asserted that students were not necessarily able to produce a formalized representation of this system. Table 2 presents the mathematical formula for finding combinations when repetition is allowed.

Table 2

Combinations and Permutations

Combinations	
Purpose: To select r things from a total of n objects when repetition is not allowed. Referred to as 'n chose r'.	Combinations Formula = $\frac{n!}{r!(n-r)!}$
Permutations	
Purpose: To select n things r times when repetition is not allowed.	Permutations Formula = n!

According to Piaget and Inhelder (1951), permutations represented a much more difficult leap in understanding for students than did combinations. Piaget and Inhelder (1951) used the term permutation to refer to the act of rearranging objects or values in an ordered fashion. A permutation of a set of objects is an arrangement of those objects into a particular order. For example, when repetitions are not allowed there are six permutations of the set {1,2,3}, namely {[1,2,3], [1,3,2], [2,1,3], [2,3,1], [3,1,2], and [3,2,1]}. Piaget and Inhelder (1951) defined a permutation as a rearrangement of any elements when order mattered and repetitions were not allowed. They noted that younger students were not expected to come up with or derive the formulaic generalization for permutations but that it did exist, which can be seen in Table 2 above.

Piaget and Inhelder (1951) found that the understanding of permutations emerged in five different ways. First, similar to combinations, students strove through trial and error to arrive at all the permutations. However, students often had difficulty understanding that several permutations could be made with the same elements. For example, students might have repeated the same permutation and not have understood that they were actually the same. Second, they might have continued to use trial and error but were likely to recognize things such as the possibility of starting twice with the same color. Third, students began looking for a system to organize their search for all permutations. Fourth, they first arrived at a systematic listing or grouping of permutations, but the systems developed often had errors or misconceptions contained in it. Finally, students arrived at a full systematic grouping or listing of all possible permutations that were also generalizable to other contexts.

Piaget and Inhelder (1951) used the term arrangements to described permutations and combinations where repetition was allowed. Thus, arrangement was a more general term and could refer to either a combination or permutation, but in either case it referred to a situation where repetition was allowed. Arrangements also have explicit formulas, which are supplied in Table 3.

Piaget and Inhelder (1951) found four ways that students understand arrangements. First, students approached arrangements by grouping elements of a set together with no evident systematic approach. Piaget and Inhelder found that students had more problems with permutations than with combinations. Second, students later began searching for a system, as was the case with combinations and permutations but as yet had not arrived at a generalizable system. Third, students discovered or constructed a generalized rule for a specific arrangement, such as how many two digit numbers could be made from the numbers 1 and 2. However, they did not understand the reason why their generalization worked and were unable to adapt their generalization to another context, such as how many two digit numbers could be made from the numbers 1, 2, and 3. Finally, students could construct a generalization and understand the reasons why their generalization worked and were able to adapt this to other settings and contexts.

Table 3

Arrangements

Arrangements (Combinations and Permutations where repetition is allowed)	
Purpose: To select r things from a total of n objects when repetition is allowed. Referred to as 'n chose r'	Combinations Formula = $\frac{(n+r-1)!}{r!(n-1)1}$
Purpose: To select n things r times when repetition is allowed.	Permutations Formula = n^r

Combinatorial Misconceptions

Previous research into understanding the combinatorial reasoning of students centered on examining the misconceptions and misunderstandings that students hold about combinatorics (Batanero et al., 1997; English, 1991; Fischbein & Gazit, 1988; Hadar & Hadass, 1981; Kahneman & Tversky, 1983). Some concern students' understanding of and approach to the structure of combinatorial problems while others pertain to general problem solving skills. Batanero et al. (1997) identified several different types of errors that students make while engaged in combinatorics problems.

1. Error of order. This type of error occurs when students do not recognize when order is important and when it is not.
2. Error of repetition. This type of error occurs when students do not recognize whether repetition is allowed.
3. Error of indistinguishable/distinguishable elements. This type of error occurs when students do not recognize that some elements in the problem are distinguishable and some are not.
4. Error of over- or under-counting. Here, students count an object more than once or some objects are not counted at all.
5. Non-systematic listing. This error occurs when students do not follow a systematic method for listing all possibilities. This was observed in adults recently introduced to combinatorics (Hadar & Hadass, 1981) and in children (English, 1991).
6. Misinterpretation of the problem. This error can occur in three ways according to Batanero et al. (1997). Changing the type of mathematical model, transforming a single problem into a compound combinatorial problem, and interpreting the words 'distribute' or 'share' as requiring a division by two in the data.
7. Misunderstanding the formula. Batanero et al. (1997) found that students will sometimes identify the structure of the problem correctly but use the incorrect formula to arrive at the solution.

8. Mistaken intuitive answer. Students only give a mistaken numerical solution with justification for the response.
9. Confusing multiplication and addition. Students will use some combination of addition and multiplication principles to arrive at an incorrect answer instead of applying a known formula or listing all possibilities.
10. Students interpret a tree diagram in an incorrect manner.

Hadar and Hadass (1981) pointed out several other errors that students have a tendency to make when solving combinatorial problems. They observed that adult students did not use a method where one variable was held constant while the other was allowed to vary and that even when students were able to fix one variable, many still struggled to generalize their solution successfully. This prevented them from thinking about abstract mathematical ideas and being able to solve other problems. Other studies have examined how adults verify their combinatorial strategy (Eizenberg & Zaslavsky, 2004). Eizenberg and Zaslavsky (2004) found that the verification of a solution strategy was an important step that helped students identify errors they may have made but that many of them did not automatically verify their solutions.

Combinatorics Tasks used to Investigate Other Areas of Mathematics

Investigating combinatorial reasoning from the standpoint of errors made by students has not been the only approach. Some researchers have utilized combinatorial reasoning as a way of studying other mathematical concepts such as justification and proof (Maher & Martino, 1996; Martino & Maher, 1999). Maher and Martino (1996) studied a student for five years and used the application of combinatorics problems to explore how she learned to construct proofs and justify her solutions. They selected combinatorics as a content area because it enabled them to investigate how a student sees patterns, heuristics, and develop organization and classification schemes. They used a series of tasks, one of which asked the student to build towers out of red and blue colored blocks. The researchers identified several important strategies that this student used to enumerate the number of towers. One of these was building a specific tower and then flipping or turning it over to find its *opposite*. Maher and colleagues developed this into a 10-year longitudinal study in which they concluded that “students build fundamental mathematical understanding, over time, through extended task-based explorations. They created models, invented notations, and justified, reorganized, and extended previous ideas and understandings to address new challenges” (Powell, 2010, p. 201). Thus, combinatorial tasks are not only essential in underpinning and preceding probabilistic reasoning, but they also provide a powerful and rich environment to investigate student thinking in other important areas of mathematics education, such as modeling and justification.

In summary, combinatorics has been identified as an important subject for students to understand because of its many connections with other fields (Eizenberg & Zaslavsky, 2004; Kavousian, 2008; Zaslavsky, Zaslavsky, & Moore, 2001). From a pedagogical perspective, the subject presents teachers with the option of giving students challenging and interesting problems that require little prior instruction from the teacher or knowledge of advanced mathematics from students (Kavousian, 2008; Lockwood, 2010; Maher & Martino, 1996). Combinatorial tasks have also been used to investigate how students justify their work (Maher, Powell, & Uptegrove, 2010), and these have been useful and productive lines of inquiry. However, much remains to be learned about this area of learning. For instance, more knowledge is needed regarding how students come to generalize their combinatorial strategies and what aspects within a combinatorial task students use to assist in generalizing it.

This study focused on using combinatorial tasks and reasoning to understand the construction of sample space better since knowledge of sample space construction, as seen through the lens of combinatorial reasoning, is a necessary pre-cursor that enables probabilistic reasoning (Jones et al., 2007; Piaget & Inhelder, 1951). The existing research, in trying to explain how students reason about combinatorics, has almost exclusively dealt with their misconceptions (Batanero et al., 1997; English, 1991, 1993; Hadar & Hadass, 1981). However, this study sought to investigate how students developed combinatorial reasoning and how it enabled them to create generalizations of combinatorial tasks.

Frameworks

As demonstrated thus far, many researchers believe that students' understandings of sample space goes through a sequence of three levels: (a) Identifying outcomes; (b) Mapping a probability onto each element of the sample space; and (c) Systematically enumerating all possible outcomes. This study focused on the last aspect of sample space construction, that of systematic enumeration and how students come to generalize a solution to a combinatorial task. To operationalize systematic enumeration, I utilized combinatorial reasoning and combinatorial tasks since this type of knowledge directly precedes probabilistic understanding (Piaget & Inhelder, 1951). Full knowledge of this type of understanding requires analysis of two separate processes. The first process is enumeration and consisted of listing all possible outcomes in a deterministic manner. The second is generalization, which requires the ability to reason about more general situations that are related to the current problem in a mathematical sense. In order to do this, I used two separate frameworks, one for enumeration and one for generalization. This section describes and summarizes each one.

Combinatorial Enumeration Strategy Framework

This framework draws from several sources in order to operationalize enumeration strategies, which made it possible to answer research question one. English (1991, 1993), in a series of two studies, examined the systematic and efficient strategies for generating solutions to combinatorial problems involving enumeration. She used a task in which students dressed toy bears in combinations of shirts and pants (a two-stage combinatorial problem) and shirts, pants, and tennis rackets (a three-stage combinatorial problem). She divided her strategies and placed them in a hierarchical order based upon the complexity of the problem. For two-stage problems, she found a sequence of five strategies for enumeration that increased in terms of complexity and thoroughness. See Table 4 for a summary of the five strategies.

Strategy 1 was the simplest of the possible solution strategies and was labeled trial-and-error (English, 1991; 1993). In this strategy, students selected items randomly and rejected those possible solutions that seemed unsuitable. Strategies 2 and 3 were transitional in nature since they were both more efficient than trial-and-error but were not nearly as efficient as strategies 4 and 5. What set strategies 2 and 3 apart from strategy 1, trial-and-error, was that it became apparent that students used a pattern of some type, usually cyclic in nature in both of these strategies.

Table 4

Enumeration Strategy Framework (English, 1991, 1993)

Strategy	Name	Description
1	Trial and Error	Students use a trial and error strategy
2	Emerging Strategy	Students use some sort of pattern but it is not fully used or developed
3	A Cyclic Pattern	Students use a cyclic pattern such as opposites that is fully utilized
4	Odometer With Errors	Students hold one variable constant but fail to fully enumerate or over enumerate
5	Odometer Complete	Students hold one variable constant and find a full enumeration

In Strategy 2, however, the pattern was only emerging and was not consistently and completely utilized for thorough enumeration. For example, if students were asked to select 3 marbles with the possibility of either red (R) or blue (B) marbles, what does a full enumeration of all possible three marble combinations look like? Using Strategy 2, the enumeration might look like Figure 4.

Clearly, there was a pattern to the first four arrangements, which Maher, Sran, and Yankelewitz (2010) defined as the *opposites strategy*, where students simply took a specific combination such Red, Red, and Blue and then created its opposite or reflection, Blue, Blue, Red. However, what made this only an emerging strategy was that it was not fully implemented. The pattern might break down at the end causing students to go back to employing a trial-and-error strategy where they choose possible combinations and look to see if the new combination they found without the use of a pattern replicate any of the previous combinations.

RRR
BBB
RRB
BBR
RBR
RBB
RBR
BRB

Figure 4. Strategy 2

In contrast, Strategy 3 was characterized by a consistent and complete pattern, which was usually cyclic. This pattern was used to generate all possible combinations and was often a full implementation of Maher et al.'s (2010) *opposites strategy*. Figure 5 demonstrates a complete enumeration using a cyclic pattern.

English (1991, 1993) suggested that Strategy 4 and 5 were the most efficient and therefore the most sophisticated. She labeled these two strategies the *odometer strategy*, named for the odometer in a motor vehicle. Maher et al. (2010) referred to the *odometer strategy* as the *staircase pattern* where colors appear in a stair step configuration. In this strategy, one variable was held constant while the other was allowed to vary. In this way, it resembled strategies 2 and 3 since it was cyclic in nature, but it incorporated an additional feature, which was a constant item or variable. This item was held constant until all possible combinations that contained it had been identified. However in strategy 4, students either failed to fully exhaust a given variable, over-selected a certain variable, thereby creating duplications, or failed to realize when an item was completely exhausted. For example, again using combinations of 3 marbles, strategy 4 might look like one of the columns in Figure 6.

RRR
BBB
RBB
BRR
RBR
BRB
RRB
BBR

Figure 5. Strategy 3

When one variable was not fully exhausted, students tended to revert to an earlier and less sophisticated strategy, such as the *opposites* or *trial-and-error*, to complete their enumeration, as demonstrated above in the first column of Figure 6 where they had not fully exhausted the constant variable. However, when one variable was over-selected, duplications of prior combinations tended to find their way into the systematic enumeration. See column 2 of Figure 6.

In comparison to Strategy 4, Strategy 5 demonstrated a complete and consistent exhaustion of all possible variables that could be held constant and successfully and completely produced all possible combinations. Defined as the *odometer strategy* by English (1991; 1993), this strategy was defined by Maher et al. (2010) as the staircase pattern because it can look like a staircase visually. See Figure 7 for an illustration of Strategy 5.

Failure to Fully Exhaust One Variable	**Over Selection of One Variable**
RRR	RRR
BRR	RBB
RBR	RRB
BBB	RRR
BBR	BRB
RRB	RBR
BRB	RBB
RBB	BRR

Figure 6. Strategy 4

Odometer	**Staircase**
BRR	BRR
RBR	BBR
RRB	BBB
BBR	BRB
BRB	RBB
RBB	RRB
RRR	RRR
BBB	RBR

Figure 7. Strategy 5

The staircase and odometer differed slightly but were both examples of holding one variable constant until full enumeration was achieved. In the *odometer strategy*, students held the number of blue marbles constant and allowed the number of red marbles to vary. However with the *staircase strategy*, students were instead holding a single position constant. For instance, marbles could occupy three positions, one for the first, second, and third marble, and if the first position was held constant and a blue marble was placed in it, and then the other two positions were allowed to vary, this would enumerate all possibilities with a blue marble in the first position. This was called the *staircase pattern* since the first three combinations BRR, BBR, and BBB resembled a staircase as the number of blue marbles moved upwards.

Generalization Framework

As students moved from an individual combinatorial problem to a class of combinatorial problems, they needed to understand how the solution evolved and was eventually generalized. The second research question, which involved generalizations of combinatorial rules, sought to investigate the reasoning required to make this next step in reasoning. Generalizations are rules that are explicit formulas, descriptions, sentences, or paragraphs that allow students to generate specific instances of a pattern, situation, or context, which are independent from earlier instances of the pattern or data (Lannin, 2005). Statements concerning generality and discovering these generalities are at the core of mathematical activity and reasoning (Kaput, 1999; Mason, 1996). However, many generalizations represent reasoning that is either incomplete or faulty for more complex contexts. Examples of this include the guess-and-check strategy for producing a generalization (Healy & Hoyles, 1999) or local tactics (Mason, 1996), in which students use specific instances of a pattern to produce a generalization rather than attempt to understand the underlying relationship in the problem or context.

Lannin (2005) used some of the research mentioned above, along with the results of his teaching experiments, to develop a framework that described a hierarchy of systematic generalization schemes that students produced when they were engaged in pattern finding activities. He divided his classification of generalizations into two categories (see Table 5). These two categories were:

1. Non-explicit generalizations
2. Explicit generalizations that produce a rule where direct calculation of one variable is possible if given the value of the other.

Table 5

Generalization Framework (Lannin, 2005)

Strategy	**Description**
Non-explicit	
Counting	Drawing a picture or constructing a model to represent the situation to count the desired attribute
Recursive	Building on the previous term or terms in the sequence to determine the subsequent term
Explicit	
Whole-Object	Using a portion as unit to construct a larger unit by multiplying (e.g. 3 sodas cost $8 so 9 sodas cost $24). There may or may not be an appropriate adjustment for over or undercounting

Table 5 (cont.)

Strategy	Description
Guess-and-Check	Guessing a rule without regard to why the rule might work. This usually involves experimenting with various operations and numbers provided within the context of the situation or problem.
Contextual	Construction of a rule based on information provided in the context of the problem and relating that to a counting technique.

Non-explicit generalizations tended to rely on less sophisticated means of generalization, such as recursively finding the next pattern. In a generalization of this type, the next iteration of a pattern directly depended on the previous result. For example, if someone received 3 apples for every dollar, they then needed to know how many apples they would get for $3 to know how many apples they would get for $4. Explicit rules were far more sophisticated (Lannin, 2005) in that they allowed for the prediction of the values of one variable based only upon the value of the other, unlike a recursive strategy, where prior values of one variable were required to find later ones. See Table 5 above for a summary of the different possible generalization strategies that students could use.

In summary, sample space is an important issue for students to understand (Jones, Langrall, & Mooney, 2007; Shaughnessy, 2003). One of the most essential aspects of the understanding of sample space is how to systematically enumerate it with combinatorics (Horvath & Lehrer, 1998; Jones et al., 2007). Researchers suggest that combinatorial reasoning is a necessary pre-cursor to probabilistic reasoning (Jones et al., 2007; Piaget & Inhelder, 1951). Furthermore, it is thought that understanding the bridge between enumerating a combinatorial problem and then generalizing it will enable greater knowledge of the process by which students construct the ability to reason probabilistically and become adept at probabilistic decision making (Jones et al., 2007). This study made use of two frameworks to investigate this connection. First, it used the framework obtained from the work of English (1991, 1993) to operationalize the enumeration process, and then employed the framework gathered from the research of Lannin (2005) to do the same for the generalization process.

CHAPTER III

STUDY DESIGN AND METHODOLOGY

As previously argued, an important concept for students to understand when learning probability is sample space (Ayres & Way, 2000; Jones et al., 2007, 1999; Shaughnessy, 1992, 2003). This understanding serves as a foundation for other important areas, such as probability and inference, which in turn will enable students to make better probabilistic decisions (Shaughnessy, 2003). Because combinatorics is one way of examining sample space (Jones et al., 2007), the purpose of this chapter is to describe the design and methodology used to address the research problem as summarized in the earlier chapters and to answer the three following research questions:

1. What combinatorial strategies do undergraduate students use to construct the sample space of a compound event?
2. When undergraduate students enumerate the sample space of a compound event, what generalizations do they generate?
3. What are the associations between the generalization rules and enumeration strategies?

Theoretical Perspective

The results and perspective taken on the learning of mathematics in this study were influenced by and drawn from the constructivist theoretical perspective. This section provides an overview of the theory, highlighting two portions of constructivist theory that are most germane to this study. The first of these was that the individual constructs knowledge. The second influential constructivist aspect was that often knowledge in a given content area is *compartmentalized* or isolated from other related topics.

Constructivism is a theory of learning that claims that knowledge is not received from some outside source but is constructed actively by the individual as he or she engages in solving problems and makes sense of the world. Constructivism maintains that knowledge is a mental construction, which is dependent on the form and content of the individual who holds it (Guba & Lincoln, 2000). In *Genetic Epistemology* (Piaget, 1970), the fundamental idea was that there was a logical organization to knowledge and to the formation of psychological processes, that is, knowledge is genetic in that it evolves with the individual. Piaget called the mental structures that form during this process schemes. Viewed through the lens of mathematics education, the process of refining schemes is called the cycle of constructive activity, which is a term used to describe how individuals abstract components of objects based on actions that are the result of some experience that causes perturbation or confusion for the individual (Confrey & Kazak, 2006).

Another constructivist concept that was of particular importance to this study was what Spiro, Feltovich, Jacobson, and Coulson (1992), in their constructivist theory of learning and instruction, referred to as the *compartmentalization of knowledge*. They described *compartmentalization of knowledge* as the isolation of conceptual elements that are, in reality, highly interdependent. They suggested that the *compartmentalization of knowledge* blocks effective learning in content areas where knowledge is tightly interconnected. Instruction that has, historically, focused on a wide scope of applications across cases or examples as much of school mathematics has can lead to misunderstandings and *compartmentalization of knowledge* (Spiro et al., 1992).

Mixed Methods

To accomplish the goals of this study, I chose to use mixed methods. Mixed methods is a procedure that allows for the collecting, analyzing, and mixing or integrating of both qualitative and quantitative data at some stage in the research process for the purpose of gaining a better understanding of the research problem (Creswell & Clark, 2011; Teddlie & Tashakkori, 2009). Mixing qualitative and quantitative data within the same study is advantageous since neither quantitative nor qualitative methods by themselves are necessarily sufficient to gain as full an understanding of a given situation as possible (Ivankova, Creswell, & Stick, 2006). When used together, qualitative and quantitative methods complement one another, allowing for a robust analysis that takes advantage of the strengths of each method (Miles & Huberman, 1994; Tashakkori, 2006).

The specific design for this study was a parallel mixed design (Teddlie & Tashakkori, 2009), which had at least two parallel and independent strands, with each strand having its own questions, data collection, and analytic techniques. The qualitative and quantitative strands were planned and implemented to answer related aspects of the mixed methods research question. Finally, inferences, based upon the integrated findings, form meta-inferences at the end of the study (Teddlie & Tashakkori, 2009). A meta-inference is a conclusion drawn from the integration of inferences and insights gained from the qualitative and quantitative strands of the study.

Research Site

I selected a university in the southeastern United States as the site for my study. This university is medium sized and research intensive with an enrollment of roughly 17,000 to 20,000 students, drawn from across the state in which it is located. It grants bachelors, masters, and doctoral degrees in a variety of fields. Students come from a variety of diverse backgrounds, and the school itself is known for its teacher education program. The institution's emphasis on teacher education made it particularly suitable for this study since its purpose was to understand better how undergraduate students, studying to become teachers, constructed sample space with combinatorial operations.

Participants

The participants in this study consisted of two classes of undergraduate students, who were enrolled in an introductory teacher education course that focused on mathematics needed to complete requirements for admission into the elementary teacher education program. One class was drawn from the spring semester of 2011 and the other from the fall semester of the same year. The objectives of this course were to deepen the mathematical content knowledge of future elementary teachers in topics such as number and operation, algebra, measurement, statistics, probability, and geometry. College algebra was a pre-requisite of this class and any undergraduate student wishing to enter the elementary teacher education program must pass this class with at least a C.

Students in these classes were invited to participate in the study. A total of 150 students from both classes agreed to be participants out of a total enrollment of 248 in both classes. These 150 participants made up the first sample, which represented a convenience sample. Out of the 150 students who agreed to participate in the first phase of the study a total of 8 agreed to participate in follow-up interviews. The 8 participants who made up the second sample used in this study were selected in a purposeful manner. Maxwell (2005) defined purposeful sampling as selecting settings, persons, or activities deliberately to elicit information that could not be gained with other sampling methods. In this case, the second sample was chosen based on the type of enumeration strategy that participants employed on the whole group activity. See Appendix A. In this study the selection of the undergraduates that participated in this portion of the study was based on whether they demonstrated growth of understanding, regression of understanding, or consistently high or low understanding across the tasks, where 15 of the 150 total students demonstrated these trends in understanding. At least one undergraduate who exhibited each of these trends was invited to become a participant in the second sample. However, only 8 of 15 students invited to participate agreed to do so. Hence, the second sample consisted of 8 participants.

Tasks

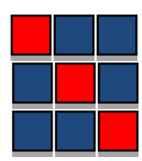
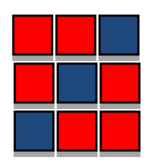
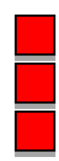
Study participants engaged in a series of either three or four tasks. Tasks 1 and 2 consisted of a sequence of tower tasks, developed by Maher, Powell, and Uptegrove (2010), in which students were asked to build towers with different color blocks. Task 3 asked them to generalize their thoughts and strategies from the first two in order to make predictions to similar tasks. All participants completed these three tasks. Task 4, however, was a new combinatorial problem and was completed only by those participants that engaged in the follow-up interview. The participants in the follow-up interview were given this combinatorial task to give them the opportunity to experience a related task that had a different context. This allowed me to see if their reasoning in the new task was related to their reasoning in the other tasks.

Task 1: 3 Block Tall Towers with Two Colors

Task 1, drawn from the longitudinal study of Maher, Powell, and Uptegrove (2010), asked participants to construct towers that were three blocks tall consisting of one or two colors. Table 6 shows a possible full solution.

Table 6

Enumeration of 3 Block Tall Towers Using Odometer

Task 1: 3 Block Tall Towers				
Number of Red Blocks	**0 Red**	**1 Red**	**2 Red**	**3 Red**
Arrangement of Towers				
Number of Towers	1	2 3 4	5 6 7	8

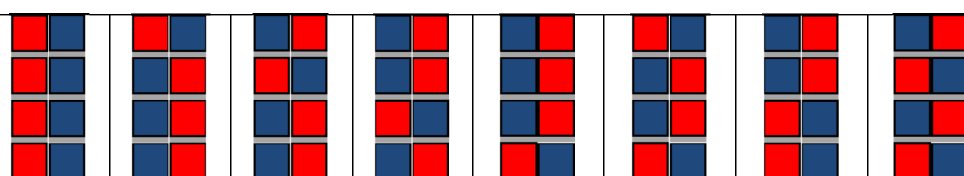
This was not the only possible solution for this task but was one suggested in the framework developed in the work of English (1991, 1993) and Maher, Powell, and Uptegrove (2010). This particular solution was an example of enumeration strategy 5, where one variable, in this case the number red blocks, was held constant while the position of the other blocks were allowed to vary until all possible arrangements of blocks were completely exhausted. English (1991, 1993) referred to this strategy as the *odometer strategy*. All worksheets collected in this study were coded for the category of enumeration strategy they represented.

Task 2: 4 Block Tall Towers with Two Colors

Task 2 was similar to Task 1 in that it asked participants to extend their understanding of towers that were 3 blocks tall to towers that were 4 blocks tall. Once again, the purpose of this task was to elicit enumeration strategies that could be classified using the framework summarized and described in detail in Chapter II. Table 7 demonstrates one of several possible complete solutions, and is an example of strategy 3, which Maher et al. (2010) called the *opposites strategy*.

Table 7

Enumeration of 4 Block Tall Towers Using Opposites

Task 2: Four Block Tall Towers								
Tower Pairs								
Number of Towers	1 2	3 4	5 6	7 8	9 10	11 12	13 14	15 16

Task 3: Generalization of the Towers Problem

Task 3 was designed to elicit the generalizations participants made from Tasks 1 and 2. It challenged them to find a solution for the number of possible towers that was one hundred blocks tall and made with either one or two colors. Where Task 2 was more complex than Task 1, Task 3 was much more complicated than either of the first two. It required that participants formalize what they had done in the prior two tasks and asked them to find a rule or strategy that would work for towers that were one hundred blocks tall without enumerating each possible tower by hand, as was possible in the prior tasks.

Sequence of the First Three Tasks

An important note in the sequence of tasks was that it supported certain generalizations that assisted in distinguishing between those participants who were using local tactics (Mason, 1996), trial and error, or those who generalized in a more contextual manner. For instance, in Task 2 the correct enumeration was 16, which generalized to 2^4. However, another possible generalization was 4^2, which made use of all the numbers that were observed in the task and gave the correct numerical enumeration of 16. Thus, one might then conclude that the generalization was to raise the number of blocks to the power of the number of colors, which led to a numeric generalization in Task 3 of 100^2 instead of 2^{100}. The structure and sequence of the tasks helped determine which participants were engaged in contextual generalization and which were simply using trial and error or local tactics.

Task 4: Three Digit Numbers

Task 4 was designed so that it was isomorphic to Tasks 1, 2, and 3. It provided participants with a set of numbers and asked them to engage in a series of three problems. The first problem within Task 4, isomorphic to Task 1, asked participants to construct all two-digit numbers that were possible with the numbers 6, 7, and 8, where a number such as 88 was allowed. Table 8 outlines what a solution to this task might look like, arranged according to the *opposites strategy*.

Table 8

Two Digit Numbers Solution

Two Digit Numbers					
66	67	68	77	78	88
	76	86		87	

The second problem in Task 2, which was isomorphic to Task 2, asked participants to find the total amount of three-digit numbers, if numbers such as 777 were allowed. This task required that participants use combinatorial operations. A possible enumeration of the sample space, organized around the *odometer strategy* can be found in Table 9. A final aspect of Task 4, which was isomorphic to Task 3, asked participants to determine how many 100-digit numbers could be made with the numbers 6, 7, and 8, of which the proper generalization was 3^{100}.

Table 9

Three Digit Numbers Solution

Three Digit Numbers				
One Number	**Two 6's**	**Two 7's**	**Two 8's**	**One of each Number**
666	667	776	886	678
777	676	767	868	687
888	766	677	688	876

Table 9 (cont.)

Three Digit Numbers				
One Number	**Two 6's**	**Two 7's**	**Two 8's**	**One of each Number**
	668	778	887	867
	686	787	878	786
	866	877	788	768

Data Sources

I used three sources of data in this study. The first came from the written records of Task 1, 2, and 3, which served as an evaluation instrument to categorize both the combinatorial enumeration strategies and generalization rules participants used. See Appendix A for the instrument, which was collected from all 150 participants in the study. The second source of data consisted of video recorded interviews, which was collected from those 8 participants that agreed to take part in the follow-up interview. These interviews were general and guided, which implied that topics and issues were specified in advance, but the interviewer decided the sequence and wording of the questions during the course of the interview (Teddlie & Tashakkori, 2009). See Appendix B for the interview protocol. During this interview, the written records of Tasks 1, 2, and 3 were returned to participants, who had completed it several weeks prior to the interview. They were asked to explain their thinking and to clarify any notation, pictures, formulas, or other explanations that they had given on the worksheet. The third source of data, collected as part of the interview, consisted of video recordings and written records of Task 4, which involved having participants engage in solving problems they had not seen prior to the interview. See Appendix C for the Instrument and Appendix B for the interview protocol. This is called a task-based interview (Goldin, 2000). Since the discussion surrounding Task 4 occurred immediately after the discussion of the first three tasks, it is possible that this discussion had impact upon the selection of which enumeration strategy or generalization rule that participants selected.

Analysis

The analysis portion of this study took place in three phases. In the first phase, quantitative data were analyzed to answer the first and second research questions, and partially the third, which was the mixed methods research question. In the second phase, qualitative data in the form of videotaped interviews were analyzed to provide a second element that served to answer the third research question. The third and final phase was mixed and involved the formation of meta-inferences using both the qualitative and quantitative analyses from the previous phases to answer the third research question completely.

Phase 1

In Phase 1, Tasks 1 and 2 were coded using the strategy framework described in Chapter 2, while Task 3 was coded using Lannin's generalization framework. The reliability of these coded categorizations was .83. The data was tabulated and analyzed using frequency tables in order to answer the first and second research questions that addressed the enumeration strategies and generalization rules participants produced when constructing sample space with combinatorial operations.

The third research question concerned associations between the enumeration strategy and generalization rule that participants constructed. To analyze the data for this portion of the study, I conducted a statistical analysis using SPSS, a widely available and well-recognized statistical analysis software program. The Chi-square test of independence was appropriate for answering questions of association between two ordinal variables. However, with the small sample sizes I obtained for this study, the assumptions of the Chi-Square test would be violated if the data were not collapsed. The Chi-Square test assumes that each cell within the frequency table is at least 5. In this study, the frequency data did not match this assumption. To counteract this, I collapsed the categories of each of the frameworks outlined in Chapter II. The new categories were theoretically supported since strategies 1 and 2 in the enumeration framework were considered less sophisticated and as non-explicit. However, the collapsing of the data was only used for the Chi-square analysis and was not used to answer the first two research questions. Table 10 illustrates how the enumeration strategy categories were collapsed for Task 1 and 2. In Table 10, category 0 was used to represent those participants that left either Task 1 or 2 blank.

Similarly, the generalization framework outlined in Chapter II was collapsed, as seen in Table 11, in a theoretical manner. This was accomplished by collapsing the non-explicit generalization categories, 1 and 2, into one category and the explicit generalization categories, 3, 4, and 5, into another, where Category 0, once again, represented participants who left this task blank.

Table 10

Collapsed Enumeration Framework

Enumeration Strategy Name	Enumeration Category	Collapsed Category
Blank	0	1
Trial and Error	1	1
Emerging Strategy	2	1
A Cyclic Pattern	3	2
Odometer With Errors	4	2
Odometer Complete	5	2

Table 11

Collapsed Generalization Framework

Category Name	Category Number	Collapsed Number
Non-explicit		
Blank	0	1
Counting	1	1
Recursive	2	1
Explicit		
Whole-Object	3	2
Guess-and-Check	4	2
Contextual	5	2

Finally, I investigated the nature of the dependence found with the chi-square test by assessing its strength with an appropriate measure of association. This was accomplished by using the Spearman Correlation Coefficient since it was appropriate for ordinal data, which the collapsed data in this study represented.

Phase 2

Phase 2 of the study was qualitative. In this phase, I asked participants to explain their reasoning in Tasks 1, 2, and 3 and also to engage in Task 4. Their responses were then analyzed for themes using constant comparative analysis (Glaser, 1965). See Appendix B for the Interview Protocol and Appendix C for Task 4 Material. I used open coding to analyze the follow-up interviews to identify emerging trends and to group them in terms of similarity. Then I used extreme case analysis, as defined by Miles and Huberman (1994), to identify any extreme cases within these emerging trends. If a trend was determined as different from other examples, I defined a new category corresponding to that extreme case. However, if a trend was not deemed to be different I left it in the original category. I repeated this process for all extreme cases identified in the follow-up interviews.

Phase 3

Phase 3 was the mixed methods portion of the analysis. Here, the quantitative conclusions found in Phase 1 were combined with the qualitative conclusions in Phase 2 to produce meta-inferences (Teddlie & Tashakkori, 2009). A meta-inference is an inference or insight gained through the integration of the qualitative and quantitative findings, which gives a broader view of what is happening in a given situation or research question. The quantitative findings found for the third research question in Phase 1 were informed by qualitative findings produced in Phase 2 to create inferences to fully answer the third research question.

Summary

In this section, I have provided a summary crosswalk that relates the research questions with the tasks that I have described earlier, the data sources used in analysis, and the methods used for analysis (see Table 12). I have then briefly described and summarized the entire research process relating how the research questions were associated with the tasks, data sources, and analyses described earlier in this chapter.

Table 12

Crosswalk

Crosswalk of Research Questions, Tasks, Data Sources, and Analyses						
	Tasks					
	Task 1	**Task 2**	**Task 3**	**Task 4**	**Data Sources**	**Analysis**
RQ1	X	X			Written Records of Tasks 1 and 2	Frequency Analysis
RQ2			X		Written Record of Task 3	Frequency Analysis
RQ3	X	X	X	X	Written Records of Tasks 1, 2, 3, and 4 Follow-Up Interview	Chi-Square Test of Independence and Constant Comparative Analysis

I addressed the first and second research questions by classifying the written records of Tasks 1, 2, and 3 using the enumeration and generalization frameworks presented in Chapter II. Then using frequency tables, I described how participants enumerated and generalized. I used mixed methods to answer the third research question, and combined both the qualitative and quantitative conclusions to form meta-inferences

(Teddlie & Tashakkori, 2009). First, I statistically analyzed the independence, using the Chi-Square test of independence between participant's enumeration strategy, as classified in the written records of Task 1 and 2, and their generalization rules, as classified from the written records of Task 3. Then, I examined the follow-up interviews for emerging trends using constant comparative analysis. Finally, I combined these two analyses to form meta-inferences that provided a robust and rigorous response to the third research question.

Validity

Creswell and Clark (2011) defined validity in mixed methods research as employing strategies that address potential issues in data collection, data analysis, and interpretation that might compromise the merging or connecting of quantitative and qualitative research strands and the conclusions drawn from this merging. Validity issues will be addressed as they relate to data collection, data analysis, and interpretation.

Data Collection

A concern in mixed methods research pertains to whether or not unequal sample sizes have been collected in the qualitative and quantitative portions of the study (Creswell & Clark, 2011). In this study, the qualitative and quantitative data represent two separate data sets. Thus, there was no concern about equality of sample sizes since the conclusions that were drawn qualitatively were found independent of and were used in a complimentary fashion to the quantitative findings.

Data Analysis

A validity concern in this study pertained to the reliability of the classification of participant work into the appropriate categories. To ensure that I properly sorted and categorized these strategies, I randomly selected 20% of the written records from Tasks 1, 2, and 3 and determined the inter-rater reliability with two other independent raters. The inter-rater reliability was found to be .83, which confirmed that the ratings or categorizations I used in research questions 1, 2, and 3 were reliable.

The third phase of analysis, involving the qualitative analysis of the follow-up interview data, provided an additional source of validity since it was used to help make sense of the quantitative results (Creswell & Clark, 2011). For instance, significant findings from the quantitative analysis were demonstrated with a quote from the interviews. The additional task of constructing two and three digit numbers also added richness and validity to the study since it provided me with an opportunity to triangulate my data. The qualitative analysis was done using constant comparative analysis. One participant in the follow-up interviews demonstrated four out of the five identified emerging trends. To ascertain the reliability of identification of these emerging trends an independent researcher watched a video of this participant and identified 87.5% of the occurrences of the emerging trends in that video.

Interpretation Issues

Creswell and Clark (2011) identified several validity threats in the interpretation stage of a study. The first concerned the idea of not directly discussing the mixed methods research questions. This study only had one mixed methods research question, which examined whether or not the combinatorial enumeration strategy used by participants was associated with the generalization rules they subsequently produced. Creswell and Clark (2011) also indicated that not relating the phases within a mixed methods research study could lead to validity issues. The phases of this study were done in a parallel manner and therefore supported each other as the parallel mixed design calls for. The quantitative findings from phase 1 were combined with the qualitative findings from phase 2 to form meta-inferences, where these meta-inferences used both qualitative and quantitative findings in a complimentary fashion. The formation of the meta-inferences did not involve changing any of the findings from earlier phases.

Ethics

The main ethical issue in this study concerned the collection of data because Tasks 1, 2, and 3 were given to all participants in the study who were enrolled in a class for which they would eventually receive a grade. Because of this I was concerned that participants might have felt obligated to participate and forced to give me access to their work. To ameliorate this issue, participants had multiple chances to decline participation. Every participant also signed a consent form that allowed me access to their work and gave me permission to invite them to be part of the follow-up interview process. I also assured them that they would have complete anonymity during all phases of the study, including the collection of the worksheet and the participant interviews. To help assure them that their grade in the class would be unaffected by their participation, the instructor of the course was only involved as a facilitator in my access to participants.

CHAPTER IV

RESULTS AND FINDINGS

Enumeration as a tool for the construction of the sample space is an important skill for students of all ages to have at their disposal. However, it is a complex task that uses multiple mathematical skills simultaneously. Primarily it concerns the concept of combinatorics for the enumeration of a sample space and the making of connections between the strategies used for enumeration and a generalized understanding of this solution. Do students connect their strategies for enumeration with their eventual generalizations, and if so what do these connections look like?

In this chapter, I will outline findings from this study to answer my research questions discussed in the preceding chapters. The first two questions were answered quantitatively. The third, the mixed methods research question, contains sections that concern quantitative findings, qualitative findings, as well as a section that integrates or mixes both findings and produces meta-inferences based upon both analyses. This chapter will conclude with a summary of the results.

Research Question 1: How Do Undergraduates Enumerate?

The first research question concerned the ways in which students utilized combinatorial reasoning to enumerate the sample space of a probability experiment. Task 1 was concerned with how participants created 3 block tall towers that used either blue or red blocks. Table 13 contains a frequency table for the enumeration strategies used by participants to solve Task 1, i.e. 3 block tall towers using two colors.

Table 13

Task 1 Enumeration Frequencies

Frequencies of Enumeration Strategy for Task 1			
Enumeration Strategy	**Frequency**	**Percent**	**Cumulative Percent**
1	26	17.3	17.3
2	29	19.3	36.7
3	55	36.7	73.3
4	12	8.0	81.3
5	28	18.7	100.0
Total	150	100.0	
Correct Enumeration	102	68.0	

Using the framework from Chapter III, I classified all 150 participants' strategies into one of the five categories contained in that framework. The distribution of those participants that chose Categories 1 and 2 was uniform and the percentage that chose to use each was approximately equal. These first two categories represent *trial-and-error* (Category 1) or *some type of pattern* (Category 2). Category 3, the use of a fully developed and implemented pattern, was used by 36.7% of participants with a cumulative percentage of 73.3% falling into one of the first 3 categories. This indicates that roughly 3/4ths of the participants used fairly unsophisticated enumeration strategies. In contrast, only 26.7% used the *odometer strategy*, categories 4 and 5, as described by English

(1991; 1993) to be the most effective and useful of the enumeration strategies. Thus, in Task 1 the majority of participants used less sophisticated enumeration strategies, suggesting that they saw no need for organization and found the task fairly easy.

Task 2, as described in Chapter 3, was completed immediately after Task 1. Table 14 describes the frequencies of the enumeration strategy participants selected to complete this task.

Table 14

Task 2 Enumeration Frequencies

Frequencies of Enumeration Strategy for Task 2			
Enumeration Strategy	**Frequency**	**Percent**	**Cumulative Percent**
0	3	2.0	2.0
1	17	11.3	13.3
2	23	15.3	28.7
3	59	39.3	68.0
4	27	18.0	86.0
5	21	14.0	100.0
Total	150	100.0	
Correct Enumeration	55	36.6	

Category 0 in Table 14 was not part of the framework originally detailed in Chapter 2 but represents those students that left Task 2 blank. Two important observations were noteworthy here. The first was the decline in the usage of the trial-and-error and the emerging patterns (Categories 1 and 2), which comprised 28.7% of the total responses in Task 2, as compared to 36.7% in Task 1, a decline of nearly 10%. Second, the percentage of students who used the sophisticated *odometer strategy* (Categories 4 and 5) increased from 26.7% in Task 1 to 32% in Task 2, an increase of 5%. However, the percentage of students that used a fully realized strategy (Category 3) remained relatively the same, at 36.7% in Task 1 and 39.3% in Task 2. Thus, some progression in the sophistication of strategies used Task 1 and Task 2 occurred as the task became more complex. However, there were limits to this progression, as shown by the lack of increase in the percentage of participants that used Category 3. This increase in sophistication did not equate into increased success of enumeration since the percentage of correctly enumerated examples of Task 2 decreased from 68% in Task 1 to 36.6% in Task 2.

Thus, the complexity of Task 2 had the tendency to elicit more sophistication in the type of enumeration strategy chosen by participants, as is evidenced by the 10% decrease in use of Categories 1 and 2, and the 5% increase in those participants that used Categories 4 and 5. However, the percentage of students that used Category 3 was nearly the same. This suggests that as the complexity of the tasks increased it had an effect on the enumeration strategies that participants decided to use. On the other hand, the selection of increasingly sophisticated enumeration strategies did not lead to an increase in the percentage of successful enumerations.

Research Question 2: How Do Undergraduates Generalize?

Lannin (2005) described a hierarchy of generalization strategies, as detailed in Chapter II. This hierarchy of five categories had two distinct subsections. Lannin characterized the first of these subsections as non-explicit. This subcategory is comprised of categories 1 and 2, described as counting and recursive respectively. However, he designated that the last three categories represented explicit generalization rules, described as whole-object, guess-and-check, and contextual. Table 15 gives the frequency of occurrences for the categories of generalization rules constructed by participants in this study.

Table 15

Task 3 Frequencies

Frequencies of Generalization Type			
Generalization Category	**Frequency**	**Percent**	**Cumulative Percent**
0	43	28.7	28.7
1	13	8.7	37.3
2	12	8.0	45.3
3	4	2.7	48.0
4	66	44.0	92.0
5	12	8.0	100.0
Total	150	100.0	
Correct Generalization	23	15.3	

Of the participants in this study a majority, 54.7%, fell into one of the explicit strategy categories, 3, 4, and 5. However, nearly a third (28.7%) did not give a response to Task 3, which involved generalization. These findings were noteworthy for several reasons. First, the generalization of an explicit rule was more sophisticated than constructing a generalization that was non-explicit. A majority of the participants actually chose to approach generalization explicitly. It was also difficult to generalize this task in a recursive manner, which would require the construction of many towers since the number of total towers increases exponentially as the height increases. Second and equally noteworthy, nearly a third of the participants in this study did not attempt the generalization. Finally, and perhaps of the greatest significance, was that even though 54.7% of participants attempted an explicit generalization, only 15.3% of all participants generalized successfully.

Thus, the majority of participants attempted the generalization task in an explicit manner, while nearly a third of all participants left the generalization task blank. One possible explanation for this might be that historically students have been asked to generate explicit rules. Therefore, they attempted to find an explicit solution even though they did not necessarily understand what it meant in context of the problem.

Research Question 3: Is There an Association Between Enumeration and Generalization?

The third research question in this study concerned whether a relationship existed between the enumeration strategies participants used to find all possible outcomes in Tasks 1 and 2 and the generalization rules they subsequently constructed in Task 3. To address this question, I investigated whether an association existed between how students chose to enumerate in Task 1 and the generalization rules they constructed in Task 3. Then, I replicated this analysis and looked for an association between the enumeration strategy used in Task 2 and the generalization rule created in Task 3. I followed this by analyzing eight follow-up participant interviews to find emerging trends that would explain the quantitative findings. Finally, I combined or mixed these findings to produce meta-inferences (Teddlie & Tashakkori, 2009). Thus, the results of the third research question will be divided into three sections: (1) Association between tasks 1, 2, and 3; (2) Analysis of the follow-up interviews; (3) Mixed Conclusions.

Association between 3 Block Towers and 100 Block Towers

To investigate the existence of an association between the enumeration strategy participants used in Task 1 and the generalization rules they created in Task 3, I used a Chi-Square test of independence. In this statistical test, the null hypothesis was that the two variables were independent of each other, or in the context of this study, that the enumeration strategy was not associated with the generalization rule that participants constructed. Table 16 is a contingency table for the enumeration strategy for Task 1 vs. the generalization rule constructed in Task 3.

Table 16

Cross Tabulation of Task 1 vs. Task 3

Cross tabulation of Task 1 Enumeration and Task 3 Generalization							
Task 1 Enumeration Strategy	**Generalization Rule**						**Total**
	0	**1**	**2**	**3**	**4**	**5**	
1	11	2	2	1	10	0	26
2	10	3	3	0	10	3	29
3	12	7	3	2	29	2	55
4	2	0	0	1	7	2	12
5	8	1	4	0	10	5	28
Total	43	13	12	4	66	12	150

A Chi-Square analysis was not appropriate since the assumptions of the Chi-Square test were violated by the data given in Table 16. The Chi-Square test assumes that the frequency in each cell is at least 5. In this case, 19 of the 25 cells were less than the minimum of 5 called for in the assumptions of the Chi-Square test. To address the violation of this assumption, I collapsed the enumeration and generalization categories, as described in Chapter III and repeated for convenience in Tables 17 and 18. The generalization categories were collapsed on a theoretical basis, as discussed in Chapter II. The non-explicit generalizations (Categories 0, 1, and 2) were collapsed into one category and the explicit generalizations (Categories 3, 4, and 5) were collapsed into a second one. Table 19 is a new cross tabulation of the data, which was collapsed according to the description in Chapter III.

Table 17

Task 1 Collapsing System

Enumeration Category	Collapsed Category
0	1
1	1
2	1
3	2
4	2
5	2

Table 18

Collapsed Generalization Framework

Category Name	Category Number	Collapsed Number
Non-explicit		
Blank	0	1
Counting	1	1
Recursive	2	1

Table 18 (cont.)

Category Name	Category Number	Collapsed Number
Explicit		
Whole-Object	3	2
Guess-and-Check	4	2
Contextual	5	2

Table 19

Cross Tabulation of Collapsed Data

Task 1 by Generalization Cross tabulation				
		Generalization		
		1.00	**2.00**	**Total**
Task 1	1.00	31	24	55
	2.00	37	58	95
Total		68	82	150

With these new categories, as shown in Table 19, the assumptions of the Chi-Square test were no longer violated. I therefore proceeded with the Chi-Square test of independence using SPSS. I also used SPSS to generate the correlation between Task 1 and Task 3 using the Spearman Correlation Coefficient, which measures the statistical dependency between two variables. Tables 20 and 21 report the results of the Chi-Square test as well as the calculation of the correlation between Task 1 and 3.

Table 20

Chi-Square of Task 1 vs. Task 3

Chi-Square Test					
	Value	**df**	**Asymp. Sig. (2-sided)**	**Exact Sig. (2-sided)**	**Exact Sig. (1-sided)**
Pearson Chi-Square	4.263	1	.039		
Fisher's Exact Test				.043	.029
Linear-by-Linear Association	4.235	1	.040		
N of Valid Cases	150				

Table 21

Correlation between Task 1 and Task 3

Correlation between Task 1 and Task 3				
Correlation Measure	**Value**	**Asymp. Std. Error**	**Approx. T**	**Approx. Sig.**
Spearman Correlation	.169	.081	2.081	.039

The results of the Chi-Square Test reported in Table 20 were significant (p = .039), and suggested that enumeration strategy in Task 1 was related to generalization rule constructed in Task 3. This indicated that the generalizations that participants produced were related to the enumeration strategies they used to find all possible outcomes. However, the calculation of the correlation between these two variables was weak at only 0.169 but statistically still significant (p = .043). An explanation of this surprisingly weak yet significant correlation will be given in the mixed findings section of this chapter. A strong correlation was expected between the enumeration strategies that students selected and the generalization rules they constructed.

Association between 4 Block Towers and 100 Block Towers

Continuing with the analysis designed to answer the third research question, I investigated the association between Task 2 and 3. Table 22 is the cross tabulated data of Task 2 vs. Task 3. Similar to the concerns raised regarding the analysis of Task 1 and Task 3, the assumptions of the Chi-Square were again violated with 24 of 36 cells having fewer than the required total of 5. Therefore, I collapsed the data in a similar manner as presented earlier in this chapter and outlined in depth in Chapter III. This theoretically collapsed data, which produced a new table of cross tabulations free from the violations of the assumptions of the Chi-Square, can be found in Table 23.

Table 22

Cross Tabulation of Task 2 vs. Task 3

Cross Tabulation of Task 2 vs. Task 3							
Task 2 Enumeration Strategy	**Generalization**						**Total**
	0	**1**	**2**	**3**	**4**	**5**	
0	2	0	0	0	1	0	3
1	7	2	2	1	5	0	17
2	6	3	0	0	12	2	23
3	12	4	5	2	32	4	59
4	9	3	3	1	10	1	27
5	7	1	2	0	6	5	21
Total	43	13	12	4	66	12	150

Table 23

Cross Tabulated Task 2 vs. Task 3 Data

Task 2 by Generalization Cross tabulation				
		Generalization		
		1.00	**2.00**	**Total**
Task 2	1.00	22	21	42
	2.00	46	61	107
	Total	68	82	150

Similar to the earlier analysis, I used SPSS to test the association between Task 2 and Task 3 and again found the correlation between these two variables to gauge the strength of the relationship using the Spearman Correlation Coefficient. Tables 24 and 25 report these findings.

Table 24

Chi-Square Test of Association between Task 2 and Task 3

Chi-Square Tests					
	Value	**df**	**Asymp. Sig. (2-sided)**	**Exact Sig. (2-sided)**	**Exact Sig. (1-sided)**
Pearson Chi-Square	.827	1	.363		
Fisher's Exact Test				.372	.233
Linear-by-Linear Association	.821	1	.365		
N of Valid Cases	150				

Table 25

Test of Correlation between Task 2 and Task 3

Symmetric Measures				
	Value	**Asymp. Std. Error**	**Approx. T**	**Approx. Sig.**
Spearman Correlation	.074	.082	.906	.367

The results shown in Table 24 suggested that the association between the enumeration strategy chosen in Task 2 and the generalization rule constructed in Task 3 was not significant (*p* = 0.363). The correlation between these two was once again weak at 0.074, but unlike Task 1 vs. Task 3, it was not significant (*p* = 0.367). This finding was even more surprising than in the prior analysis. Not only was the relationship not statistically significant, the strength of it was even weaker than in the prior analysis. This finding, like the earlier one, will be given more attention and explanation in the mixed findings section of this chapter.

Conclusions

The quantitative analysis summarized in the preceding sections demonstrated that for Task 1 and 2, the participants used more advanced enumeration strategies on Task 2 than 1. Furthermore, a statistical association existed between the type of enumeration strategy that they used in Task 1 and the generalization rule that they produced in Task 3, as shown in Table 20. However, while the strength of this association was weak, no association existed between the enumeration strategies that participants selected in Task 2 and the generalizations that they constructed in Task 3. The weakness of the association between Task 1 and 2, 0.168, and the lack of association between Task 2 and 3 suggested that there was very little if any relationship between the enumeration strategies that participants selected and the generalizations that they constructed. Quantitative analysis alone was unable to answer why this was the case. However, it was still a very important finding, which will be addressed in greater depth in the mixed findings portion of this chapter. The mixed findings used these quantitative conclusions and the qualitative conclusions in the next section to form a meta-inference to explain why this important and surprising finding occurred.

Analysis of Follow-up Interviews

This section reports the findings of the qualitative analysis conducted on the follow-up interviews. I interviewed eight participants in order to understand their work on Tasks 1, 2, and 3. After they completed the exposition of their thinking on these tasks, I asked them to engage in Task 4, a closely related task. Using constant comparative analysis with open coding and extreme case analysis (Miles & Huberman, 1994), I identified five emerging trends, summarized in Table 26, while Table 27 reports which of the eight participants expressed each trend. The remainder of this section is devoted to the discussion of each trend and to comparing and contrasting the range of responses that I observed in each. Examples were chosen to clearly demonstrate the range within each emerging trend. Although participants often demonstrated a trend multiple times, I did not include every occurrence of it, but instead selected the most illustrative example.

Table 26

Emerging Trends

Emerging Trend	#	Description
Recognizing the Need for Organization	A	Constructing a strategy to provide organization for one of the enumeration tasks.
Recalling a Formula	B	Remembers a formula or strategy from prior study.
Using an Error	C	Noticing an error, which assists in better generalization.
Found Formula first	D	Finding a formula first and then building or drawing out the towers to verify that the formula is correct.
Relationship in Tasks	E	Seeing a relationship between Tasks

Table 27

Trends by Participant

Trend	Jack	Kathryn	Mary	Mike	Melinda	Rachel	Sara	Tonya
1	X		X	X	X		X	X
2	X	X		X	X	X		
3	X	X						
4	X		X		X			
5		X	X					

Emerging Trend A: Recognizing the need for organization. One of the most prevalent and important trends observed was that, as participants engaged in describing their reasoning, they recognized the increased complexity of the current task in comparison to prior tasks. This often happened when participants moved from 3 block tall towers in Task 1 to 4 block tall towers in Task 2. They realized that they needed to organize their enumeration in such a way that it was less likely to duplicate or omit a possible tower. Six of the eight participants exhibited this realization. In this section, I present data in the form of transcripts from six of the eight participants, Jack, Mary, Mike, Melinda, Sara, and Tonya, each demonstrating in different ways the need to organize their enumerations and how their realizations affected their ability to generalize.

Participant Jack. Jack demonstrated four out of the five total emerging trends. His was a very illustrative case in how organizing an enumeration made it possible to generalize correctly. Before he engaged in the interview, he was unable to find a correct generalization in Task 3, 100 block towers. He began by describing his work on Task 1 and noticed during the conversation that in his original enumeration he had skipped a tower. He then described what he had noticed, which made him believe he had missed a tower.

> **Jack:** Actually I think I missed one or two. I think it's 8 [He originally had drawn out 6 total towers] now that I look back at it. I forgot the sandwich one.
>
> **Interviewer:** So what did you see that made you think that you had left some out?
>
> **Jack:** I was doing like how I look at colors. But I could tell that you do this [He pointed at the bottom block of the towers BBB BBR BRR and indicated that you could put a blue on the bottom of BRR instead of the R and get BRB and have something different [(See Figure 8 for an illustration)] and put a blue down here and be different than before. Because you start with a blue and then it could be either blue or red and you can go back and be either blue or red.
>
> **Interviewer:** Take me back through that.
>
> **Jack:** Ok, so the first one is either blue or red.
>
> **Interviewer:** You mean that the first block is either blue or red?
>
> **Jack:** Yeah, the first block is either blue or red and the next blocks will either be blue or red. You can go blue [He indicated the first block of a tower], blue/red [He indicated the second block in that tower, holding the first one constant], then

blue/red [He indicated the third block in that tower again holding the first one constant]. So it's like a tree.

Interviewer: Ok I think I see that. Why don't you draw that out on here [See Figure 9]?

Jack: So it's like blue then my options are blue or red and then my options from that could be blue red and blue red [Here he was drawing out the 'legs' of this tree diagram]. So then when you do red it can be blue red [He drew out the initial steps in the two legs] then blue red and blue red [He finished drawing out the legs of the tree diagram]. So I messed up and left these out [He circled the ones he had left out on his paper]. So it's really 8.

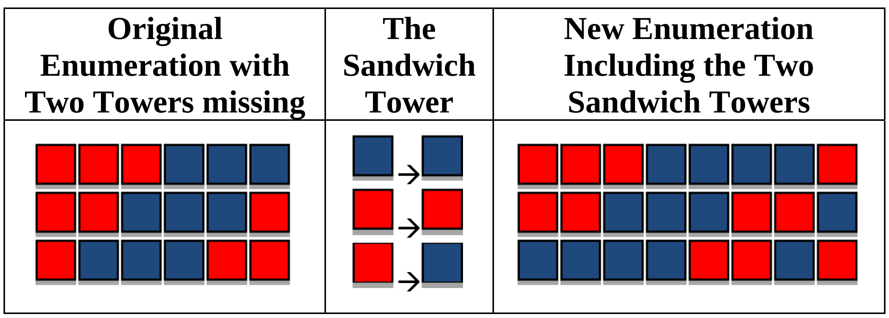


Figure 8. Jack's Sandwich Tower

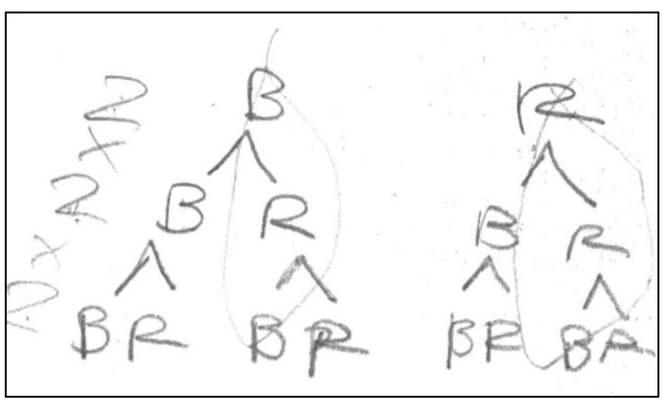


Figure 9. Jack's Tree Diagram

In the discussion above Jack formulated a tree diagram strategy, an example of the *odometer strategy*, which enabled him to organize his enumeration and prevented him from making duplications or skipping a tower. It is important to note that he was not able to generalize before he had constructed the tree diagram and that this new organizational strategy gave him a way to enumerate correctly, which in turn provided a way for him to generalize. After he discovered his new organizational system, he went back and corrected his work in Tasks 1 and 2 and supplied the following reasoning for his generalization.

Interviewer: So, what is your new answer?

Jack: 2 to the nth, where nth is the number of blocks in each tower.

Interviewer: This would be what [I indicated Task 3]?

Jack: 2 to the 100^{th}.

Interviewer: How did you realize that this [Task 2] was the 2^4 and that this [Task 3] was 2^{100}?

Jack: Well, when I first looked at this [Task 1], I asked myself how can I get 8 from 2, and I thought 2 * 2 * 2 since there are two options, two options, and two options.

Interviewer: What do you mean two options?

Jack: Blue or red. So, each option is 2 so you multiply it by 2 and by 2. So, I thought 2^3 equals 8. And then I saw how this one [Task 2] was 16 and if you do 2 * 2 * 2 * 2, that is 16, and if you do 2^4, that is also 16. So, then I just thought it would be 2^n.

Thus, Jack recognized the need for organizing his enumeration and did so via the tree diagram. This proved to be instrumental in enabling him to complete the tasks, which provided the structure for him to extend his reasoning not only to a correct generalization, but also beyond that to the mathematical reasoning that undergirded his generalization.

Participant Tonya. Tonya was the second participant who demonstrated recognizing the need for organization. She stated that she worried about leaving out or repeating a combination of numbers in Task 4. This fact was instrumental in enabling her to generalize correctly since it helped her to fully and accurately enumerate Task 4, which she correctly generalized. The following transcript is evidence of her realizations as she verbally explained them to me.

Interviewer: Why do you feel like you need to order them?

Tonya: Because I might be confused as to what I've used and what I haven't.

Interviewer: OK

Tonya: 6, 7, 8. I could do 666, 667…oh this is kind of like what I did with the blocks.

Interviewer: In what way?

Tonya: Because if this [666] were the blocks it would be all blue. Well, these only have 2 colors and these have 3 numbers, so it's kind of the same thing.

Interviewer: Tell me about your ordering there.

Tonya: Ok, here I started with the 6's [666]. Then I dropped the last 6 and moved to the next number [667]. So, these [666, 667, and 668] are the numbers I can do with a 66 as the first two numbers. So, then I moved onto 67 and put a 7 on the end [677], then kept the 67 and put an 8 [668] on the end. But now I can do…I'm probably leaving something out…now I'm going to move to the 68.

Interviewer: Are you talking about the first two digits?

Tonya: Yeah.

Interviewer: Is that what you are ordering on?

Tonya: Yes. So, I would do 686, 687, and since this is in with my six, I'm starting them all with a six. Then I have 688. Well that works. I was thinking I could do 678, but I looked back and I've already done it here. I'm moving to the next number so I start with 777, 776, 778 then 767, 767…I'm kind of lost in my pattern now.

Interviewer: Tell me what you need to start with.

Tonya: I'm kind of following these [She indicated the numbers that started with a 6].

Interviewer: So you started with the 777.

Tonya: Then I dropped the last one and 776 and 778 and that's all I could do with those. So then I moved to 76. So then I should move to 768 but I could do 766 too.

Interviewer: Then where would you move to next?

Tonya: Then I would move to 78 and would do 786, 787, and 788. Yeah that's what I would do. 888, 886, 887, 868, 867, 866, 877, 878, and 876. These are out of order too. I started with the 8, dropped it and moved it.

Interviewer: When you said they were out of order, what did you mean?

Tonya: Because I did the same in these top two…well, no I didn't. Never mind. Ok I'm trying to do them in the same order and they're ending up not in the same order. See now I'm missing one.

Interviewer: Why do you think you are missing one?

Tonya: Because if all of these can equal the same amount, then the last one should have the same amount too.

Interviewer: OK. Retrace your steps.

Tonya: Oh, I should've gone to all 86's and done all the 86's that I could've done.

Interviewer: So, do you think those are all of them?

Tonya: I'm going to go with yes.

Tonya used her organizational scheme, generated in Task 4, to help her see when she had missed a number in her enumeration. At first, she was confused by how to do this but as she continued to work through Task 4, it became increasingly clear to her how she ought to organize her enumeration in such a way that she did not leave out or repeat any three digit numbers. This realization enabled her to go on and correctly generalize because she knew that she had enumerated correctly because her scheme enabled her to not repeat or leave out any three-digit numbers.

Participant Melinda. Melinda provided a third example of the emerging trend of recognizing the need for an organization. Like Jack and Tonya, her realization led her to a correct generalization. Although her realization was different from Jack's and Tonya's, she recognized that the odometer strategy was superior to the opposites strategy. As she described the increasing complexity of the tasks, she discussed her thinking when moving from Task 1 to Task 2. She compared her strategy in Task 1 (opposites) to her strategy in Task 2 (odometer) and then went on to generalize correctly in Task 3 and Task 4.

> **Interviewer:** Do you feel this way [grouping by opposites] is better than this way [Odometer] when you have the physical blocks?
>
> **Melinda:** Um, I guess I feel like this [the Odometer] would be better because when you get to having 4 [blocks], it's more difficult than when you have 3 blocks in a tower because there are 4 now and there are a lot more options and it is easy to get confused if you don't keep track of what you are doing. That's why this is easier because you can visibly see the blocks shifting.

Melinda's acknowledgment of the inherent increasing complexity of the tasks enabled her to realize that the *odometer strategy*, or in her words, being able to 'visibly see the blocks shift', a visual trait possessed by the *odometer strategy*, caused her to shift from using the *opposites strategy*. This shift, and the complete implementation that it enabled, was valuable and helped her to generalize correctly.

Participant Mary. Mary, like the previously mentioned participants, realized during the interview that she needed to organize her enumeration, but her strategy did not enable correct generalization as it did for the participants discussed thus far. Mary correctly enumerated 3 block tall towers but quickly realized, when she began working on 4 block tall towers, the need for organization. The following transcript relates her thoughts as she was building 4 block tall towers and illustrates her realization of the complexity of the task and her response to that complexity.

> **Mary:** Can I rearrange these?
>
> **Interviewer:** Please feel free to. Just explain to me how you are rearranging them, and why you feel it is necessary to rearrange them.
>
> **Mary:** Because you start with this one [RRBR], you can have one blue in this whole red setting. Then you can have 2 blues in this one [RBBR] and then you can have 3 blues in this one [RBBB] and then 4 blues. Therefore, you have all the possibilities where blue can go. And then in this one you have red ones [RRRR], then 3 red and 1 blue [BRRR], then two [BBRR], and then one [She first built BRBR and then realized that she needed to have BBRB also but she did not carry her organizational scheme through to the end because she left out BBBR] and then you do that by one, wait a minute there is supposed to be one. And those are all your different possibilities.
>
> **Interviewer:** Ok, so that's what, 12 of those?
>
> **Mary:** Yes 12.
>
> **Interviewer:** So, you can't make any more because they would be duplicates?
>
> **Mary:** Yeah.

Thus, Mary realized in her statements regarding the number of red and blue blocks, contained in a given tower, the strength of systematically organizing her enumeration. However, unlike Jack, Tonya, and Melinda she did not arrive at the correct enumeration because she had not completely exhausted all possible arrangements of blocks. Her enumeration strategy was not as strong as Jack's tree diagram strategy, because she only found 12 total towers instead of 16. Nonetheless, although she recognized the need for organization, she was not able to construct the correct generalization because she had left out four tower combinations.

Participant Mike. Mike was the fifth participant who demonstrated the emerging trend of recognizing the need for organization. His realization was similar to Tonya's, but unlike Tonya, who was concerned with both repeating herself and leaving out combinations, he was only worried about repeating himself. As Mike explained his reasoning on Task 4, he became convinced that he should be concerned with not repeating a number combination. In the following transcript Mike was trying to find all possible three-digit numbers that could be made with 6, 7, and 8 and was describing how to list them.

> **Mike:** Um, just going down the line I will do 676 and I'll just stick with one number rather than going down the line like that. 667 then 678…no wait.
>
> **Interviewer:** Why did you change your mind there?
>
> **Mike:** Because I'm afraid that if I start in with the 8 and don't finish with all the possibilities that have a 7, I'm going to make repeats down the line because it looks like this is going to be a long one. [He listed all the numbers that started with a 6] Now I'm going to move onto the 7's and keep the 7 as the first number. I think this will help make it so that it won't go on as a long line, and I can keep up with what's repeating and what isn't because unlike with the red and blue

> blocks, where you were working with 2 unique variables, here you're working with three unique variables that are going to be constantly changing. [He listed all the numbers that contained 7's] With this one I'm going to change it up a little. I'm going to include the repeats and just go through the line and see where I repeated things.
>
> **Interviewer:** What do you mean include the repeats?
>
> **Mike:** In order to keep it consistent with what I was doing up here, like saying 676 if I wanted to go down the line and I would need to go 767 and I've already . . . no wait, I haven't done that. Scratch that last thought! That would not be a problem in this.

Mike realized that without organization of his enumeration, he would be likely to construct numbers that he had already recorded. He even stated that he would include the repeats and then noticed that when he did so, his organizational method created a situation with no repeated numbers. However, he did not arrive at a correct generalization in either 100 block tall towers or 100 digit numbers. Thus, his realization did not lead to a correct generalization because his concern for repeating himself did not include a concern for not leaving out combinations as well.

Participant Sara. Sara, like Jack, recognized the need for organization but developed a different organizational scheme than he did. However, unlike Jack she did not carry her newly constructed organizational scheme through all the tasks. Thus, unlike Jack, she was not able to provide the structure necessary to aid her in correctly generalizing. As she engaged in 4 block tall towers, she decided that it was more complex and needed more organization than did 3 block tall towers. This realization carried over into how she solved the first part of Task 4 when she constructed a number line strategy to aid in how she organized her enumeration in order to avoid creating repeated entries.

> **Interviewer:** Tell me about what you mean thinking about it on a number line.
>
> **Sara:** So, in your mind you think about starting with a 6. So if you think on a number line you have your 66, your 67, and your 68. Those are the only numbers that you can make with a six. Then you go forth and go to your 7 and you have 76, 77, and 78, and those are the only numbers that you can make that start with a 7. Then you move onto to 8 and you have your 86, 87, and 88. So you think about where does this number fall on the number line. This falls before this and this is next and so forth.
>
> **Interviewer:** You said you know that you've found them all because these [66, 67, and 68] are all the numbers that start with a 6. What about this [76, 77, and 78]?
>
> **Sara:** Like when I thought about it so, we're starting with 7, and then let's go back and 76, 77, and 78. That's all the given numbers that they've given you to work with are 6, 7, and 8 so you can't go 79 or 80. You have to go up to the 8's.

Sara was able to use her number line strategy when asked to find two digit numbers that consisted of the numbers 6, 7, and 8. However, she did not carry this organization method through to part b of Task 4, where she had to list 3 digit numbers that were made up of 6, 7, and 8. Thus, she was not able to generalize in Task 3, 100 black tall towers, or in the final part of Task 4, 100 digit numbers, because she did not use her organization scheme throughout all the tasks.

Summary of recognizing the need for organization. Noticing the need for organization was important for many of the participants in this study. However, a continuum of responses existed, which assisted generalization at one end and hindered it on the other. Jack, Melinda, and Tonya were able to use their realization to help them find a correct generalization. Each of them developed some form of the odometer strategy to assist them. Their formulations differed, but in each case development of some type of organizational plan or scheme enhanced their ability to generalize correctly. For instance, Jack developed the tree diagram, which provided an organizational scheme that ensured that he neither replicated nor left out any combination of towers or numbers. Melinda and Tonya's realizations about ordering their enumerations in a systematic manner helped them in counting and generalizing sample space. However, not all of those who recognized the need for organization were able to generalize correctly in Task 3, 100 block tall towers, or Task 4, 100 digit numbers, as was the case with Sara, Mary, and Mike. For instance, Sara did not use her organizational scheme throughout all of Task 4, which caused her to not be able to generalize. Neither Sara nor Mike consistently recognized or used their organizational strategies throughout all the tasks. Thus, whenever the participants recognized organization and consistently implemented it, this helped facilitate generalization, but if they did not consistently use it, they were unable to generalize.

Emerging Trend B: Recalling a formula. Many of the participants in the interview stated that they recalled that a formula existed for how to generalize a task, such as in Task 3 or 4. As was the case in Trend A, a continuum of responses existed that either helped or hindered generalization. In some cases, recalling this formula aided in generalization. Either participants remembered it explicitly or were able to reconstruct it because they remembered aspects of the formula but did not recall it in its entirety. In other cases, it hindered their generalization and caused them to struggle with finding a formula. Five of the eight participants stated that they recalled that a formula existed for a problem of this type. Jack, Kathryn, Mike, Melinda, and Rachel all gave verbal accounts of how this either helped or hindered them in their attempts to generalize in Tasks 3 and 4.

Participant Jack. As introduced earlier, Jack stated that he remembered doing a tree diagram in school. Recalling a strategy that he had used in prior educational experiences helped him enumerate correctly in Tasks 3 and 4. In the following transcript he described what made him think about using the tree diagram.

> **Interviewer:** I'm curious what made you think of this [The tree diagram] now but you didn't think of it in class?
>
> **Jack**: I was around people and they were all doing it like that [He indicated how he had originally done it in Task 2 part b, which was not his tree diagram], and now I remember when we used to have to do this in high school, we had to do a tree, or when in middle school, we had to do probability. It was always like heads or tails and heads or tails. It made me think of what I do . . . it helps me not double count . . . With this way [He indicated his original formulation again] it's hard to double check yourself, but with this way [He indicated the tree diagram] there's only two options you have. So, you can just go tree because you can easily mistake rewriting one or leaving one out. But with the tree diagram you can clearly see all the trees or branches on the trees.

In the account above, Jack described how he recalled a strategy that he remembered from middle and high school. This recollection helped him fully enumerate in an organized manner. Because he realized that he need to organize his enumeration this was related to Trend A. Once he remembered the tree diagram strategy, he was quickly able to generalize in Task 3, 100 block tall towers. He then used this same strategy in Task 4, 100 digit numbers, which was instrumental in helping him generalize quickly and efficiently.

Participant Melinda. Melinda, like Jack, remembered doing a similar problem in the past. However, her memory of the past problem included a formula as well as the mathematical reasoning behind it, whereas Jack only recalled an enumeration strategy. In this case memory of a formula, which included the mathematical understanding of why the formula worked, enabled her to generalize from the outset in Task 1. For instance, when asked if she had started out in Task 1 by writing 2 * 2 * 2, or if she had enumerated and then constructed a generalization, she stated that she wrote out the 2 * 2 * 2 and then checked that her answer was correct by drawing out the enumeration.

Interviewer: My first question is, did you do this part first [I indicated 2 * 2 * 2]?

Melinda: Yeah.

Interviewer: You did that from the outset?

Melinda: Yeah.

Interviewer: Tell me about that first.

Melinda: All right, I remember with combinations. I don't remember when that was, but back in school somewhere.

Interviewer: Back in high school?

Melinda: Probably middle school. I remember that you would multiply the number of choices you had. Like how many you had of the choices. For example, in this there are two different colors of blocks, and you want 3 blocks in a tower. So, because of the different options, and there are two of them, you would multiply 2 * 2 * 2, basically, two to the third power to figure out how many different combinations that you have. Afterward, just to show that I was right, I listed the different possibilities.

In the above interview excerpt, Melinda explained that she remembered the formula for how to generalize a problem and went as far as to explain how the formula worked mathematically. This recollection, which included the mathematical reasoning behind the formula, made the generalization in Task 3 very quick and easy. In a follow-up question, I asked if there was anything she would like to add to what she had already said. She then described recalling a problem in middle school where her class had been asked to construct the number of outfits that could be made with a combination of colored shirts and pants. She remembered that the solution involved multiplying the number of shirts and the number of pants together to find the answer. This recollection was important in enabling her to generalize quickly and correctly. Melinda's recollection of the correct formula, which included the contextual reasoning behind it, made it easy for her to generalize and describe how she had arrived at her generalization.

Participant Kathryn. Kathryn, on the other hand, did not remember the correct formula, as did Melinda. Nor did she remember an enumeration method like Jack did. Instead, she thought she remembered that a formula existed but was unable to reconstruct it. This resulted in the formulation of an unsuccessful generalization.

> **Interviewer:** What did you think about the rule for 100 blocks?
>
> **Kathryn:** Well, I looked at the one with 3 blocks and the one with 4 blocks and with the 3 blocks tall is 3 times 2 plus 2, and that's what it ended up being, which is 6 plus 2 is 8, and with the 4 block tall towers it was 4 times 2 plus 2. So, I figured that must've been a pattern. So with 100 you could do 100 times 2 plus 2, which is 202.
>
> **Interviewer:** You tested your rule out on the 3 and 4 block towers and it gave you what you expected?
>
> **Kathryn:** Yes, so if it was 3 blocks tall I did 3 times 2 plus 2, and if it was 4 blocks tall, I did 4 times 2 plus 2, and that was the answer I found for both of those. So I figured for the 100 block tall one, it could be 100 times 2 plus 2 because that's what the pattern was for the other ones.

> **Interviewer:** Do you have anything else that you'd like to share? Something that you may have thought about as you were doing it, that you think I might find interesting?
>
> **Kathryn:** It reminded me of a problem we did in class where we were trying to figure out how many different outfits you could make with a certain number of shirts and pants, and how you could pair them together, but I couldn't remember how we had done that. So, I came up with my own way.

Kathryn found that the enumeration for Task 1 gave her a total of 8 towers and that the enumeration of Task 2 gave a total 10 towers. Thus, she constructed a generalization, where the total number of towers was found by multiplying the height of the tower by two and then adding two. She then stated that she recalled that a formula for this existed and that she remembered a similar example from middle school as Melinda had. This example also involved the construction of combinations with pants and shirts. However, unlike Melinda she did not remember the correct formula, which led her to construct a formula that did not take into account the context or reasoning of the problem. Instead, she found a rule that fitted the data and assumed that this must be the formula she could not remember.

Participant Mike. Like Kathryn, Mike also recalled that a formula existed but could not remember it. Although he was very close to being able to generalize, the fact that he knew there was a formula kept him from figuring out what the generalization was, and how it could be obtained with the data that he had at his disposal.

> **Mike:** So, the first one says how many towers can you create that are three blocks tall and are made from the red and blue blocks. You can have three red and three blue (He built RRR and BBB) for your first two combinations. I can't remember the equation that goes to this. So, I did it completely visually. So, basically you go through the whole process of figuring out how many combinations there are. So,

you just pick one set either red or blue. You don't want to have to do this particular strategy for every tower (He was describing setting up the odometer strategy for tower construction) especially for the 100-block towers. That's the one I didn't answer because although it doesn't require the equation that I don't know, but it makes it easier.

Interviewer: If you know there are 8 towers when there are 3 blocks and 16 towers when there are 4 blocks, does that help you any knowing how many there would be if you had 100 blocks?

Mike: Yeah, it's doubling every time you add a block. Right?

Interviewer: What do you think?

Mike: I don't know but it seems that way.

Interviewer: Well, if that were the case what would it mean if there were towers that were 100 blocks tall?

Mike: That would mean…I don't know.

Interviewer: It's okay if you can't think of anything. Feel free to talk your way through it if you want to also.

Mike: I'm fairly certain this isn't right. If I divide 4 into 100, you would get 25. If you're doubling that, you would get 50, but I don't think that is right.

Interviewer: What are you trying to do when you are dividing 4 into 100?

Mike: Um, I'm trying to get it so that I get back to something I'm familiar with, like this problem [Task 2], where I was working with 4 blocks.

Interviewer: What would it be if you had towers that were 5 blocks tall?

Mike: It would be 24, I think.

Interviewer: So how did you get 24?

Mike: Um, based on the first problem it was…no, wait that would be right. It's multiply by 2 so it would be 32.

Interviewer: What about for 6?

> **Mike:** It would 72 if it would be going by the same thing. No wait . . . that's 64. Never mind, sorry.
>
> **Interviewer:** If we extrapolated out to 100 and we didn't want to multiply by 2 every time, what would we do? If not, that's okay, at least you can figure out that it's going to double every time.
>
> **Mike:** Um, I swear it's just staring me in the face.
>
> **Interviewer:** We'll skip it and move on to the next task and maybe you'll have some thought on it that comes later.

Mike did a lot of thoughtful work in his reasoning process. Yet, in the end he was unable to generalize due to his focus on recalling a formula. While describing his reasoning, he realized that the pattern within the block tasks meant that the total number of towers doubled as the height was increased by one. Even with this realization, he was still trying to recall a formula when the reasoning for how to construct it was 'staring him in the face,' as he put it. He did not understand that recalling the formula was not necessary since he had already formulated the reasoning behind the formula. It appeared the act of trying to recall the formula and being unable to do so caused much of his inability to generalize.

Participant Rachel. Rachel also thought that she recalled a formula, but unlike Mike and the other participants, she assumed that the formula she remembered was correct. She never questioned the veracity of her recollection. Therefore, she did not draw out the towers to confirm her generalization.

> **Interviewer:** Is there anything you would like to add? Anything you thought of along the way that you think I might find interesting?

> **Rachel:** Well, we worked on this in class, but I couldn't remember how to do it. I just took the numbers and multiplied them together because I just figured that's what you would do.
>
> **Interviewer:** Did you not think about trying to draw something out on the paper there?
>
> **Rachel:** No I didn't. I didn't even think of it. I just took the numbers and multiplied them and figured that would be the answer.

Rachel simply assumed that the formula she recalled was correct and did not even try to enumerate to verify that it was actually correct. Thus, her recollection of an incorrect formula, along with her belief in it, caused her not only to generalize unsuccessfully but also not to attempt an enumeration to begin with.

Summary of recalling a formula. Being able to recall that a formula existed was either beneficial or problematic, depending on participants' ability to generalize successfully. In the case of Jack and Melinda, it helped them arrive quickly and efficiently at a generalization. Jack recalled a specific enumeration strategy, the tree diagram, which aided him in enumeration and in his subsequent successful generalization. Melinda recalled the exact formula and the mathematical reasoning behind it, which allowed her to generalize successfully. However, Mike expended time and mental energy trying to remember a formula, which he thought he knew but was unable to recall. Although Kathryn and Rachel also recalled that a formula existed, this hindered them even more than it did Mike. They both pursued incorrect solution strategies, which did not take into account the data or possible explanations available to them from the information they had at their disposal. Instead, they either assumed what they had done was correct, in the case of Rachel, or constructed a generalization that was based upon incorrect enumeration data from Task 1 and 2, as was the case with Kathryn. Thus, the ability to generalize successfully related to the ways in which a formula was recalled. If a helpful strategy or the correct formula was remembered, then this led participants to successful generalizations. If, on the other hand, they recalled an incorrect formula or spent an inordinate amount of time trying to recall it, this served to hinder their ability to generalize successfully.

Emerging Trend C: Using an error. Two of the eight participants, Jack and Kathryn, recognized that they had made an error in one of the enumeration tasks. This realization then led both participants to successful generalizations in Task 3.

Participant Jack. In the following transcript, Jack had just finished talking about Task 1, in which he developed a new organizational strategy, the tree diagram, discussed in depth in the earlier emerging trends. Immediately upon being asked to move to Task 2, he realized that his prior work was incorrect.

Jack: So, I messed up this problem too.

Interviewer: How do you know that you messed it up?

Jack: Because now there are four blocks and you can start with blue or red, but you'll have four options from that [Here Jack was referring to his tree diagram].

Interviewer: What will those four options be?

Jack: Actually it's not 4 [He began drawing out all of the options talking quietly to himself]. So, 8 and then 16. [He counted the last row of his tree diagram]

Interviewer: Why did you just count up these last ones like that?

Jack: The starting color being blue so that means that the end product with the starting color being blue is that many [He indicated the last row of color blocks]. Then you could just do it again when you start with red so it's this [He indicated that there were 8] times two. Because that just makes sense to me. If you start with blue and you have 8 and then you start with red you also have 8. I think I only got 14, so I missed two of them. Probably the between ones but that's OK. That's how I would get 16.

Interviewer: I thought it was interesting how you just counted these and got 8. How did you know that by just counting these that there would be 8 total towers?

Jack: Oh, because this is block 1 and block 2 and block 3 and block 4 [See Figure 10 for an illustration of his work] so by the fourth block that's your tower. And so I would say BBBB is a tower and I could just say that this is one, so 1, 2, 3, 4, 5, 6, 7, and 8. So, now I can fix my formula because that is wrong [He indicated Task 3 and crossed out his previous answer].

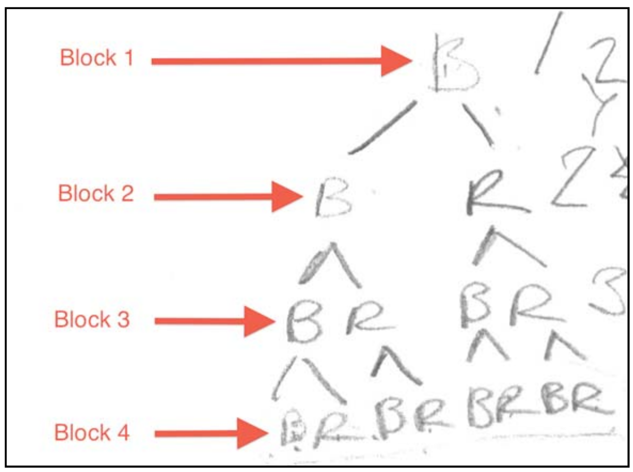


Figure 10. Jack's Second Tree Diagram

Once Jack realized that he had made an error in both Task 1 and 2, it caused him to rethink his reasoning. The error that he noticed helped him to recall a prior strategy as illustrated in a previous emerging trend, which in turn aided him in successfully generalizing in Task 3 and being able to generalize quickly in Task 4.

Participant Kathryn. Kathryn, like Jack, realized that she had made an error or errors, but for her it did not occur until she was working on Task 4. Once she detected an error, she was able to go back and correct her work on the prior tasks. Her realization demonstrated the power of noticing an error and the potential it has for self-correction.

Kathryn: I think that I did the blocks wrong.

Interviewer: Ok, which one? The 3 blocks or the 4 blocks?

Kathryn: Um, the 4 blocks.

Interviewer: Ok, so what made you think that?

Kathryn: Because, um there are different . . . Can I use them?

Interviewer: Yes, please do.

Kathryn: Because I realized that I can put two in the middle, like two blues in the middle with two reds and so on and so forth. I just realized that there are more ways to do it with four blocks.

Interviewer: What about this [I indicated task 4] made you think of that?

Kathryn: Because when you had three, what I was doing was just putting 7 in each position. But then I realized that you had to put two 7's in each one to get the answer.

Interviewer: Like when you wrote this 677 over here? [I was referring to when she was doing her first ordering of Task 4 part b where she had 677 out of order and this made her realize that she had left some combinations out.]

Kathryn: Yeah, that's when I realized I had to put two each in there also and so that kind of coordinates with these over here [She indicated the blocks]. You can't just have one of each in there. You have to put two of each in there too.

Interviewer: Why don't you make it? I think it's easier to see it.

Kathryn: So, red on the bottom. Yeah, I don't have any of this one.

Interviewer: That kind of matches your description of how you knew that you had them all because this doesn't replicate?

Kathryn: Right.

Interviewer: What else would there be?

Kathryn There would be two reds in the middle, two blues on top and two reds on the bottom, two reds on the top with two blues on the bottom, and um…I think that's all of them with the two's.

Interviewer: These are all of your two's [I indicated RBBR BRRB RRBB BBRR] so there's what 10…14?

Kathryn: So, now I have each with four red that has three of each and reds that have two of each and reds that have one of each. I feel like there should be more of these [She indicated those towers that had 2 red and 2 blue].

Interviewer: Okay, why?

Kathryn: I guess not because there are four of this kind [1 red and 3 blue]. Oh . . . there is another one [She built RBRB] and then the opposite [BRBR].

Interviewer: Okay, so how many do you have now?

Kathryn: 16.

Interviewer: Glance back at your write-up of that first problem. Does that change your answer for the hundred block towers?

Kathryn: Yes. Because now that I have found all of them for this, and there are 16 and when it was just 3 blocks tall there 8 it doubled from 8 to 16. So now for each one I think it would double instead.

Interviewer: Similar to the number problems?

Kathryn: Yeah that tripled for each one. Because there were three numbers and for this one it doubles because there were only two colors. So for five blocks I think it would be 32.
Interviewer: And then for 6.

Kathryn: It would 64.

Kathryn came to a very important realization in Task 4. She noticed, after working on it, that she had done all the preceding work in Tasks 1, 2, and 3 incorrectly. This motivated her to go back and completely rework what she had done earlier and develop a correct generalization.

Summary of noticing an error. Both Kathryn and Jack were able to utilize an error in their original enumerations in a self-corrective manner to reformulate their earlier work. In some cases, such as in Jack's, it stimulated the recall of a strategy that in turn enabled successful generalization. In other cases, like Kathryn's, it caused the participants to completely reconsider their prior mathematical reasoning, which also led to successful generalization. Thus, the benefit of noticing an error depended on what that observation inspired. Although it was beneficial in both cases, it enabled both participants that demonstrated it to successfully generalize.

Emerging Trend D: Finding a formula first. This Emerging Trend was similar to recalling a formula but was different in one very important way. This trend involved finding a formula before engaging in either a physical construction of the towers and forcing the enumeration to fit the formula. It was different from Emerging Trend B because Trend B involved trying to recall a formula that would fit the data gathered from enumeration. Jack, Mary, and Melinda all demonstrated this strategy. After having found a formula for the total number of towers, they then drew or built an enumeration to verify that their formulas were correct.

Participant Jack. In the following transcript, Jack described this process in response to my question about which part of Task 1, 3 block tall towers, he had done first. He indicated that he actually generalized before enumerating.

> **Interviewer:** First of all, did you do this part [I indicated part a of Task 1, which asked how many towers there would be] or this part [I indicated part b of Task 1, which asked him to draw out how many towers there were] first?
>
> **Jack:** I did this part [He indicated part a of Task 1] first.
>
> **Interviewer:** You did this [I indicated part b of Task 1] as verification of this answer [part a of Task 1].
>
> **Jack:** Yeah, I drew it out to make sure that this answer [part a] was right.
>
> **Interviewer:** So talk to me about what you did right here [part b].
>
> **Jack:** So, I started with red. I feel like red was presented first. So, I started with red all the way down. Then I started with two reds and decided to go with a blue next. Then I did one red and two blue. Then I did the same process for the blues and did all blues. Then two blues and a red. Then I did two blues and one red.
>
> **Interviewer:** And how did you know that this was all of them?
>
> **Jack:** Actually I think I missed one or two. I think it's 8 now that I look back at it. I forgot the sandwich one.

Jack demonstrated the trend of constructing a formula first when he indicated that he did part a of Task 1, which asked how many total three blocks towers there were. However, it is important to note that this tendency, at least for Jack, was linked to noticing an error, Emerging Trend C. In his initial work, which was done in class several weeks prior to the interview described above, Jack's original generalization was wrong since he described first finding a formula and then using a picture or drawing to convince himself that he was right. Then during the interview he realized that he had made errors, Emerging Trend C, and recognized the need for organization, Emerging Trend A.

Participant Mary. Mary also demonstrated the same tendency as was described in Jack's transcript. She first found a formula of 3 * 2 = 6 and then used her drawing of the enumeration to verify that there were only 6 towers. This generalization was incorrect. Her process of enumerating, which was conducted in an effort to verify her formula, only resulted in her confirming her prior incorrect generalization. In the following transcript Mary described her reasoning and how she arrived at the generalization she eventually settled on.

> **Mary:** Well, first it says how many different towers I did that [she answered that there were 6 total towers in part a] because there are 2 different towers and 3 blocks. So that's 2 * 3 = 6.
>
> **Interviewer:** That's how you got the 6 here [In part a of Task 1]?
>
> **Mary:** Yes.
>
> **Interviewer:** Did you do this [part a of Task 1, which asked for how many towers there are] or this [part b of Task 1, which asked her to draw out how many towers there were] first?
>
> **Mary:** I did this [part a].
>
> **Interviewer:** You did this [part a] first and then did this [part b] to verify?
>
> **Mary:** Yes. So, I did it with three of those [BBB] and you have that [RRR] and then you have that [BBR]…
>
> **Interviewer:** A quick question. You picked these [BBB and RRR] first, then how did you pick this one [RBB] as the next one to move to?
>
> **Mary:** Well, you have your three here. So, you have all your of the same color. Then you switch them up and have two of the same and one that is different and then you have your red blue red and those aren't the same as those. This [BBR] is

> the reverse of this [RBB] and they aren't the same. Then . . . this [BRB] is the opposite of this [RBR]. And that's how I got all of my possibilities.
>
> **Interviewer:** How do you know that is all the possible towers?
>
> **Mary:** I don't know.
>
> **Interviewer:** Okay, so how would you argue that you have them all if you had to convince someone?
>
> **Mary:** Okay . . . the question is 'How many towers can be built with red or blue blocks if the towers are three blocks tall?' So, using that you can use the same ones, then you have the opposite so that's another 2, and then you have more opposites and that's another two. If you did it another way you would have duplicates.
>
> **Interviewer:** So, what you're thinking of is that you would just think of random towers and anything that you might make would now be a duplicate?
>
> **Mary:** Yes.

Mary was unable to give a convincing argument as to how she knew that she had built all possible 3 block tall towers. She used a mixture of the *opposites* and the *odometer strategy*. However, unlike Jack, she did not notice that she had made an error. Instead, she sought to verify her formula from part a, which was 3 * 2 = 6.

Participant Melinda. Melinda also generalized before enumerating. However, as shown earlier, she recalled the correct formula, Emerging Trend B, and was able to use this to generalize successfully. The fact that she recalled the reasoning behind the generalization enabled her to draw out a solution that verified her formula.

Summary of finding a formula first. Jack, Mary, and Melinda first found a formula and used a picture of the enumeration to verify that their formula was correct. Jack then saw an error, Emerging Trend C, and went on to find a correct generalization.

Mary did not see an error in her formulation in part a of Task 1 and used her drawing to convince herself that she was correct but was unable to supply a satisfactory answer. Therefore, finding a formula and then using enumeration to verify it was detrimental for Mary. In Melinda's case, she recalled the correct formula and the reasoning behind it (Emerging Trend B). Thus, the strategy of using a picture or drawing to help verify a formula can be detrimental if the correct formula was not recalled or if a realization about an error was not observed.

Emerging Trend E: Using a relationship between the tasks. The fifth and final trend observed dealt with participants noticing a relationship or relationships between Tasks 1, 2, 3, or 4. For one of the participants, the relationship that they observed was helpful in the construction of an enumeration strategy, but for another the connection hindered their generalization. Kathryn and Mary discussed the relationships they noticed between the tasks and explained how their observations influenced their approach to solving them. Kathryn used a relationship to help her construct an enumeration strategy, but Mary used an incorrect similarity between tasks to construct an unsuccessful generalization.

Participant Kathryn. Kathryn recognized that a strategy that could be used for problems of this type was to allow numbers or colors to occupy all possible positions, which is a good description of the odometer strategy. In the following transcript excerpt, she described her reasoning,

> **Interviewer:** Were there any similarities between the tower problems and the number problems?

> **Kathryn:** Yeah, I tried to get the numbers in each different spot like I did with the blocks. I tried to get the colors in different spots in the towers to make sure that I was including all of them. So, bottom, middle, and top level [Here she was indicating the position of the red block in the 4 block tall towers], and here it is left, middle, and right [She was indicating the placement of the digits within a number].
>
> **Interviewer:** It was putting things in certain positions, is that how you were doing it?
>
> **Kathryn:** Yeah.

Kathryn observed that a similarity between Tasks 2 and 4 involved how she arranged them to ensure that all possibilities had been found. What she described was the *odometer strategy*, which turned out to help her enumerate more efficiently. However, in the long run this did not enable her to generalize successfully.

Participant Mary. Mary also noticed a similarity among the tasks, but unlike Kathryn she recognized a numerical relationship between the answers. The numerical relationship that she observed caused her to construct an incorrect generalization in Task 3.

> **Interviewer:** How many towers do you think we would have if each tower were 100 blocks tall?
>
> **Mary:** When you had four blocks, it increased from 8 to 12. So, you're increasing by 4. So, you would take 100 and multiply it by 4, which would give you 400 towers. So, each time you go up and add a block you increase by 4. That would be the rule.
>
> **Interviewer:** Okay, good job.

Mary realized in her answers to Task 1 and 2 that when a block was added to the height of a tower, the total number of towers increased by four. Thus, she constructed a generalization of 400 because she thought that each addition of a block would increase the number of towers by 4. Therefore, her generalization was to multiply 100 by 4, which resulted in her finding that there were 400 total towers.

Summary of using a relationship. Both Kathryn and Mary used similarities between the tasks, which then led them to enumeration strategies and generalizations. In Kathryn's case, it helped her formulate an enumeration strategy that was beneficial, while Mary used a relationship in the outcomes of the tasks that caused her to formulate an incorrect generalization. The most important thing I noticed in both of these situations was less about what they had done correctly or incorrectly but was more about participants demonstrating conclusions and strategies that could be developed as they observed similarities between the tasks, whatever those similarities might have been. Thus, using similarities between the tasks was helpful in aiding participants in generating enumeration strategies, but if the correct relationships were not used, then an unsuccessful generalization was often the result.

Conclusions from the Interviews

All eight participants in this study demonstrated at least one of the emerging trends described above. These trends illustrated some of the strategies and generalizations that participants used as they engaged in solving combinatorics problems. Some students recognized the need to organize their combinatorial enumerations because of the complexity of the problem. Other students remembered bits and pieces from past educational experiences. In some cases, this was beneficial because it enabled quick and accurate generalization, as was the case with Jack and Melinda. In others cases, it hindered generalization because the wrong formula was remembered, as demonstrated by Kathryn and Rachel. In the case of Mike, the simple act of trying to remember a formula caused him to be unable to think through the problem. Detecting an error in either an enumeration or generalization was also a very powerful tool that enabled deeper understanding of the mathematics content involved in solving a combinatorial problem. However, if the participants simply found a formula and then enumerated to check the veracity of it and either did not recall it correctly or notice an error in it, then this had the tendency to hinder generalization. Finally, if the participants were able to use relationships between tasks, they were able to develop helpful and efficient enumeration strategies. However, if the relationships they noticed were incorrect, it led them down a path that did not help them to generalize.

Mixed Conclusions

The strength of a mixed methods study is that both qualitative and quantitative analyses can be used to help answer research questions. In this study, the third research question could not be completely answered by only using one type of analysis. The quantitative findings suggested that very little relationship existed between the strategies participants used to enumerate the sample space and the subsequent generalizations they found. The qualitative findings also lacked an overarching reason for why the trends observed were important. This section will use both the quantitative and qualitative findings to form meta-inferences. The emerging trends detailed in the previous section were used to explore and give meaning to the lack of relationship or association between enumeration strategies and generalization rules that the quantitative analysis indicated was present.

The lack of any real association between enumeration strategies and generalization rules suggests that participants were often *compartmentalizing* how they went about engaging in the tasks or that the enumeration strategies they selected had little if any relationship to the generalization rules that they constructed. This was born out in Emerging Trend D, Finding a Formula First, where Jack, Mary, and Melinda all seemingly *compartmentalized* their solutions, but only Melinda was able to produce the correct generalization since she also recalled a formula, Emerging Trend B. Mike also *compartmentalized* when he spent time trying to recall a formula, when in fact he had all the information necessary to solve the problem at his disposal.

I observed that participants often recognized the need for organization, Emerging Trend A, and sometimes remembered that there should be a formula that would help them to generalize, Emerging Trend B. It took the increased complexity of either Task 2 or Task 3 to provide an impetus for remembering this fact. Those participants that exhibited the desire to organize their enumerations demonstrated a range of tendencies that either supported or hindered successful generalization. Participants who both systematically enumerated and consistently implemented their systematic enumerations were able to successfully generalize, as was the case with Jack, Tonya, and Melinda. Those participants who did not carry through their systematic enumeration were less likely to generalize successfully, as demonstrated by Mary, Mike, and Sara. Participants also revealed a continuum of responses when trying to recall a formula, which at one end resulted in a successful generalization but at the other an unsuccessful one. Jack, for instance, only recalled a helpful enumeration strategy whereas Melinda remembered the exact formula. They both were able to go on to generalize successfully. However, recalling that a formula existed did not always help with generalization. Sometimes it hindered the process, which was the case with Kathryn and Rachel who thought there was a formula but either remembered an incorrect one or simply assumed that they remembered it correctly and therefore did not bother checking their answers.

Another trend that addressed the lack of relationship observed in the second Chi-Square analysis was Emerging Trend C, in which participants noticed an error in prior work. If they saw, either through physical construction or mental realization, that some of their prior work was incorrect, this helped them change their minds about what the generalization rule should have been. This often occurred in Task 2 or 4, as was the case with Kathryn, who was able to see her mistakes, trace back through her work, and identify where she had gone wrong. She then proceeded to correct her mathematical understanding. Emerging Trend D also helped to explain the lack of relationship identified above because it pointed out that using a relationship between tasks and first finding a formula, Emerging Trend E, could both help or hinder depending on which relationships were noticed and which formula was found.

In the quantitative analysis, I found that the association between Tasks 1 and 3 was significant yet weak. The analysis also indicated that Tasks 2 and 3 were independent of each other and were also weakly correlated. The emerging trends observed and identified in the follow-up interviews yielded some explanations for why the correlation between Tasks 1 and 3 was significant, yet weak. Participants in these interviews first described finding a formula but were not necessarily considering why the formula worked, Emerging Trend D. This suggested that, while participants might be successfully generalizing, they were not making sense of the generalizations they constructed. In this case, making sense of generalizations meant observing that there was a relationship between the strategies they used to enumerate the sample space and the generalizations that they constructed about other similar sample spaces. Participants who used one of the more sophisticated enumeration strategies, such as the *odometer strategy*, were more likely to generalize successfully. This was the case with Jack and Melinda, who were both able to organize, Emerging Trend A, and recall a formula, Emerging Trend B. However, Mike demonstrated that remembering a formula was not always helpful since he spent a lot of time attempting to recall a formula but was unable to do so even after he saw the doubling pattern in the three and four block tall tower tasks, Tasks 1 and 2. Thus, the weak correlations observed in the quantitative analysis were explained by the qualitative findings, which suggested that participants were not making sense of the generalizations they were constructing and that they had compartmentalized the strategies they used to enumerate as well as the generalizations they constructed.

Conclusions

The purpose of this study was to investigate one of the precursors of probabilistic reasoning. To investigate this, I examined the connections between the enumeration of a sample space and the generalization to other similar sample spaces. I found an increase in the sophistication of the enumeration strategies employed by participants as they moved from Task 1 to Task 2, but this increase in sophistication did not increase the proportion of successful enumerations. I also found that the generalizations they produced in Task 3 were somewhat dependent on the enumeration strategies they employed in Task 1 but were only weakly correlated. However, the generalizations participants found in Task 3 were discovered to be independent of the strategies they employed in Task 2. The growth or increase in sophistication of enumeration strategies and the lack of practical significance in the association between enumeration strategies and generalization rules was indicative of how the participants may separate their knowledge. They *compartmentalized* their actions in constructing sample space and how they then mathematically generalized to other similar sample spaces. It seemed that they viewed these two processes as separate and as having little relationship to each other. When the participants in this study attempted to generalize, one of their first actions was trying to recall a formula from past educational experiences. In some instances this proved helpful, but in others it proved to be a hindrance. However in either case, participants attempted to rely not on their own mathematical reasoning abilities but on their recall of past formulas and experiences with mathematics content. These conclusions have implications, which the following chapter will discuss.

CHAPTER V

IMPLICATIONS AND FUTURE DIRECTIONS

This chapter will discuss the implications of the findings of this study. Specifically, it will be organized around two major findings. The first major finding was that participants who organized their enumeration of sample space in a systematic manner were better able to count the sample space and generalize about its total size. Another major finding was that participants tended to *compartmentalize* their understandings of reasoning about the processes of enumeration and generalization, making few connections between the two. Finally, this chapter will conclude by discussing the limitations and future directions of this study.

Organized Enumeration Supports Generalization

In studies conducted by English (1991, 1993), students were able to develop systematic strategies for generating solutions to unique combinatorial problems. Her studies used tasks that involved the dressing of toy bears to model different combinations, which demonstrated that students could “develop and modify their solution strategies, detect and correct errors, and develop generative procedures on their own” (English, 1993, p. 270). Noting, as did this study, that the most sophisticated strategy used by students was the odometer strategy, she recommended that the development of the odometer strategy be best handled, not through direct instruction, but through allowing students to grapple with meaningful, hands-on combinatorial problems. Jones, Langrall, Thornton, and Mogill (1999) found that utilizing these types of instructional practices contributed to the success that many students had with the construction of sample space problems. The findings of this study concur with those of English (1991, 1993) and Jones, Langrall, Thornton, and Mogill (1999) and provide evidence for organized enumeration supporting generalization.

Doerr and English (2003) found that a modeling approach where students engaged in a series of tasks, designed to build on understanding formed through prior experience, helped students build and create generalizable and reusable systems or models for selecting, ranking, and weighting data. Although the tasks used and the mathematical content was different in Doerr and English's (2003) study, the process that led to generalizable results was similar to the findings of this study. Some of my participants developed a systematic enumeration system, which helped them then construct generalizations that were reusable in other tasks. Researchers have also demonstrated, that given the opportunity to invent and revise strategies, generalizations are powerful in promoting statistical reasoning (Lehrer & Romberg, 1996; Lehrer & Schauble, 2000). The findings of this study agree and harmonize with those of these researchers because participants in my study demonstrated the ability to revise strategies and generalizations, which assisted them in completing and understanding the tasks.

However, the implications of the findings in this study go beyond the learning of statistics and probability and concern mathematical reasoning. For instance, Stacey (1989) found that as students engaged in the process of generalizing a linear relationship, substantial inconsistency existed in the choice of the model students selected. Students who began a question correctly often adopted a simpler, yet incorrect model for more difficult parts of the question. A similar result, regarding the generalization abilities of participants, was observed in this study where they often would adopt a mathematical formula or model for generalization. However, this formula was often incorrect leading to errors in later more complex tasks and there was generally an overreliance on these remembered formulas.

Statements about generality and its discovery are at the very core of mathematical activity (Kaput, 1999; Mason, 1996). Furthermore, generalizing a numeric situation is viewed as one way of assisting students in the transition to formal algebra (Lee, 1996). Generalization can provide a connection to referential contexts that can aid student understanding of symbolic representations (Lannin, 2005). However, many generalizations represent faulty reasoning where students incorrectly apply multiplication and ratio concepts (Stacey, 1989) or use a guess-and-check strategy to construct a generalization (Healy & Hoyles, 1999). Mason (1996) suggested that students often used local tactics in an attempt to find a generalization. He defined local tactics as a type of guess and check strategy using the numbers in the problem statement in various mathematical configurations, such as multiplying or dividing them. The current study found some of these same tendencies since participants attempted to generalize an expression or formula from the data. They generalized by simply manipulating the mathematical operations with little or no concern for whether or not these mathematical operations made any sense in context of the problem. The majority of participants attempted to generalize explicitly, where many of the observed generalizations were examples of Mason's (1996) local tactics.

Separation of Understanding of Enumeration and Generalization

A second major finding in this study was that participants tended to *compartmentalize* or separate their knowledge of the enumeration process and the generalization process, viewing each process as separate and disconnected. Batanero, Navarro-Pelayo, and Godino (1997) linked the difficulty of constructing sample space with the lack of combinatorial knowledge. English (1991, 1993), using a similar set of tasks employed in this study, found that students were able to develop and modify their solution strategies, detect and correct errors, and develop generative procedures on their own. The trends observed in this study, including the detection of errors and development of enumeration procedures, concur with observations made by English. However, this study adds to the body of knowledge by suggesting that students *compartmentalize* how they understand the processes of combinatorial enumeration and generalization. This *compartmentalization of knowledge* about mathematics has ramifications for how students approach probabilistic situations since without a complete and full enumeration of the sample space, they will be unable to construct and comprehend the concept of the probability distribution (Jones et al., 2007).

Schoenfeld (1992) in his review of literature on problem solving suggested that students had some general tendencies when they tried to solve unique mathematics problems. One of the most pervasive of these was that they believed there existed only one-way in which to solve a problem. This was usually the rule or formula that a teacher had most recently demonstrated for students. A second important tendency, noted by Schoenfeld (1992), was that students did not expect to understand the mathematics they were learning but rather expected to memorize a formula to be recalled and applied when needed. Schoenfeld's (1992) review of literature was conducted nearly twenty years ago, but little has changed in the area of research on problem solving since that time (Lesh & Zawojewski, 2007). Participants in this study were also engaged in problem solving activities and they, like the students that Schoenfeld (1992) referred to, very often attempted to recall formulas in a rote fashion, which rarely resulted in a successful strategy for the construction of sample space.

It is widely known that students of all age groups often use mathematically superficial reasoning when solving different kinds of tasks (Palm, Boesen, & Lithner, 2006; Palm, 2002; Verschaffel, Greer, & de Corte, 2000). Studies on secondary and undergraduate students go even further. They indicate that reasoning which focuses on past, familiar, or remembered educational experiences is dominant over reasoning based on mathematical properties or the context of the current problem, even when it may lead to progress (Bergqvist, Lithner, & Sumpter, 2003; Lithner, 2000, 2003). These researchers further suggested that students seldom made attempts at constructing their own solution strategies based on their own reasoning. Therefore, it becomes crucial for students to find solution procedures to imitate, with the decision about what procedure or formula to choose often being mathematically superficial (Lithner, 2000, 2003). The reliance on such mathematically superficial reasoning is not likely to be efficient for the learning of advanced thinking about mathematical concepts and ideas (Palm et al., 2006).

In addition, it is likely to have an even more detrimental effect on solutions, which deal with non-routine tasks where no easily recalled solution procedures are directly available to students. A large body of researchers demonstrated that many students of different age groups have difficulties solving non-routine tasks (Boesen, Lithner, & Palm, 2005; A. Schoenfeld, 1985; Selden, Selden, & Mason, 1994; Verschaffel et al., 2000). The findings of the aforementioned researchers suggest that the focus on remembering procedures, only superficially related to the task, limits the likelihood of finding a successful solution especially when the procedure is forgotten or a mistake is made in the procedure. The findings of the current study agree with the observations made by the researchers outlined above. Many participants attempted to generalize in an explicit manner yet spent time and mental effort on recalling formulas or procedures. As described by participants in the interview process, many believed that recalling a formula would assist them in solving the task at hand. However, rather than using the data they had and their own mathematical reasoning abilities to construct a generalization they instead attempted to recall formulas and procedures.

This concentration on recalled formulas, or the replication of procedures previously demonstrated to students, might have a great deal to do with the classroom environments students have experienced. Within the mathematics education literature, there seems to be agreement that classrooms which foster mathematical reasoning are environments where students are encouraged to be curious about mathematics and to develop their mathematical intuitions and analytic abilities (Franke, Kazemi, Battey, & Lester, 2007). However, research into the area of classroom practice demonstrates that many teachers, usually because of inexperience with classrooms of this type, depend on models of classroom instruction that rely on the evaluation of student answers rather than on the strategies students use to arrive at those answers (Hiebert et al., 2003; Hiebert & Stigler, 2000; National Research Council, 2001; Stigler & Hiebert, 1997, 2009). Many times, the teacher assumes responsibility for solving the problems and stating the formula as opposed to allowing students to struggle with the mathematical concepts as suggested by Hiebert and Grouws (2007). This tendency gives students little opportunity to reason about and to discuss connections between mathematical ideas and concepts. Instead, students are expected to simply respond with the next step in a given procedure used to solve the problem. This lack of ability to reason about mathematical concepts and reliance upon remembered or recalled facts and formulas were evident in the results of this study. Thus, an important implication for teachers and their pedagogical decision-making is that their classroom practices promote the making of connections between mathematical ideas and the discussion surrounding sample space enumeration.

The *compartmentalization of knowledge* surrounding sample space enumeration and generalization, described in this study, was also indicative of the chasm that exists between conceptual and procedural understanding. Hiebert and Grouws (2007) suggested that these two, conceptual and procedural knowledge, are two of the most valued learning goals in school mathematics. Of the two, conceptual understanding seemed to be the least well represented because my participants often spent time trying to recall formulas or ways of doing the problem as opposed to finding new and creative solutions that did not involve past recollection of rote formulas or memorized procedures. Also, as stated earlier, they seemed to make few connections between the mathematical features of enumeration and generalization. Hiebert and Grouws (2007) suggested that two features of classroom mathematics teaching facilitate conceptual understanding. First, they recommended that teachers give explicit attention to the connections among mathematical ideas, facts, and procedures. Other researchers have made the same argument (e.g. Gamoran, 2001; Hiebert, 2003; National Research Council, 2001).

By attending to mathematical connections, Hiebert and Grouws (2007) recommended that those concepts be treated in an explicit and public way. This might entail discussing the mathematical meanings that underlie procedures or the posing of questions about how different solution strategies are similar or different (Fennema & Romberg, 1999). For instance, Fuson and Briars (1990) found that making connections between the physical and written representations of addition and multi-digit multiplication problems helped students gain in conceptual understanding of those concepts. Some teachers were able to do this with a classroom demonstration followed by a classroom discussion while others used targeted small group activities to facilitate the same understandings. The implication of my study is that a similar process and structure should be applied to the construction of sample space because participants were supposed to make explicit connections between generalized formulas and physically enumerated representations, but generally did not do so.

Another feature of mathematics instruction, which facilitates conceptual understanding, is that students should be allowed to struggle with the underlying mathematics in a problem or task (Ball, 1993; Heaton, 2000; Hiebert & Grouws, 2007;

Lampert, 2003; Schoenfeld, 1985). By allowing students to struggle, Hiebert and Grouws (2007) meant that students should expend effort to make sense and meaning out of the mathematics contained in a problem. This occurs by solving problems that are within reach but requires thought about mathematical ideas that are not well formed. Hiebert and Grouws (2007) were echoing the thoughts of earlier mathematics education researchers, such as Polya (1957) and Stein and Lane (1996). For example, Stein and Lane (1996) found that tasks, which placed higher demands on students, resulted in greater conceptual development and understanding of mathematical concepts. For the students in Stein and Lane's (1996) study, the most challenging tasks involved conjecturing and pattern finding while the least challenging were those tasks that just required application of a procedure. The findings of my study concur with the recommendations of Hiebert and Grouws (2007) and the findings of Stein and Lane (1996) outlined above. Participants who struggled with the tasks during the interview were able to develop conceptual understanding by creating useful ways of organizing their enumerations, which in turn helped facilitate generalization.

The findings of this study suggest that when students are introduced to the concept of sample space and its construction, they should be given the opportunity first to think and reason about the mathematical concepts that underlie it. An emphasis upon the formula or procedure often causes difficultly with solving problems at a later date because it fails to allow students to form conceptual understanding of the mathematical content. Another implication for the teaching of sample space is that the classroom practices of teachers reflect the implications suggested above. In other words, a classroom should be set up to provide and support student exploration of sample space. This will promote conceptual understanding and not encourage the adoption of superficial mathematical solutions, which do not take into account the context and the mathematical characteristics of a problem.

Final Thoughts

All participants in this study were prospective undergraduate pre-service teachers. Thus, none had been exposed in their elementary or secondary schooling to the Common Core State Standards Initiative [CCSSI] (Council of Chief State School Officers, 2010) currently being implemented in many states across the nation. Had the mathematics curriculum that they were exposed to been shaped by these standards such as making sense of problems, looking for and making use of structure, modeling with mathematics, and looking for and expressing regularity in repeated reasoning, their ability to interact with the mathematics and construct sample space might then have been different. If participants in this study had been historically encouraged to make sense out of the mathematics within a problem and not been urged simply to respond to a stimuli, and recall a definition or a formula, they might then have seen more connections between the tasks. Not all participants looked for and made use of the structure within the tasks, as suggested by the CCSSI, which also proved to be a hindrance in their ability to make sense of the mathematics within the problems. More research on elementary and secondary students who have been exposed to the Common Core State Standards is needed to address this question.

Limitations and Future Directions

One limitation of this study was that it was conducted with prospective elementary teachers. A way to mitigate this limitation might be to replicate it with elementary students. This would provide the ability to compare and contrast the results with younger students, who may or may not be as influenced by their prior educational experiences as elementary pre-service teachers were. Because of the lack of experience with the content, it is possible that elementary students may not produce the same enumeration strategies or generalization rules. One of the theoretically surprising results found in this study was that there existed little relationship between the enumeration strategies and the subsequent generalization rules participants found. To investigate this further, it might be possible to look at only those participants who completed the generalization task. Then the chi-square analyses could be repeated, which would in turn determine if there is still a lack of relationship between the enumeration strategies and generalization rules of those students that completed Task 3.

In a future study, I would like to investigate how to incorporate the findings and implications from this study into a professional development program that emphasizes reasoning in order to ascertain how teachers learn about sample space construction, how they understand their students learning regarding sample space, and how teacher understanding of sample space relates to the learning of their students. All participants in this study were prospective elementary pre-service teachers. Because sample space construction is partially a prerequisite skill for probabilistic reasoning, the tasks used in this study could be implemented in their professional development program to support teacher content knowledge (Ball, Thames, & Phelps, 2008; Shulman, 1986) about this important area. If the study were to be replicated on elementary students, as suggested earlier, the findings from that additional study could be combined with the findings from this study to address teacher content knowledge, teacher pedagogical content knowledge, and knowledge of student reasoning about sample space construction and its generalization (Ball et al., 2008; Shulman, 1986). The integrated results could then inform the construction of a professional development program, or at least that part of a professional development program that involves the understanding of mathematical content that supports probabilistic reasoning.

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APPENDIX A

DATA COLLECTION INSTRUMENT 1

1. Suppose you have blocks that can be put together to make a tower. These blocks come in ***two*** colors, red and blue.
   a. How many ***different towers*** can you create that are ***three blocks tall*** that are made of ***red or blue colored blocks***?

   b. Draw a picture or diagram, or describe in words how you found the answer.

   c. How do you ***know*** that you have found all the towers? ***Convince me*** that you are ***correct.***

2. Suppose we now want to build towers that are ***4 blocks*** tall, still only using red or blue colored blocks.
    a. How many ***different*** towers can you build with ***two colors*** that are four ***blocks tall***?

    b. Draw a picture or diagram, or describe in words how you found the answer.

    c. How do you ***know*** that you have found all of the towers? ***Convince me*** that you are ***correct***?

    d. Find a ***rule*** that would work if you wanted to know how many towers you can make with red or blue blocks that are ***100 blocks*** tall.

APPENDIX B

INTERVIEW PROTOCOL

Interview Protocol:

Thanks for meeting with me today. I was looking through the work you did in class and was intrigued by your responses to the problems. I will not tell the instruction about our discussions. Therefore, you do not need to worry about your for the class. Take a moment to look back over your responses. (Hand back the papers they used for the pre-assessment have some blocks on hand for them to use.)

Tasks 1, 2, and 3

1. Tell me how you solved the first problem with the towers that were 3 blocks tall.
    a. What did you think the problem was asking you to do?
    b. How did you get started?
    c. Can you explain your approach?
    d. How did you convince yourself that you had found all possible combinations?
    e. How could you convince a Friend?
2. Now that you have looked over you previous work, can you think of other ways that you could have solved it even after finding this particular strategy? Explain to me how you found your second strategy.
3. How do you know that you have found all towers that are 3 blocks tall?
4. Look at the second problem that dealt with towers that were 4 blocks tall.
5. Tell me about the strategy you used to find how many towers there were?
6. Was the strategy you used here different or similar from the strategy you used for towers that were 3 blocks tall? Tell me how it was different or similar.
7. How do you know that you have found all possible tower combinations? Convince me that you are right.
8. Tell me about the rule you found for finding out how many towers there were if they were 100 blocks tall.
    a. Did you use any of the previous strategies?

Give them the new worksheet with the follow-up problems, and give them dice to use on this problem as well.

Task 4

1. Look over this problem. (Hand it to them and also offer the number tiles.)
    a. What do you think the problem is asking you to do?
2. Tell me how you decided to list the two digit numbers in the way that you did.
3. Now that you have thought of one way of doing it, can you think of another?
4. How do you know that you have found all possible combinations of numbers? Convince that you are right.

5. Tell me how you decided to list the three digit numbers in the way that you did.
6. Now that you have thought of one way of doing it, can you think of another?
7. How do you know that you have found all possible combinations of numbers? Convince that you are right.
8. How is this approach the same or different from the one you found in the towers problems?
9. Tell me how you determined how many 100-digit numbers that are that are made up of the numbers 6, 7, and 8.
10. How do you know that is all of them?

APPENDIX C

FOLLOW-UP INTERVIEW TASK

Follow-up Interview Work Sheet Name: ___________________

1. Suppose you have tiles that have the numbers 6, 7, and 8 on them. How many ***2 digit numbers*** can be made with these numbers ***if a number like 77 is allowed***.
    a. List all possible 2-digit numbers made up of the numbers 6, 7, and 8.

    b. How do you ***know*** that you have found all possible 2-digit numbers? ***Convince me*** that you are correct.

2. Suppose you have tiles that have the numbers 6, 7, and 8 on them. How many ***3 digit numbers*** can be made with these numbers ***if a number like 777 is allowed***.
    a. List all possible 3-digit numbers made up of the numbers 6, 7, and 8.

    b. How do you ***know*** that you have found all possible 3-digit numbers? ***Convince me*** that you are correct.

3. Now suppose you want to make 100-digit numbers with 6, 7, and 8.
    a. How many different 100-digit numbers could be produced?

    b. How do you know that you have found them all? ***Convince me*** you are correct.