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Motivated by the problem of optimizing sensor network covers, we generalize the persistent homology of simplicial complexes over a single radial parameter to the context of multiple radial parameters. The persistent homology of so-called multiradial (multi)filtrations is identified as a special case of multidimensional persistence. Specifically, we exhibit that the persistent homology of (multi)filtrations corresponds to both generalized persistence modules of the form $\mathbf{Z}^N_{\geq 0} \to \mathbf{Mod}_R$ and (multi)graded modules over a polynomial ring. The stability of persistence barcodes/diagrams of multiradial filtrations is derived, along with explicit bounds associated to perturbations in both radii and vertex position. A strengthening of the Vietoris-Rips lemma of [DSG07, p. 346] to the setting of multiple radial parameters is obtained. We also use the categorical framework of [BdSS15] to show the persistent homology modules of multiradial (multi)filtrations are stable.

MULTIRADIAL (MULTI)FILTRATIONS AND PERSISTENT HOMOLOGY

by

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Approved by

____
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For my family

APPROVAL PAGE

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CHAPTER I

INTRODUCTION

Topological data analysis combines methods from algebraic topology and statistical theory to associate topological invariants to point cloud data. A typical problem in data analysis is to classify an unknown space $\mathbb{U}$ from a finite point sample S embedded in an ambient space $\mathbb{A}$. The motivation to infer topological features comes from the need to learn intrinsic properties of the unknown space $\mathbb{U}$. This is in contrast to extrinsic geometric properties, such as distance or curvature, that may have been imposed on S through a topological embedding.

Persistent homology is the masthead of the collection of tools comprising topological data analysis. Early formalizations of persistence theory can be found in [Fro92], [FD95], [Rob99], and [ELZ02]. The charm of persistence is in its ability to associate a system of topological invariants to a dataset in the following sense. Suppose we have a filtered simplicial complex $K = \{K^\bullet\}$ which is a family of subcomplexes where K^p is a subcomplex of K^q whenever $p \leq q \in P$. Assume H is a homology theory, which can be thought of as a mechanism for transforming topological spaces into algebraic objects, such as vector spaces, and continuous maps into homomorphisms, or linear maps. This is made precise using the concept of functoriality from category theory. Persistent homology captures the homological structure of a filtration through the direct sum of algebraic objects derived from topological spaces.

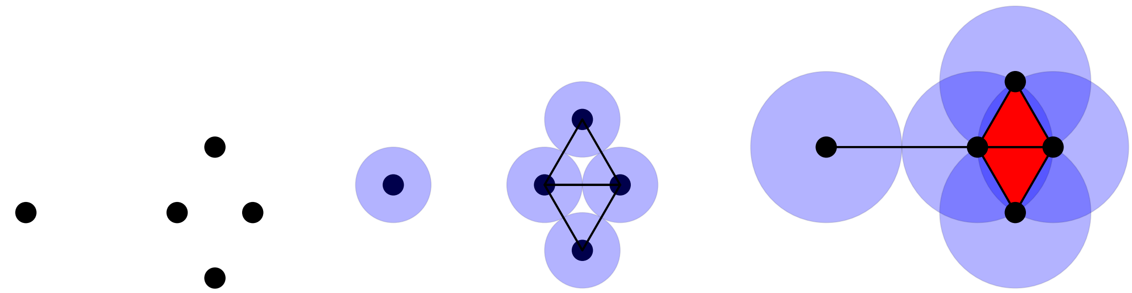

Figure 1. Simplicial Filtration over a Single Radial Parameter

The applications of persistence theory are widespread over many scientific disciplines. Examples include: analysis of wireless sensor networks [DSG06], [DSG07], [AC15], [BGK15]; identification of breast cancer subtypes [NLC11]; preclinical identification of neurological injury [NPL+15]; analysis of protein folding and conformation [XW15], [CBPC13]; analysis of the cerebral circulatory system [BMM+14]. Persistent homology's success as an exploratory tool is due to the stable and simple summaries called persistence barcodes/diagrams which exist in the case $\bigoplus H(K^\bullet)$ is singly graded [ZC05]. In addition, there is a robust statistical theory built around these visual topological summaries [MMH11], [TMMH14], [Bub15], [ACE+15].

The typical persistence pipeline converts a point cloud into a simplicial complex that is filtered along a single radial parameter; see [Ghr08]. Using the algebraic and categorical language established in [CZ09] and [BdSS15], we will develop a theoretical framework for computing the persistent homology of a simplcial complex that is filtered over multiple radial parameters. The motivation for this task comes from the coverage problem for wireless sensor networks, in which the question is to qualify the extent of domain coverage by a collection of sensing regions; see [DSG07]. We hope that the concepts developed in this thesis will help inform planned implementation for optimizing network covers using computational commutative algebra.

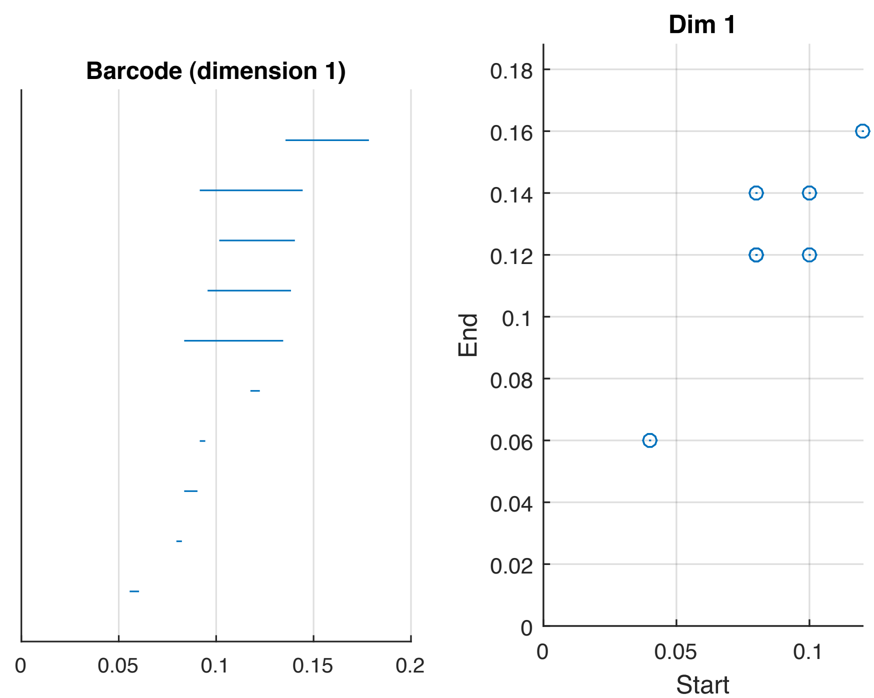

Figure 2. Topological Summaries Computed By [TVJA14]

In Chapter II, we provide a review of simplicial homology and the persistent homology of simplicial filtrations. Our content with simplicial homology is due to the following theorem.

Theorem 1.1 (Eilenberg-Steenrod Theorem [ES52, pp. 100-101])**.** *Let H and H' be two homology theories defined on admissible categories containing all triangulable pairs and their maps. If H and H' have isomorphic coefficient groups, then H and H' are isomorphic.*

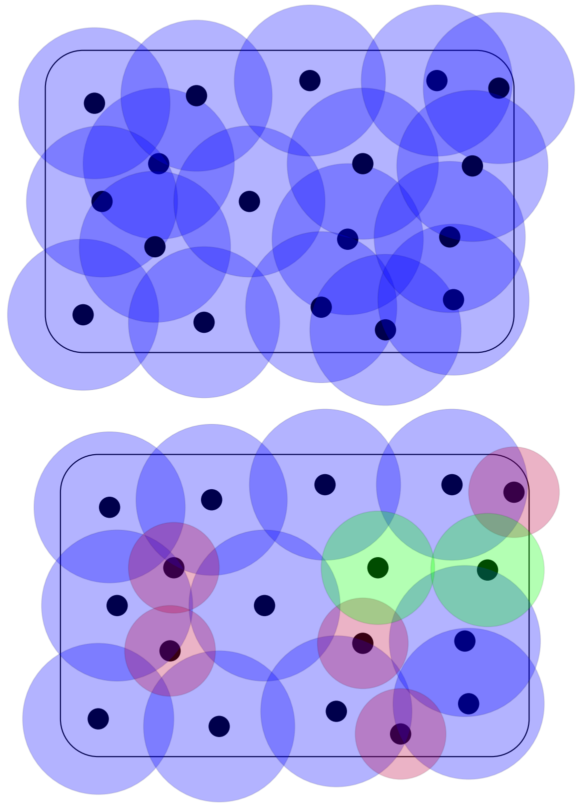

Figure 3. Illustration of Sensor Network Optimization

By **Theorem 1.1**, various homology theories populating algebraic topology are isomorphic up to triangulable topological spaces. This allows us to favor simplicial homology since it is the most computationally effective for homology inference from point clouds.

We devote Chapters III-VI to the development of persistent homology of multiradial filtrations, that is, the persistent homology of simplicial filtrations of nerve and flag complexes over multiple radial parameters. We identify the persistence of multiradial (multi)filtrations with functors from a preordered set to the category of modules over a polynomial ring. Further, the persistence of multiradial (multi)filtrations is identified with nonnegatively multigraded modules over a polynomial ring. This distinguishes the persistent homology of multiradial mulit-filtrations as a specific case of multidimensional persistent homology [CZ09]. The stability of the bottleneck distance is established for persistence barcodes/diagrams summarizing multiradial filtrations along with explicit bounds associated to vertex and radii perturbations. We generalize the Vietoris-Rips lemma proved in [DSG07, p. 346] to the case of multiple radial parameters. Finally, we use the categorical results established in [BdSS15] to prove the stability of the interleaving distance on persistent homology modules obtained from multiradial (multi)filtrations.

CHAPTER II

REVIEW OF PERSISTENT HOMOLOGY

This section will provide a brisk review of the persistent homology of simplicial filtrations and will be devoid of proof. For a more extensive introduction, we direct the reader to the included appendices. A **category C** is defined by the following:

(1) a set $\text{obj}(\mathbf{C})$ whose elements are referred to as objects;

(2) a set $\hom(\mathbf{C})$ consisting of small sets $\hom_{\mathbf{C}}(X, Y)$ of morphisms $X \to Y$;

(3) there exists a binary operation, or composition,

$$\hom_{\mathbf{C}}(X, Y) \times \hom_{\mathbf{C}}(Y, Z) \to \hom_{\mathbf{C}}(X, Z) \colon (f, g) \mapsto g \circ f;$$

(4) there exists a morphism $\text{id}_X \in \hom_{\mathbf{C}}(X, X)$ for all $X \in \text{obj}(\mathbf{C})$;

(5) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \hom_{\mathbf{C}}(W, X)$, $g \in \hom_{\mathbf{C}}(X, Y)$, and $h \in \hom_{\mathbf{C}}(Y, Z)$;

(6) $\text{id}_Y \circ f = f$ and $g \circ \text{id}_Y = g$ for each $f \in \hom_{\mathbf{C}}(X, Y)$ and $g \in \hom_{\mathbf{C}}(Y, Z)$.

Similar to groups and group homomorphisms, we can define a notion of homomorphism between categories. Suppose **C** and **D** are categories. A (covariant) **functor** from **C** to **D** is defined by the following:

(1) A mapping $\text{obj}(\mathbf{C}) \to \text{obj}(\mathbf{D})$; we denote the image of $X \in \mathbf{C}$ by $F(X)$;

(2) A mapping $\hom_{\mathbf{C}}(X, Y) \to \hom_{\mathbf{D}}(F(X), F(Y))$; we denote the image of $f \in \hom_{\mathbf{C}}(X, Y)$ by $F(f)$;

(3) $F(g \circ f) = F(g) \circ F(f)$ for any $f \in \hom_{\mathbf{C}}(X, Y)$ and $g \in \hom_{\mathbf{C}}(Y, Z)$;

(4) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any $X \in \mathbf{C}$.

We can also define a notion of homomorphism between functors. Suppose we have two functors $F, G\colon \mathbf{C} \to \mathbf{D}$. A **natural transformation** $\tau\colon F \Rightarrow G$ is a set of morphisms $\{\tau_X\colon F(X) \to G(X)\}_{X \in \mathrm{obj}(\mathbf{C})}$ of $\mathbf{D}$ such that $(\tau_Y \circ F(f)) = (G(f) \circ \tau_X)$ for every morphism $f\colon X \to Y$ in $\mathbf{C}$.

Homology is an algebraic tool that converts abstract simplicial complexes into algebraic objects and simplicial maps into corresponding homomorphisms. This is made precise using the machinery of functors, that is, we have a homology functor $(H_n \circ C_\bullet)$ for each integer $n \geq 0$. By way of notation, let **AbSimp** be the category of abstract simplicial complexes and $\mathbf{Mod}_R$ be the category of R-modules where R is a commutative ring with unit.

Proposition 2.1 (Proposition D.12). *For all $n \in \mathbb{Z}$, $(H_n \circ C_\bullet)\colon$ **AbSimp** $\to \mathbf{Mod}_R$ is a functor.*

As discussed in the Introduction, persistent homology uses the homology functor to associate an algebraic object to an abstract simplicial complex derived from a point cloud. Two popular simplicial complexes used in topological data analysis are the Čech complex and the Vietoris-Rips complex. Suppose $U = \{U_i\}_{i \in I}$ is a cover of some topological space X. The **Čech complex, or *nerve*, of** U is the abstract simplicial complex defined by

$$\check{C}(U) := \left\{ \{U_{i_0}, U_{i_1}, \ldots, U_{i_n}\} \,\middle|\, \bigcap_{j=0}^{n} U_{i_j} \neq \varnothing \right\}.$$

The **Vietoris-Rips**, or **Rips**, **complex of** U is the abstract simplicial complex defined by

$$R(U) := \left\{ \{U_{i_0}, U_{i_1}, \ldots, U_{i_n}\} \,\middle|\, U_{i_j} \cap U_{i_k} \neq \varnothing \text{ for } j \neq k \text{ and } 0 \leq j,k \leq n \right\}.$$

Given an abstract simplicial complex K and a preordered set P, a P**-filtration of** K is a family $\{K^\bullet\}$ of subcomplexes of K satisfying $\iota_p^q \colon K^p \hookrightarrow K^q \in \{K^\bullet\}$ whenever $p \leq q \in P$, where ι_p^q is a simplicial inclusion map. The n**th persistent homology module**

$$\left\{ (H_n \circ C_\bullet)(K^\bullet), (H_n \circ C_\bullet)\left(\iota_p^q\right) \right\} := \left(\{(H_n \circ C_\bullet)(K^\bullet)\}, \{(H_n \circ C_\bullet)\left(\iota_p^q\right)\} \right)$$

of $\{K^\bullet\}$ is, simply put, the homology of the P-filtration $\{K^\bullet\}$. When $P = \mathbb{Z}_{\geq 0}^N$, persistent homology modules can be given the structure of a module over the polynomial ring $R[x_1, \ldots, x_N]$; see **Proposition 3.15**. Plainly speaking, persistent homology modules correspond to an external direct sum $\mathcal{H}_n(K) := \bigoplus (H_n \circ C_\bullet)(K^\bullet)$ which summarizes the homological structure of a $\mathbb{Z}_{\geq 0}^N$-filtered simplicial complex.

In the case that the preordered set $P = \mathbb{Z}_{\geq 0}$, the persistent homology module $\bigoplus (H_n \circ C_\bullet)(K^\bullet)$ is singly-graded. We say that $\mathcal{H}_n(K)$ is of **finite type** provided each component of $\mathcal{H}_n(K)$ is finitely generated and the R-module homomorphisms $(H_n \circ C_\bullet)\left(\iota_{p_0}^q\right)$ are R-module isomorphisms for all $q \geq p_0$ and some $p_0 \in \mathbb{Z}_{\geq 0}$.

Proposition 2.2 (Structure theorem for persistent homology modules of finite type; see **Proposition E.18**). *Suppose M is a persistent homology module of finite type. Then M uniquely decomposes as*

$$M \cong \left(\bigoplus_{i=0}^{\kappa_0} \Sigma^{\xi_i} R[x]\right) \oplus \left(\bigoplus_{j=0}^{\kappa_1} \Sigma^{\zeta_j} R[x] \Big/ \left(x^{d_j}\right)\right)$$

where $\xi_i, \zeta_j, \kappa_0, \kappa_1 \in \mathbb{Z}_{\geq 0}$, x^{d_j} are homogeneous elements in $R^{d_j}[x]$ with $x^{d_j} \mid x^{d_{j+1}}$.

The structure theorem for persistent homology modules allows us to define **persistence barcodes and persistence diagrams** which are multisets of subintervals of $[0, +\infty]$ or points in $[0, +\infty) \times [0, +\infty]$, respectively. Plainly speaking, given the decomposition provided by the structure theorem, the persistence barcode bcode_n of $\mathcal{H}_n(K)$ is defined to be $\{[\xi_i, +\infty)\}_{i=0}^{\kappa_0} \cup \{[\zeta_j, d_j - \zeta_j)\}_{j=0}^{\kappa_1}$ and the persistence diagram dgm_n of $\mathcal{H}_n(K)$ is defined to be $\{(\xi_i, +\infty)\}_{i=0}^{\kappa_0} \cup \{(\zeta_j, d_j - \zeta_j)\}_{j=0}^{\kappa_1}$; see Appendix E. It follows from the structure theorem that persistence barcodes and persistence diagrams are invariants of persistent homology modules derived from simplicial filtrations; see **Proposition E.22** and **Corollary E.23**.

Persistent homology can be *categorified* in the following way; see [BdSS15]. Assume that **P** is a preordered set and **D** is an arbitrary category. A **generalized persistence module** is a functor $\mathbf{P} \to \mathbf{D}$. Due to the following proposition, we are now capable of considering categories of persistence modules having the form $\mathbf{P} \to \mathbf{D}$.

Proposition 2.3 (Corollary A.15). *Suppose* **P** *is a preordered set and* **D** *is an arbitrary category. Taking objects to be functors* $\mathbf{P} \to \mathbf{D}$ *and morphisms as natural transformations between said functors forms a functor category denoted by* $\mathbf{D}^{\mathbf{P}}$.

CHAPTER III

PERSISTENT HOMOLOGY OF MULTIRADIAL FILTRATIONS

The standard Čech and Vietoris-Rips complexes are a standard tool in persistent homology; see **Definition E.4**. But, persistence theory has traditionally focused only on the Čech and Vietoris-Rips complexes derived from covers consisting of balls of a single fixed radius. At this point, we would like to extend the definitions of the Čech and Vietoris-Rips complexes to the case of covers comprising balls of different radius.

Definition 3.1. Suppose we have a covers $X = \{X_i\}_{i \in I}$ and $Y = \{Y_j\}_{j \in J}$ of a topological space Z. We say X is a **precise refinement of** Y if $I = J$ and $X_i \subseteq Y_i$ for all $i \in I$. Suppose $X \subseteq \mathbb{X}$ where $(\mathbb{X}, \mathrm{d})$ is some metric space. Take n to be a nonnegative integer. Let $\varepsilon_i \in \mathbb{R}$ and $\varepsilon_i \leq \varepsilon_j$ for all $0 \leq i \leq j \leq n$ with $\varepsilon_0 = 0$. We will denote the set of functions $X \to (0, \infty)$ by ${\mathbb{R}_{>0}}^X$. With the intuition that we are *weighting* the balls in our cover, an element of ${\mathbb{R}_{>0}}^X$ may be called a **weight function**. Fixing $\mathbf{r} \in {\mathbb{R}_{>0}}^X$, we will now consider the cover $F = \{\bar{B}_{\varepsilon_n \mathbf{r}(x)}(x)\}_{x \in X}$ of X. Similar to **Definition E.4**, set $F^i = \{\bar{B}_{\varepsilon_i \mathbf{r}(x)}(x)\}_{x \in X}$ for $0 \leq i \leq n$ and $F^i = F^n$ whenever $i > n$. Clearly, F^i is a precise refinement of F^n. We will call $\check{C}_{\varepsilon_i \mathbf{r}}(X) := \check{C}_{F^i}(F)$ the **multiradial Čech complex of** X **at scale** $\varepsilon_i \mathbf{r}$ for $i \geq 0$. Similarly, $R_{\varepsilon_i \mathbf{r}}(X) := R_{F^i}(F)$ is called the **multiradial Vietoris-Rips complex of** X **at scale** $\varepsilon_i \mathbf{r}$ for $i \geq 0$. By **Definition C.18**, $\left\{\check{C}_{\varepsilon_i \mathbf{r}}(X)\right\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\left\{R_{\varepsilon_i \mathbf{r}}(X)\right\}_{i \in \mathbb{Z}_{\geq 0}}$ are $\mathbb{Z}_{\geq 0}$-filtrations of $\check{C}(F)$ and $R(F)$, respectively. We will refer to filtrations of the form $\left\{\check{C}_{\varepsilon_i \mathbf{r}}(X)\right\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\left\{R_{\varepsilon_i \mathbf{r}}(X)\right\}_{i \in \mathbb{Z}_{\geq 0}}$ as **multiradial $\mathbb{Z}_{\geq 0}$-filtrations**.

Lemma 3.2 (Proposition E.9). *Given a $\mathbb{Z}_{\geq 0}$-filtered oriented abstract simplicial complex K, the nth persistent homology module $\mathcal{H}_n(K) \in \mathbf{Mod}_R^{\mathbf{Z}_{\geq 0}}$ is a generalized persistence module.*

Proposition 3.3. *Suppose K is a multiradial $\mathbb{Z}_{\geq 0}$-filtered Čech or Vietoris-Rips complex. Then $\mathcal{H}_n(K) \in \mathbf{Mod}_R^{\mathbf{Z}_{\geq 0}}$ is a generalized persistence module.*

Proof. This follows immediately from **Lemma 3.2** since multiradial $\mathbb{Z}_{\geq 0}$-filtered simplicial complexes are $\mathbb{Z}_{\geq 0}$-filtered simplicial complexes. □

Proposition 3.4. *The persistence barcodes/diagrams in* $\mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}$ *and* $\mathrm{Dgms}_{\mathbb{Z}^2_{\geq 0}}$ *are invariants of the isomorphism classes of nth persistent homology modules $\mathcal{H}_n(K)$ where K is a finite multiradial $\mathbb{Z}_{\geq 0}$-filtered Čech or Vietoris-Rips complex.*

Proof. This is a simple consequence of the fact that persistence barcodes/diagrams are invariants of the isomorphism classes of persistent homology modules derived from simplicial $\mathbb{Z}_{\geq 0}$-filtrations. See **Proposition E.22**, **Lemma D.11**, and **Corollary E.23** for more details. □

We will now introduce the concept of multifiltrations.

Definition 3.5. Let K be an arbitrary abstract simplicial complex. A $\mathbb{Z}^N_{\geq 0}$**-filtration of** K is a family $\{K^{\mathbf{v}}\}_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}}$ of subcomplexes of K so that $K^{\mathbf{u}} \subseteq K^{\mathbf{v}}$ whenever $\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. Note that $\leq$ is the product order of $\mathbb{Z}^N_{\geq 0}$ where $(u_1, \ldots, u_N) \leq (v_1, \ldots, v_N)$ if and only if $u_i \leq v_i$ for each $0 \leq i \leq N$.

Lemma 3.6. *$\mathbb{Z}^N_{\geq 0}$-filtering is a functor $\mathbf{Z}^N_{\geq 0} \to \mathbf{AbSimp}$.*

Proof. Assume K is a finite $\mathbb{Z}^N_{\geq 0}$-filtered abstract simplicial complex. We define $\mathbf{v} \mapsto K^{\mathbf{v}}$ and $\left(\mathbf{u} \xrightarrow{\leq} \mathbf{v}\right) \mapsto \left(K^{\mathbf{u}(0)} \xhookrightarrow{\iota} K^{\mathbf{v}(0)}\right)$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. By way of notation, we will denote the inclusion map $\iota\colon K^{\mathbf{u}(0)} \hookrightarrow K^{\mathbf{v}(0)}$ by $\iota_{\mathbf{u}}^{\mathbf{v}}$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}_{\geq 0}^N$. Suppose that $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \operatorname{obj}\left(\mathbf{Z}_{\geq 0}^N\right)$ and take $v \in K^{\mathbf{u}(0)}$. Observe

$$\begin{aligned} \iota_{\mathbf{u}}^{\mathbf{w}}(v) &= v \\ &= \iota_{\mathbf{v}}^{\mathbf{w}}(v) \\ &= \left(\iota_{\mathbf{v}}^{\mathbf{w}} \circ \iota_{\mathbf{u}}^{\mathbf{v}}\right)(v). \end{aligned}$$

Since $v \in K^{\mathbf{u}(0)}$ is arbitrary, composition is preserved. Also,

$$\begin{aligned} \iota_{\mathbf{u}}^{\mathbf{u}}(v) &= v \\ &= \operatorname{id}_{K^{\mathbf{u}(0)}}(v) \end{aligned}$$

which proves identities are preserved since $v \in K^{\mathbf{u}(0)}$. □

Next, we will see that multifiltrations allow us to parameterize families of multiradial $\mathbb{Z}_{\geq 0}$-filtrations over the individual radii associated to the vertices of an abstract simplicial complex.

Definition 3.7. Suppose $X \subseteq \mathbb{X}$ where $(\mathbb{X}, \mathrm{d})$ is some metric space. Suppose that $\mathbf{r_u}, \mathbf{r_v} \in {\mathbb{R}_{\geq 0}}^X$ so that $\mathbf{r_u}(x) \leq \mathbf{r_v}(x) \in \mathbb{R}_{\geq 0}$ for any $\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}_{\geq 0}^N$. We will set $F^{\mathbf{u}} := \left\{\bar{B}_{\mathbf{r_v}(x)}(x)\right\}_{x \in X}$ for each $\mathbf{v} \in \mathbb{Z}_{\geq 0}^N$. For convenience, we will assume there exists some $\mathbf{b} \in \mathbb{Z}_{\geq 0}^N$ so that $\mathbf{r_v}(x) \leq \mathbf{r_b}(x) \in \mathbb{R}_{\geq 0}$ for any $\mathbf{v} \in \mathbb{Z}_{\geq 0}^N$ and each $x \in X$. Set $F := \{\bar{B}_{\mathbf{r_b}}(x)\}_{x \in X}$. Since $\bar{B}_{\mathbf{r_u}(x)}(x) \subseteq \bar{B}_{\mathbf{r_v}(x)}(x)$ for each $x \in X$ whenever $\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}_{\geq 0}^N$, $F^{\mathbf{u}}$ precisely refines $F^{\mathbf{v}}$ whenever $\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}_{\geq 0}^N$. Even further, $F^{\mathbf{v}}$ precisely refines F given our definition of F.

We will call $\check{C}_{\mathbf{r_v}}(X) := \check{C}_{F^{\mathbf{v}}}(X)$ the **multiradial Čech complex of X at scale $\mathbf{r_v}$** for $\mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. Similarly, we call $R_{\mathbf{r_v}}(X) := R_{F^{\mathbf{v}}}(X)$ the **Vietoris-Rips complex of X at scale $\mathbf{r_v}$** for $\mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. Note that $\{\check{C}_{\mathbf{r_v}}(X)\}_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}}$ and $\{R_{\mathbf{r_v}}(X)\}_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}}$ are $\mathbb{Z}^N_{\geq 0}$-filtrations of $\check{C}(F)$ and $R(F)$, respectively. This follows from the fact that $\{\check{C}_{\mathbf{r_v}}(X)\}_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}}$ and $\{R_{\mathbf{r_v}}(X)\}_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}}$ are both families of precise refinements indexed by $\mathbb{Z}^N_{\geq 0}$; see **Definition C.18**. We will refer to $\mathbb{Z}^N_{\geq 0}$-filtrations of the types just mentioned as **multiradial $\mathbb{Z}^N_{\geq 0}$-filtrations**.

We will next identify the homology of multiradial filtrations as both generalized persistence modules and multigraded modules.

Definition 3.8. An N-graded ring is a ring R where R is isomorphic to a direct sum of abelian groups

$$R \cong \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} R^{\mathbf{v}}$$

such that $R^{\mathbf{u}} \cdot R^{\mathbf{v}} \subseteq R^{\mathbf{u}+\mathbf{v}}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. We say that the elements of $R^{\mathbf{v}}$ are **homogeneous of degree v**. An (nonnegatively) N-graded R-module is a module M over an N-graded ring R where M is isomorphic to a direct sum decomposition of abelian groups

$$M \cong \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} M^{\mathbf{v}}$$

such that $R^{\mathbf{u}} \cdot M^{\mathbf{v}} \subseteq M^{\mathbf{u}+\mathbf{v}}$ for all $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^N_{\geq 0}$.

Example 3.9. Let $R[x_1, \ldots, x_N]$ be the commutative ring of polynomials in N variables with coefficients in R. We will make use of **multidegree** notation:

$$x^{\mathbf{v}} := x_1^{v_1} \cdots x_N^{v_N}$$

for any $\mathbf{v} = (v_1, \ldots, v_N) \in \mathbb{Z}^N_{\geq 0}$. Set $R^{\mathbf{v}}[x_1, \ldots, x_N] := R \cdot x^{\mathbf{v}}$ for $\mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. Define the function $\vartheta \colon \bigoplus_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} R^{\mathbf{v}}[x_1, \ldots, x_N] \to R[x_1, \ldots, x_N]$ by

$$\vartheta \left([r_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \right) := \sum_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} r_{\mathbf{v}} x^{\mathbf{v}}.$$

Since $\sum\limits_{\mathbf{v} \in \mathbb{Z}^{\mathbf{N}}_{\geq \mathbf{0}}} r_{\mathbf{v}} x^{\mathbf{v}} \neq \sum\limits_{\mathbf{v} \in \mathbb{Z}^{\mathbf{N}}_{\geq \mathbf{0}}} s_{\mathbf{v}} x^{\mathbf{v}}$ implies $[r_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \neq [s_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}}$, it is easy to see ϑ is well-defined.

Let $[r_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}}, [s_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \in \bigoplus_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} R^{\mathbf{v}}[x_1, \ldots, x_N]$. Observe

$$\begin{aligned}
\vartheta \left([r_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} + [s_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \right) &= \vartheta \left([(r_{\mathbf{v}} + s_{\mathbf{v}}) x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \right) \\
&= \sum_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} (r_{\mathbf{v}} + s_{\mathbf{v}}) \, x^{\mathbf{v}} \\
&= \left(\sum_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} r_{\mathbf{v}} x^{\mathbf{v}} \right) + \left(\sum_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} s_{\mathbf{v}} x^{\mathbf{v}} \right) \\
&= \vartheta \left([r_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \right) + \vartheta \left([s_{\mathbf{v}} x^{\mathbf{v}}]_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \right).
\end{aligned}$$

Thus ϑ is a group homomorphism.

Suppose $[r_{\mathbf{v}}x^{\mathbf{v}}]_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} \neq [s_{\mathbf{v}}x^{\mathbf{v}}]_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} \in \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} R^{\mathbf{v}}[x_1,\dots,x_N]$. Then there exists some $\mathbf{u} \in \mathbb{Z}^N_{\geq 0}$ such that $r_{\mathbf{u}} \neq s_{\mathbf{v}}$. Thus

$$\sum_{\mathbf{v}\in\mathbb{Z}^{\mathbf{N}}_{\geq \mathbf{0}}} r_{\mathbf{v}}x^{\mathbf{v}} \neq \sum_{\mathbf{v}\in\mathbb{Z}^{\mathbf{N}}_{\geq \mathbf{0}}} s_{\mathbf{v}}x^{\mathbf{v}}$$

and hence ϑ in injective. Now suppose $\sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} r_{\mathbf{v}}x^{\mathbf{v}} \in R[x_1,\dots,x_N]$ is arbitrary. Then

$$\vartheta\left([r_{\mathbf{v}}x^{\mathbf{v}}]_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}}\right) = \sum_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} r_{\mathbf{v}}x^{\mathbf{v}}$$

which shows ϑ is surjective. Altogether, we have that ϑ is a group isomorphism and $R[x_1,\dots,x_N] \cong \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} R^{\mathbf{v}}[x_1,\dots,x_N]$.

Finally, observe

$$\begin{aligned}(rx^{\mathbf{u}})\cdot(sx^{\mathbf{v}}) &= (rs)\cdot x^{\mathbf{u}+\mathbf{v}} \\ &\in R^{\mathbf{u}+\mathbf{v}}[x_1,\dots,x_N].\end{aligned}$$

Altogether, $R[x_1,\dots,x_N] \cong \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} R^{\mathbf{v}}[x_1,\dots,x_N]$ is an N-graded ring.

The following lemma is needed by technicality. The interested reader can find more details in **Definition A.1**.

Lemma 3.10. *If X and Y are small sets, then $X \times Y$ is a small set.*

Proof. Fix $y \in Y$. It follows that $X \times \{y\} \in \mathfrak{U}$ since the function

$$X \to X \times \{y\} \text{ defined by } x \mapsto (x, y)$$

is surjective; see **Definition A.1**(4). Notice $\bigcup_{y \in Y} X \times \{y\} = X \times Y$ and therefore $X \times Y \in \mathfrak{U}$ by **Definition A.1**(2), that is, $X \times Y$ is a small set. □

Proposition 3.11. *Taking objects to be N-graded R-modules and morphisms to be R-homomorphisms $f\colon M \to N$ satisfying $f(M^{\mathbf{v}}) \subseteq N^{\mathbf{v}}$ for any $\mathbf{v} \in \mathbb{Z}^N_{\geq 0}$, we can form the category $\mathbf{Mod}_R^N$ of N-graded R-modules.*

Proof. Notice that composition of morphisms is well-defined because composition of R-homomorphisms is well-defined. Clearly, $\mathrm{id}_M \in \hom_{\mathbf{Mod}_R^N}(M, M)$ is just the identity R-homomorphism on M. Associativity of composition holds since composition of R-homomorphisms is associative. Assume $f \in \hom_{\mathbf{Mod}_R^N}(M_1, M_2)$, $g \in \hom_{\mathbf{Mod}_R^N}(M_2, M_3)$, $\mathbf{v} \in \mathbb{Z}^N_{\geq 0}$, and take $m_1 \in M_1$ and $m_2 \in M_2$. Observe

$$\begin{aligned}(\mathrm{id}_{M_2} \circ f)\,(m_1) &= m_1 \\ &= f(m_1)\end{aligned}$$

and

$$\begin{aligned}(g \circ \mathrm{id}_{M_2})\,(m_2) &= m_2 \\ &= g(m_2).\end{aligned}$$

With everything else being routine, we need to prove that $\hom_{\mathbf{Mod}_R^N}(M_1, M_2)$ is a small set where $M_1, M_2 \in \operatorname{obj}(\mathbf{Mod}_R^N)$ are arbitrary. Identifying a function with its graph, it suffices to show $2^{X\times Y}$ is a small set. By **Lemma 3.10**, $X \times Y$ is a small set and hence $2^{X\times Y}$ is a small set by **Definition A.1**(3). Thus

$$\hom_{\mathbf{Mod}_R^N}(M_1, M_2) \subseteq 2^{X\times Y}$$

is a small set by **Lemma A.2**. Finally, $\mathbf{Mod}_R^N$ is a category. □

Definition 3.12. Suppose K is a multifiltered abstract simplicial complex. We define the N**-graded** n**th persistent homology module** $\mathcal{H}_n^N(K)$ by

$$\mathcal{H}_n^N(K) := \left\{ H_n\left(K^{\mathbf{u}}\right), (H_n \circ C_\bullet)\left(K^{\mathbf{u}(0)} \overset{\iota}{\hookrightarrow} K^{\mathbf{v}(0)}\right)\right\}_{\mathbf{u}\leq\mathbf{v}\in\mathbb{Z}_{\geq 0}^N}.$$

By way of notation, $\iota_{\mathbf{u}}^{\mathbf{v}}\colon K^{\mathbf{u}(0)} \hookrightarrow K^{\mathbf{v}(0)}$ is the image of $\mathbf{u} \overset{\leq}{\to} \mathbf{v}$ under multifiltration. Also, we define $H_n^{\mathbf{u}}(K) := H_n\left(K^{\mathbf{u}}\right)$ for each $\mathbf{u} \in \mathbb{Z}_{\geq 0}^N$. Given a commutative ring R with unit, we will denote the category of R-chain complexes by $\mathbf{Comp}_R$. For a review of chain complexes and there homology, see **Appendix B**.

Lemma 3.13 (Proposition D.12). *For all $n \in \mathbb{Z}$, $(H_n \circ C_\bullet)\colon \mathbf{AbSimp} \to \mathbf{Mod}_R$ is a functor.*

Proposition 3.14. *Suppose K is a multiradial $\mathbb{Z}_{\geq 0}^N$-filtered Čech or Vietoris-Rips complex. Then the N-graded nth persistent homology module $\mathcal{H}_n^N(K) \in {\mathbf{Mod}_R^N}^{\mathbb{Z}_{\geq 0}^N}$ is a generalized persistence module.*

Proof. Recall that $\mathbb{Z}_{\geq 0}^N$-filtering is a functor $\mathbf{Z}_{\geq 0}^N \to \mathbf{AbSimp}$ by **Lemma 3.6**. Also, $\mathbf{Comp}_R \xrightarrow{(H_n\circ C_\bullet)} \mathbf{Mod}_R$ is a functor for each $n \in \mathbb{Z}_{\geq 0}$ by **Lemma 3.13**. It follows that $\mathcal{H}_n^N(K) := \left\{H_n^{\mathbf{u}}(K), (H_n \circ C_\bullet)(\iota_{\mathbf{u}}^{\mathbf{v}})\right\}_{\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}_{\geq 0}^N}$ is a functor since (covariant) functors are closed under composition. More precisely, $\mathcal{H}_n^N(K)\colon \mathbf{Z}_{\geq 0}^N \to \mathbf{Mod}_R$ is a functor defined by $\mathbf{u} \xmapsto{\mathcal{H}_n^N} H_n^{\mathbf{u}}(K)$ and $\left(\mathbf{u} \xrightarrow{\leq} \mathbf{v}\right) \xmapsto{\mathcal{H}_n^N} (H_n \circ C_\bullet)(\iota_{\mathbf{u}}^{\mathbf{v}})$ for any $\mathbf{u}, \mathbf{v} \in \operatorname{obj}\left(\mathbf{Z}_{\geq 0}^N\right)$ and each $n \in \mathbb{Z}_{\geq 0}$. The following diagram summarizes the functor $\mathcal{H}_n^N$:

$$\mathbf{Z}_{\geq 0}^N \longrightarrow \mathbf{AbSimp} \longrightarrow \mathbf{Comp}_R \longrightarrow \mathbf{Mod}_R, \qquad \mathcal{H}_n^N\colon \mathbf{Z}_{\geq 0}^N \to \mathbf{Mod}_R$$

□

Proposition 3.15. *Suppose*

$$\mathcal{H}_n^N(K) := \left\{H_n^{\mathbf{u}}(K), (H_n \circ C_\bullet)(\iota_{\mathbf{u}}^{\mathbf{v}})\right\}_{\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}_{\geq 0}^N}.$$

Overloading notation, we will set

$$\mathcal{H}_n^N(K) := \bigoplus_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} H_n^{\mathbf{u}}(K).$$

Then $\mathcal{H}_n^N(K)$ is a multigraded $R[x_1, x_2, \ldots, x_N]$-module where

$$\left(\sum_{k=0}^{\ell} r_k x^{\mathbf{v_k}}\right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} := \sum_{k=0}^{\ell} \left(r_k \cdot \Sigma^{\mathbf{v}_k}\left[(H_n \circ C_\bullet)\left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k}\right)(\gamma_{\mathbf{u}})\right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N}\right)$$

and

$$sx^{\mathbf{0}} \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} := \left[\Sigma^{\mathbf{0}}(s \cdot \gamma_{\mathbf{u}})\right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} = (s \cdot \gamma_{\mathbf{u}})_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N}$$

for any $\sum_{k=0}^{\ell} r_k x^{\mathbf{v}_k} \in R[x_1, \ldots, x_N]$ *with* $\ell \geq 1$ *an integer, for every* $s \in R$*, and for any* $[\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \in \mathcal{H}^N_n(K)$*. Recall* $\Sigma^{(\cdot)}$ *is the shift map on multigrading defined by*

$$[\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}} \xmapsto{\Sigma^{\mathbf{v}}} [\gamma'_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}}$$

where $\gamma'_{\mathbf{u}} = \mathbf{0}$ *whenever* $\mathbf{u} \not\geq \mathbf{v}$ *and* $\gamma'_{\mathbf{u}} = \gamma_{\mathbf{u}-\mathbf{v}}$ *provided* $\mathbf{u} \geq \mathbf{v}$.

Proof. We will start by verifying $\mathcal{H}^N_n(K)$ satisfies the definition of an $R[x_1, \ldots, x_N]$-module; see **Definition B.4**. Thus we need to show the following for each $x, y \in \mathcal{H}^N_n(K)$ and $r, s \in R[x_1, \ldots, x_N]$:

(1) $r(x + y) = rx + ry$;

(2) $(r + s)x = rx + sx$;

(3) $(rs)x = r(sx)$;

(4) $1_R x = x$.

Let us take the following as arbitrary:

$$\sum_{k=0}^{\ell} c_k x^{\mathbf{v}_k}, \sum_{k=0}^{m} d_k x^{\mathbf{v}_k} \in R[x_1, \ldots, x_N]$$

and

$$[\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}}, [\eta_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \in \mathcal{H}^N_n(K).$$

(1)

$$\left(\sum_{k=0}^{\ell} c_k x^{\mathbf{v}_k}\right) \cdot \left([\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} + [\eta_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}}\right)$$

$$
\begin{aligned}
&= \sum_{k=0}^{\ell} \left(c_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}} + \eta_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&= \sum_{k=0}^{\ell} \left(\Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (c_k \cdot \gamma_{\mathbf{u}} + c_k \cdot \eta_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&= \sum_{k=0}^{\ell} \left(\Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (c_k \cdot \gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right. \\
&\qquad \left. + \Sigma^{\mathbf{v_k}} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (c_k \cdot \eta_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&= \sum_{k=0}^{\ell} \left(\Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (c_k \cdot \gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&\qquad + \sum_{k=0}^{\ell} \left(\Sigma^{\mathbf{u}} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (c_k \cdot \eta_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&= \sum_{k=0}^{\ell} \left(c_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&\qquad + \sum_{k=0}^{\ell} \left(c_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\eta_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&= \left(\sum_{k=0}^{\ell} c_k x^{\mathbf{v}_k} \right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} + \left(\sum_{k=0}^{\ell} c_k x^{\mathbf{v}_k} \right) \cdot [\eta_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N}.
\end{aligned}
$$

(2)

$$
\begin{aligned}
&\left(\sum_{k=0}^{\ell} (c_k x_k^{\mathbf{v}}) + \sum_{k=0}^{m} (d_k x^{\mathbf{v}_k}) \right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \\
&= \left(\sum_{k=0}^{\max\{\ell,m\}} c_k x^{\mathbf{v}_k} + d_k x^{\mathbf{v}_k} \right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \\
&= \left[\sum_{k=0}^{\max\{\ell,m\}} (c_k + d_k)\, x^{\mathbf{v}_k} \right] \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \\
&= \sum_{k=0}^{\max\{\ell,m\}} \left((c_k + d_k) \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right) \\
&= \sum_{k=0}^{\max\{\ell,m\}} \left(c_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}_{\geq 0}^N} \right.
\end{aligned}
$$

$$
\begin{aligned}
&\qquad\qquad + d_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \Bigg) \\
&= \sum_{k=0}^{\ell} \left(c_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \right) \\
&\qquad + \sum_{k=0}^{m} \left(d_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \right) \\
&= \left(\sum_{k=0}^{\ell} c_k x^{\mathbf{v}_k} \right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} + \left(\sum_{k=0}^{m} d_k x^{\mathbf{v}_k} \right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} .
\end{aligned}
$$

(3)

$$
\begin{aligned}
&\left(\sum_{k=0}^{\ell} (c_k x^{\mathbf{v}_k}) \cdot \sum_{k=0}^{m} (d_k x^{\mathbf{v}_k}) \right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} = \left[\sum_{q=0}^{\ell+m} \left(\sum_{p=0}^{q} c_p d_{q-p} \right) x^{\mathbf{v}_q} \right] \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \\
&= \sum_{q=0}^{\ell+m} \left(\left(\sum_{p=0}^{q} c_p d_{q-p} \right) \Sigma^{\mathbf{v}_q} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_q} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \right) \\
&= \sum_{q=0}^{\ell+m} \left(\left(\sum_{p=0}^{q} c_p d_{q-p} \right) \Sigma^{\mathbf{v}_{p+q-p}} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_{p+q-p}} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \right) \\
&= \sum_{q=0}^{\ell+m} \left(\left\{ \left(\sum_{p=0}^{q} c_p d_{q-p} \right) x^{\mathbf{v}_p} \right\} \cdot \Sigma^{\mathbf{v}_{q-p}} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_{q-p}} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \right) \\
&= \left(\sum_{k=0}^{\ell} c_k x_k^{\mathbf{v}} \right) \cdot \left(\sum_{k=0}^{m} d_k \cdot \Sigma^{\mathbf{v}_k} \left[(H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}_k} \right) (\gamma_{\mathbf{u}}) \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \right) \\
&= \left(\sum_{k=0}^{\ell} c_k x^{\mathbf{v}_k} \right) \cdot \left(\left(\sum_{k=0}^{m} d_k x^{\mathbf{v}_k} \right) \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \right) .
\end{aligned}
$$

(4)

$$
\begin{aligned}
1_R x^{\mathbf{0}} \cdot [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} &= \Sigma^{\mathbf{0}} \left[1_R \cdot \gamma_{\mathbf{u}} \right]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} \\
&= [\gamma_{\mathbf{u}}]_{\mathbf{u} \in \mathbb{Z}^N_{\geq 0}} .
\end{aligned}
$$

Thus $\mathcal{H}_n(K)$ is an $R[x_1, \ldots, x_N]$-module.

To prove $\mathcal{H}_n(K)$ is $\mathbb{Z}^N_{\geq 0}$-graded, take $rx^{\mathbf{v}} \in R^{\mathbf{v}}$ and $\gamma \in H^{\mathbf{u}}_n(K)$ as arbitrary. Observe

$$\begin{aligned} rx^{\mathbf{v}} \cdot \gamma &= r \cdot (H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}}\right) (\gamma) \\ &= (H_n \circ C_\bullet) \left(\iota_{\mathbf{u}}^{\mathbf{u}+\mathbf{v}}\right) (r \cdot \gamma) \\ &\in H^{\mathbf{u}+\mathbf{v}}_n(K). \end{aligned}$$

This shows that $R^{\mathbf{v}}[x_1, \ldots, x_N] \cdot H^{\mathbf{u}}_n(K) \subseteq H^{\mathbf{u}+\mathbf{v}}_n(K)$ for any $\mathbf{u}, \mathbf{v} \in \mathbb{Z}^N_{\geq 0}$ and each $n \in \mathbb{Z}_{\geq 0}$. Thus $\mathcal{H}^N_n(K)$ is $\mathbb{Z}^N_{\geq 0}$-graded. □

Our overloading of notation for N-graded persistent homology modules is justified by the following proposition.

Definition 3.16. Suppose $\mathbf{C}$ and $\mathbf{D}$ are categories and let $F\colon \mathbf{C} \to \mathbf{D}$ is a functor. We say F is **faithful** provided $F(f)$ is injective for each $f \in \hom_{\mathbf{C}}(X, Y)$ where $X, Y \in \mathbf{C}$ are arbitrary. Also, we say F is **full** provided $F(f)$ is surjective for each $f \in \hom_{\mathbf{C}}(X, Y)$ where $X, Y \in \mathbf{C}$ are arbitrary. The functor F is said to be **fully faithful** provided F is both full and faithful. A functor $F\colon \mathbf{C} \to \mathbf{D}$ is an **isomorphism of categories** provided F is fully faithful and $F\colon \mathrm{obj}(\mathbf{C}) \to \mathrm{obj}(\mathbf{D})$ is a set bijection. Given an isomorphism of categories $F\colon \mathbf{C} \to \mathbf{D}$, we say $\mathbf{C}$ and $\mathbf{D}$ are isomorphic.

Proposition 3.17 ([Les15, p. 8])**.** *$\mathbf{Mod}_{R[x_1,\ldots,x_N]}{}^{\mathbb{Z}^N_{\geq 0}}$ and $\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}$ are isomorphic as categories.*

Proof. Define the functor $\mho\colon \mathbf{Mod}^N_{R[x_1,\ldots,x_N]} \to \mathbf{Mod}_R{}^{\mathbb{Z}^N_{\geq 0}}$ by

$$\mho(M) = F\colon \mathbf{Z}^N_{\geq 0} \to M$$

and

$$\mho\left(M \xrightarrow{f} N\right) = f|_{M^\bullet} = \{f|_{M^{\mathbf{v}}}\}_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}}$$

where $F(\mathbf{v}) = M^{\mathbf{v}}$. Let $F, G\colon \mathbf{Z}^N_{\geq 0} \to \mathbf{Mod}_R$, $\mathbf{u}, \mathbf{v} \in \operatorname{obj}\left(\mathbf{Z}^N_{\geq 0}\right)$, and $m \in F(\mathbf{u})$ be arbitrary. Also, assume $f\colon \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} F(\mathbf{v}) \to \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} G(\mathbf{v}) \in \hom\left(\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}\right)$. Notice

$$\begin{aligned}
\left(\mho_{\mathbf{v}}(f) \circ F(\mathbf{u} \leq \mathbf{v})\right)(m) &= f|_{F_{\leq}(\mathbf{v})}\left(F\left(\mathbf{u} \leq \mathbf{v}\right)(m)\right) \\
&= G\left(\mathbf{u} \leq \mathbf{v}\right)\left(f|_{F_{\leq}(\mathbf{u})}(m)\right) \\
&= \left(G(\mathbf{u} \leq \mathbf{v}) \circ \mho_{\mathbf{u}}(f)\right)(m).
\end{aligned}$$

This shows $\mho(f)$ is indeed a natural transformation. Now observe

$$\begin{aligned}
\mho(g \circ f) &= (g|_{M^\bullet} \circ f|_{M^\bullet}) \\
&= \mho(g) \circ \mho(f)
\end{aligned}$$

and

$$\begin{aligned}
\mho\left(\mathrm{id}_M\right) &= \mathrm{id}|_{M^\bullet} \\
&= \mathrm{id}_{\mathbf{Mod}_R^{\mathbf{Z}^N_{\geq 0}}}.
\end{aligned}$$

This shows $\mho$ is well-defined.

Suppose $f, g \in \hom(M, P)$ for $M, P \in \operatorname{obj}\left(\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}\right)$. Thus $f(m) \neq g(m)$ for some $m \in M$ and hence $f|_{M^{\mathbf{v}}} \not\equiv g|_{M^{\mathbf{v}}}$ for some $\mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. Thus $\mho(f) \not\equiv \mho(g)$

which shows $\mho$ is faithful. Now suppose $\tau\colon F \Rightarrow G$ is a natural transformation for $F, G \in \text{obj}\left(\mathbf{Mod}_R{}^{\mathbb{Z}^N_{\geq 0}}\right)$. Define $M, P \in \text{obj}\left(\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}\right)$ by $M = \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} F(\mathbf{v})$ and $P = \bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} G(\mathbf{v})$. Also, define $\Lambda\colon M \to P$ by $\Lambda(m) = \tau_{\mathbf{u}}(m)$ where $m \in M^{\mathbf{u}}$ for some $\mathbf{u} \in \mathbb{Z}^N_{\geq 0}$. By construction, $\mho(\Lambda) = \tau$ and hence $\mho$ is fully faithful. Suppose $M \neq P \in \text{obj}\left(\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}\right)$. Then $M^{\mathbf{v}} \neq P^{\mathbf{v}}$ for some $\mathbf{v} \in \mathbb{Z}^N_{\geq 0}$. Now suppose $F\colon \mathbf{Z}^N_{\geq 0} \to \mathbf{Mod}_R$. Let $M \in \text{obj}\left(\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}\right)$ be defined by $\bigoplus_{\mathbf{v}\in\mathbb{Z}^N_{\geq 0}} F(\mathbf{v})$. Clearly, $\mho(M) = F$. Thus $\mho$ induces a bijection on obj $\left(\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}\right)$. Therefore $\mathbf{Mod}_R{}^{\mathbb{Z}^N_{\geq 0}}$ and $\mathbf{Mod}^N_{R[x_1,\ldots,x_N]}$ are isomorphic as categories. $\square$

CHAPTER IV

STABILITY OF PERSISTENCE DIAGRAMS OF MULTIRADIAL $\mathbb{Z}_{\geq 0}$-FILTRATIONS

The stability of persistence diagrams of multiradial filtrations is a special case of the results described in [CSEH07, p. 108]. The goal of this section is to use the framework established in Appendix F to provide specific bounds, in terms of vertex position and radial size, for the stability of persistence diagrams summarizing multiradial filtrations. To this end, we will interpret a multiradial filtration as a real-valued continuous function.

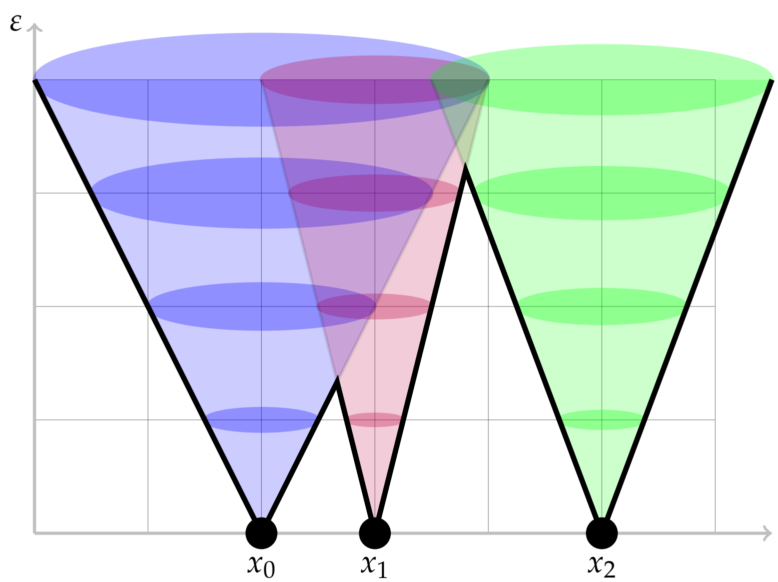

Figure 4. Illustration of Entry Function

Definition 4.1. Suppose $X \subseteq \mathbb{X}$ where $(\mathbb{X}, \mathrm{d}_{\mathbb{X}})$ is a metric space. Take $\mathbf{r} \in \mathbb{R}_{>0}{}^X$ to be a weight function. With the intuition of enlarging cover sets, we define the **entry function** $f_{X,\mathbf{r}} : \mathbb{X} \to \mathbb{R}$ by

$$f_{X,\mathbf{r}}(x) = \inf_{y \in X} \left\{ \frac{\mathrm{d}(x,y)}{\mathbf{r}(y)} \right\}.$$

For what follows, we will use the notation $\bar{f}_{X,\mathbf{r}}$ to denote the restricted function $f_{X,\mathbf{r}}|_C$ where $\mathbb{X} \supseteq C \supseteq X$.

Proposition 4.2. *Suppose $f_{X,\mathbf{r}}$ is an entry function and $\varepsilon \geq 0$. Then*

$$f_{X,\mathbf{r}}^{-1}((-\infty, \varepsilon]) = \bigcup_{x \in X} \bar{B}_{\varepsilon \mathbf{r}}(x).$$

Proof. Let $\varepsilon \in \mathbb{R}_{\geq 0}$ be arbitrary. Take $p \in f_{X,\mathbf{r}}^{-1}((-\infty, \varepsilon])$. It follows that

$$\inf_{y \in X} \left\{ \frac{\mathrm{d}\,(p,y)}{\mathbf{r}(y)} \right\} \leq \varepsilon.$$

Thus $p \in \bar{B}_{\varepsilon \mathbf{r}}(y_0)$ for some $y_0 \in X$. Hence

$$p \in \bar{B}_{\varepsilon \mathbf{r}}(y_0) \subseteq \bigcup_{y \in X} \bar{B}_{\varepsilon \mathbf{r}}(y).$$

Now suppose $p \in \bigcup_{y \in X} \bar{B}_{\varepsilon \mathbf{r}}(y)$. Then $p \in \bar{B}_{\varepsilon \mathbf{r}}(y_0)$ for some $y_0 \in X$. Hence

$$\inf_{y \in X} \left\{ \frac{\mathrm{d}\,(p,y)}{\mathbf{r}(y)} \right\} \leq \varepsilon$$

which implies $p \in f_{X,\mathbf{r}}^{-1}((-\infty, \varepsilon])$. By the arbitrariness of p,

$$f_{X,\mathbf{r}}^{-1}((-\infty, \varepsilon]) = \bigcup_{x \in X} \bar{B}_{\varepsilon\mathbf{r}}(x).$$

$\square$

Lemma 4.3 (Corollary C.16). *If F is a finite collection of closed subsets of a compact Hausdorff space X such that every nonempty intersection of sets in F is contractible, then $\check{C}(F) \simeq \bigcup_{D \in F} D$.*

Corollary 4.4. *Suppose $f_{X,\mathbf{r}}$ is an entry function where $X \subseteq \mathbb{X}$ has finite cardinality N and $\mathbb{X}$ is compact. Then*

$$f_{X,\mathbf{r}}^{-1}((-\infty, \varepsilon]) \simeq \check{C}_{\varepsilon\mathbf{r}}(X)$$

for some integer $n \geq 0$ and any $\varepsilon \in \mathbb{R}_{\geq 0}$.

Proof. Without loss of generality, suppose $X = \{x_1, \ldots, x_N\}$. The conclusion follows immediately from **Proposition 4.2** and **Lemma 4.3** since

$$f_{X,\mathbf{r}}^{-1}((-\infty, \varepsilon]) = \bigcup_{i=1}^{N} \bar{B}_{\varepsilon\mathbf{r}}(x_i).$$

$\square$

Lemma 4.5 (Lemma D.11). *If K is a finite abstract simplicial complex, then $H_n(K)$ is finitely generated for each $n \in \mathbb{Z}$.*

Proposition 4.6. *Suppose $f_{X,\mathbf{r}}$ is an entry function for a set X with finite cardinality N. Then $\bar{f}_{X,\mathbf{r}}$ is continuous and tame.*

Proof. Recall that metrics $\mathbb{X} \times \mathbb{X} \to \mathbb{R}$ are continuous real-valued functions. Thus $f_{X,\mathbf{r}}$ is continuous as the composition of a continuous function after a quotient of continuous functions. Following from **Corollary 4.4**, $f_{X,\mathbf{r}}$ has at most 2^N homological critical values. Now, recall **Lemma 4.5**, which says the homology of finite simplicial complexes is finitely generated. Thus, $H_n\left(f^{-1}(-\infty, \varepsilon]\right)$ is finitely generated for each $\varepsilon \in \mathbb{R}$ by **Corollary 4.4**. □

With respect to entry functions, we will prove two different versions of stability under the supremum norm.

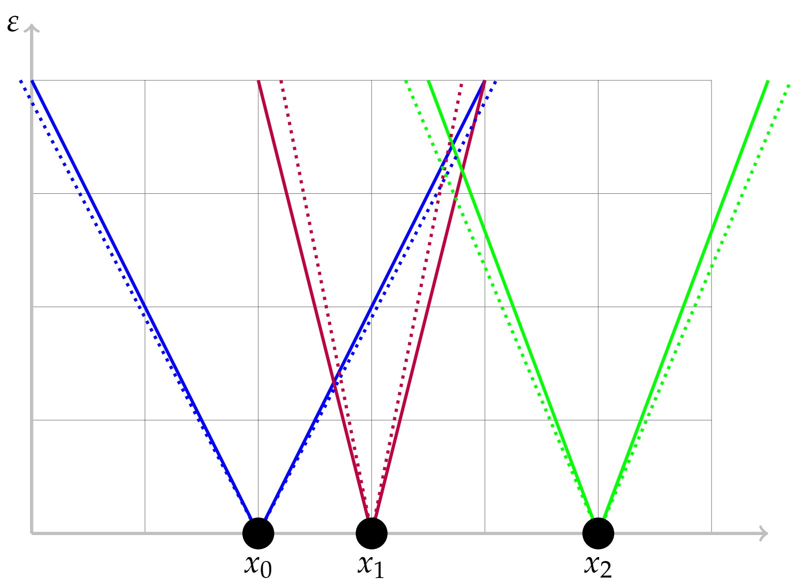

Figure 5. Illustration of Weight Stability

Proposition 4.7 (Weight stability). *Suppose $X \subseteq \mathbb{X}$ has finite cardinality N. Let $\mathbf{r}, \mathbf{r}' \in {\mathbb{R}_{>0}}^X$ be weight functions and assume $\bar{f}_{X,\mathbf{r}}, \bar{f}_{X,\mathbf{r}'} \colon C \to \mathbb{R}$ are entry functions restricted to a compact set $C \supseteq X$. Then*

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty < \varepsilon$$

whenever

$$\|\mathbf{r} - \mathbf{r}'\|_\infty < \delta = \frac{\varepsilon \cdot \min\{\mathbf{r}(x) \cdot \mathbf{r}'(x) \mid x \in X\}}{\operatorname{diam}(C)}$$

for every $\varepsilon > 0$.

Proof. It suffices to show that

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty \leq \frac{\|\mathbf{r} - \mathbf{r}'\|_\infty \operatorname{diam}(C)}{\min\{\mathbf{r}(x) \cdot \mathbf{r}'(x) \mid x \in X\}}.$$

To begin,

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty = \max_{y \in C}\{|\bar{f}_{X,\mathbf{r}}(y) - \bar{f}_{X,\mathbf{r}'}(y)|\}$$

since $\bar{f}_{X,\mathbf{r}}$ and $\bar{f}_{X,\mathbf{r}'}$ are continuous functions with compact domain. It follows that there exists some $y_0 \in C$ such that

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty = |\bar{f}_{X,\mathbf{r}}(y_0) - \bar{f}_{X,\mathbf{r}'}(y_0)|$$

by [Rud76, p. 89, Theorem 4.16]. Then we have

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty = \left| \min_{x \in X} \left\{ \frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{\mathbf{r}(x)} \right\} - \min_{x \in X} \left\{ \frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{\mathbf{r}'(x)} \right\} \right|.$$

Since X is a finite set, there exist $x_j, x_k \in X$ so that

$$\left| \min_{x \in X} \left\{ \frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{\mathbf{r}(x)} \right\} - \min_{x \in X} \left\{ \frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{\mathbf{r}'(x)} \right\} \right| = \left| \frac{\mathrm{d}_{\mathbb{X}}(x_j, y_0)}{\mathbf{r}(x_j)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}'(x_k)} \right|.$$

It is either the case that $\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) = \mathrm{d}_{\mathbb{X}}(x_k, y_0)/\mathbf{r}'(x_k)$ or, without loss of generality, $\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) > \mathrm{d}_{\mathbb{X}}(x_k, y_0)/\mathbf{r}'(x_k)$. If

$$\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) = \mathrm{d}_{\mathbb{X}}(x_k, y_0)/\mathbf{r}'(x_k),$$

then $\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty = 0$ and we are done. Now, suppose

$$\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) > \mathrm{d}_{\mathbb{X}}(x_k, y_0)/\mathbf{r}'(x_k).$$

Since $\mathrm{d}_{\mathbb{X}}(x, y_0)/\mathbf{r}(x) \geq \mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j)$ for all $x \in X$, it must hold that

$$\frac{\mathrm{d}_{\mathbb{X}}(x_j, y_0)}{\mathbf{r}(x_j)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}'(x_k)} \leq \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}(x_k)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}'(x_k)}.$$

Therefore

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty = \left| \frac{\mathrm{d}_{\mathbb{X}}(x_j, y_0)}{\mathbf{r}(x_j)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}'(x_k)} \right|$$

$$\leq \left| \frac{d_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}(x_k)} - \frac{d_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}'(x_k)} \right|$$
$$= \left| \frac{[\mathbf{r}'(x_k) - \mathbf{r}(x_k)] \cdot d_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}(x_k) \cdot \mathbf{r}'(x_k)} \right|.$$

Finally,

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X,\mathbf{r}'}\|_\infty \leq \left| \frac{[\mathbf{r}'(x_k) - \mathbf{r}(x_k)] \cdot d_{\mathbb{X}}(x_k, y_0)}{\mathbf{r}(x_k) \cdot \mathbf{r}'(x_k)} \right|$$
$$\leq \frac{\|\mathbf{r} - \mathbf{r}'\|_\infty \cdot \operatorname{diam}(C)}{\min\{\mathbf{r}(x) \cdot \mathbf{r}'(x) \mid x \in X\}},$$

as desired. □

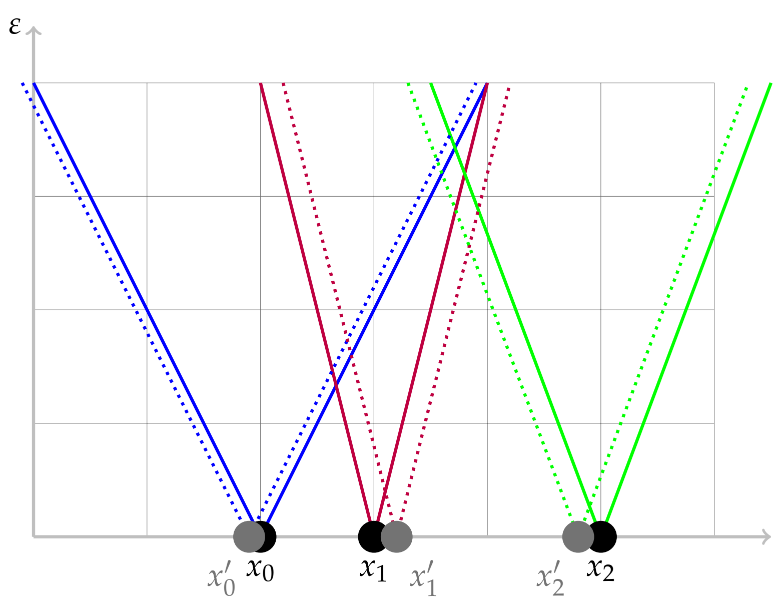

Figure 6. Illustration of Vertex Perturbation Stability

Proposition 4.8 (Vertex perturbation stability). *Suppose $X, X' \subseteq \mathbb{X}$ have finite cardinality N, $\mathbf{r} : X \to (0, \infty)$, and suppose that $\eta : X' \to X$ is a set bijection. Let $f_{X',\mathbf{r}\circ\eta}$ be the entry function on X' induced by $\mathbf{r} \circ \eta$. Also, let $\bar{f}_{X,\mathbf{r}}, \bar{f}_{X',\mathbf{r}\circ\eta} \colon C \to \mathbb{R}$ be entry functions restricted to a compact set $C \supseteq (X \cup X')$. Then $\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty < \varepsilon$ whenever $\max_{x\,\in\,X'}\{\mathrm{d}_{\mathbb{X}}\,(x, \eta(x))\} < \delta = \varepsilon \cdot \min_{x\,\in\,X}\{\mathbf{r}(x)\}$ for every $\varepsilon > 0$.*

Proof. We will proceed by showing

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty \leq \frac{\max\{\mathrm{d}_{\mathbb{X}}(x, \eta(x)) \mid x \in X'\}}{\min\{(\mathbf{r}\circ\eta)(x) \mid x \in X'\}}.$$

Since $\bar{f}_{X,\mathbf{r}}$ and $\bar{f}_{X',\mathbf{r}'}$ are continuous functions with compact domain,

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty = \max\left\{|\bar{f}_{X,\mathbf{r}}(x) - \bar{f}_{X',\mathbf{r}\circ\eta}(x)| \,\middle|\, x \in C\right\}.$$

By [Rud76, p. 89, Theorem 4.16], there exists some $y_0 \in C$ so that

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty = \left|\min_{x\,\in\,X}\left\{\frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{\mathbf{r}(x)}\right\} - \min_{x\,\in\,X'}\left\{\frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{(\mathbf{r}\circ\eta)(x)}\right\}\right|.$$

The finiteness of X and X' implies the existence of $x_j \in X$ and $x_k \in X'$ such that

$$\left|\min_{x\,\in\,X}\left\{\frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{\mathbf{r}(x)}\right\} - \min_{x\,\in\,X'}\left\{\frac{\mathrm{d}_{\mathbb{X}}(x, y_0)}{(\mathbf{r}\circ\eta)(x)}\right\}\right| = \left|\frac{\mathrm{d}_{\mathbb{X}}(x_j, y_0)}{\mathbf{r}(x_j)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{(\mathbf{r}\circ\eta)(x_k)}\right|.$$

Now it is either the case that $\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) = \mathrm{d}_{\mathbb{X}}(x_k, y_0)/(\mathbf{r}\circ\eta)(x_k)$ or, without loss of generality, $\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) > \mathrm{d}_{\mathbb{X}}(x_k, y_0)/(\mathbf{r}\circ\eta)(x_k)$. In the first case, we have that $\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty = 0$ and we are done. To continue, suppose that

$$\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) > \mathrm{d}_{\mathbb{X}}(x_k, y_0)/(\mathbf{r}\circ\eta)(x_k).$$

Since $\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) \leq \mathrm{d}_{\mathbb{X}}(x, y_0)/\mathbf{r}(x)$ for all $x \in X$, it must be the case that $\mathrm{d}_{\mathbb{X}}(x_j, y_0)/\mathbf{r}(x_j) \leq \mathrm{d}_{\mathbb{X}}(\eta(x_k), y_0)/(\mathbf{r}\circ\eta)(\eta(x_k))$. Therefore

$$\frac{\mathrm{d}_{\mathbb{X}}(x_j, y_0)}{\mathbf{r}(x_j)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{(\mathbf{r}\circ\eta)(x_k)} \leq \frac{\mathrm{d}_{\mathbb{X}}(\eta(x_k), y_0)}{(\mathbf{r}\circ\eta)(x_k)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{(\mathbf{r}\circ\eta)(x_k)}.$$

This implies

$$\begin{aligned}
\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty &= \left|\frac{\mathrm{d}_{\mathbb{X}}(x_j, y_0)}{\mathbf{r}(x_j)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{(\mathbf{r}\circ\eta)(x_k)}\right| \\
&\leq \left|\frac{\mathrm{d}_{\mathbb{X}}(\eta(x_k), y_0)}{(\mathbf{r}\circ\eta)(x_k)} - \frac{\mathrm{d}_{\mathbb{X}}(x_k, y_0)}{(\mathbf{r}\circ\eta)(x_k)}\right| \\
&= \frac{|\mathrm{d}_{\mathbb{X}}(\eta(x_k), y_0) - \mathrm{d}_{\mathbb{X}}(x_k, y_0)|}{(\mathbf{r}\circ\eta)(x_k)} \\
&\leq \frac{\mathrm{d}_{\mathbb{X}}(\eta(x_k), x_k)}{(\mathbf{r}\circ\eta)(x_k)} \\
&\leq \frac{\max\{\mathrm{d}_{\mathbb{X}}(x, \eta(x)) \mid x \in X'\}}{\min\{(\mathbf{r}\circ\eta)(x) \mid x \in X'\}}
\end{aligned}$$

as desired. □

Proposition 4.9 (Combined stability)**.** *Suppose $X, X' \subseteq \mathbb{X}$ of common cardinality N, $\mathbf{r} : X \to (0,\infty)$ and $\mathbf{r}' : X' \to (0,\infty)$ are weight functions, and $\eta : X' \to X$ is a set bijection. Also, let $\bar{f}_{X,\mathbf{r}}, \bar{f}_{X',\mathbf{r}'}$ denote the entry functions restricted to a compact set $C \supseteq (X \cup X')$. Then*

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}'}\|_\infty < \varepsilon$$

whenever

$$\max_{x\in X'}\{\mathrm{d}_{\mathbb{X}}(x,\eta(x))\} + \|(\mathbf{r}\circ\eta) - \mathbf{r}'\|_\infty < \delta$$
$$= 2\cdot\min\left\{\frac{\varepsilon}{2}\cdot\min_{x\in X'}\{\mathbf{r}(x)\}, \frac{\varepsilon\cdot\min\{(\mathbf{r}\circ\eta)(x)\cdot\mathbf{r}'(x)\mid x\in X'\}}{2\cdot\mathrm{diam}(C)}\right\}$$

for every $\varepsilon > 0$.

Proof. Let $\varepsilon > 0$. By **Proposition 4.7**,

$$\|\bar{f}_{X',\mathbf{r}\circ\eta} - \bar{f}_{X',\mathbf{r}'}\|_\infty < \frac{\varepsilon}{2}$$

whenever

$$\|(\mathbf{r}\circ\eta) - \mathbf{r}'\|_\infty < \frac{\varepsilon\cdot\min\{(\mathbf{r}\circ\eta)(x)\cdot\mathbf{r}'(x)\mid x\in X'\}}{2\cdot\mathrm{diam}(C)}.$$

Also,

$$\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty < \frac{\varepsilon}{2}\text{ whenever }\max_{x\in X'}\{\mathrm{d}_{\mathbb{X}}(x,\eta(x))\} < \frac{\varepsilon}{2}\cdot\min_{x\in X}\{\mathbf{r}(x)\}$$

by **Proposition 4.8**. Therefore, if we require

$$\max_{x\in X'}\{\mathrm{d}_{\mathbb{X}}(x,\eta(x))\} + \|(\mathbf{r}\circ\eta) - \mathbf{r}'\|_\infty < \delta$$
$$= 2\cdot\min\left\{\frac{\varepsilon}{2}\cdot\min_{x\in X'}\{\mathbf{r}(x)\}, \frac{\varepsilon\cdot\min\{(\mathbf{r}\circ\eta)(x)\cdot\mathbf{r}'(x)\mid x\in X'\}}{2\cdot\mathrm{diam}(\mathrm{C})}\right\},$$

then we have

$$\begin{aligned}\|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}'}\|_\infty &\leq \|\bar{f}_{X,\mathbf{r}} - \bar{f}_{X',\mathbf{r}\circ\eta}\|_\infty + \|\bar{f}_{X',\mathbf{r}\circ\eta} - \bar{f}_{X',\mathbf{r}'}\|_\infty \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

and we are done. □

Our results on stability of entry functions provides us with corresponding statements for their persistence diagrams. We direct the reader to review the definition of persistence barcodes bcode_n and persistence diagrams dgm_n given in **Definition F.5**.

Lemma 4.10 (Bottleneck Stability; [CSEH07]; **Theorem F.9).** *Suppose $\mathbb{X}$ is a triangulable space and assume $f, g\colon \mathbb{X} \to \mathbb{R}$ are tame continuous functions. If $(f - g)$ is bounded, then*

$$\mathrm{d}_B\left(\mathrm{dgm}_n(f), \mathrm{dgm}_n(g)\right) \leq \|f - g\|_\infty$$

where d_B is the bottleneck distance for persistence barcodes/diagrams, $\|\cdot\|_\infty$ is the supremum norm for bounded real-valued functions and $n \geq 0$ is an arbitrary integer.

Corollary 4.11. *Suppose $X, X' \subseteq \mathbb{X}$ of common size N,*
$\mathbf{r} : X \to (0, \infty)$ and $\mathbf{r}' : X' \to (0, \infty)$ are weight functions, and $\eta : X' \to X$ is a set bijection. Then the following hold

(1) For every $\varepsilon > 0$ we have

$$\mathrm{d}_B(\mathrm{dgm}_n(\bar{f}_{X,\mathbf{r}}), \mathrm{dgm}_n(\bar{f}_{X,\mathbf{r}'})) < \varepsilon$$

whenever

$$\|\mathbf{r} - \mathbf{r}'\|_\infty < \delta = \frac{\varepsilon \cdot \min\{\mathbf{r}(x) \cdot \mathbf{r}'(x) \mid x \in X\}}{\mathrm{diam}(C)}$$

(2) For every $\varepsilon > 0$ we have

$$\mathrm{d}_B(\mathrm{dgm}_n(\bar{f}_{X,\mathbf{r}}), \mathrm{dgm}_n(\bar{f}_{X',\mathbf{r}\circ\eta})) < \varepsilon$$

whenever

$$\max_{x \in X'}\{\mathrm{d}_{\mathbb{X}}(x, \eta(x))\} < \delta = \varepsilon \cdot \min_{x \in X}\{\mathbf{r}(x)\}.$$

(3) For every $\varepsilon > 0$ we have

$$\mathrm{d}_B(\mathrm{dgm}_n(\bar{f}_{X,\mathbf{r}}), \mathrm{dgm}_n(\bar{f}_{X',\mathbf{r}'})) < \varepsilon$$

whenever

$$\max_{x \in X'}\{\mathrm{d}_{\mathbb{X}}(x, \eta(x))\} + \|(\mathbf{r} \circ \eta) - \mathbf{r}'\|_\infty < \delta$$
$$= 2 \cdot \min\left\{\frac{\varepsilon}{2} \cdot \min_{x \in X'}\{\mathbf{r}(x)\}, \frac{\varepsilon \cdot \min_{x \in X'}\{(\mathbf{r} \circ \eta)(x) \cdot \mathbf{r}'(x)\}}{2 \cdot \mathrm{diam}(C)}\right\}.$$

Proof. This is an immediate consequence of **Lemma 4.10** and **Proposition 4.6**. □

We will close this section by showing a particular case of *interpolation* between persistence barcodes/diagrams.

Lemma 4.12. *Let $\mathbb{X}$ be a compact metric space. Assume $X, Y \subseteq \mathbb{X}$ have finite cardinalities and suppose $\mathbf{r} \in {\mathbb{R}_{\geq 0}}^{X \cup Y}$ is a weight function satisfying the property that $\mathbf{r}(y) = 0$ for each $y \in Y$. Then $H_n\left(\check{C}_{\mathbf{r}}(X \cup Y)\right) \cong H_n\left(\check{C}_{\mathbf{r}}(X)\right)$ for each $n \geq 1$.*

Proof. First, assume $Y = \{y\} \subseteq \mathbb{X}$. It is either the case $y \in \bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)$ or $y \notin \bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)$. If $y \in \bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)$, then

$$\bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x) = \{y\} \cup \bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)$$

By **Theorem C.16**, **Proposition 4.2**, and **Corollary 4.4**,

$$\begin{aligned} H_n\left(\check{C}_{\mathbf{r}}(X \cup Y)\right) &\cong H_n\left(\{y\} \cup \bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)\right) \\ &= H_n\left(\bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)\right) \\ &= H_n\left(\check{C}_{\mathbf{r}}(X)\right) \end{aligned}$$

for any $n \geq 0$.

Now, if $y \notin \bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)$, then

$$\begin{aligned} H_n\left(\check{C}_{\mathbf{r}}(X \cup Y)\right) &\cong H_n\left(\{y\} \cup \bigcup_{x \in X} \bar{B}_{\mathbf{r}(x)}(x)\right) \\ &\cong H_n\left(\check{C}_{\mathbf{r}}(X)\right) \oplus H_n\left(\{y\}\right) \\ &\cong H_n\left(\check{C}_{\mathbf{r}}(X)\right) \end{aligned}$$

for each $n \geq 1$. Thus $H_n\left(\check{C}_{\mathbf{r}}(X \cup Y)\right) \cong H_n\left(\check{C}_{\mathbf{r}}(X)\right)$ when $n \geq 1$ and Y is a singleton. The conclusion holds by induction on the cardinality of Y. □

Proposition 4.13. *Let $\mathbb{X}$ be a compact metric space. Assume $X, Y \subseteq \mathbb{X}$ have finite cardinalities and suppose $\mathbf{r} \in \mathbb{R}_{>0}{}^{X\cup Y}$ is a weight function. Let $\mathbf{r}' \in \mathbb{R}_{>0}{}^{X\cup Y}$ satisfy $\mathbf{r}'(x) = \mathbf{r}(x)$ for each $x \in X$ and $\mathbf{r}'(y) \leq \mathbf{r}(y)$ for each $y \in Y$. Also, define $\mathbf{s} \in \mathbb{R}_{>0}{}^{X\cup Y}$ by $\mathbf{s}|_X \equiv \mathbf{r}|_X$ and $\mathbf{s}'|_Y \equiv 0$. Then, for every $\varepsilon > 0$ and $n \geq 1$, there exists a $\delta > 0$ such that $\mathrm{d}_B\left(\mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{s}}\right), \mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{r}}\right)\right) < \varepsilon$ whenever $\|\mathbf{s} - \mathbf{r}\|_\infty < \delta$.*

Proof. Fix an integer $n \geq 1$. By **Corollary 4.11** (1), there exists a $\delta_0 > 0$ for any $\varepsilon_0 > 0$ such that

$$\mathrm{d}_B\left(\mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{r}}\right), \mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{r}'}\right)\right) < \varepsilon_0$$

whenever

$$\|\mathbf{r} - \mathbf{r}'\|_\infty < \frac{\delta_0}{2}.$$

By **Lemma 4.12**, there exists a $\delta_1 > 0$ for any $\varepsilon_1 > 0$ such that

$$\mathrm{d}_B\left(\mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{s}}\right), \mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{r}'}\right)\right) < \varepsilon_1$$

whenever

$$\|\mathbf{s} - \mathbf{r}'\|_\infty < \frac{\delta_1}{2}.$$

Now, fix an arbitrary $\varepsilon > 0$ and choose $\varepsilon_0, \varepsilon_1 > 0$ so that $\varepsilon_0, \varepsilon_1 < \varepsilon/2$. By the triangle inequality,

$$\mathrm{d}_B\left(\mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{s}}\right), \mathrm{dgm}_n\left(f_{X\cup Y,\mathbf{r}}\right)\right) < \varepsilon$$

whenever

$$\|\mathbf{s} - \mathbf{r}\|_\infty < \delta.$$

□

CHAPTER V

MULTIRADIAL VIETORIS-RIPS LEMMA

In this section, we prove a stronger version of the Vietoris-Rips lemma stated in [DSG07, p. 346]. For convenience, we provide the original Vietoris-Rips lemma.

Lemma 5.1 (Vietoris-Rips lemma; [DSG07, p. 346]; **Lemma E.5**). *Suppose we have that* $\{x_0, \ldots, x_N\} = X \subset \mathbb{R}^d$ *and* $\varepsilon, \varepsilon' > 0$. *If* $(\varepsilon/\varepsilon') \geq \sqrt{2d/(d+1)}$, *then*

$$R_{\varepsilon'}(X) \subseteq \check{C}_{\varepsilon}(X) \subseteq R_{\varepsilon}(X).$$

In words, we generalize the Vietoris-Rips lemma to the context of Čech and Vietoris-Rips filtrations over multiple radial parameters. We would like to point out to the reader that this implies the stability of persistence diagrams of multiradial filtrations of affine Vietoris-Rips complexes by the results of the previous section.

Theorem 5.2 (Multiscale Vietoris-Rips Lemma)**.** *Let* $X \subseteq \mathbb{R}^d$ *be of finite cardinality* N. *Assume* $\mathbf{r} : X \to (0, \infty)$ *is a weight function. Then*

$$R_{\varepsilon'\mathbf{r}}(X) \subseteq \check{C}_{\varepsilon\mathbf{r}}(X) \subseteq R_{\varepsilon\mathbf{r}}(X)$$

whenever $\varepsilon, \varepsilon' > 0$ *and* $\varepsilon/\varepsilon' \geq \sqrt{2d/(d+1)}$.

Proof. The second containment $\check{C}_{\varepsilon\mathbf{r}}(X) \subseteq R_{\varepsilon\mathbf{r}}(X)$ follows from the fact that the multiscale Rips complex is the flag complex of the Čech complex. To show that $R_{\varepsilon\mathbf{r}}(X)$, we suppose there is some finite collection $\{x_k\}_{k=0}^{\ell} \subseteq \mathbb{R}^d$ so that $\|x_i -$

$x_j\|_2 \le \varepsilon'(\mathbf{r}(x_i) - \mathbf{r}(x_j))$ whenever $i \neq j$. Define a function $f : \mathbb{R}^d \to \mathbb{R}$ by

$$f(y) = \max_{0 \le j \le \ell} \left\{ \frac{\|x_j - y\|_2}{\mathbf{r}(x_j)} \right\}.$$

Clearly, f is continuous and $|f(y)| \to \infty$ as $\|y\|_2 \to \infty$. Thus f attains a minimum on some compact set containing $\operatorname{conv}(\{x_k\}_{k=0}^{\ell})$. It follows that f attains an absolute minimum, say y_0, on $\mathbb{R}^d$. By a reordering of vertices if needed, we may assume $f(y_0) = \frac{1}{\mathbf{r}(x_j)}\|x_j - y_0\|_2^2$ for some subcollection $\{x_j\}_{j=0}^{n} \subseteq \{x_k\}_{k=0}^{\ell}$ and $f(y_0) > \frac{1}{\mathbf{r}(x_j)}\|x_j - y_0\|_2^2$ for $\{x_j\}_{j=n+1}^{\ell}$. Let $g(y) = \max_{0 \le j \le n} \left\{ \frac{1}{\mathbf{r}(x_j)}\|x_j - y\|_2 \right\}$ and $h(y) = \max_{n+1 \le j \le \ell} \left\{ \frac{1}{\mathbf{r}(x_j)}\|x_j - y\|_2 \right\}$.

Now we wish to show that $y_0 \in \operatorname{conv}(\{x_j\}_{j=0}^{n})$. To this end, we apply Farkas' lemma; see [HUL12, p. 59]. Either $y_0 \in \operatorname{conv}(\{x_j\}_{j=0}^{n})$ or there is a $v \in \mathbb{R}^d$ such that $v \cdot x_j \ge 0$ for all $0 \le j \le n$ and $v \cdot y_0 < 0$. Thus we need only show that there is no $v \in \mathbb{R}^d$ so that $v \cdot (x_j - y_0) > 0$ for $0 \le j \le n$. By way of contradiction, suppose otherwise. Since

$$\|x_j - (y_0 + \lambda v)\|_2^2 = \|x_i - y_0\|_2^2 - 2\lambda v \cdot (x_j - y_0) + \lambda^2 \|v\|_2^2$$

for each $0 \le j \le n$, it follows that $g(y_0 - \lambda v) < f(y_0)$ for all $\lambda \in (0, \lambda_1)$ where $\lambda_1 = \min_{0 \le j \le n} 2v \cdot (x_j - y_0)/\|v\|_2^2$. Since $h(y)$ is continuous and $h(y_0) < f(y_0)$, there exists a λ_2 so that $h(y_0 + \lambda v) < f(y_0)$ for $\lambda \in [0, \lambda_2)$. Thus there exists a $\lambda > 0$ such that $f(y_0 + \lambda v) = \max\{g(y_0 + \lambda v), h(y_0 + \lambda v)\} < f(y_0)$, a contradiction to the minimality of y_0.

By Carathéodory's theorem, see [HUL12, p. 29], and reordering of vertices if necessary, there exists some subcollection of vertices $\{x_i\}_{i=0}^m$, where $0 < i \leq \min\{d, n\}$, such that y_0 is in conv $\left(\{x_i\}_{i=0}^m\right)$. It is not possible that $i = 0$. If so, then $y_0 = x$ and $f(y_0) = \frac{1}{\mathbf{r}(x_0)}\|x_0 - y_0\|_2 = 0$ and f is identically zero. Since σ contains a vertex $x_1 \neq x_0$, it follows that

$$f(y_0) = f(x_0) > \frac{1}{\mathbf{r}(x_1)}\|x_1 - x_0\|_2 > 0,$$

a contradiction.

By way of notation, let $\hat{x}_j = x_j - y_0$. Note that

$$\|\hat{x}_j\|_2^2 = \mathbf{r}(x_j)^2 f(y_0)^2. \tag{5.1}$$

Take $a_0, a_1, \ldots, a_m \in \mathbb{R}_{\geq 0}$ so that

$$\sum_{i=0}^m a_i = 1 \text{ and } y_0 = \sum_{i=0}^m a_i x_i.$$

Then $\sum_{i=0}^m a_i \hat{x}_i = 0$. By relabeling, we may assume that $a_0 \mathbf{r}(x_0) \geq \mathbf{r}(x_i) a_i$ when $i > 0$. Then we obtain

$$\hat{x}_0 = -\sum_{i=1}^m \frac{a_i}{a_0}\hat{x}_i,$$

and so

$$\mathbf{r}(x_0)^2 f(y_0)^2 = \|\hat{x}_0\|_2^2 = -\sum_{i=1}^m \frac{a_i}{a_0}\hat{x}_0 \cdot \hat{x}_i.$$

Among the indices $1, 2, \ldots, m$, there is some ι such that

$$\frac{1}{d}\mathbf{r}(x_0)^2 f(y_0)^2 \leq \frac{1}{m}\mathbf{r}(x_0) f(y_0)^2 \leq -\frac{a_\iota}{a_0}\hat{x}_0\hat{x}_\iota.$$

Putting (1) and (2) together, we find

$$\begin{aligned} f(y_0)^2\left(\mathbf{r}(x_0)^2 + \frac{2a_0\mathbf{r}(x_0)^2}{a_\iota d} + \mathbf{r}(x_\iota)^2\right) &\leq \|\hat{x}_0\|_2^2 - 2\hat{x}_0\hat{x}_\iota + \|\hat{x}_\iota\|_2^2 \\ &= \|\hat{x}_0 - \hat{x}_\iota\|_2^2 \\ &= \|x_0 - x_\iota\|_2^2 \\ &\leq (\varepsilon'(\mathbf{r}(x_0) + \mathbf{r}(x_\iota)))^2. \end{aligned}$$

We will now show that

$$\frac{(\mathbf{r}(x_0)^2 + \mathbf{r}(x_\iota)^2)^2}{\mathbf{r}(x_0)^2 + \frac{2a_0\mathbf{r}(x_0)^2}{a_\iota d} + \mathbf{r}(x_\iota)^2} \leq \frac{2d}{d+1}.$$

It suffices to show

$$(d - 1 + 4\frac{a_0}{a_\iota})\mathbf{r}(x_0)^2 - 2(d+1)\mathbf{r}(x_0)\mathbf{r}(x_\iota) + (d-1)\mathbf{r}(x_\iota)^2 \geq 0.$$

Since $\frac{a_0}{a_\iota} \geq \frac{\mathbf{r}(x_\iota)}{\mathbf{r}(x_0)}$, we get

$$\begin{aligned} &(d - 1 + 4\frac{a_0}{a_\iota})\mathbf{r}(x_0)^2 - 2(d+1)\mathbf{r}(x_0)\mathbf{r}(x_\iota) + (d-1)\mathbf{r}(x_\iota)^2 \\ &\qquad \geq (d - 1 + 4\frac{\mathbf{r}(x_\iota)}{\mathbf{r}(x_0)})\mathbf{r}(x_0)^2 - 2(d+1)\mathbf{r}(x_0)\mathbf{r}(x_\iota) + (d-1)\mathbf{r}(x_\iota)^2 \end{aligned}$$

$$= (d-1)(\mathbf{r}(x_0) - \mathbf{r}(x_\iota))^2$$

$$\geq 0$$

as desired. Our assumption that $\varepsilon \geq \varepsilon' \sqrt{2d/(d+1)}$ implies $f(y_0) \leq \varepsilon$ and thus

$$y_0 \in \bigcap_{i=0}^{m} \bar{B}_{\varepsilon \mathbf{r}(x_i)}(x_i).$$

Therefore $\sigma \in \check{C}_{\varepsilon \mathbf{r}}(X)$ and we are done. □

CHAPTER VI

STABILITY OF N-GRADED PERSISTENT HOMOLOGY MODULES OF MULTIRADIAL $\mathbb{Z}^N_{\geq 0}$-FILTRATIONS

This section is devoted to showing the interleaving metric defined in [BdSS15] is stable on the persistent homology of multiradial $\mathbb{Z}^N_{\geq 0}$-filtered Čech and Vietoris-Rips complexes. We will provide a quick review of the necessary results from [BdSS15].

Definition 6.1. Suppose $\mathbf{P} = (P, \leq)$ is a preordered set. A **translation on** $(P, \leq)$ is a function $\Gamma\colon P \to P$ such that $\Gamma(p) \leq \Gamma(q)$ whenever $p \leq q$ and $p \leq \Gamma(p)$ for any $p, q \in P$. It can be shown that a translation $\Gamma\colon \mathbf{P} \to \mathbf{P}$ is a functor and there exists a natural transformation $\eta_\Gamma\colon \mathrm{id} \Rightarrow \Gamma$; see [BdSS15, p. 1510]. We denote the set of all translations on $\mathbf{P}$ by $\mathbf{Trans_P}$. Let $\Gamma, K \in \mathbf{Trans_P}$ and $F, G \in \mathbf{D^P}$ for some category $\mathbf{D}$. A (Γ, K)**-interleaving between** F **and** G is a pair of natural transformations $\varphi\colon F \Rightarrow (G \circ \Gamma)$ and $\psi\colon G \Rightarrow (F \circ K)$ such that

$$(\psi \circ \Gamma) \circ \varphi = F \circ \eta_{(K \circ \Gamma)}$$

and

$$(\varphi \circ K) \circ \psi = G \circ \eta_{(\Gamma \circ K)}.$$

We say F and G are (Γ, K)**-interleaved** provided the existence of a (Γ, K)-interleaving between F and G. A **sublinear projection** is a function $\omega\colon \mathbf{Trans_P} \to [0, +\infty]$

satisfying the properties $\omega(\mathrm{id}_{\mathbf{P}}) = 0$ for the identity translation $\mathrm{id}_{\mathbf{P}} = \mathrm{id}_P$ and $\omega(\Gamma_0 \circ \Gamma_1) \leq \omega(\Gamma_0) + \omega(\Gamma_1)$ for any $\Gamma_0, \Gamma_1 \in \mathbf{Trans_P}$.

Let $\varepsilon \in [0, +\infty)$. We say $\Gamma \in \mathbf{Trans_P}$ is an **ε-translation** provided $\omega(\Gamma) \leq \varepsilon$. Generalized persistence modules $F, G \in \mathbf{d^P}$ are **ε-interleaved with respect to** ω provided F and G are (Γ, K)-interleaved for some ε-translations $\Gamma, K \in \mathbf{Trans_P}$. Set

$$\Upsilon_{F,G} := \{\varepsilon \in [0, +\infty) \mid F \text{ and } G \text{ are } \varepsilon\text{-interleaved with respect to } \omega\}$$

The **interleaving distance** $\mathrm{d} := \mathrm{d}^{\omega} \colon \mathrm{obj}\left(\mathbf{D^P}\right) \times \mathrm{obj}\left(\mathbf{D^P}\right) \to [0, +\infty]$ is defined by

$$\mathrm{d}(F, G) := \begin{cases} \inf \Upsilon_{F,G} & \text{if } \Upsilon_{F,G} \neq \varnothing \\ +\infty & \text{if } \Upsilon_{F,G} = \varnothing \end{cases}.$$

Theorem 6.2 ([BdSS15, p. 1516])**.** *The interleaving distance* $\mathrm{d} := \mathrm{d}^{\omega}$ *is an extended pseudometric on* $\mathrm{obj}\left(\mathbf{D^P}\right)$ *for any sublinear projection* $\omega \colon \mathbf{Trans_P} \to [0, +\infty]$.

We are finally ready to state the relevant stability theorem from [BdSS15]. The following theorem says that the interleaving metric is 1-Lipschitz.

Theorem 6.3 ([BdSS15, p. 1517])**.** *Suppose* $\mathbf{P}$ *is a preordered set,* $F, G \in \mathrm{obj}\left(\mathbf{D^P}\right)$, $\omega \colon \mathbf{Trans_P} \to [0, +\infty]$ *is a sublinear projection, and* $H \colon \mathbf{D} \to \mathbf{E}$ *is an arbitrary functor. Then* $\mathrm{d}\left((H \circ F), (H \circ G)\right) \leq \mathrm{d}(F, G)$.

The next definition will yield our primary source of sublinear projections.

Definition 6.4. A **Lawvere metric space** is a pair (P, d_P) where P is a set and $\mathrm{d}_P \colon P \times P \to [0, +\infty]$ satisfies $\mathrm{d}_P(p, p) = 0$ and $\mathrm{d}_P(p, r) \leq \mathrm{d}_P(p, q) + \mathrm{d}_P(q, r)$ for any $p, q, r \in P$.

Proposition 6.5 ([BdSS15, p. 1518])**.** *Suppose* $\mathbf{P} = (P, \leq)$ *is a preordered set and* (P, d_P) *is a Lawvere metric space. Then* $\omega := \omega_{\mathrm{d}_P} \colon \mathbf{Trans}_P \to [0, +\infty]$ *defined by*

$$\omega_{\mathrm{d}_P}(\Gamma) = \sup_{p \in P} \{\mathrm{d}_P(p, \Gamma(p))\}$$

is a sublinear projection.

Definition 6.6. Assume K is a multiradial $\mathbb{Z}^N_{\geq 0}$-filtered Čech or Vietoris-Rips complex. Referencing **Lemma 3.6**, we will denote the $\mathbb{Z}^N_{\geq 0}$-filtration of K as a functor $K^\bullet \colon \mathbf{Z}^N_{\geq 0} \to \mathbf{AbSimp}$.

Proposition 6.7. *Suppose K and L are both multiradial* $\mathbb{Z}^N_{\geq 0}$*-filtered Čech or Vietoris-Rips complexes and take* $\mathrm{d}_{\mathbb{Z}^N_{\geq 0}}$ *to be the metric induced by the standard euclidean norm restricted to* $\mathbb{Z}^N_{\geq 0}$*. Then the interleaving distance* $\mathrm{d} \colon \mathbf{D}^{\mathbf{P}} \to [0, +\infty]$ *is an extended pseudometric and*

$$\mathrm{d}\left(\mathcal{H}^N_n(K), \mathcal{H}^N_n(L)\right) \leq \mathrm{d}(K, L).$$

Proof. By **Proposition 6.5**, $\omega := \omega_{\mathrm{d}_{\mathbb{Z}^N_{\geq 0}}} : \mathbf{Trans}_{\mathbf{Z}^N_{\geq 0}} \to [0, +\infty]$ defined by

$$\omega_{\mathrm{d}_{\mathbb{Z}^N_{\geq 0}}}(\Gamma) = \sup_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} \left\{\mathrm{d}_{\mathbb{Z}^N_{\geq 0}}(\mathbf{v}, \Gamma(\mathbf{v}))\right\}$$

is a sublinear projection. It follows that $\mathrm{d} := \mathrm{d}^\omega$ is an extended pseudometric on $\mathrm{obj}\left(\mathbf{AbSimp}^{\mathbf{Z}^N_{\geq 0}}\right)$. Recall that $(H_n \circ C_\bullet) : \mathbf{AbSimp} \to \mathbf{Mod}_R$ is a functor for any $n \in \mathbb{Z}_{\geq 0}$; see **Lemma 3.13**. Using the convention in **Definition 6.6**, we can say

$K^\bullet, L^\bullet \colon \mathbf{Z}^N_{\geq 0} \to$ **AbSimp** are arbitrary functors in obj $\left(\mathbf{AbSimp}^{\mathbf{Z}^N_{\geq 0}}\right)$. Thus, with $\mathrm{d} := \mathrm{d}^\omega$,

$$\begin{aligned} \mathrm{d}\left(\mathcal{H}^N_n(K), \mathcal{H}^N_n(L)\right) &= \mathrm{d}\left((H_n \circ C_\bullet) \circ K^\bullet, (H_n \circ C_\bullet) \circ L^\bullet\right) \\ &\leq \mathrm{d}(K, L) \end{aligned}$$

for any $n \in \mathbb{Z}_{\geq 0}$ by **Theorem 6.3**. □

Therefore N-graded persistent homology modules of multiradial filtrations are 1-Lipschitz.

CHAPTER VII

CONCLUSION AND FUTURE DIRECTIONS

This thesis has been devoted to building a vocabulary for the persistent homology of simplicial filtrations of nerve complexes, and their flag complexes, with multiple radial parameters. We have identified the persistent homology of multiradial filtrations as a specific case of multidimensional persistent homology, established various stability results for the persistence diagrams and (multiparameter) persistent homology modules summarizing multiradial (multi)filtrations, and generalized the Vietoris-Rips lemma of [DSG07, p. 346].

Now that we have established a parlance, we have several directions for future research. With reference to **Proposition 4.13**, we intend to develop implementation for exploratory data analysis using *weighted* persistence barcodes/diagrams. More precisely, we wish to develop software based around the functionality of computing persistence barcodes/diagrams with respect to different weight functions, either customized directly by a client or by a client-provided density function defined on the dataset. A goal would be to provide tools to carry out significance testing for a collection of weighted barcodes derived from a single dataset. We have had premature success in computing these weighted barcodes by *retrofitting* JavaPlex [TVJA14].

Another objective would be to generalize our multiradial Vietoris-Rips lemma. Currently, the original and multiradial versions of the Vietoris-Rips lemma specify that the Čech and Vietoris-Rips complexes are affine. In other words, the Vietoris-

Rips lemma only applies to Čech and Vietoris-Rips complexes having vertex sets in euclidean space. We would like to broaden this hypothesis to the case of nerve and flag complexes of covers over arbitrary compact metric spaces; see **Corollary C.16**.

The last direction we will mention is our original goal for developing implementation for sensor network covers. Recalling **Definition 3.7**, suppose we have $X = \{x_1, \ldots, x_N\} \subseteq \mathbb{X}$, $\mathbf{r} \in {\mathbb{R}_{>0}}^X$ is a weight function, and take $\mathbf{v} = \sum_{i=1}^{N} \mathrm{proj}_{\mathbf{e}_i}(\mathbf{v}) \in \mathbb{Z}^N_{\geq 0}$, where, of course, $\mathrm{proj}_{\mathbf{e}_i}(\mathbf{v}) := \frac{\mathbf{v} \bullet \mathbf{e}_i}{\mathbf{e}_i \bullet \mathbf{e}_i} \cdot \mathbf{e}_i$. Define $\mathbf{r_v}(x_i) = \mathbf{r}(x_i) \cdot \|\mathrm{proj}_{\mathbf{e}_i}(\mathbf{v})\|_2$ for each $0 \leq i \leq N$. Note that

$$\mathbf{r_u}(x_i) \leq \mathbf{r_v}(x_i)$$

$$\text{whenever}$$

$$\mathbf{u} \leq \mathbf{v} \in \mathbb{Z}^N_{\geq 0}$$

for each $0 \leq i \leq N$. For coverage optimization, we would like to consider multifiltered complexes $\check{C}_{\mathbf{r}}(X) := \left\{\check{C}_{\mathbf{r_v}}(X)\right\}_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}}$ and $R_{\mathbf{r}}(X) := \{R_{\mathbf{r_v}}(X)\}_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}}$.

With respect to the weight functions described in the previous paragraph, the N-graded persistent homology modules $H_n^N\left(\check{C}_{\mathbf{r}}(X)\right) = \bigoplus_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} H_n\left(\check{C}_{\mathbf{r_v}}(X)\right)$ and $H_n^N\left(R_{\mathbf{r}}(X)\right) = \bigoplus_{\mathbf{v} \in \mathbb{Z}^N_{\geq 0}} H_n\left(R_{\mathbf{r_v}}(X)\right)$ for $n = 0, 1$ would summarize all homological configurations for a planar sensor network. Our aim is to provide implementation for optimizing a given network cover, probably using [AB]. Conceptually, we would like to choose an optimal multigraded component $H_n\left(R_{\mathbf{r}_{\mathbf{v}_0}}(X)\right)$ of $H_n^N\left(R_{\mathbf{r}}(X)\right)$ which minimizes a cost function $\mathrm{Cost}(\mathbf{v})$ given for a particular sensor network. *Friis transmission equation* suggests $\mathrm{Cost}(\mathbf{v})$ be quadratic, or polynomial at least, in the radii of the sensing regions; see [BTB$^+$06], [BIV16], and [Sha13]. We also point out that the fixed weight function $\mathbf{r}$ can be interpreted as a setting of fixed radial increments for the *sensors* in X. Provided we can select an optimal component $H_n\left(R_{\mathbf{r}_{\mathbf{v}_0}}(X)\right)$ for fixed weight function $\mathbf{r}$, it is worthwhile to investigate the possibility of a *continuum* solution as $\mathbf{r}(x_i) \to 0$ for each $0 \leq i \leq N$.

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APPENDIX A
CATEGORY THEORY

We will make light use of category theoretic concepts to motivate the development of persistent homology. Again, this is done with the intuition that we can replace a topological description of a space with an algebraic one. We will primarily base our discussion on [Bor94]. To avoid potential pathologies of the size of sets, we include the following definitions and axioms. This is done merely for completeness; see [Bor94, chapter 1] for more information.

Definition A.1. A **Grothendieck universe** is a set $\mathfrak{U}$ satisfying:

(1) if $x \in y$ and $y \in \mathfrak{U}$, then $x \in \mathfrak{U}$;

(2) if $I \in \mathfrak{U}$ and $x_i \in \mathfrak{U}$ for every $i \in I$, then $\bigcup_{i \in I} x_i \in \mathfrak{U}$;

(3) if $x \in \mathfrak{U}$, then $2^x \in \mathfrak{U}$ where 2^x denotes the power set of x;

(4) if $x \in \mathfrak{U}$ and $f \colon x \to y$ is surjective, then $y \in \mathfrak{U}$;

(5) $\mathbb{N} = \{0, 1, 2, \ldots\} \in \mathfrak{U}$.

Lemma A.2. *If $y \in \mathfrak{U}$ and $x \subseteq y$, then $x \in \mathfrak{U}$.*

Proof. By **Definition A.1** (5), $\varnothing = 0 \in \mathbb{N}$ and $\mathbb{N} \in \mathfrak{U}$. This implies $\varnothing \in \mathfrak{U}$. Now, suppose $\varnothing \neq x \subseteq y$ and $y \in \mathfrak{U}$. Take an arbitrary $p \in x$. Define the set function $f \colon y \to x$ by

$$f(t) = \begin{cases} t & \text{if } t \in x \\ p & \text{if } t \notin x \end{cases}.$$

It is easy to see that f is surjective. By **Definition A.1** (4), $x \in \mathfrak{U}$ as desired. $\square$

We will axiomatically guarantee the existence of Grothendieck universes.

Axiom 1. *For every set x, there exists some universe $\mathfrak{U}$ such that $x \in \mathfrak{U}$.*

Definition A.3. For a fixed universe $\mathfrak{U}$, elements of $\mathfrak{U}$ will be called **small sets** and subsets of $\mathfrak{U}$ are called **sets**.

Without dwelling on pathology, we move on to categories.

Definition A.4. A **category C** is defined by the following:

(1) a set $\text{obj}(\mathbf{C})$ whose elements are referred to as objects;

(2) a set $\text{hom}(\mathbf{C})$ consisting of small sets $\text{hom}_{\mathbf{C}}(X, Y)$ of morphisms $X \to Y$;

(3) there exists a binary operation, or composition,

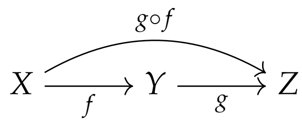
$$\text{hom}_{\mathbf{C}}(X, Y) \times \text{hom}_{\mathbf{C}}(Y, Z) \to \text{hom}_{\mathbf{C}}(X, Z) \colon (f, g) \mapsto g \circ f;$$

(4) there exists a morphism $\text{id}_X \in \text{hom}_{\mathbf{C}}(X, X)$ for all $X \in \text{obj}(\mathbf{C})$;

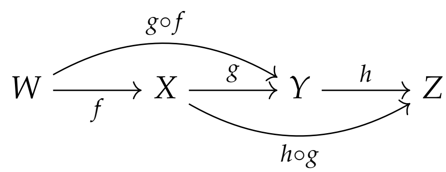
(5) $h \circ (g \circ f) = (h \circ g) \circ f$ for all $f \in \text{hom}_{\mathbf{C}}(W, X)$, $g \in \text{hom}_{\mathbf{C}}(X, Y)$, and $h \in \text{hom}_{\mathbf{C}}(Y, Z)$;

(6) $\text{id}_Y \circ f = f$ and $g \circ \text{id}_Y = g$ for any $f \in \text{hom}_{\mathbf{C}}(X, Y)$ and $g \in \text{hom}_{\mathbf{C}}(Y, Z)$.

Intuitively, we can think of categories as a collection of diagrams with some sort of restriction on the shapes presented by the diagrams. We summarize (3) by saying the following diagram commutes:



We extend this by saying a diagram **commutes** or is **commutative** if any composites sharing the same source and target are equal. With this convention, (5) gives us the following commutative diagram:



Lastly, we say **C** is a **small category** if obj($\mathbf{C}$) is a small set.

Example A.5. A **preordered set** $(P, \leq)$, or *proset*, is a set P with a binary relation $\leq$ that is both reflexive and transitive. Familiar examples are the integers $\mathbb{Z}$ and real numbers $\mathbb{R}$ with the standard total ordering. We can simultaneously view P as a category $\mathbf{P}$; let $\text{obj}(\mathbf{P}) = P$ and define $p \xrightarrow{\leq} q = p \to q \in \text{hom}_{\mathbf{P}}(p, q)$ if and only if $p \leq q$. The reflexivity of $\leq$ implies the existence of identity $p \xrightarrow{\text{id}} p = p \to p \in \text{hom}_{\mathbf{P}}(p, p)$ for all $p \in \text{obj}(\mathbf{P})$. The transitivity of $\leq$ guarantees a well-defined composition between morphisms: if $p \to q$ and $q \to r$ are morphisms in $\mathbf{P}$ then $p \leq q$ and $q \leq r$ which implies $p \leq r$; thus $p \to r \in \text{hom}_{\mathbf{P}}(p, r)$.

For associativity, suppose $p, q, r, s \in \text{obj}(\mathbf{P})$ are arbitrary with $p \leq q \leq r \leq s$. Then

$$(p \xrightarrow{\leq} q \xrightarrow{\leq} r) \xrightarrow{\leq} s = p \xrightarrow{\leq} (q \xrightarrow{\leq} r \xrightarrow{\leq} s),$$

which gives us associativity of composition. Finally, we have

$$p \xrightarrow{\leq} q \xrightarrow{\text{id}} q = p \xrightarrow{\leq} q$$

and

$$q \xrightarrow{\text{id}} q \xrightarrow{\leq} r = q \xrightarrow{\leq} r$$

for $p, q, r \in \text{obj}(\mathbf{P})$ with $p \leq q \leq r$. Thinking of $n = \{0, 1, \ldots, n-1\}$, $\mathbb{Z}$, and $\mathbb{R}$ as preordered sets with the standard total ordering, we will notate their categorical counterparts as $\mathbf{n}$, $\mathbf{Z}$, and $\mathbf{R}$. Another relevant preordered set is $(\mathbb{R}^d, \leq)$ where $(a_1, a_2, \ldots, a_d) \leq (b_1, b_2, \ldots, b_d)$ if and only if $a_i \leq b_i$ for each $0 \leq i \leq d$. The preorder $\leq$ is reflexive and transitive.

- Note $a_i \leq a_i$ for all $0 \leq i \leq d$ which implies $(a_1, a_2, \ldots, a_d) \leq (a_1, a_2, \ldots, a_d)$ for all $(a_1, a_2, \ldots, a_d) \in \mathbb{R}^d$. This shows $\leq$ is reflexive since $(a_1, a_2, \ldots, a_d)$ is arbitrary in $\mathbb{R}^d$.

- Suppose $(a_1, a_2, \ldots, a_d) \leq (b_1, b_2, \ldots, b_d)$ and $(b_1, b_2, \ldots, b_d) \leq (c_1, c_2, \ldots, c_d)$. This implies $a_i \leq b_i \leq c_i$ for all $0 \leq i \leq d$. In particular, $a_i \leq c_i$ for all $0 \leq i \leq d$ and thus $(a_1, a_2, \ldots, a_d) \leq (c_1, c_2, \ldots, c_d)$. This shows $\leq$ is transitive since $(a_1, a_2, \ldots, a_d), (b_1, b_2, \ldots, b_d), (c_1, c_2, \ldots, c_d) \in \mathbb{R}^d$ are arbitrary.

In this context, we call $\leq$ the **product order**. Given the preordered sets $(\mathbb{R}^d, \leq)$ and $\mathbb{Z}^d, \leq$ where $\leq$ is product order, we notate the associated categories by $\mathbf{R}^d$ and $\mathbf{Z}^d$.

Notice that

$$\bigcup_{p \in \text{obj}(\mathbf{P})} \text{hom}_{\mathbf{P}}(p, p) = \bigcup_{p \in \text{obj}(\mathbf{P})} \{\text{id}_p\}$$

is a small set by **Definition A.1** (2). It is easy to see that the set function

$$\bigcup_{p\in \mathrm{obj}(\mathbf{P})} \{\mathrm{id}_p\} \to \mathbf{P}\colon \mathrm{id}_p \mapsto p$$

is surjective. Thus **P** is a small set. Therefore, any preordered set is a small category.

We will quickly list other friendly examples of categories.

Example A.6. (1) The category **Set** of sets with morphisms being set functions.

(2) The category $\mathbf{Vect}_{\mathbb{F}}$ of vector spaces over the field $\mathbb{F}$ with morphisms being linear mappings.

(3) The category **Grp** of groups with morphisms being group homomorphisms.

(4) The category **Top** of topological spaces with morphisms being continuous mappings.

Now given categories, we can define a sort of *homomorphism* between them.

Definition A.7. Suppose **C** and **D** are categories. A (covariant) **functor** from **C** to **D** is defined by the following:

(1) A mapping $\mathrm{obj}(\mathbf{C}) \to \mathrm{obj}(\mathbf{D})$; we denote the image of $X \in \mathbf{C}$ by $F(X)$;

(2) A mapping $\mathrm{hom}_{\mathbf{C}}(X,Y) \to \mathrm{hom}_{\mathbf{D}}(F(X),F(Y))$; we denote the image of $f \in \mathrm{hom}_{\mathbf{C}}(X,Y)$ by $F(f)$;

(3) $F(g \circ f) = F(g) \circ F(f)$ for any $f \in \mathrm{hom}_{\mathbf{C}}(X,Y)$ and $g \in \mathrm{hom}_{\mathbf{C}}(Y,Z)$;

(4) $F(\mathrm{id}_X) = \mathrm{id}_{F(X)}$ for any $X \in \mathbf{C}$.

Intuitively, a functor $\mathbf{C} \to \mathbf{D}$ is a mapping between categories that takes objects to objects and morphisms to morphisms in a way that respects the categorical structure, that is, preserves composition and identity morphisms. In some sense, the diagrams of the source category $\mathbf{C}$ are mapped to diagrams in $\mathbf{D}$ in a way that preserves diagrammatic shape.

Example A.8. If $\mathbf{C}$ is a category, we define the identity functor $\mathrm{id}\colon \mathbf{C} \to \mathbf{C}$ by $\mathrm{id}(c) = c$ and $\mathrm{id}(f) = f$ for every $c \in \mathrm{obj}(\mathbf{C})$ and every morphism f. It is easy to check this satisfies **Definition** A.7.

Example A.9. Suppose $F\colon \mathbf{C} \to \mathbf{D}$ and $G\colon \mathbf{D} \to \mathbf{E}$ are functors. By pointwise composition, it is obvious $(G \circ F)\colon \mathbf{C} \to \mathbf{E}$ provides us with a well-defined mapping of objects and morphisms. Let $f \in \hom_{\mathbf{C}}(X, Y)$ and $g \in \hom_{\mathbf{C}}(Y, Z)$ be arbitrary morphisms. Notice that

$$\begin{aligned}(G \circ F)(g \circ f) &= G(F(g) \circ F(f)) \\ &= (G \circ F)(g) \circ (G \circ F)(f).\end{aligned}$$

Also,

$$\begin{aligned}(G \circ F)(\mathrm{id}_X) &= G(\mathrm{id}_{F(X)}) \\ &= \mathrm{id}_{(G \circ F)(X)}.\end{aligned}$$

Thus the composition of (covariant) functors is closed.

Example A.10. The *forgetful* functor $\mathcal{F}\colon \mathbf{Top} \to \mathbf{Set}$ assigns to each $X \in \mathrm{obj}(\mathbf{Top})$ the underlying set X without its topological structure. For each continuous map

$f \in \hom_{\mathbf{Top}}(X, Y)$, $\mathcal{F}$ assigns the corresponding set function that evaluates identically to f. Let us check (3) and (4) of **Definition** A.7.

(3) Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are arbitrary continuous maps. Then

$$\mathcal{F}(g \circ f) = g \circ f = \mathcal{F}(g) \circ \mathcal{F}(f).$$

(4) Suppose $X \in \mathbf{Top}$ is arbitrary. Then

$$\mathcal{F}(\mathrm{id}_X) = \mathrm{id}_X = \mathrm{id}_{\mathcal{F}(X)}$$

where id_X denotes the identity map on X. There are similar forgetful functors $\mathbf{Grp} \to \mathbf{Set}$ and $\mathbf{Vect}_{\mathbb{F}} \to \mathbf{Set}$.

Example A.11. Suppose $f\colon X \to Y$ is a set function. Define the *direct image mapping* $\tilde{f}\colon 2^X \to 2^Y$ by

$$\tilde{f}(S) = \{f(x) \mid x \in X\}.$$

We obtain a functor $\wp\colon \mathbf{Set} \to \mathbf{Set}$ by mapping X to 2^X and $f\colon X \to Y$ to $\tilde{f}\colon 2^X \to 2^Y$ for any $X, Y \in \mathrm{obj}(\mathbf{Set})$ and each $f \in \hom_{\mathbf{Set}}(X, Y)$. The functor $\wp$ is called the *power set functor*.

Given two functors with the same source and target, we can define a sort of *morphism* between them that preserves the structure of the underlying categories.

Definition A.12. Suppose we have two functors $F, G\colon \mathbf{C} \to \mathbf{D}$. A **natural transformation** $\tau\colon F \Rightarrow G$ is a set of morphisms $\{\tau_X\colon F(X) \to G(X)\}_{X \in \mathrm{obj}(\mathbf{C})}$ of $\mathbf{D}$ such that $(\tau_Y \circ F(f)) = (G(f) \circ \tau_X)$ for every morphism $f\colon X \to Y$ in **C**. This means diagram (1.1)

$$\begin{array}{ccc} F(X) & \xrightarrow{F(f)} & F(Y) \\ {\scriptstyle \tau_X}\downarrow & & \downarrow{\scriptstyle \tau_Y} \\ G(X) & \xrightarrow[G(f)]{} & G(Y) \end{array} \tag{1.1}$$

commutes for every **C**-morphism $f\colon X \to Y$. In some sense, τ maps the **D**-morphisms determined by F to **D**-morphisms determined by G in a way that respects both **C** and **D**.

Example A.13. With reference to the identity functor and the power set functor defined in **Examples** A.8 and A.11, we define the family of functions $\{\varsigma_X\colon X \to 2^X\}_{X \in \mathbf{Set}}$ by

$$\varsigma_X(x) = \{x\}.$$

Notice that, for any sets X and Y and any set function $f\colon X \to Y$,

$$\begin{aligned} (\varsigma_Y \circ \mathrm{id}(f))(x) &= \{x\} \\ &= \mathrm{id}(\tilde{f})(\{x\}) \\ &= (\mathrm{id}(\tilde{f}) \circ \varsigma_X)(x). \end{aligned}$$

Thus $\varsigma\colon \mathrm{id} \Rightarrow \wp$ is a natural transformation.

If we are careful, natural transformations actually take on the role of morphisms in so-called **functor categories**.

Proposition A.14. *Suppose* $\mathbf{C}$ *is a small category and* $\mathbf{D}$ *is an arbitrary category. Taking objects to be functors* $\mathbf{C} \to \mathbf{D}$ *and morphisms as natural transformations between said functors forms a functor category denoted by* $\mathbf{D}^{\mathbf{C}}$*.*

Proof. Recall that a natural transformation $\tau \colon F \Rightarrow G$ is a set of $\mathbf{D}$-morphisms $\{\tau_X\}_{X \in \mathbf{C}}$. It is easy to see that

$$\tau \subseteq H := \bigcup_{X \in C} \hom_D \left(F(X), G(X)\right).$$

for any $X \in \mathbf{C}$ and natural transformation $\tau \colon F \Rightarrow G$. By **Definition A.1** (2), H is a small set since $\mathbf{C}$ is a small category. It follows from **Definition A.1** (3) that 2^H is a small set. Since $\tau \in 2^H$ for any natural transformation $\tau \colon F \Rightarrow G$,

$$\hom_{\mathbf{D}^{\mathbf{C}}}(F, G) \subseteq 2^H$$

and hence $\hom_{\mathbf{D}^{\mathbf{C}}}(F, G)$ is a small set by **Definition A.1** (1). Thus $\hom(\mathbf{D}^{\mathbf{C}})$ is well-defined.

We define composition of natural transformations by **vertical composition**: $(v \circ \tau)_X := v_X \circ \tau_X$ where $X \in \mathbf{C}$, $\tau \colon F \Rightarrow G$ and $v \colon G \Rightarrow H$ are natural transformations, and $F, G, H \colon \mathbf{C} \to \mathbf{D}$ are functors. It is easy to see the vertical composition is well-defined and associative as a point-wise composition of $\mathbf{D}$-morphisms. Clearly, the identity morphism $\mathrm{id}_F \colon F \Rightarrow F$ is the natural transformation consisting of $\{\mathrm{id}_{F(X)} \colon F(X) \to F(X)\}_{X \in \mathbf{C}}$. Altogether, $\mathbf{D}^{\mathbf{C}}$ is a category. □

Corollary A.15. *Suppose* $\mathbf{P}$ *is a preordered set and* $\mathbf{D}$ *is an arbitrary category. Taking objects to be functors* $\mathbf{P} \to \mathbf{D}$ *and morphisms as natural transformations between said functors forms a functor category denoted by* $\mathbf{D}^{\mathbf{P}}$.

Proof. By **Example A.5**, $\mathbf{P}$ is a small category. Thus $\mathbf{D}^{\mathbf{P}}$ is a functor category by **Proposition A.14**. □

In Appendix E, we will develop persistent homology modules as objects in a particular kind of functor category.

APPENDIX B
MODULES AND CHAIN COMPLEXES

We will need to recall some basic module theory for our development of persistent homology. Our discussion is primarily based on [Rot02, DF04]. For what follows, we will assume that $(R, +, \cdot)$ is a nonzero commutative ring with unit 1_R and identity 0_R. By way of notation, we will let $\cong$ denote isomorphisms of groups, modules, et cetera, $\simeq$ will denote homotopy equivalence, and $\approx$ will denote homeomorphism equivlence.

Definition B.1. Recall that an **ideal in** R is an additive subgroup I such that $ra \in I$ whenever $a \in I$ and $r \in R$. Suppose $A = \{s_1, s_2, \ldots, s_n\} \subseteq R$. The set

$$AR := \{r_1 s_1 + r_2 s_2 + \cdots + r_n s_n \mid r_i \in R\}$$

is an ideal in R called the **ideal** AR **generated by** $A \subseteq R$. An ideal $(a) := aR$ generated by $a \in R$ is called a **principal ideal**. R is a **principal ideal domain**, or PID, if R is an integral domain in which every ideal in R is a principal ideal.

Example B.2. Let us mention some trivial ideals. The trivial subgroup $\{0_R\}$ is an ideal of R since $0_R r = 0_R \in \{0_R\}$ for all $r \in R$. Also, R is an ideal in itself since $sr \in R$ for all $r, s \in R$ by closure of R under multiplication as a ring. It is easy to see that $\{0_R\}$ is principal since $0r = 0$ for all $r \in R$.

Example B.3. Consider the ring $R[x]$ of polynomials over a single indeterminate with coefficients in R. Let I be a nonzero ideal in $R[x]$ and take $p \in I$ to be a nonzero monic polynomial of minimal degree. Given any $f \in I$, $f = qp + r$ where

$r = 0$ or $\deg(r) < \deg(p)$ and $q, r \in I$ by the division algorithm; see [Rot02, p. 132]. Thus $r = f - qp \in I$ since I is an ideal. However, this means $r = 0$ since $\deg(r) < \deg(p)$ is a contradiction to our choice of p otherwise. Thus $g = qp \in (p)$ for all $g \in I$. Therefore all ideals of $R[x]$ are principal and have the form (x^k) for some $k \geq 0$. This shows $R[x]$ is a PID.

With the intuition of a vector space over a field of scalars, we can think of a module as a *vector space over a ring of scalars*:

Definition B.4. An R**-module** is an abelian group $(M, +)$ equipped with an operation $R \times M \to M$ such that the following hold for all $r, s \in R$ and $x, y \in M$:

(1) $r(x + y) = rx + ry$;

(2) $(r + s)x = rx + sx$;

(3) $(rs)x = r(sx)$;

(4) $1_R x = x$.

Suppose M is an R-module and $x \in M$. A **submodule** N **of** M is a subgroup N of M satisfying $rn \in N$ for any $r \in R$. We define the **cyclic submodule generated by x** to be $\langle x \rangle := \{rx \mid r \in R\}$. More generally, if $X \subseteq M$, then we define the **submodule generated by** X to be

$$\langle X \rangle := \left\{ \sum_{i=1}^{n} r_i x_i \,\middle|\, r_i \in R, x_i \in X, 0 \leq n < +\infty \right\}.$$

A module M is **finitely generated** if there exists a finite subset $X \subseteq M$ with $M = \langle X \rangle$. Suppose N is a submodule of an R-module M. We can define the

quotient module M/N to be the quotient group M/N equipped with the *scalar multiplication*

$$r(m+N) = rm + N$$

where $r \in R$ and $m \in M$. For completeness, let us recall the notion of a free R-module. Suppose M is an R-module and $N \subseteq M$. We say M is **free on** $B \subseteq M$ if $m = \sum_{i \in I} r_i b_i$ where $r_i \in R$ and $b_i \in B \subseteq M$ are unique for some at most finite index set I and all $0 \neq m \in M$. We say that the **rank of M** is $\operatorname{rk}(M) := |B|$ and call B a **basis for M**. We will also write $M = \langle B \rangle$.

Take $I \neq \varnothing$ to be an indexing set and assume M_i is an R-module for every $i \in I$. The **direct sum of modules** $\bigoplus_{i \in I} M_i$ **over I** is the R-module defined as follows:

$$\prod_{i \in I} \{m_i\} \in \bigoplus_{i \in I} M_i$$
$$\text{with the action} \tag{2.1}$$
$$r \cdot \prod_{i \in I} \{m_i\} := \prod_{i \in I} \{r \cdot m_i\}$$

for any $r \in R$ and $m_i \in M_i$ with the stipulation that $m_i = 0_{M_i}$ for all but finitely many $i \in I$. Note that (2.1) indicates the direct sum of the abelian groups M_i. This does define an R-module indeed; observe the following where $r, s \in R$ and $(m_i)_{i \in I}, (n_i)_{i \in I} \in \bigoplus_{i \in I} M_i$.

(1)

$$\begin{aligned} r \cdot \left[(m_i)_{i \in I} + (n_i)_{i \in I}\right]_{i \in I} &= r \cdot (m_i + n_i)_{i \in I} \\ &= (r \cdot [m_i + n_i])_{i \in I} \\ &= (r \cdot m_i + r \cdot n_i)_{i \in I} \end{aligned}$$

$$= (r \cdot m_i)_{i \in I} + (r \cdot n_i)_{i \in I}$$
$$= r \cdot (m_i)_{i \in I} + r \cdot (n_i)_{i \in I}.$$

(2)

$$\begin{aligned}(r+s) \cdot (m_i)_{i \in I} &= ([r+s] \cdot m_i)_{i \in I} \\ &= (r \cdot m_i + s \cdot m_i)_{i \in I} \\ &= (r \cdot m_i)_{i \in I} + (s \cdot m_i)_{i \in I} \\ &= r \cdot (m_i)_{i \in I} + s \cdot (n_i)_{i \in I}.\end{aligned}$$

(3)

$$\begin{aligned}(rs) \cdot (m_i)_{i \in I} &= ([rs] \cdot m_i)_{i \in I} \\ &= (r \cdot [s \cdot m_i])_{i \in I} \\ &= r \cdot (s \cdot m_i)_{i \in I} \\ &= r \cdot [s \cdot (m_i)_{i \in I}].\end{aligned}$$

(4)

$$\begin{aligned}1_R \cdot (m_i)_{i \in I} &= (1_R \cdot m_i)_{i \in I} \\ &= (m_i)_{i \in I}.\end{aligned}$$

Lemma B.5. *If M is free and $|\mathrm{rk}(M)| < \infty$, then M is finitely generated.*

Proof. Since M is free and has finite rank, there exists some $B = \{b_i\}_{i=0}^{n} \subseteq M$ such that $\sum_{i=0}^{n} r_i b_i$ for all $0 \neq m \in M$. Hence $M = \langle B \rangle$ and therefore M is finitely generated. □

Lemma B.6. *If N is a submodule of a finitely generated module M, then N is finitely generated.*

Proof. Let $Y \subseteq M$ be a finite set and assume $M = \langle Y \rangle$. Since N is a submodule of M, there is some set $X \subseteq Y$ such that $N = \langle X \rangle$. Hence $X \subseteq M$ is a finite set and therefore N is finitely generated. □

Lemma B.7. *If N is a submodule of a finitely generated module M, then M/N is finitely generated.*

Proof. By **Lemma B.6**, $M = \langle Y \rangle$ and $N = \langle X \rangle$ for some finite sets $X \subseteq Y \subseteq M$. Say $X = \{x_i\}_{i=0}^{n}$ and $Y = \{x_i\}_{i=0}^{m}$ with $m < n$ and define

$$Y' := \{x_i + N \,|\, m < i \leq n\}.$$

In the case that $m = n$, $M/N = \{0\}$ is trivial. Obviously, $\langle Y' \rangle$ is finitely generated. Take an arbitrary element of $\langle X \rangle$, say $\sum_{i=m+1}^{n} r_i \cdot (x_i + N) \in \langle X \rangle$. Then

$$\sum_{i=m+1}^{n} r_i \cdot (x_i + N) = \sum_{i=m+1}^{n} (r_i x + N) \in M/N$$

and hence $\langle Y' \rangle \subseteq M/N$.

Now, take an arbitrary $m + N \in M/N$ and $n \in N$. It follows that $m = \sum_{i=m+1}^{n} r_i x_i$ where $r_i \in R$ are arbitrary. Observe

$$\begin{aligned} m + n &= \left(\sum_{i=m+1}^{n} r_i x_i \right) + n \\ &= (r_{m+1} x_{m+1} + n) + \left(\sum_{i=m+2}^{n} r_i x_i \right) \\ &= (r_{m+1} x_{m+1} + n) + \left(\sum_{i=m+2}^{n} r_i x_i + 0_N \right) \in \langle Y' \rangle. \end{aligned}$$

Since $n \in N$ was arbitrary, $M/N = \langle Y' \rangle$. Therefore M/N is finitely generated. □

Example B.8. As before, let R be a commutative ring. Then R is an R-module since for $p, q, r, s \in R$

(1) $r(x+y) = rx + ry$ holds by left distributivity of R;

(2) $(r+s)x$ holds by right distributivity of R;

(3) $(rs)x = r(sx)$ holds by associativity of multiplication in R;

(4) $1_R x = x$ since R has a multiplicative identity by definition.

Example B.9. Clearly, any vector space over a field $\mathbb{F}$ is an $\mathbb{F}$-module.

Definition B.10. Suppose M and N are are R-modules. A function $f\colon M \to N$ is an R**-homomorphism** if, for all $x, y \in M$ and all $r \in R$, the following are satisfied:

(1) $f(x+y) = f(x) + f(y)$;

(2) $f(rx) = rf(x)$.

We say f is an R**-isomorphism** and write $M \cong N$ if f is also a bijection. We respectively define the **kernel of** f and **image of** f by

$$\ker(f) := \{x \in M \mid f(x) = 0\}$$

and

$$\operatorname{im}(f) := f(M) = \{y \in N \mid f(x) = y \text{ for some } x \in M\}.$$

Example B.11. Suppose M is an R-module. The identity map $\mathrm{id}_M\colon M \to M$ is an R-homomorphism. Take arbitrary $x, y \in M$ and $r \in R$. Note that

$$\mathrm{id}_M(x+y) = x + y = \mathrm{id}_M(x) + \mathrm{id}_M(y).$$

Also,

$$\mathrm{id}_M(rx) = rx = r \cdot \mathrm{id}_M(x).$$

Example B.12. Suppose $f\colon X \to Y$ and $g\colon Y \to Z$ are R-homomorphisms and take $x, y \in X$ and $r \in R$. Note the following:

(1)

$$\begin{aligned}(g \circ f)(x+y) &= g(f(x)+f(y)) = g(f(x)) + g(f(y)) \\ &= (g \circ f)(x) + (g \circ f)(y);\end{aligned}$$

(2) $(g \circ f)(rx) = g(rf(x)) = rg(f(x)) = r(g \circ f)(x)$.

By arbitrariness of $x, y \in X$ and $r \in R$, $(g \circ f)$ is an R-homomorphism.

Example B.13. Suppose $f\colon V \to W$ is an R-homomorphism. Then $\ker(f)$ is a submodule of V and $\mathrm{im}(f)$ is a submodule of W.

Example B.14. Let $f\colon V \to W$ be a linear map between vector spaces over the field $\mathbb{F}$. Then f is an $\mathbb{F}$-homomorphism.

Example B.15. Taking objects to be R-modules and morphisms to be R-homomorphisms, we can form the category of R-modules $\mathbf{Mod}_R$. By the previous example, composition of morphisms is well-defined. Associativity follows from associativity of composition and the identity morphism is just the identity map from **Example** B.11.

In Appendix E, we shall see that persistent homology can be understood through the structure of graded finitely generated modules.

Definition B.16. A (nonnegatively) **graded ring** R is a ring with a direct sum decomposition of abelian groups

$$R \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} R^i$$

such that $R^i R^j \subseteq R^{i+j}$ for all $i, j \in \mathbb{Z}_{\geq 0}$. We say the elements of R^i are **homogeneous of degree** i.

Example B.17. Of particular interest for us in Appendix E, we can grade the polynomial ring $R[x]$ by degree. Setting $R^i[x] := Rx^i$ for all $i \geq 0$, we have that

$$rx^i \cdot sx^j = (rs)x^{i+j} \in R^{i+j}[x] := Rx^{i+j}.$$

for any homogeneous elements $rx^i \in Rx^i$ and $sx^j \in Rx^j$. This gives the following decomposition:

$$R[x] \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} Rx^i.$$

Definition B.18. A (nonnegatively) **graded module** is an R-module M, R is a graded ring, with a direct sum decomposition of abelian groups

$$M \cong \bigoplus_{i \in \mathbb{Z}_{\geq 0}} M^i$$

such that $R^i M^j \subseteq M^{i+j}$.

The standard structure theorem for finitely generated modules over PIDs provides us with a *factorization* of modules up to isomorphism.

Theorem B.19 (Structure theorem for finitely generated modules)**.** *Suppose R is a PID. Then every finitely generated R-module M decomposes uniquely into a direct sum of cyclic R-modules*

$$M \cong R^f \oplus \left(\bigoplus_{i=0}^{m} R/r_i R \right)$$

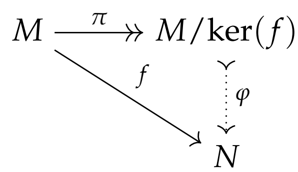
for some $f, m \in \mathbb{Z}_{\geq 0}$ and $r_i \in R$ satisfying $r_i \mid r_{i+1}$.

Theorem B.19 says that every finitely generated module over a PID R decomposes uniquely, up to isomorphism, into the direct sum of a finitely generated *free* module and a finitely generated *torsion* module. The interested reader can find the proof for **Theorem** B.19 in [DF04, p. 463] or [Row06, p. 68].

There is also a structure theorem for graded finitely generated modules. For the proof, we recall the first isomorphism theorem for modules:

Theorem B.20 (First isomorphism theorem for modules)**.** *Let M and N be R-modules and suppose $f\colon M \to N$ is an R-module homomorphism. Then $\ker(f)$ is a submodule of M, $f(M)$ is a submodule of N, and $M/\ker(f) \cong f(M)$.*

Proof. Consider M and N as abelian groups. By the first isomorphism theorem for groups, $\ker(f)$ is a subgroup of M, $f(M)$ is a subgroup of N, and both $M/\ker(f)$ and $f(M)$ are isomorphic as abelian groups. This is summarized by the following commutative diagram:



Take $x \in \ker(f)$; then

$$f(rx) = rf(x) = r0_M = 0$$

for all $r \in R$ which implies $f(rx) \in \ker(f)$. Hence $x \in \ker(f)$ is a submodule of M. Let $y \in f(M)$; then $y = f(x)$ for some $x \in M$ and

$$ry = rf(x) = f(rx) \in f(M)$$

for all $r \in R$. Thus $f(M)$ is a submodule of N. Since f is an R-homomorphism,

$$\varphi(r(x + \ker(f))) = \varphi(rx + \ker(f)) = f(rx)$$
$$rf(x) = r\varphi(x + \ker(f));$$

this shows φ is an R-homomorphism. Thus $M/\ker(f) \cong f(M)$ as R-modules. □

We would like to note that the proof for the following theorem is quite similar to the proof in the ungraded case; see [DF04].

Theorem B.21 (Structure theorem for graded finitely generated modules)**.** *Suppose M is a graded finitely generated module over a graded PID R. Then M is isomorphic to a* *direct sum of the form:*

$$\left(\bigoplus_{i=0}^{\kappa_0} \Sigma^{\xi_i} R\right) \oplus \left(\bigoplus_{j=0}^{\kappa_1} \Sigma^{\zeta_j} R \,/\, (d_j)\right)$$

where $\xi_i, \zeta_j, \kappa_0, \kappa_1 \in \mathbb{Z}_{\geq 0}$*,* $(d_j) := d_j R$ *are homogeneous elements so that* $d_j \mid d_{j+1}$*. The map* Σ^k *gives a k-shift upward in grading for* $k \in \mathbb{Z}_{\geq 0}$*, that is to say,*

$$(\eta_i)_{i \in \mathbb{Z}_{\geq 0}} \overset{\Sigma^k}{\longmapsto} (\eta_i')_{i \in \mathbb{Z}_{\geq 0}}$$

where $\eta_i' = 0$ *whenever* $i < k$ *and* $\eta_i' = \eta_{i-k}$ *for each* $i \geq k$*.*

Proof. Suppose $\{m_i\}_{i=0}^n$ is a set of generators for M of minimal cardinality $n \geq 0$. Let R^{n+1} be the free R-module of rank $(n+1)$ with basis $\{b_i\}_{i=0}^n$. Define the mapping $f \colon R^{n+1} \to M$ by $f(b_i) = m_i$ for all $0 \leq i \leq n$. **Extending by linearity**, we obtain an R-module homomorphism; this means we set

$$f\left(\sum_{i=0}^{n} r_i b_i\right) = \sum_{i=0}^{n} r_i f(b_i)$$

for all $r_i \in R$. By the first isomorphism theorem for modules,

$$R^{n+1} / \ker(f) \cong M.$$

By [DF04, p. 60, Theorem 4], we can choose another basis $\{y_i\}_{i=0}^n$ of R^{n+1} so that $\{a_i y_i\}_{i=0}^m$ is a basis of $\ker(f)$ for some $\{a_i\}_{i=0}^m \subseteq R$ with $a_0 \mid a_1 \mid \cdots \mid a_m$ and $0 \leq$

$m \leq n$. This implies

$$\begin{aligned} M &\cong R^{n+1} / \ker(f) \\ &\cong \left(\bigoplus_{i=0}^{n} R y_i \right) \Big/ \left(\bigoplus_{i=0}^{m} R a_i y_i \right). \end{aligned}$$

We will use the first isomorphism theorem for modules to identify the last quotient module above. Define the surjective R-module homomorphism

$$\pi\colon \bigoplus_{i=0}^{n} R y_i \to \left(\bigoplus_{m+1}^{n} \Sigma^{\deg(y_i)} R \right) \oplus \left(\bigoplus_{0}^{m} \Sigma^{\deg(y_i)} R / (a_i) \right)$$

defined by

$$\pi((\alpha_0 y_0, \alpha_1 y_1, \ldots, \alpha_n y_n)) = (\alpha_0 \bmod(a_0), \alpha_1 \bmod(a_1), \ldots, \alpha_n \bmod(a_n))$$

where $\deg(y_i)$ is the degree induced by the grading of R. By definition of π,

$$\begin{aligned} \ker(\pi) &\cong \bigoplus_{i=0}^{m} R a_i y_i \\ &\cong \ker(f). \end{aligned}$$

Thus, by the first isomorphism theorem, $\operatorname{im}(\pi) \cong M$, that is,

$$M \cong \left(\bigoplus_{i=m+1}^{n} \Sigma^{\deg(y_i)} R \right) \oplus \left(\bigoplus_{i=0}^{m} \Sigma^{\deg(y_i)} R / (a_i) \right),$$

as desired. □

The next well-known proposition is included for completeness and will become relevant in Appendix D.

Proposition B.22. *Suppose R is a PID and let $f\colon M \to N$ be an R-module homomorphism between free R-modules M and N with rank n and m, respectively. Assume that f is described by an $m \times n$ matrix A. Then there exists invertible matrices P and Q such that*

$$\begin{aligned} \mathsf{PAQ} &= \mathsf{D} \\ &= \begin{bmatrix} d_1 & 0 & 0 & \cdots & 0 & \cdots & 0 \\ 0 & d_2 & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots & \ddots & 0 \\ 0 & 0 & \cdots & d_k & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 & 0 & \cdots & 0 \end{bmatrix} \end{aligned}$$

where $d_1, \ldots, d_k \in R$ satisfy $d_1 \mid d_2 \mid \cdots \mid d_k$.

The matrix D is said to be the **Smith normal form of** A. **Proposition** B.22 guarantees that any matrix representation of an R-module homomorphism can be reduced to a Smith normal form using row and column reduction operations. The proof of **Proposition** B.22 can be found in [Row06, p. 66]. Even though it is in the context of free abelian groups, [Mun84, s. 1.11] gives an accessible account of this result as well.

We will use sequences of R-modules in our treatment of persistent homology.

Definition B.23. An R**-chain complex** $(C_\bullet, \partial_\bullet)$ is a sequence of R-modules and R-homomorphisms

$$\cdots \xleftarrow{\partial_n} C_n \xleftarrow{\partial_{n+1}} C_{n+1} \xleftarrow{\partial_{n+2}} C_{n+2} \xleftarrow{\partial_{n+3}} \cdots$$

where $(\partial_n \circ \partial_{n+1}) \equiv 0$ for $n \in \mathbb{Z}$. The R-homomorphism ∂_n is called the n**th boundary map**. Using less structure, we note other treatments will define chain complexes as sequences of abelian groups with group homomorphisms; see [Rot98, c. 5] or [Mun84, s. 1.5].

Example B.24. A trivial example of a chain complex is the *zero complex* $\{0_\bullet, \partial_\bullet\}$ where the R-modules $\{0\}$ and R-homomorphisms $\partial_\bullet \colon \{0\} \to \{0\}$ are trivial. In this case, $(\partial_n \circ \partial_{n-1})(0) = \partial_n(\partial_{n-1}(0)) = 0$.

Next, we will define the proper morphisms between R-chain complexes.

Definition B.25. Suppose $(C_\bullet, \partial_\bullet)$ and $(C'_\bullet, \partial'_\bullet)$ are chain complexes. A **chain map**

$$f_\bullet \colon (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial'_\bullet)$$

is a sequence of maps $f_n \colon C_n \to C'_n$ making the following diagram commute for all $n \in \mathbb{Z}$:

$$\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \downarrow{\scriptstyle f_{n+1}} & & \downarrow{\scriptstyle f_n} & & \downarrow{\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow[\partial_{n+1}]{} & C'_n & \xrightarrow[\partial_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}$$

If we define composition componentwise

$$\{g_\bullet\} \circ \{f_\bullet\} := \{g_\bullet \circ f_\bullet\},$$

then we can construct the category $\mathbf{Comp}_R$ of R-chain complexes with morphisms being chain maps. Associativity of morphisms follows from associativity of composition and the identity morphism is the sequence $\{\mathrm{id}_\bullet\}$ of identity maps.

Using sequences of modules, we can construct homology modules in a general setting.

Definition B.26. Suppose the $(C_\bullet, \partial_\bullet)$ is a chain complex. Then the **R-module of n-cycles** $Z_n(C_\bullet) := \ker(\partial_n)$ and the **R-module of n-boundaries** $B_n(C_\bullet) := \mathrm{im}(\partial_{n+1})$. We define the **$n$th homology R-module of** $(C_\bullet, \partial_\bullet)$ to be

$$H_n(C_\bullet) := Z_n(C_\bullet)/B_n(C_\bullet).$$

The **nth Betti number β_n of** $(C_\bullet, \partial_\bullet)$ is $\mathrm{rk}(H_n(C_\bullet))$.

Example B.27. Suppose $f\colon M \to N$ is an R-homomorphism. We can construct a chain complex $(C_\bullet, \partial_\bullet)$ by setting $C_0 = N$, $C_1 = M$, $\partial_1 = f$, and letting $C_i = \{0\}$ and $\partial_i\colon \{0\} \to \{0\}$ be trivial for all $i \neq 1$. Cancellation of boundary maps is satisfied since

$$(\partial_0 \circ \partial_1)(x) = \partial_0(\partial_1(x)) = 0$$

and

$$(\partial_1 \circ \partial_2)(0) = \partial_1(\partial_2(0)) = 0$$

for all $x \in M$ and all other compositions are trivial. If $n = 1$, then

$$\begin{aligned} H_n(C_\bullet) &= Z_n(C_\bullet)/B_n(C_\bullet) = \ker(f)/\{0\} \\ &= \ker(f). \end{aligned}$$

If $n \neq 0$, then

$$\begin{aligned} H_n(C_\bullet) &= Z_n(C_\bullet)/B_n(C_\bullet) = \ker(\partial_0)/\mathrm{im}(f) \\ &= N/\mathrm{im}(f). \end{aligned}$$

If $n \neq 0, 1$, then

$$\begin{aligned} H_n(C_\bullet) &= Z_n(C_\bullet)/B_n(C_\bullet) = \{0\}/\{0\} \\ &= \{0\}. \end{aligned}$$

In this case, we can say with certainty that $\beta_n = 0$ for all $n \neq 0, 1$.

Proposition B.28. *For all $n \in \mathbb{Z}$, $H_n \colon \mathbf{Comp}_R \to \mathbf{Mod}_R$ is a functor.*

Proof. Suppose $(C_\bullet, \partial_\bullet)$ is an R-chain complex. By definition, H_n maps $(C_\bullet, \partial_\bullet)$ to $H_n(C_\bullet)$. Let $f_\bullet \colon (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial'_\bullet)$ be a chain map and for morphisms define $H_n(f_\bullet) \colon H_n(C_\bullet) \to H_n(C'_\bullet)$ by

$$\begin{aligned} H_n(f_\bullet)([z]) &= H_n(f_\bullet)(z + B_n(C_\bullet)) \\ &= f_n(z) + B_n(C'_\bullet) \end{aligned}$$

for any $[z] \in H_n(C_\bullet)$. We will show H_n is well-defined on chain maps; we will need to show $f_n(z) \in Z_n(C'_\bullet)$ and that $f_n(z)$ is independent of the choice of representative $z \in Z_n(C_\bullet)$. Note that the following diagram commutes since $f_\bullet$ is a chain complex:

$$
\begin{array}{ccccccccc}
\cdots & \longrightarrow & C_{n+1} & \xrightarrow{\partial_{n+1}} & C_n & \xrightarrow{\partial_n} & C_{n-1} & \longrightarrow & \cdots \\
 & & \Big\downarrow {\scriptstyle fn+1} & & \Big\downarrow {\scriptstyle f_n} & & \Big\downarrow {\scriptstyle f_{n-1}} & & \\
\cdots & \longrightarrow & C'_{n+1} & \xrightarrow[\partial_{n+1}]{} & C'_n & \xrightarrow[\partial_n]{} & C'_{n-1} & \longrightarrow & \cdots
\end{array}
\tag{2.2}
$$

If $z \in Z_n(C_\bullet)$, then $\partial_n(z) = 0_{C_{n-1}}$ by definition of $Z_n(C_\bullet)$. Given the commutativity of diagram of the diagram above,

$$
\begin{aligned}
(\partial'_n \circ f_n)(z) = (f_{n-1} \circ \partial_n)(z) = f_{n-1}(0) \\
= 0.
\end{aligned}
$$

Thus $f_n(z) \in Z_n(C'_\bullet)$ for any $z \in Z_n(C_\bullet)$.

Next, assume that $z - y \in B_n(C_\bullet)$, that is, z and y are **homologous**, meaning they are representatives of the same homology class. This means $z - y = \partial_{n+1}(c)$ for some $c \in C_{n+1}$. Thus,

$$
\begin{aligned}
f_n(z - y) &= f_n(\partial_{n+1}(c)) \\
&= (\partial_{n+1} \circ f_{n+1})(c) \in B_n(C_\bullet)
\end{aligned}
$$

by commutativity. Assume $f_\bullet\colon (C_\bullet, \partial_\bullet) \to (C'_\bullet, \partial'_\bullet)$ and $g_\bullet\colon (C'_\bullet, \partial'_\bullet) \to (C''_\bullet, \partial''_\bullet)$ are chain maps with a well-defined composition $(g_\bullet \circ f_\bullet)$. Then, for any $[z] \in H_n(C_\bullet)$,

$$\begin{aligned} H_n(g_\bullet \circ f_\bullet)([z]) &= (g_n \circ f_n)(z) + B_n(C''_\bullet) \\ &= H_n(g_\bullet)(f_n(z) + B_n(C'_\bullet) \\ &= H_n(g_\bullet)H_n(f_\bullet)([z]). \end{aligned}$$

Lastly, it is clear that $H_n\left(\mathrm{id}_{(C_\bullet, \partial_\bullet)}\right) = \mathrm{id}_{(C_\bullet, \partial_\bullet)}$ is the identity. Therefore H_n is a functor between $\mathbf{Comp}_R$ and $\mathbf{Mod}_R$. □

APPENDIX C
SIMPLICIAL COMPLEXES

As mentioned in the introduction, we are looking to approximate complicated topological spaces with simpler ones. Simplicial complexes are a standard tool used for this purpose as they are built up by incidence relations between simple discrete pieces. In most cases, we will be interested in purely combinatorial abstract simplicial complexes. However, we will establish intuition by first discussing simplicial complexes embedded in euclidean space. Our treatment is motivated largely by [Rot98].

Definition C.1. Suppose $A \subseteq \mathbb{R}^d$. We say A is **affine** if $(1-t)p + tq \in A$ for any $p, q \in A$ and any $t \in \mathbb{R}$. Assume $S = \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$ and let A be the affine set they span, that is, for all $q \in A$, $q = (1-t)x_i + tx_j$ for some $t \in \mathbb{R}$ and $x_i, x_j \in S$. A function $f\colon A \to \mathbb{R}^d$ is **affine** provided

$$f\left(\sum_{i=0}^{n} t_i p_i\right) = \sum_{i=0}^{n} t_i f(p_i)$$

for $t_0, t_1, \ldots, t_n \in \mathbb{R}$ with $\sum_{i=0}^{n} t_i = 1$. We define an **affine combination of the points** $x_0, x_1, \ldots, x_n$ to be a point $p = \sum_{i=0}^{n} t_i x_i$ where $\sum_{i=0}^{n} t_i = 1$. A **convex combination of** $x_0, x_1, \ldots, x_n \in \mathbb{R}^d$ is an affine combination $p = \sum_{i=0}^{n} t_i x_i$ where $t_i \geq 0$ for each $0 \leq i \leq n$. The **convex hull of** $S = \{x_0, x_1, \ldots, x_n\}$ is defined by

$$\mathrm{conv}(S) := \left\{ p \in \mathbb{R}^d \,\middle|\, p \text{ is a convex combination of } x_0, x_1, \ldots, x_n \right\}.$$

We say S is **affinely independent** if the set $\{x_i - x_0 | 1 \leq i \leq n\} \subseteq \mathbb{R}^d$ is linearly independent. The restriction $f|_{\text{conv}(S)}$ of an affine map $f\colon A \to \mathbb{R}^d$ will also be referred to as an **affine map**.

The following proposition is useful when dealing with convex hulls.

Proposition C.2. *Let $S = \{x_0, x_1, \ldots, x_n\} \subseteq \mathbb{R}^d$ be affinely independent. For any point $p \in \text{conv}(S)$, p is uniquely expressible as a convex combination of the points in S.*

Proof. Take $p \in \text{conv}(S)$ and suppose

$$p = \sum_{i=0}^{n} t_i x_i = \sum_{i=0}^{n} t_i' x_i$$

where $\sum_{i=0}^{n} t_i = 1 = \sum_{i=0}^{n} t_i'$, and $t_i, t_i' \geq 0$ for each $0 \leq i \leq n$. Consequently,

$$\sum_{i=0}^{n} (t_i - t_i') = 0 = \sum_{i=0}^{n} (t_i - t_i') x_i.$$

This allows the following manipulation:

$$\begin{aligned}
\sum_{i=1}^{n} (t_i - t_i')(x_i - x_0) &= \left(\sum_{i=1}^{n} (t_i - t_i') x_i \right) - \left(\sum_{i=1}^{n} (t_i - t_i') x_0 \right) \\
&= -x_0 \sum_{i=1}^{n} (t_i - t_i') \\
&= 0.
\end{aligned}$$

Affine independence of S gives us the linear independence of $\{x_i - x_0\}_{i=1}^{n}$, and thus $t_i = t'_i$ for each $1 \leq i \leq n$. Hence,

$$0 = \sum_{i=0}^{n}(t_i - t'_i)x_i$$
$$= (t_0 - t'_0)x_0$$

and thus $t_0 = t'_0$. Therefore the weights t_i are unique for each $0 \leq i \leq n$ and for any $p \in \text{conv}(S)$. □

Definition C.3. Suppose that $S = \{x_0, x_1, \ldots, x_n\} \subset \mathbb{R}^d$ is affinely independent. We define the ***n*-simplex** $(x_0, x_1, \ldots, x_n)$ **with vertices** $x_0, x_1, \ldots, x_n$ to be the convex hull of S and say that σ is spanned by S. If $\sigma \subset \mathbb{R}^d$ is an n-simplex, we denote its vertices by $\text{vert}(\sigma)$. A ***k*-face of** σ is a k-simplex τ where $\text{vert}(\tau) \subseteq \text{vert}(\sigma)$ and $0 \leq k \leq n$. The **dimension of** σ is $\dim(\sigma) := |\text{vert}(\sigma)|$. Note that we retrieve a total ordering of $\text{vert}(\sigma)$ if we define $x_i \preceq_\sigma x_j$ if and only if $i \leq j$ for all $x_i, x_j \in \text{vert}(\sigma)$. Thus the total ordering $\preceq_\sigma$ allows us to distinguish n-simplices sharing the same vertex set. Suppose σ is an n-simplex and $\tau = (y_0, \ldots, y_m)$ is an m-simplex with $0 \leq m \leq n$. If $y_i \in \text{vert}(\sigma)$ for each $0 \leq i \leq m$ and $y_i \preceq_\sigma y_j$ whenever $0 \leq i \leq j \leq m$, then we call τ an ***m*-face of** σ. Similarly, we say that σ is an ***n*-coface of** τ.

With restrictions on incidence relations, simplices are combined to build more complicated structures.

Definition C.4. Suppose K is a collection of simplices in $\mathbb{R}^d$. We say K is a **(affine) simplicial complex** if the following are satisfied:

(1) if $\sigma \in K$, then $\tau \preceq_\sigma \sigma$ implies $\tau \in K$;

(2) if $\sigma, \tau \in K$, then either $\sigma \cap \tau = \varnothing$ or $\sigma \cap \tau \preceq_\tau \tau$ and $\sigma \cap \tau \preceq_\sigma \sigma$.

A **subcomplex of K** is a simplicial complex L where $L \subseteq K$. The **dimension of a simplicial complex** is $\dim(K) := \sup_{\sigma \in K}(\dim(\sigma))$. Let K be a simplicial complex. Then the j-skeleton $K^{(j)}$ of K is defined by $\sigma \in K^{(j)}$ when $\sigma \in K$ and $\dim(\sigma) \leq j$ for $j \geq -1$. In particular, notice $K^{(-1)} = \varnothing$. We will say that a simplicial complex K is **finite** provided $|K^{(0)}| < \infty$; otherwise, K is **infinite**. If K and L are simplicial complexes, then $f : K^{(0)} \to L^{(0)}$ is a **simplicial map** provided $\{f(x_0), f(x_1), \ldots, f(x_n)\}$ spans a simplex in L whenever $\{x_0, x_1, \ldots, x_n\}$ spans a simplex in K.

Using simplicial maps as morphisms, we can form a category of affine simplicial complexes denoted by **Simp**.

Proposition C.5. *Taking objects to be affine simplicial complexes and morphisms to be simplicial maps, we can form the category of affine simplcial complexes* **Simp**.

Proof. This is simple to check. Composition of morphisms is given by composition of simplicial maps as set functions. Associativity of morphisms is due to associativity of simplicial maps. Finally, the identity morphism is just the identity map $\mathrm{id}\colon K^{(0)} \to K^{(0)}$ for any $K \in \mathrm{obj}(\mathbf{Simp})$. □

In practice, simplicial complexes are useful as discrete approximations of topological spaces. It is possible to associate an underlying topological space to arbitrary simplicial complexes; see the discussion of *CW-topology* in [MS82, pp. 289-290]. With an eye towards computation, we will restrict ourselves to finite simplicial complexes. With this in mind, we will discuss obj(**Simp**) as containing only finite simplicial complexes without any change in notation.

Definition C.6. Suppose K is a finite simplicial complex in $\mathbb{R}^d$ and X is a topological space. We call the set

$$|K| := \bigcup_{\sigma \in K} \sigma \subseteq \mathbb{R}^d$$

the **underlying space of K**. The space X is said to be a **polyhedron** or **triangulable** if there exists a homeomorphism $f : |L| \to X$ for some simplicial complex L. We will say that (L, f) provides a **triangulation of** X.

Underlying gives us a tool for associating diagrams involving simplicial and topological objects.

Proposition C.7. *Underlying defines a functor* $|\cdot|\colon \mathbf{Simp} \to \mathbf{Top}$.

Proof. With respect to objects, let underlying map K to $|K|$. We will need to take some care with the mapping of morphisms. Suppose we have a simplical map $f\colon K^{(0)} \to L^{(0)}$ and take an arbitrary $\sigma = (x_0, x_1, \ldots, x_n) \in K$. Consider the restricted map $f|_{\mathrm{vert}(\sigma)}\colon \{x_0, x_1, \ldots, x_n\} \to \{f(x_0), f(x_1), \ldots, f(x_n)\}$. By the definition of simplicial map, $\{f(x_0), f(x_1), \ldots, f(x_n)\}$ spans a unique affine simplex $\tau \in L$. This gives us a well-defined map

$$\tilde{f}_\sigma\colon \{x_0, x_1, \ldots, x_n\} \to |L|\colon \tilde{f}_\sigma(x_i) = (\iota_{|L|} \circ f|_{\mathrm{vert}(\sigma)})(x_i)$$

where $\iota_{|L|}$ is the inclusion map into $|L|$.

Now we define an affine map $T_\sigma\colon \sigma \to |L|$ by

$$T_\sigma\left(\sum_{i=0}^{n} t_i x_i\right) = \sum_{i=0}^{n} t_i \tilde{f}_\sigma(x_i)$$

where $\sum_{i=0}^{n} t_i x_i$ is a convex combination in σ. Take note that $T_\sigma(x_i) = \tilde{f}_\sigma(x_i)$ for each $0 \leq i \leq n$; by **Proposition C.2**, T_σ is the unique affine function with this property. Suppose σ_0 and σ_1 are arbitrary simplices in K satisfying $\sigma_0 \cap \sigma_1 \neq \varnothing$. Then, by definition of simplicial complex, $\sigma_0 \cap \sigma_1 \in K$ and, as above, we have a unique affine function $T_{\sigma_0 \cap \sigma_1}$ that agrees with $\tilde{f}_\sigma$ on the vertices of the face $\sigma_0 \cap \sigma_1$. The uniqueness of $T_{\sigma_0 \cap \sigma_1}$ guarantees that T_{σ_0} and T_{σ_1} agree on overlaps, that is,

$$T_{\sigma_0}|_{\sigma_0 \cap \sigma_1} = T_{\sigma_1}|_{\sigma_0 \cap \sigma_1}.$$

By [Rot98, p. 14, Lemma 1.1], there exists a unique continuous map $|K| \to |L|$ which we will denote by $|f|$. The map $|f|$ will be defined as the image of the morphism f with respect to the underlying functor and is referred to as **piecewise linear**.

It is easy to see that, for simplicial maps $f\colon K \to L$ and $g\colon L \to M$,

$$\begin{aligned}
\widetilde{(g \circ f)}_\sigma &= \iota_{|M|} \circ (g \circ f)|_{\mathrm{vert}(\sigma)} \\
&= \iota_{|M|} \circ g|_{\mathrm{vert}(f(\sigma))} \circ \iota_{|L|}|_{\mathrm{vert}(f(\sigma))} \circ f|_{\mathrm{vert}(\sigma)} \\
&= \tilde{g}_{f(\sigma)} \circ \tilde{f}_\sigma.
\end{aligned}$$

This implies

$$\sum_{i=0}^{n} t_i \widetilde{(g \circ f)}_\sigma(x_i) = \sum_{i=0}^{n} t_i (\tilde{g}_{f(\sigma)} \circ \tilde{f}_\sigma)(x_i)$$

where $\sum_{i=0}^{n} t_i x_i$ is a convex combination in σ. Hence $|g \circ f| = |g| \circ |f|$. Now consider the identity simplicial map $\mathrm{id}_K \colon K \to K$. Notice that

$$\begin{aligned}\tilde{f}_\sigma &= \iota_{|K|} \circ \mathrm{id}_{\mathrm{vert}(\sigma)} \\ &= \mathrm{id}_{\mathrm{vert}(\sigma)}\end{aligned}$$

and thus $T_\sigma = \mathrm{id}_\sigma$ for each $\sigma \in K$. Therefore $|\mathrm{id}_K| = \mathrm{id}_{|K|}$. Altogether underlying is a functor from **Simp** into **Top**. □

Quite often, it is convenient to work with a purely combinatorial description of a simplicial complex. In some sense, one can worry about *realizing* it as a topological space when the need arises.

Definition C.8. Suppose V is a set. An **abstract simplicial complex** K is a collection of finite empty subsets of V satisfying

(1) if $v \in V$, then $\{v\} \in K$;

(2) if $\sigma \in K$ and $\tau \subseteq \sigma$, then $\tau \in K$.

We say K is **finite** when V is finite; we say K is **infinite** otherwise. We will naturally set $\mathrm{vert}(K) := V$ and call $\sigma \in K$ an **n-simplex** whenever $|\mathrm{vert}(\sigma)| = n + 1$. We also say $\dim(\sigma) = n$ in this case. In the context that K and L are abstract simplicial complexes, a **simplicial map** is a function $f : K^{(0)} \to L^{(0)}$ satisfying the property that $\{f(v_0), f(v_1), \ldots, f(v_n)\} \in L$ whenever $\{v_0, v_1, \ldots, v_n\} \in \mathrm{vert}(K)$. As in the affine case, the **dimension of** σ is $\dim(\sigma) := |\mathrm{vert}(\sigma)|$. We also extend **Definitions** C.3-C.4 to the case of abstract simplicial complexes.

As with affine simplicial complexes, we can form a category of abstract simplicial complexes.

Proposition C.9. *Abstract simplicial complexes and simplicial maps between them forms a category* **AbSimp**.

Proof. This is another routine check. Composition of morphisms is given by composition of simplicial maps as set functions. Associativity of morphisms is due to associativity of simplicial maps. Finally, the identity morphism is just the identity map $\mathrm{id}\colon K^{(0)} \to K^{(0)}$ for any $K \in \mathrm{obj}(\mathbf{AbSimp})$. □

It should be clear that every affine simplicial complex K determines an abstract simplicial complex K': take $\{x_0, x_1, \ldots, x_n\} \in K'$ and collect the vertices $x_0, x_1, \ldots, x_n \in \mathrm{vert}(K')$ for each simplex $(x_0, x_1, \ldots, x_n) \in K$. As mentioned earlier, it is possible to reverse this correspondence and *realize* an arbitrary abstract simplicial complex. Again, we will favor intuition and limit ourselves to finite abstract simplicial complexes, even in the context of **AbSimp**, but we encourage the reader to investigate the general construction found in [Rot98, p. 197] or [MS82, p. 290]. Our approach is sufficient for the computation of simplicial homology using finite-dimensional linear algebra.

Definition C.10. We define the **standard** n**-simplex** $\Delta^n := (\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_{n+1}) \subseteq \mathbb{R}^{n+1}$ where $\mathbf{e}_i \in \mathbb{R}^{n+1}$ is the ith standard basis vector. Let K be a finite abstract simplicial complex and define

$$|\{v_{i_0}, v_{i_1}, \ldots, v_{i_n}\}| := \left(\mathbf{e}_{i_0}, \mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}\right)$$

for $\{v_{i_0}, v_{i_1}, \ldots, v_{i_n}\} \in K$. We let the **standard geometric realization of** K be the topological subspace

$$|K| := \bigcup_{\sigma \in K} |\sigma|$$

in ambient euclidean space. Further, a **geometric realization of** K is any topological space $L \approx |K|$. We will denote any geometric realization of K by $|K|$.

Proposition C.11. *The construction of standard geometric realization provides a functor* $|\cdot|\colon \mathbf{AbSimp} \to \mathbf{Simp}$.

Proof. Given the construction of the standard geometric realization, it is easy to see that $|K|$ is a subcomplex of Δ^n where K is an abstract simplicial complex with $\dim(K) = n + 1$. Thus the mapping of objects $K \mapsto |K|$ is well-defined. Suppose that $f\colon K^{(0)} \to L^{(0)}$ is a simplicial mapping of abstract simplicial complexes where $K^{(0)} = \{u_i\}_{i=0}^m$ and $L^{(0)} = \{v_j\}_{j=0}^n$. Define $|f|\colon |K|^{(0)} \to |L|^{(0)}$ by

$$|f|(\mathbf{e}_i) = \mathbf{e}_j \text{ whenever } f(u_i) = v_j$$

where $|K|^{(0)} = \{\mathbf{e}_i\}_{i=0}^m$ and $|L|^{(0)} = \{\mathbf{e}_j\}_{j=0}^n$. The mapping of morphisms is well-defined as a consequence of the total ordering of vertices in K and L.

Suppose $f\colon K \to L$ and $g\colon L \to M$ are simplicial maps with $K^{(0)} = \{u_i\}_{i=0}^{p}$, $L^{(0)} = \{v_j\}_{j=0}^{q}$, and $M^{(0)} = \{w_k\}_{k=0}^{r}$. Notice that

$$\begin{aligned}(|g| \circ |f|)\,(\mathbf{e}_i) &= |g|(\mathbf{e}_j)\\ &= \mathbf{e}_k\\ &= |g \circ f|(\mathbf{e}_i)\end{aligned}$$

provided $(g \circ f)(u_i) = g(v_j) = w_k$. It follows that $|g \circ f| = |g| \circ |f|$. Also, it is clear that $|\mathrm{id}_K| = \mathbf{e}_i = \mathrm{id}_{|K|}(\mathbf{e}_i)$ for any $0 \leq i \leq p$. Thus $|\mathrm{id}_K| = \mathrm{id}_{|K|}$. This proves the functoriality of geometric realization. □

In practice, it is typical to not distinguish between abstract simplicial complexes and affine counterparts. **Proposition C.11** quickly gives us the following corollary.

Corollary C.12. *Composition of underlying after geometric realization provides a functor* $\|\cdot\|\colon \mathbf{AbSimp} \to \mathbf{Top}$.

Proof. This follows immediately from functoriality of both underlying and standard geometric realization and **Example A.9**. □

Next, we will provide some important examples of abstract simplicial complexes. Let $\mathbb{X}$ be a topological space. Recall that a cover of a subspace $X \subseteq \mathbb{X}$ is an indexed family of subsets $\{X_i\}_{i \in I}$ of $\mathbb{X}$ such that $X \subseteq \bigcup_{i \in I} X_i$. We will say that a cover is closed provided all sets in the cover are closed.

Definition C.13. We say a cover is **good** if each nonempty intersection of members of the cover is contractible. Suppose $U = \{U_i\}_{i \in I}$ is a cover of some topological space X. The **Čech complex, or *nerve*, of** U is the abstract simplicial complex defined by

$$\check{C}(U) := \left\{ \{U_{i_0}, U_{i_1}, \ldots, U_{i_n}\} \,\middle|\, \bigcap_{j=0}^{n} U_{i_j} \neq \varnothing \right\}.$$

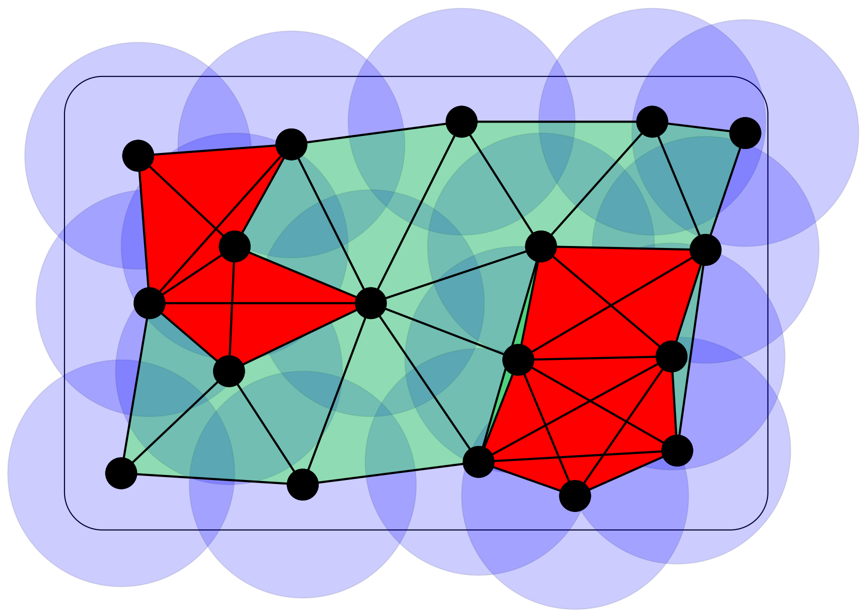


Figure 7. Picture of Nerve of a Closed Cover

The following result is of great importance.

Theorem C.14 (Čech theorem or nerve lemma)**.** *If G is an open cover of a paracompact Hausdorff space X such that every nonempty intersection of cover sets in G is contractible, then* $\check{C}(G)$ *has the homotopy type of X.*

The nerve lemma tells us we can approximate a paracompact Hausdorff space with an abstract simplicial complex constructed from a good cover. For proof, see

[Hat02, p. 459, Corollary 4G.3] or [Koz07, p. 269, Theorem 15.21]. We note that both proofs omit showing the existence of a partition of unity for paracompact spaces; this is done in [MS82, p. 325] and [Dug66, p. 170]. However, we are interested in using closed covers for our treatment of persistent homology. That leads us to reformulate **Theorem** C.14.

Theorem C.15. *If F is a finite closed cover of a compact Hausdorff space X such that every nonempty intersection of cover sets in F is contractible, then $\check{C}(F)$ has the homotopy type of X.*

Corollary C.16. *If F is a finite collection of closed subsets of a compact Hausdorff space X such that every nonempty intersection of sets in F is contractible, then $\check{C}(F) \simeq \bigcup_{D \in F} D$.*

These reformulations follow from the nerve lemma and proof can be found in [Flo57, pp. 319, 332]. For a short discussion of the nerve lemma in the combinatorial context, we welcome the reader to [Bjo95, s. 10].

Suppose $F = \{X_i\}_{i \in I}$ is a cover of a topological space X. The **Vietoris-Rips**, or **Rips**, **complex of** F is the abstract simplicial complex defined by

$$R(F) := \left\{ \{X_{i_0}, X_{i_1}, \ldots, X_{i_n}\} \;\middle|\; X_{i_j} \cap X_{i_k} \neq \varnothing \text{ for } j \neq k \text{ and } 0 \leq j,k \leq n \right\}.$$

It follows immediately by definition that the Vietoris-Rips complex is an example of a flag complex:

Definition C.17. An abstract simplicial complex K with vertex set V is a **flag complex** provided

$$\{v_0, v_1, \ldots, v_n\} \in K \text{ if and only if } \{v_i, v_j\} \in K$$

for any $v_0, v_1, \ldots, v_n \in V$ and $0 \leq i, j \leq n$ with $i \neq j$.

We can use the concept of precise refinements to construct subcomplexes of the Čech and Vietoris-Rips complexes.

Definition C.18. Suppose we have a covers $X = \{X_i\}_{i \in I}$ and $Y = \{Y_j\}_{j \in J}$ of a topological space Z. We say X is a **precise refinement of** Y if $I = J$ and $X_i \subseteq Y_i$ for all $i \in I$. So, in the case X is a precise refinement of Y, we can define the following subcomplex of $\check{C}(Y)$:

$$\check{C}_X(Y) := \left\{ \{Y_{i_0}, \ldots, Y_{i_n}\} \,\middle|\, \bigcap_{j=0}^{n} X_{i_j} \neq \varnothing \right\}.$$

We can also define a similar subcomplex of $R(Y)$:

$$R_X(Y) := \left\{ \{Y_{i_0}, \ldots, Y_{i_n}\} \,\middle|\, X_{i_j} \cap X_{i_k} \neq \varnothing \text{ for } j \neq k \text{ and } 0 \leq j, k \leq n \right\}.$$

The fact nonempty intersection of cover sets in X implies nonempty intersection of cover sets in Y guarantees $\check{C}_X(Y) \subseteq \check{C}(Y)$ and $R_X(Y) \subseteq R(Y)$ are well-defined subcomplexes.

Suppose $S = \{S_i\}_{i \in I}$ is a collection of nonempty subsets of some metric space $(\mathbb{X}, \mathrm{d})$. The **Voronoi cell** V_i **of the set** S_i is the set

$$V_i := \{p \in \mathbb{X} \mid \mathrm{d}(S_i, p) \leq \mathrm{d}(S_j, p) \text{ for any } S_j \in S \text{ with } S_i \neq S_j\}.$$

where $\mathrm{d}(S_i, p) := \inf_{q \in S}\{q, p\}$. We call the set of cells $V = \{V_i\}_{i \in I}$ the **Voronoi diagram of** S. The **Delaunay complex** $D(V)$ **of** V is defined to be the nerve $\check{C}(V)$, that is,

$$D(V) := \left\{ \{V_{i_0}, V_{i_1}, \ldots, V_{i_n}\} \,\middle|\, \bigcap_{j=0}^{n} V_{i_j} \neq \varnothing \right\}.$$

Our definitions are based on the treatment given in [Zom07, Koz07]. Delaunay complexes satisfy precise minimal *roughness* and *volume* properties; see [Rip90, Pow92, DS89]. These allow Delaunay complexes to provide heuristically favorable triangulations of polygonal regions and finite point sets. Discussion of algorithms for computing Voronoi diagrams and Delaunay Complexes can be found in [dBCvKO08, cc. 7, 9].

APPENDIX D

SIMPLICIAL HOMOLOGY

In this section, we will develop the homology theory of abstract simplicial complexes. This will allow us to infer the homology of triangulable topological spaces through finite point samples. The primary reference for this section is [Rot98, c. 7].

Definition D.1. Suppose K is an abstract simplicial complex and $\preceq$ is a partial ordering of $\text{vert}(K)$. We say $(K, \preceq)$ is an **oriented abstract simplicial complex** provided $\preceq$ is a total ordering when restricted to $\text{vert}(\sigma)$ for any $\sigma \in K$. We will define the **orientation of** $\sigma = \{v_{i_0}, \ldots, v_{i_n}\} \in K$ to be the equivalence class $[\sigma] = [v_{i_0}, \ldots, v_{i_n}] := [\{v_{i_0}, \ldots, v_{i_n}\}]$ given by $\{v_{i_0}, \ldots, v_{i_n}\} \sim \left\{v_{i_{\pi(0)}}, \ldots, v_{i_{\pi(n)}}\right\}$ if and only if $\text{sgn}(\pi) = 1$ where π is a permutation of $\{0, \ldots, n\}$.

Suppose K is an oriented abstract simplicial complex. Define

$$\text{alt}\left(K^{(n)} \setminus K^{(n-1)}\right) := \left\{[v_{i_{\pi(0)}}, \ldots, v_{i_{\pi(n)}}] \in \left(K^{(n)} \setminus K^{(n-1)}\right)/\sim \,\middle|\, \text{sgn}(\pi) = +1\right\}.$$

The n**th simplicial chain module** $C_n(K, R)$ **of** K is the free R-module with basis $\text{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$ where $n \geq 0$. In this context, we define

$$[v_{i_{\pi(0)}}, \ldots, v_{i_{\pi(n)}}] := \text{sgn}(\pi)[v_{i_0}, \ldots, v_{i_n}] = -[v_{i_0}, \ldots, v_{i_n}]$$

where π is an odd permutation of $\{0, 1, \ldots, n\}$. We will typically abbreviate $C_n(K, R)$ as $C_n(K)$ when R is understood. An element of $c \in C_n(K, R)$ has the form $c = \sum_i r_i[\sigma_i]$ where $r_i \in R$ and each σ_i is an n-simplex. We call the elements of $C_n(K)$

simplicial n-chains. Our development of simplicial chains is based on free modules; for a group theoretic treatment, see [Rot98, p. 143].

Note that n-chains are just *formal R-linear combinations* of oriented n-simplices. Next we define the simplicial boundary maps as operators that map an oriented $(n+1)$-simplex to an R-linear combination of oriented n-simplices. For what follows, assume K is an oriented abstract simplicial complex.

Definition D.2. Define the n**th simplicial boundary map** $\partial_n \colon C_n(K) \to C_{n-1}(K)$ by

$$\partial_n\left(\left[v_{j_0}, \ldots, v_{j_n}\right]\right) := \sum_{i=0}^{n}(-1)^i[\sigma]^{\hat{\imath}} := \sum_{i=0}^{n}(-1)^i\left[v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_n}\right]$$

and

$$\partial_n\left(0_{C_n(K)}\right) := 0_{C_{n-1}(K)}$$

where $[\sigma] = \left[v_{j_0}, \ldots, v_{j_n}\right] \in \operatorname{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$, $n \geq 1$, and $\hat{v}_{j_i}$ indicates the absence of v_{j_i} from the $(n-1)$-simplex. For an n-chain $c = \sum_i r_i[\sigma_i] \in C_n(K)$ we extend ∂_n linearly, that is,

$$\partial_n(c) = \sum_i r_i \partial_n([\sigma_i]).$$

We will use the notation $[\sigma]^{\hat{\imath}_0, \ldots, \hat{\imath}_m}$ to indicate the absence of multiple vertices.

Simplicial boundary maps satisfy the cancellation property mentioned in **Definition B.23**.

Lemma D.3. $(\partial_{n-1} \circ \partial_n)(c) = 0$ *for any* $n \geq 2$ *and* $c \in C_n(K)$.

Proof. Recall that $C_n(K)$ is generated by alt $\left(K^{(n)} \setminus K^{(n-1)}\right)$ for each $n \geq 1$. Moreover, ∂_n was extended linearly on alt $\left(K^{(n)} \setminus K^{(n-1)}\right)$ by definition. Thus, it suffices to show $(\partial_{n-1} \circ \partial_n)(\sigma) = 0$ for every $[\sigma] \in$ alt $\left(K^{(n)} \setminus K^{(n-1)}\right)$ for each $n \geq 2$. Observe

$$\begin{aligned}
(\partial_{n-1} \circ \partial_n)([\sigma]) &= \partial_{n-1}\left(\sum_{i=0}^{n}(-1)^i[\sigma]^{\hat{\imath}}\right) \\
&= \sum_{i=0}^{n}(-1)^i\left[\sum_{j=0}^{i-1}(-1)^j[\sigma]^{\hat{\imath},\hat{\jmath}} + \sum_{j=i+1}^{n}(-1)^j[\sigma]^{\hat{\imath},\hat{\jmath}}\right] \\
&= \sum_{i=0}^{n}(-1)^i\left[\sum_{j=0}^{i-1}(-1)^j[\sigma]^{\hat{\imath},\hat{\jmath}} + \sum_{j=i}^{n-1}(-1)^{j+1}[\sigma]^{\hat{\imath},\hat{\jmath}}\right] \\
&= \sum_{i=0}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}[\sigma]^{\hat{\imath},\hat{\jmath}} + \sum_{i=0}^{n}\sum_{j=i}^{n-1}(-1)^{i+j}[\sigma]^{\hat{\imath},\hat{\jmath}} \\
&= \sum_{\substack{0 \leq j \leq i-1 \\ 0 \leq i \leq n}}(-1)^{i+j}[\sigma]^{\hat{\imath},\hat{\jmath}} + \sum_{\substack{0 \leq i \leq j-1 \\ i+1 \leq j \leq n}}[\sigma]^{\hat{\imath},\hat{\jmath}}. \qquad (4.1)
\end{aligned}$$

In the last equality above, all pairs $(i, j) \in n \times n$ occur in both sums with opposite signs $(-1)^{i+j}$ and $(-1)^{i+j+1}$ where $0 \leq i \neq j \leq n$. Thus $(\partial_{n-1} \circ \partial_n)([\sigma]) = 0$ for any oriented $[\sigma] \in$ alt $\left(K^{(n)} \setminus K^{(n-1)}\right)$ as desired. □

From **Lemma D.3**, we immediately have the following:

Corollary D.4. $\operatorname{im}(\partial_n) \subseteq \ker(\partial_{n-1})$ *for all* $n \geq 2$.

Making the stipulation that $C_n(K) = \{0\}$ and $\partial_n \equiv 0$ for all $n \leq 0$, we see that $(C_\bullet(K), \partial_\bullet)$ is a chain complex.

Corollary D.5. *Suppose K is an oriented abstract simplicial complex. Take $C_n(K) = \{0\}$ for all $n \leq -1$ and $\partial_n \equiv 0$ for all $n \leq 0$. Then $(C_\bullet(K), \partial_\bullet)$ is an R-chain complex.*

Proof. By **Lemma D.3**, $(\partial_{n-1} \circ \partial_n) \equiv 0$ for each $n \geq 2$. Moreover, $(\partial_{n-1} \circ \partial_n) \equiv 0$ for each $n \leq 1$ since $\partial_n \equiv 0$ for all $n \leq 0$. Therefore $(C_\bullet(K), \partial_\bullet)$ is an R-chain complex. □

Definition D.6. We will call the chain complex discussed in **Corollary D.5** a **simplicial R-chain complex**. It is worth demonstrating the existence of chain maps between simplicial chain complexes. Much of the effort put into defining $C_n(K)$ was to guarantee simplicial chain maps are well-defined. If $f\colon K^{(0)} \to L^{(0)}$ is a simplicial map between oriented simplicial complexes, define $f_\bullet\colon C_\bullet(K) \to C_\bullet(L)$ to be the sequence of maps $f_n\colon C_n(K) \to C_n(L)$ defined by, for each $n \geq 0$ and $[v_{i_0}, \ldots, v_{i_n}] \in \text{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$,

$$f_n\left([v_{i_0}, \ldots, v_{i_n}]\right) := \begin{cases} \left[f\left(v_{i_0}\right), \ldots, f\left(v_{i_n}\right)\right] & \text{if } f\left(v_{i_k}\right) \neq f\left(v_{i_\ell}\right) \text{ for each } k \neq \ell \\ 0_{C_n(L)} & \text{if } f(v_{i_k}) = f(v_{i_\ell}) \text{ for some } k \neq \ell \end{cases}$$

where $0 \leq k, \ell \leq n$. We extend f_n linearly over $C_n(K)$ for each $n \geq 0$. Of course, we will define $f_n \equiv 0$ for every $n \leq -1$. We will call $f_\bullet$ a **simplicial chain map** and we prove $f_\bullet$ is indeed a chain map in the next lemma.

Lemma D.7. *If $f\colon K^{(0)} \to L^{(0)}$ is a simplicial map between oriented simplicial complexes, then $f_\bullet\colon C_\bullet(K) \to C_\bullet(L)$ is a chain map.*

Proof. It suffices to show $(f_n \circ \partial_n) \equiv (\partial_n \circ f_{n-1})$ for each $n \in \mathbb{Z}$ and for every $\left[v_{j_0}, \ldots, v_{j_n}\right] \in \text{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$. If $f\left(v_{j_k}\right) \neq f\left(v_{j_\ell}\right)$ for each $0 \leq k \neq \ell \leq n$, then

$$\begin{aligned}(f_n \circ \partial_n)\left(\left[v_{j_0}, \ldots, v_{j_n}\right]\right) &= f_n\left(\sum_{i=0}^{n}(-1)^i\left[v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_n}\right]\right) \\ &= \sum_{i=0}^{n}(-1)^i f_n\left(\left[v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_n}\right]\right) \\ &= \sum_{i=0}^{n}(-1)^i\left[f\left(v_{j_0}\right), \ldots, \widehat{f\left(v_{j_i}\right)}, \ldots, f\left(v_{j_n}\right)\right] \\ &= \partial_n\left(\left[f\left(v_{j_0}\right), \ldots, f\left(v_{j_i}\right), \ldots, f\left(v_{j_n}\right)\right]\right) \\ &= (\partial_n \circ f_n)\left(\left[v_{j_0}, \ldots, v_{j_i}, \ldots, v_{j_n}\right]\right).\end{aligned}$$

If $f\left(v_{j_k}\right) = f\left(v_{j_\ell}\right)$ for some $0 \leq k \neq \ell \leq n$, then

$$\begin{aligned}(f_n \circ \partial_n)\left(\left[v_{j_0}, \ldots, v_{j_n}\right]\right) &= f_n\left(\sum_{i=0}^{n}(-1)^i\left[v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_n}\right]\right) \\ &= \sum_{i=0}^{n}(-1)^i f_n\left(\left[v_{j_0}, \ldots, \hat{v}_{j_i}, \ldots, v_{j_n}\right]\right) \\ &= \sum_{i=0}^{n}(-1)^i \cdot 0_{C_{n-1}(L)} \\ &= 0_{C_{n-1}(L)} \\ &= \partial_n\left(0_{C_n(L)}\right) \\ &= (\partial_n \circ f_n)\left(\left[v_{j_0}, \ldots, v_{j_i}, \ldots, v_{j_n}\right]\right).\end{aligned}$$

Since $\left[v_{j_0}, \ldots, v_{j_n}\right] \in \operatorname{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$ was arbitrary, we are done. □

Proposition D.8. *$C_\bullet(\,\cdot\,)\colon$ **AbSimp** $\to$ **Comp**$_R$ is a functor.*

Proof. Suppose $K, L, M \in \mathrm{obj}(\mathbf{AbSimp})$, $f \in \hom(K, L)$, and $g \in \hom(L, M)$ are arbitrary. We define $K \overset{C_\bullet}{\longmapsto} C_\bullet(K)$ and $f \overset{C_\bullet}{\longmapsto} f_\bullet$. For preservation of composition, it is sufficient to show $(g \circ f)_n \equiv (g_n \circ f_n)$ on $\mathrm{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$ for each $n \in \mathbb{Z}$. Take $\sigma = \left[v_{i_0}, \ldots, v_{i_n}\right] \in \mathrm{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$. Notice

$$\begin{aligned}(g \circ f)_n\left(\left[v_{i_0}, \ldots, v_{i_n}\right]\right) &= \left[(g \circ f)\left(v_{i_0}\right), \ldots, (g \circ f)\left(v_{i_n}\right)\right] \\ &= g_n\left(\left[f\left(v_{i_0}\right), \ldots, f\left(v_{i_n}\right)\right]\right) \\ &= (g_n \circ f_n)\left(\left[v_{i_0}, \ldots, v_{i_n}\right]\right).\end{aligned}$$

Since $\left[v_{i_0}, \ldots, v_{i_n}\right] \in \mathrm{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$ is arbitrary, $C_\bullet$ preserves composition. Observe that

$$\begin{aligned}\mathrm{id}_n\left(\left[v_{i_0}, \ldots, v_{i_n}\right]\right) &= \left[\mathrm{id}_K\left(v_{i_0}\right), \ldots, \mathrm{id}_K\left(v_{i_n}\right)\right] \\ &= \left[v_{i_0}, \ldots, v_{i_n}\right] \\ &= \mathrm{id}_{C_n(K)}\left(\left[v_{i_0}, \ldots, v_{i_n}\right]\right).\end{aligned}$$

Again, $\sigma \in \mathrm{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$ is arbitrary, so identity morphisms are preserved. Altogether, $\mathbf{AbSimp} \xrightarrow{C_\bullet} \mathbf{Comp}_R$ is a functor. □

Given $(C_\bullet, K)$ is a chain complex for every oriented abstract simplicial complex K, we can define simplicial versions of the cycle, boundary, and homology modules from **Definition B.26**.

Definition D.9. Given the simplicial chain complex $(C_\bullet(K), \partial_\bullet)$, we have the **simplicial R-module of n-cycles** $Z_n(K) := Z_n(C_\bullet(K)) := \ker(\partial_n)$ and the **simplicial**

R**-module of** n**-boundaries** $B_n(K) := B_n(C_\bullet(K)) := \mathrm{im}(\partial_{n+1})$. We define the n**th simplicial homology** R**-module of** $(C_\bullet, \partial_\bullet)$ to be

$$H_n(K) := H_n(C_\bullet(K)) := Z_n(C_\bullet(K))/B_n(C_\bullet(K)).$$

The n**th simplicial Betti number** β_n **of** $(C_\bullet(K), \partial_\bullet)$ is $\mathrm{rk}\,(H_n(K))$.

Lemma D.10. *If K is a finite oriented abstract simplicial complex, then $C_n(K)$ is finitely generated for each $n \in \mathbb{Z}$.*

Proof. Since $\mathrm{vert}(K)$ is a finite set, $\left|K^{(n)} \setminus K^{(n-1)}\right| < \infty$. Since every permutation of $\{0, 1, \ldots, n\}$ is either even or odd, $\left(K^{(n)} \setminus K^{(n-1)}\right)/\sim$ is a finite set. Moreover, $\mathrm{alt}\left(K^{(n)} \setminus K^{(n-1)}\right)$ is a finite set. Hence $C_n(K)$ has finite rank and is finitely generated by **Lemma B.5**. □

Lemma D.11. *If K is a finite oriented abstract simplicial complex, then $H_n(K)$ is finitely generated for each $n \in \mathbb{Z}$.*

Proof. By **Lemma D.10**, $C_n(K)$ in finitely generated. Note that $Z_n(C_\bullet(K))$ is a submodule of $C_n(K)$ and hence $Z_n(C_\bullet(K))$ is finitely generated by **Lemma B.6**. Using **Lemma D.4**, it follows that $B_n(C_\bullet(K))$ is a submodule of $Z_n(C_\bullet(K))$. Thus $B_n(C_\bullet(K))$ is finitely generated by **Lemma B.6**. Therefore the quotient module $H_n(K) := Z_n(C_\bullet(K))/B_n(C_\bullet(K))$ is finitely generated by **Lemma B.7**. □

Next, we will prove the homology of abstract simplicial complexes is functorial. Due to [ES52, pp. 100-101], it is worth noticing that the simplicial homology of an abstract simplicial complex K is isomorphic to the singular homology of $|K|$; see

[Rot98, c. 4]. Ultimately, simplicial homology is independent of the orientation imposed on abstract simplicial complexes.

Proposition D.12. *For all* $n \in \mathbb{Z}$, $(H_n \circ C_\bullet)\colon \mathbf{AbSimp} \to \mathbf{Mod}_R$ *is a functor.*

Proof. Since $\mathbf{AbSimp} \xrightarrow{C_\bullet} \mathbf{Comp}_R$ and $\mathbf{Comp}_R \xrightarrow{H_n} \mathbf{Mod}_R$ are functors for any integer $n \in \mathbb{Z}$, their composition $\mathbf{AbSimp} \xrightarrow{(H_n \circ C_\bullet)} \mathbf{Mod}_R$ is a functor; see **Example A.9**, **Proposition B.28**, and **Proposition D.8**. □

APPENDIX E

PERSISTENT HOMOLOGY

Persistent homology parameterizes Betti numbers on a preordered index set P, producing a function $\beta_n \colon P \to \mathbb{Z}_{\geq 0}$. Homology only computes the number of homology classes, or *holes*, in a topological space or simplicial complex. Persistent homology is stronger in the sense that a *size* or *persistence* is associated to homology classes. Naively, a persistent homology module is a family $\mathcal{M} = \{M^i, \varphi^i\}_{i \in P}$ of R-modules M^i where $\varphi^i \colon M^i \to M^{i+1}$ are R-module homomorphisms. We will settle our discussion inside the categorical framework formalized in [BdSS15]. The primary references for this section are [ZC05, ELZ02, CZ09].

Definition E.1. Recall **Corollary A.15**. Let $\mathbf{P}$ be a preordered set and $\mathbf{D}$ be an arbitrary category. A **generalized persistence module** is a functor $\mathcal{M} \in \mathbf{D}^{\mathbf{P}}$. For our development of persistent homology, $\mathbf{P}$ will be taken to be $\mathbf{Z}_{\geq 0}$.

Using the concept of filtrations, we can fromalize the parameterization of Betti numbers on $\mathbb{Z}_{\geq 0}$.

Definition E.2. Given an abstract simplicial complex K, a $\mathbb{Z}_{\geq 0}$**-filtration** $\{K^i\}_{i \in \mathbb{Z}_{\geq 0}}$ **of** K is a sequence of subcomplexes of K, that is, $K^i \subseteq K^j$ for all $i \leq j$. In this context, we say K is $\mathbb{Z}_{\geq 0}$**-filtered**.

Lemma E.3. *$\mathbb{Z}_{\geq 0}$-Filtering provides a functor $\mathbf{Z}_{\geq 0} \to$ **AbSimp**.*

Proof. Suppose K is a $\mathbb{Z}_{\geq 0}$-filtered abstract simplicial complex. We define $i \mapsto K^i$ and $\left(i \xrightarrow{\leq} j\right) \mapsto \left(K^{i^{(0)}} \overset{\iota}{\hookrightarrow} K^{j^{(0)}}\right)$ for any $i, j \in \operatorname{obj}(\mathbf{Z}_{\geq 0})$. By way of notation, we will denote the inclusion map $\iota \colon K^{i^{(0)}} \hookrightarrow K^{j^{(0)}}$ by ι_i^j for any $i, j \in \mathbb{Z}_{\geq 0}$. Suppose that $i, j, k \in \operatorname{obj}(\mathbf{Z}_{\geq 0})$ and take $v \in K^{i^{(0)}}$. Observe

$$\begin{aligned}\iota_i^k(v) &= v \\ &= \iota_j^k(v) \\ &= \left(\iota_j^k \circ \iota_i^j\right)(v).\end{aligned}$$

Since $v \in K^{i^{(0)}}$ is arbitrary, composition is preserved. Also,

$$\begin{aligned}\iota_i^i(v) &= v \\ &= \mathrm{id}_{K^{i^{(0)}}}(v)\end{aligned}$$

which proves identities are preserved since $v \in K^{i^{(0)}}$. □

Definition E.4. We will describe two standard $\mathbb{Z}_{\geq 0}$-filtrations that are used in topological data analysis. These $\mathbb{Z}_{\geq 0}$-filtrations make use of the precise refinement concept discussed in **Example C.18** and are parameterized by radial distance. Suppose $X \subseteq \mathbb{X}$ where $(\mathbb{X}, \mathrm{d})$ is some metric space. Take an integer $n \geq 0$. We will assume $\varepsilon_i \in \mathbb{R}$ and $\varepsilon_i \leq \varepsilon_j$ for all $0 \leq i \leq j \leq n$ with $\varepsilon_0 = 0$. Consider the cover $F = \{\bar{B}_{\varepsilon_n}(x)\}_{x \in X}$ of X. Let $F^i = \{\bar{B}_{\varepsilon_i}(x)\}_{x \in X}$ for $0 \leq i \leq n$ and $F^i = F^n$ when $i > n$. It is clear that F^i is a precise refinement of F^n. We will refer to $\check{C}_{\varepsilon_i}(X) := \check{C}_{F^i}(F)$ as the **Čech complex of X at scale ε_i** for $i \geq 0$. Similarly, $R_{\varepsilon_i}(X) := R_{F^i}(F)$ is the **Vietoris-Rips complex of X at scale ε_i**. Thus, by the discussion in **Example C.18**, $\{\check{C}_{\varepsilon_i}(X)\}_{i \in \mathbb{Z}_{\geq 0}}$ and $\{R_{\varepsilon_i}(X)\}_{i \in \mathbb{Z}_{\geq 0}}$ are $\mathbb{Z}_{\geq 0}$-filtrations of $\check{C}(F)$ and $R(F)$, respectively. Filtrations of the type just described will be referred to as **radial $\mathbb{Z}_{\geq 0}$-filtrations**.

Čech complexes constructed on finite point sets are expensive to compute. For a simplex $\{v_0, \ldots, v_n\}$ and scale $\varepsilon > 0$, one would have to check the containment of the vertices $v_0, \ldots, v_{n-1}$ and v_n in a minimum enclosing ball of radius at least ε. For containment in the corresponding Vietoris-Rips complex, one would only have to check containment of pairs of vertices v_i and v_j in a minimum enclosing ball of radius at least ε for $0 \leq i \neq j \leq n$. The next lemma tells us that the Vietoris-Rips complex is a *good* approximation of the Čech complex.

Lemma E.5 (Vietoris-Rips lemma). *Suppose* $\{x_0, \ldots, x_N\} = X \subset \mathbb{R}^d$ *and* $\varepsilon, \varepsilon' > 0$. *If* $(\varepsilon/\varepsilon') \geq \sqrt{2d/(d+1)}$, *then*

$$R_{\varepsilon'}(X) \subseteq \check{C}_{\varepsilon}(X) \subseteq R_{\varepsilon}(X).$$

The proof of the Vietoris-Rips lemma can be found in [DSG07, p. 346]. The special case when $X \subseteq \mathbb{R}^2$ is proven in [DSG06, p. 1207]. We will prove a stronger version of the Vietoris-Rips lemma in chapter V.

Given a $\mathbb{Z}_{\geq 0}$-filtration $\{K^i\}_{i \in \mathbb{Z}_{\geq 0}}$, we can associate a simplicial R-chain complex to each K^i. Thus we also can talk about the cycle, boundary, and homology R-modules of K^i.

Definition E.6. Suppose we have a $\mathbb{Z}_{\geq 0}$-filtration $\{K^i\}_{i \in \mathbb{Z}_{\geq 0}}$ of K. Then we utilize the notation

$$(C^i_\bullet(K), \partial^i_\bullet) := (C_\bullet(K^i), \partial_\bullet), Z^i_n(K) := Z_n(K^i),$$
$$B^i_n(K) := B_n(K^i), H^i_n(K) := H_n(K^i),$$
$$\text{and } \beta^i_n := \mathrm{rk}(H_n(K^i)).$$

By filtering an abstract simplicial complex, we can use functoriality to encode the *persistence* of different homology classes over the filtration indices. For what follows, assume K is a filtered oriented abstract simplicial complex.

Proposition E.7. *Constructing chain complexes over a $\mathbb{Z}_{\geq 0}$-filtered abstract simplicial complex provides a functor* $\mathbf{Z}_{\geq 0} \to \mathbf{Comp}_R \colon \left\{K^i\right\}_{i \in \mathbf{Z}_{\geq 0}} \mapsto \left\{C_\bullet\left(K^i\right)\right\}_{i \in \mathbb{Z}_{\geq 0}}$.

Proof. $\mathbb{Z}_{\geq 0}$-Filtering an abstract simplicial complex is a functor $\mathbf{Z}_{\geq 0} \to \mathbf{AbSimp}$ by **Lemma E.3** and $\mathbf{AbSimp} \xrightarrow{C_\bullet} \mathbf{Comp}_R$ is a functor by **Proposition A.9**. It follows that their composition is a functor $\mathbf{Z}_{\geq 0} \to \mathbf{AbSimp} \to \mathbf{Comp}_R$; see **Example D.8**. □

Definition E.8. We define the n**th persistent homology module**

$$\mathcal{H}_n(K) := \left\{H_n^i(K), \left(H_n \circ f_n^i\right)\right\}_{i \in \mathbb{Z}}$$

of K to be the family of nth simplicial homology R-modules of $\{C_\bullet^i(K), \partial_\bullet^i\}_{i \in \mathbb{Z}}$ along with the collection of R-module homomorphisms

$$\left(H_n \circ f_n^i\right) : H_n^i(K) \to H_n^{i+1}(K).$$

We say that $\mathcal{H}_n(K)$ is of **finite type** if each R-module $\mathcal{H}_n^i(K)$ is finitely generated and the R-module homomorphisms $\left(H_n \circ f_n^i\right)$ are R-module isomorphisms for all $i \geq j$ and some $j \in \mathbb{Z}_{\geq 0}$. For nonnegative integers $j > i$, the i,j**-persistent** n**th homology module of** K is

$$H_n^{i,j}(K) := \operatorname{im}\left(H_n \circ f_n^{i,j}\right)$$

where $\left(H_n \circ f_n^{i,j}\right) : H_n^i(K) \to H_n^{i+j}(K)$ is the R-homomorphism induced by the inclusion map $K^{i^{(0)}} \xrightarrow{\iota} K^{j^{(0)}}$. Specifically, $\left(H_n \circ f_n^{i,j}\right)$ is the image of the inclusion map ι under the functor from **Proposition D.12**. The ***i*,*j*-persistent *n*th Betti number of** K is defined to be $\beta_n^{i,j} := \mathrm{rk}\left(H_n^{i,j}(K)\right)$. Informally, the homology classes of $H_n^{i,j}(K)$ represent the n-dimensional *holes* present in the simplicial subcomplexes $\{K^k\}_{k=i}^{j}$ that initially appear in the simplical complex K^i. The Betti number $\beta_n^{i,j}$ simply counts the homology classes, or *holes*, generating $H_n^{i,j}(K)$.

Proposition E.9. *Given a $\mathbb{Z}_{\geq 0}$-filtered oriented abstract simplicial complex K, the nth persistent homology module $\mathcal{H}_n(K) \in \mathbf{Mod}_R{}^{\mathbf{Z}_{\geq 0}}$ is a generalized persistence module.*

Proof. Recall that $\mathbf{Comp}_R \xrightarrow{(H_n \circ C_\bullet)} \mathbf{Mod}_R$ is a functor for each $n \in \mathbb{Z}_{\geq 0}$ by **Proposition D.12**. It follows that $\mathcal{H}_n(K) := \left\{H_n^i(K), \left(H_n \circ f_n^i\right)\right\}_{i \in \mathbb{Z}}$ is a functor since (covariant) functors are closed under composition. More precisely, we are composing $(H_n \circ C_\bullet)$ after the functor from **Proposition E.7**. Thus $\mathcal{H}_n(K)\colon \mathbf{Z}_{\geq 0} \to \mathbf{Mod}_R$ is a functor defined by $i \overset{\mathcal{H}_n}{\longmapsto} H_n^i(K)$ and $\left(i \xrightarrow{\leq} j\right) \overset{\mathcal{H}_n}{\longmapsto} \left(H_n \circ f_n^{i,j}\right)$ for any i and j in $\mathrm{obj}\left(\mathbf{Z}_{\geq 0}\right)$ and each $n \in \mathbb{Z}_{\geq 0}$. The following diagram summarizes the functor $\mathcal{H}_n$:

$$\mathbf{Z}_{\geq 0} \longrightarrow \mathbf{AbSimp} \longrightarrow \mathbf{Comp}_R \longrightarrow \mathbf{Mod}_R \qquad \left(\mathbf{Z}_{\geq 0} \xrightarrow{\mathcal{H}_n} \mathbf{Mod}_R\right)$$

□

Next we will show that finite type persistent homology modules can be endowed with a graded $R[x]$-module structure. Using the structure theorem for graded finitely generated modules, we will show that persistent homology modules encapsulate all the i,j-persistent homology modules. Thus the nth persistent homology module provides a complete summary of the changes in the nth homology of a filtered simplicial complex. For brevity, we direct the reader to [ZC05, pp. 258-259] for a discussion of the *persistence complex*, which provides a single $R[x]$-module whose homology is identical to the collection of all nth persistent homology modules. In other words, the persistence complex eliminates the parameter n.

Definition E.10. Consider the R-modules $H_n^i(K)$ for $i \geq 0$. With respect to the abelian groups of homology classes, we will overload notation by defining

$$\mathcal{H}_n(K) := \bigoplus_{i \in \mathbb{Z}_{\geq 0}} H_n^i(K). \tag{5.1}$$

Take the polynomial ring $R[x]$ to be graded by degree.

Proposition E.11. *If $\mathcal{H}_n(K)$ is of finite type, then $\mathcal{H}_n(K)$ is a graded finitely generated $R[x]$-module where*

$$\left(\sum_{k=0}^{\ell} r_k x^k\right) \cdot (\gamma_i)_{i \in \mathbb{Z}_{\geq 0}} := \sum_{k=0}^{\ell} \left[r_k \cdot \left(\Sigma^k \left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i \in \mathbb{Z}_{\geq 0}} \right]$$

and

$$sx^0 \cdot (\gamma_i)_{i \in \mathbb{Z}_{\geq 0}} := \left(\Sigma^0 (s \cdot \gamma_i)\right)_{i \in \mathbb{Z}_{\geq 0}} = (s \cdot \gamma_i)_{i \in \mathbb{Z}_{\geq 0}}$$

for any $\sum_{k=0}^{\ell} r_k x^k \in R[x]$ *with* $\ell \geq 1$*, every* $s \in R$*, and any* $(\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} \in \mathcal{H}_n(K)$*. Recall* $\Sigma^{(\,\cdot\,)}$ *is the shift map on grading.*

Proof. We will start by verifying $\mathcal{H}_n(K)$ satisfies **Definition B.4**. Let us take the following as arbitrary: $\sum_{k=0}^{\ell} c_k x^k, \sum_{k=0}^{m} d_k x^k \in R[x]$ and $(\gamma_i)_{i\in\mathbb{Z}_{\geq 0}}, (\eta_i)_{i\in\mathbb{Z}_{\geq 0}} \in \mathcal{H}_n(K)$.

(1)

$$\begin{aligned}
&\left(\sum_{k=0}^{\ell} c_k x^k\right) \cdot \left[(\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} + (\eta_i)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i + \eta_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[\Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(c_k \cdot \gamma_i + c_k \cdot \eta_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[\Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(c_k \cdot \gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}} + \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(c_k \cdot \eta_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[\Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(c_k \cdot \gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&\qquad + \sum_{k=0}^{\ell} \left[\Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(c_k \cdot \eta_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&\qquad + \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\eta_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \left(\sum_{k=0}^{\ell} c_k x^k\right) \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} + \left(\sum_{k=0}^{\ell} c_k x^k\right) \cdot (\eta_i)_{i\in\mathbb{Z}_{\geq 0}}.
\end{aligned}$$

(2)

$$\left[\sum_{k=0}^{\ell} \left(c_k x^k\right) + \sum_{k=0}^{m} \left(d_k x^k\right)\right] \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}}$$

$$
\begin{aligned}
&= \left(\sum_{k=0}^{\max\{\ell,m\}} c_k x^k + d_k x^k\right) \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \left[\sum_{k=0}^{\max\{\ell,m\}} (c_k + d_k)\, x^k\right] \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \sum_{k=0}^{\max\{\ell,m\}} \left[(c_k + d_k) \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\max\{\ell,m\}} c_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}} \\
&\qquad + d_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&\qquad + \sum_{k=0}^{m} \left[d_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \left(\sum_{k=0}^{\ell} c_k x^k\right) \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} + \left(\sum_{k=0}^{m} d_k x^k\right) \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}}.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\left(\sum_{k=0}^{\ell} \left(c_k x^k\right) \cdot \sum_{k=0}^{m} \left(d_k x^k\right)\right) \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} &= \left[\sum_{q=0}^{\ell+m} \left(\sum_{p=0}^{q} c_p d_{q-p}\right) x^q\right] \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \sum_{q=0}^{\ell+m} \left[\left(\sum_{p=0}^{q} c_p d_{q-p}\right) \Sigma^q \left(\left(H_n \circ f_n^{i,i+q}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{q=0}^{\ell+m} \left[\left(\sum_{p=0}^{q} c_p d_{q-p}\right) \Sigma^{p+q-p} \left(\left(H_n \circ f_n^{i,i+p+q-p}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{q=0}^{\ell+m} \left[\left\{\left(\sum_{p=0}^{q} c_p d_{q-p}\right) x^p\right\} \Sigma^{q-p} \left(\left(H_n \circ f_n^{i,i+q-p}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \left(\sum_{k=0}^{\ell} \left(c_k x^k\right)\right) \cdot \left[\sum_{k=0}^{m} d_k \cdot \Sigma^k \left(\left(H_n \circ f_n^{i,i+k}\right)(\gamma_i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \left(\sum_{k=0}^{\ell} \left(c_k x^k\right)\right) \cdot \left[\left(\sum_{k=0}^{m} d_k x^k\right) \cdot (\gamma_i)_{i\in\mathbb{Z}_{\geq 0}}\right].
\end{aligned}
$$

(4)

$$\begin{aligned} 1_R x^0 \cdot (\gamma_i)_{i \in \mathbb{Z}_{\geq 0}} &= \Sigma^0 \left(1_R \gamma_i\right)_{i \in \mathbb{Z}_{\geq 0}} \\ &= (\gamma_i)_{i \in \mathbb{Z}_{\geq 0}}. \end{aligned}$$

Thus $\mathcal{H}_n(K)$ is an $R[x]$-module.

Since $\mathcal{H}_n(K)$ is of finite type, $H_n^i(K)$ is finitely generated for each $i \geq 0$. Take $\Gamma^i = \{\gamma_p\}_{p=0}^{q(i)}$ to be the finite generating set of $H_n^i(K)$. By way of notation, let $\left(\gamma_i^{\delta_{ij}}\right)_{i \in \mathbb{Z}_{\geq 0}}^r$ be the sequence $(\gamma_i)_{i \in \mathbb{Z}_{\geq 0}}$ with $\gamma_i = 0_{H_n^i(K)}$ for all $i \neq j$ and $\gamma_i = \gamma_r \in \Gamma^j$ whenever $i = j$ where $0 \leq r \leq q(j)$ and $j \geq 0$. We also define

$$\hat{\Gamma}^s := \bigcup_{0 \leq r \leq q(s)} \left\{ \left(\gamma_i^{\delta_{is}}\right)_{i \in \mathbb{Z}_{\geq 0}}^r \right\}$$

for each $s \geq 0$. It is easy to see that

$$\begin{aligned} \langle \hat{\Gamma}^i \rangle &\cong \langle \Gamma^i \rangle \\ &\cong H_n^i(K) \end{aligned}$$

as abelian groups for each $i \geq 0$.

Since $\mathcal{H}_n(K)$ has finite type, there exists some $B \geq 0$ such that $H_n^B(K) \cong H_n^i(K)$ for each $i \geq B$. Let $\hat{\Gamma} = \bigcup_{i=0}^B \hat{\Gamma}^i$. Note that $\hat{\Gamma}$ has finite cardinality as a finite union of finite sets. Thus $\langle \hat{\Gamma} \rangle$ is finitely generated submodule of $\mathcal{H}_n(K)$. By means of showing $\mathcal{H}_n(K)$ is finitely generated, it suffices to show that $(\eta_i)_{i \in \mathbb{Z}_{\geq 0}} \in \langle \hat{\Gamma} \rangle$ when there exists some finitely many indices $B < i_0 < i_1 < \cdots < i_m$ so that

$$\eta_{i_\ell} = \sum_{p=0}^{q(i_\ell)} \left[\left(\sum_{s=0}^{t} r_s^{i_\ell} x^{d_s^{i_\ell}} \right) \cdot \gamma_p \right] \neq 0_{H_n^{i_\ell}(K)}$$

for $0 \leq \ell \leq m$. By induction on B, we see that, whenever $\eta_i = 0_{H_n^B(K)}$ for each $i > B$,

$$(\eta_i)_{i \in \mathbb{Z}_{\geq 0}} = \sum_{j=0}^{B} \left[\sum_{p=0}^{q(j)} f_p^j \left(\gamma_i^{\delta_{ij}} \right)_{i \in \mathbb{Z}_{\geq 0}} \right]$$

with $f_p^j \in R[x]$. Since $H_n^{i_\ell}(K) \cong H_n^B(K)$ for each $0 \leq \ell \leq m$, induction on i_ℓ implies that

$$\begin{aligned} &(\eta_i)_{i \in \mathbb{Z}_{\geq 0}} \\ &= \sum_{j=0}^{B} \left[\sum_{p=0}^{q(j)} f_p^j \left(\gamma_i^{\delta_{ij}} \right)_{i \in \mathbb{Z}_{\geq 0}} \right] + \sum_{\ell=0}^{m} \left\{ \sum_{p=0}^{q(B)} \left[\left(\sum_{s=0}^{t_{i_\ell}} r_s^{i_\ell} x^{d_s^{i_\ell} + i_\ell - B} \right) \cdot \left(\gamma_i^{\delta_{iB}} \right)^p_{i \in \mathbb{Z}_{\geq 0}} \right] \right\} \\ &\in \langle \hat{\Gamma} \rangle. \end{aligned}$$

Thus $\mathcal{H}_n(K)$ is finitely generated by $\hat{\Gamma}$.

To prove $\mathcal{H}_n(K)$ is graded, take $rx^\ell \in R^\ell[x]$ and $\gamma \in H_n^m(K)$ as arbitrary. Observe

$$\begin{aligned} rx^\ell \cdot \gamma &= r \left(H_n \circ f_n^{m+\ell} \right) (\gamma) \\ &= \left(H_n \circ f_n^{m+\ell} \right) (r \cdot \gamma) \end{aligned}$$

$$\in H_n^{m+\ell}(K).$$

This shows that $R^\ell[x]H_n^m(K) \subseteq H_n^{m+\ell}(K)$ for any $\ell, m, n \in \mathbb{Z}_{\geq 0}$. Thus $\mathcal{H}_n(K)$ is graded. Altogether, $\mathcal{H}_n(K)$ is a graded finitely generated $R[x]$-module. □

Proposition E.11 allows us to identify $\mathcal{H}_n(K)$ as a graded finitely generated $R[x]$-module. It follows that $\mathcal{H}_n(K)$ can be described uniquely, up to isomorphism, using the structure theorem for graded finitely generated modules.

Proposition E.12 (Structure theorem for nth persistent homology modules of finite type)**.** *Suppose $\mathcal{H}_n(K)$ is of finite type. Then $\mathcal{H}_n(K)$ uniquely decomposes as*

$$\mathcal{H}_n(K) \cong \left(\bigoplus_{i=0}^{\kappa_0} \Sigma^{\xi_i} R[x] \right) \oplus \left(\bigoplus_{j=0}^{\kappa_1} \Sigma^{\zeta_j} R[x] \Big/ \left(x^{d_j} \right) \right)$$

where $\xi_i, \zeta_j, \kappa_0, \kappa_1 \in \mathbb{Z}_{\geq 0}$, x^{d_j} are homogeneous elements in $R^{d_j}[x]$ with $x^{d_j} \mid x^{d_{j+1}}$.

Proof. By **Proposition E.11**, $\mathcal{H}_n(K)$ is a graded finitely generated $R[x]$-module. The conclusion follows immediately from **Theorem B.21**. □

The decomposition of persistent homology modules is very compelling. It allows us to define simple *visual* summaries that have become very popular for classification purposes in topological data analysis.

Definition E.13. A **multiset** is a pair (S, μ_S) where S is an arbitrary set and

$$\mu_S \colon S \to \bar{\mathbb{Z}}_{\geq 0} = \mathbb{Z}_{\geq 0} \cup \{+\infty\}$$

is a set function. A **persistence interval** is a half-open interval $[b, d) \subseteq \bar{\mathbb{R}}_{\geq 0}$ with $b < d$ where $\bar{\mathbb{R}}_{\geq 0} = \mathbb{R}_{\geq 0} \cup \{+\infty\}$. Suppose $m \geq 0$ is an arbitrary integer and

$$P = \{[b_i, d_i)\}_{i=0}^{m} \subseteq 2^{\bar{\mathbb{R}}_{\geq 0}}$$

is a finite collection of persistence intervals. We call a multiset (P, μ_P) a persistence barcode provided $\sum_{p \in P} \mu_P(p) < +\infty$ and denote the **set of all persistence barcodes** by Bcodes. We will let $\text{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ denote the set of persistence barcodes (P, μ_P) such that $P = \{[b_i, d_i)\}_{i=0}^{m}$ where $0 \leq b_i < d_i \in \bar{\mathbb{Z}}_{\geq 0}$ for each $0 \leq i \leq m$.

Now suppose $Q \subseteq \mathbb{R}_{\geq 0} \times \bar{\mathbb{R}}_{\geq 0}$ with the stipulations that

$$\Delta := \{(c, c) \mid c \in \mathbb{R}_{\geq 0}\} \subseteq Q,$$

$|Q \setminus \Delta| < +\infty$, and if $(b, d) \in Q \setminus \Delta$, then $0 \leq b < d \in \bar{\mathbb{R}}_{\geq 0}$. We say that $\left(Q, \mu_Q\right)$ is a **persistence diagram** provided $\sum_{q \in Q \setminus \Delta} \mu_Q(q) < +\infty$ and $\mu_Q(q) = +\infty$ whenever $q \in \Delta$. We denote the **set of all persistence diagrams** by Dgms. Similar to $\text{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}}$, we will let $\text{Dgms}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ denote the set of persistence diagrams (Q, μ_Q) such that $Q \setminus \Delta = \{(b_i, d_i)\}_{i=0}^{m}$ where $0 \leq b_i < d_i \in \bar{\mathbb{Z}}_{\geq 0}$ for each $0 \leq i \leq m$. As a matter of fact, Bcodes and Dgms are isomorphic as sets.

Proposition E.14. *There exists a set bijection* $\text{dgm}\colon \text{Bcodes} \to \text{Dgms}$.

Proof. Define $\text{dgm}(\,\cdot\,)\colon \text{Bcodes} \to \text{Dgms}$ by

$$\text{dgm}\left((P, \mu_P)\right) = (Q, \mu_Q)$$

for $P = \{[b_i, d_i)\}_{i=0}^{m}$ where $Q \setminus \Delta = \{(b_i, d_i)\}_{i=0}^{m}$, and $\mu_P([b_i, d_i)) = \mu_Q((b_i, d_i))$ for each $0 \leq i \leq m$. Suppose $(P, \mu_P), (P', \mu_{P'}) \in$ Bcodes and

$$\mathrm{dgm}\left((P, \mu_P)\right) = (Q, \mu_Q) \neq \mathrm{dgm}\left((Q, \mu_Q)\right) = (Q', \mu_{Q'}) \in \mathrm{Dgms}.$$

Then either $Q \neq Q'$ or $\mu_Q((b_i, d_i)) \neq \mu_{Q'}((b_i, d_i))$ for some $(b_i, d_i) \in Q, Q'$. This implies that $P \neq P'$ or $\mu_P([b_i, d_i)) \neq \mu_{P'}([b_i, d_i))$ for some $[b_i, d_i) \in P, P'$. Thus $(P, \mu_P) \neq (P', \mu_{P'})$. By arbitrariness of $(P, \mu_P), (P', \mu P') \in$ Bcodes, dgm is injective. Now assume $(Q, \mu_Q) \in$ Dgms and $Q \setminus \Delta = \{(b_i, d_i)\}_{i=0}^{m}$. Clearly,

$$\mathrm{dgm}\left((P, \mu_P)\right) = (Q, \mu_Q)$$

where $P = \{[b_i, d_i)\}_{i=0}^{m}$. Hence dgm is surjective by the arbitrariness of (Q, μ_Q). Therefore the function dgm: Bcodes $\to$ Dgms is a bijection. We will use the notation bcode $:= \mathrm{dgm}^{-1}$ when appropriate. $\square$

Corollary E.15. *The restricted set function* $\mathrm{dgm}|_{\mathrm{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}}}(\,\cdot\,)\colon \mathrm{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}} \to \mathrm{Dgms}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ *is a bijection.*

Proof. Take $(Q, \mu_Q) \in \mathrm{dgm}\left(\mathrm{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}}\right)$. Then there exists some persistence barcode $(P, \mu_P) \in \mathrm{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ such that $\mathrm{dgm}((P, \mu_P)) = (Q, \mu_Q)$ and $P = \{[b_i, d_i)\}_{i=0}^{m}$ with $0 \leq b_i < d_i \in \bar{\mathbb{Z}}_{\geq 0}$ for each $0 \leq i \leq m$. It follows that $Q \setminus \Delta = \{(b_i, d_i)\}_{i=0}^{m}$ with $0 \leq b_i < d_i \in \bar{\mathbb{Z}}_{\geq 0}$ for each $0 \leq i \leq m$. Hence $(Q, \mu_Q) \in \mathrm{Dgm}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ and therefore $\mathrm{dgm}\left(\mathrm{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}}\right) \subseteq \mathrm{Dgms}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ by arbitrariness of (Q, μ_Q). Now take $(Q, \mu_Q) \in \mathrm{Dgms}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ where $Q \setminus \Delta = \{(b_i, d_i)\}_{i=0}^{m}$ for some arbitrary integer $m \geq 0$. Take $(P, \mu_P) \in \mathrm{Bcodes}_{\bar{\mathbb{Z}}^2_{\geq 0}}$ so that $P = \{[b_i, d_i)\}_{i=0}^{m}$. Then $\mathrm{dgm}(P, \mu_P) = (Q, \mu_Q)$.

Thus $\mathrm{dgm}\left(\mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}\right) \supseteq \mathrm{Dgms}_{\mathbb{Z}^2_{\geq 0}}$ by arbitrariness of $(Q, \mu_Q) \in \mathrm{Dgms}_{\mathbb{Z}^2_{\geq 0}}$. Therefore $\mathrm{dgm}\left(\mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}\right) = \mathrm{Dgms}_{\mathbb{Z}^2_{\geq 0}}$. The fact that $\mathrm{dgm}|_{\mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}}$ is a bijection follows from **Proposition E.15**. □

Definition E.16. Let R be an arbitrary graded ring. Generalizing **Definition E.8**, we say a **persistent homology R-module of finite type** is a family $\{M^i, f^i\}_{i\in\mathbb{Z}}$ of R-modules with R-module homomorphisms $f^i \colon M^i \to M^{i-1}$ where $M^i = \{0\}$ and $f^i \equiv 0$ for each $i < 0$, M^i is finitely generated for each $i \in \mathbb{Z}$, and there exists some $i \geq 0$ such that f^j are R-module isomorphisms for all $j \geq i$. We define $\mathcal{H}$ to be the **set of persistent homology $R[x]$-modules of finite type**; we emphasize that $\mathcal{H}$ is not a small set. Suppose M is a persistent homology $R[x]$-module of finite type. Overloading notation, define

$$M := \bigoplus_{i\in\mathbb{Z}_{\geq 0}} M^i$$

and

$$f_i^{i+j} := (f^{i+j} \circ f^{i+j-1} \circ \cdots \circ f^i)$$

for $i < j \in \mathbb{Z}_{\geq 0}$. This allows us to generalize **Proposition E.11**.

Proposition E.17. *If M is a persistent homology module of finite type, then M is a graded finitely generated $R[x]$-module where*

$$\left(\sum_{k=0}^{\ell} r_k x^k\right) \cdot (m^i)_{i\in\mathbb{Z}_{\geq 0}} := \sum_{k=0}^{\ell} \left[r_k \cdot \left(\Sigma^k f_i^{i+k}(m^i)\right)_{i\in\mathbb{Z}_{\geq 0}} \right]$$

and

$$sx^0 \cdot (m_i)_{i\in\mathbb{Z}_{\geq 0}} := \left(\Sigma^0(s \cdot m_i)\right)_{i\in\mathbb{Z}_{\geq 0}} = (s \cdot m^i)_{i\in\mathbb{Z}_{\geq 0}}$$

for any $\sum_{k=0}^{\ell} r_k x^k \in R[x]$ *with* $\ell \geq 1$*, every* $s \in R$*, and any* $(m^i)_{i\in\mathbb{Z}_{\geq 0}} \in M$*. Recall* $\Sigma^{(\,\cdot\,)}$ *is the shift map on grading.*

Proof. We begin by verifying M satisfies **Definition B.4**. Let us take the following as arbitrary: $\sum_{k=0}^{\ell} c_k x^k, \sum_{k=0}^{m} d_k x^k \in R[x]$ and $(m^i)_{i\in\mathbb{Z}_{\geq 0}}, (n^i)_{i\in\mathbb{Z}_{\geq 0}} \in M$.

(1)

$$\begin{aligned}
&\left(\sum_{k=0}^{\ell} c_k x^k\right) \cdot \left[(m^i)_{i\in\mathbb{Z}_{\geq 0}} + (n^i)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(f_i^{i+k}(m^i + n^i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[\Sigma^k \left(f_i^{i+k}(c_k \cdot m^i + c_k \cdot n^i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[\Sigma^k \left(f_i^{i+k}(c_k \cdot m^i)\right)_{i\in\mathbb{Z}_{\geq 0}} + \Sigma^k \left(f_i^{i+k}(c_k \cdot n^i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[\Sigma^k \left(f_i^{i+k}(c_k \cdot m^i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] + \sum_{k=0}^{\ell} \left[\Sigma^k \left(f_i^{i+k}(c_k \cdot n^i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(f_i^{i+k}(m^i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] + \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(f_i^{i+k}(n^i)\right)_{i\in\mathbb{Z}_{\geq 0}}\right] \\
&= \left(\sum_{k=0}^{\ell} c_k x^k\right) \cdot (m^i)_{i\in\mathbb{Z}_{\geq 0}} + \left(\sum_{k=0}^{\ell} c_k x^k\right) \cdot (n^i)_{i\in\mathbb{Z}_{\geq 0}}.
\end{aligned}$$

(2)

$$\left[\sum_{k=0}^{\ell} \left(c_k x^k\right) + \sum_{k=0}^{m} \left(d_k x^k\right)\right] \cdot (m^i)_{i\in\mathbb{Z}_{\geq 0}}$$

$$
\begin{aligned}
&= \left(\sum_{k=0}^{\max\{\ell,m\}} c_k x^k + d_k x^k \right) \cdot (m^i)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \left[\sum_{k=0}^{\max\{\ell,m\}} (c_k + d_k)\, x^k \right] \cdot (m^i)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \sum_{k=0}^{\max\{\ell,m\}} \left[(c_k + d_k) \cdot \Sigma^k \left(f_i^{i+k}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \right] \\
&= \sum_{k=0}^{\max\{\ell,m\}} c_k \cdot \Sigma^k \left(f_i^{i+k}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} + d_k \cdot \Sigma^k \left(f_i^{i+k}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \sum_{k=0}^{\ell} \left[c_k \cdot \Sigma^k \left(f_i^{i+k}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \right] + \sum_{k=0}^{m} \left[d_k \cdot \Sigma^k \left(f_i^{i+k}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \right] \\
&= \left(\sum_{k=0}^{\ell} c_k x^k \right) \cdot \left(m^i \right)_{i\in\mathbb{Z}_{\geq 0}} + \left(\sum_{k=0}^{m} d_k x^k \right) \cdot \left(m^i \right)_{i\in\mathbb{Z}_{\geq 0}}.
\end{aligned}
$$

(3)

$$
\begin{aligned}
\left(\sum_{k=0}^{\ell} \left(c_k x^k \right) \cdot \sum_{k=0}^{m} \left(d_k x^k \right) \right) \cdot (m^i)_{i\in\mathbb{Z}_{\geq 0}} &= \left[\sum_{q=0}^{\ell+m} \left(\sum_{p=0}^{q} c_p d_{q-p} \right) x^q \right] \cdot (m^i)_{i\in\mathbb{Z}_{\geq 0}} \\
&= \sum_{q=0}^{\ell+m} \left[\left(\sum_{p=0}^{q} c_p d_{q-p} \right) \Sigma^q \left(f_i^{i+q}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \right] \\
&= \sum_{q=0}^{\ell+m} \left[\left(\sum_{p=0}^{q} c_p d_{q-p} \right) \Sigma^{p+q-p} \left(f_i^{i+p+q-p}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \right] \\
&= \sum_{q=0}^{\ell+m} \left[\left\{ \left(\sum_{p=0}^{q} c_p d_{q-p} \right) x^p \right\} \Sigma^{q-p} \left(f_i^{i+q-p}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \right] \\
&= \left(\sum_{k=0}^{\ell} \left(c_k x^k \right) \right) \cdot \left[\sum_{k=0}^{m} d_k \cdot \Sigma^k \left(f_i^{i+k}(m^i) \right)_{i\in\mathbb{Z}_{\geq 0}} \right] \\
&= \left(\sum_{k=0}^{\ell} \left(c_k x^k \right) \right) \cdot \left[\left(\sum_{k=0}^{m} d_k x^k \right) \cdot \left(m^i \right)_{i\in\mathbb{Z}_{\geq 0}} \right].
\end{aligned}
$$

(4)

$$1_R x^0 \cdot \left(m^i\right)_{i\in\mathbb{Z}_{\geq 0}} = \Sigma^0 \left(1_R m^i\right)_{i\in\mathbb{Z}_{\geq 0}}$$
$$= \left(m^i\right)_{i\in\mathbb{Z}_{\geq 0}}.$$

Thus M is an $R[x]$-module.

Since M is of finite type, M^i is finitely generated for each $i \geq 0$. We will take $\Gamma^i = \{m^p\}_{p=0}^{q(i)}$ to be the finite generating set of M^i. By way of notation, let $\left(m^{i^{\delta_{ij}}}\right)^r_{i\in\mathbb{Z}_{\geq 0}}$ be the sequence $(m^i)_{i\in\mathbb{Z}_{\geq 0}}$ with $m^i = 0_{M^i}$ for all $i \neq j$ and $m^i = m^r \in \Gamma^j$ whenever $i = j$ where $0 \leq r \leq q(j)$ and $j \geq 0$. We also define

$$\hat{\Gamma}^s := \bigcup_{0\leq r\leq q(s)} \left\{ \left(m^{i^{\delta_{is}}}\right)^r_{i\in\mathbb{Z}_{\geq 0}} \right\}$$

for each $s \geq 0$. It is easy to see that

$$\langle \hat{\Gamma}^i \rangle \cong \langle \Gamma^i \rangle$$
$$\cong M^i$$

as abelian groups for each $i \geq 0$.

Since M has finite type, there exists some $B \geq 0$ such that $M^B \cong M^i$ for each $i \geq B$. Let $\hat{\Gamma} = \bigcup_{i=0}^{B} \hat{\Gamma}^i$. Note that $\hat{\Gamma}$ has finite cardinality as a finite union of finite sets. Thus $\langle \hat{\Gamma} \rangle$ is finitely generated submodule of M. By means of showing M is finitely generated, it suffices to show that $\left(n^i\right)_{i\in\mathbb{Z}_{\geq 0}} \in \langle \hat{\Gamma} \rangle$ when there exists some finitely many indices $B < i_0 < i_1 < \cdots < i_\alpha$ so that

$$n^{i_\ell} = \sum_{p=0}^{q(i_\ell)} \left[\left(\sum_{s=0}^{t} r_s^{i_\ell} x^{d_s^{i_\ell}} \right) \cdot m^p \right] \neq 0_{H_n^{i_\ell}(K)}$$

for $0 \leq \ell \leq \alpha$. By induction on B, we see that, whenever $n^i = 0_{M^B}$ for each $i > B$,

$$\left(n^i\right)_{i\in\mathbb{Z}_{\geq 0}} = \sum_{j=0}^{B} \left[\sum_{p=0}^{q(j)} \varphi_p^j \left(m^{i^{\delta_{ij}}}\right)_{i\in\mathbb{Z}_{\geq 0}} \right]$$

with $\varphi_p^j \in R[x]$. Since $M^{i_\ell} \cong M^B$ for each $0 \leq \ell \leq \alpha$, induction on i_ℓ implies that

$$\begin{aligned}\left(n^i\right)_{i\in\mathbb{Z}_{\geq 0}} &= \sum_{j=0}^{B} \left[\sum_{p=0}^{q(j)} \varphi_p^j \left(m^{i^{\delta_{ij}}}\right)_{i\in\mathbb{Z}_{\geq 0}} \right] \\ &\quad + \sum_{\ell=0}^{\alpha} \left\{ \sum_{p=0}^{q(B)} \left[\left(\sum_{s=0}^{t_{i_\ell}} r_s^{i_\ell} x^{d_s^{i_\ell} + i_\ell - B} \right) \cdot \left(m^{i^{\delta_{iB}}}\right)^p_{i\in\mathbb{Z}_{\geq 0}} \right] \right\} \\ &\in \langle \hat{\Gamma} \rangle.\end{aligned}$$

Thus M is finitely generated by $\hat{\Gamma}$.

To prove M is graded, take $rx^k \in R^k[x]$ and $m \in M^\ell$ as arbitrary. Observe

$$\begin{aligned} rx^k \cdot m &= r \cdot f^{\ell+k}(m) \\ &= f^{\ell+k}(r \cdot m) \\ &\in M^{\ell+k}. \end{aligned}$$

This shows that $R^k[x]M^\ell \subseteq M^{k+\ell}$ for any $k, \ell \in \mathbb{Z}_{\geq 0}$. Thus M is graded. Altogether, M is a graded finitely generated $R[x]$-module. □

By **Proposition E.17**, persistent homology modules can be described uniquely, up to isomorphism, using the structure theorem for graded finitely generated modules.

Proposition E.18 (Structure theorem for persistent homology modules of finite type)**.** *Suppose M is a persistent homology module of finite type. Then M uniquely decomposes as*

$$M \cong \left(\bigoplus_{i=0}^{\kappa_0} \Sigma^{\xi_i} R[x] \right) \oplus \left(\bigoplus_{j=0}^{\kappa_1} \Sigma^{\zeta_j} R[x] \Big/ \left(x^{d_j} \right) \right)$$

where $\xi_i, \zeta_j, \kappa_0, \kappa_1 \in \mathbb{Z}_{\geq 0}$, x^{d_j} are homogeneous elements in $R^{d_j}[x]$ with $x^{d_j} \mid x^{d_{j+1}}$.

Proof. By **Proposition E.17**, M is a graded finitely generated $R[x]$-module. The conclusion follows immediately from **Theorem B.21**. □

We would like to point out to the reader that **Propositions E.11 and E.12** now follow as corollaries to **Propositions E.17 and E.18**.

Definition E.19. Suppose $M \in \mathcal{H}$ is a persistent homology module and consider the unique decomposition

$$M \cong \left(\bigoplus_{i=0}^{\kappa_0} \Sigma^{\xi_i} R[x] \right) \oplus \left(\bigoplus_{j=0}^{\kappa_1} \Sigma^{\zeta_j} R[x] \Big/ \left(x^{d_j} \right) \right)$$

from **Proposition E.18**. Overloading notation, define the set function

$$\mathrm{bcode}(\,\cdot\,)\colon \mathcal{H} \to \mathrm{Bcodes}$$

by

$$\mathrm{bcode}(M) := \left\{[\xi_i, +\infty)\right\}_{i=0}^{\kappa_0} \cup \left\{[\zeta_j, d_j - \zeta_j)\right\}_{j=0}^{\kappa_1}.$$

The function bcode is well-defined because of the uniqueness of the decomposition guaranteed by **Proposition E.18**. Making use of the previously defined function dgm: Bcodes $\to$ Dgms, we define the set function

$$\mathrm{dgm}(\,\cdot\,)\colon \mathcal{H} \xrightarrow{\mathrm{bcode}} \mathrm{Bcodes} \xrightarrow{\mathrm{dgm}} \mathrm{Dgms}$$

by

$$\mathrm{dgm}(M) := (\mathrm{dgm} \circ \mathrm{bcode})(M).$$

Lemma E.20. *The set function* $\mathcal{H} \xrightarrow{\mathrm{bcode}|_{\mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}}} \mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}$ *is surjective.*

Proof. Take an arbitrary $(P, \mu_P) \in \mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}$. Without loss of generality, we will assume

$$P = \left\{[\xi_i, +\infty)\right\}_{i=0}^{\kappa_0} \cup \left\{[\zeta_j, d_j - \zeta_j)\right\}_{j=0}^{\kappa_1}$$

where $\xi_i, \zeta_j, \kappa_0, \kappa_1 \in \mathbb{Z}_{\geq 0}$, and $d_j \leq d_j + 1$ for each $0 \leq i \leq \kappa_0$ and each $0 \leq j \leq \kappa_1$. Let

$$M = \left(\bigoplus_{i=0}^{\kappa_0} \Sigma^{\xi_i} R[x] \right) \oplus \left(\bigoplus_{j=0}^{\kappa_1} \Sigma^{\zeta_j} R[x] \Big/ \left(x^{d_j} \right) \right).$$

Fixing an integer $\alpha \geq 0$, we define the sets

$$\xi_\alpha \coloneqq \{\xi_i \,|\, \alpha \geq \xi_i \text{ and } 0 \leq i \leq \kappa_0\}$$

and

$$\zeta_\alpha \coloneqq \left\{\zeta_j \,\middle|\, \zeta_j \leq \alpha \leq (d_j - \zeta_j) \text{ and } 0 \leq j \leq \kappa_1 \right\}.$$

Now for any $\beta \in \mathbb{Z}$, let

$$M^\beta \coloneqq \begin{cases} \langle \xi_\beta \cup \zeta_\beta \rangle & \text{if } \beta \geq 0 \\ \{0\} & \text{if } \beta < 0 \end{cases}.$$

If $\beta > 0$, then define $f^\beta \colon M^\beta \to M^{\beta-1}$ by

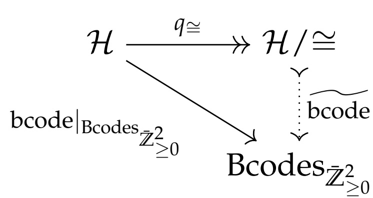
$$f^\beta(m) = \begin{cases} m & \text{if } m \in \left(\xi_{\beta-1} \cup \zeta_{\beta-1}\right) \cap \left(\xi_\beta \cup \zeta_\beta\right) \\ 0_{M^{\beta-1}} & \text{if } m \notin \left(\xi_{\beta-1} \cup \zeta_{\beta-1}\right) \cap \left(\xi_\beta \cup \zeta_\beta\right) \end{cases}$$

and extending by linearity. If $\beta \leq 0$, then we define $f^\beta \equiv 0_{M^{\beta-1}}$. Thus, with burden on notation, $M \coloneqq \{M^\beta, f^\beta\}_{\beta \in \mathbb{Z}}$ is a persistent homology module of finite type and, by construction, $\text{bcode}|_{\text{Bcodes}_{\mathbb{Z}^2_{\geq 0}}}(M) = (P, \mu_P)$. Therefore $\text{bcode}|_{\text{Bcodes}_{\mathbb{Z}^2_{\geq 0}}}$ is surjective. $\square$

Definition E.21. With respect to module isomorphism, it is easy to check that $\cong$ is an equivalence relation on $\mathcal{H}$. Let $\mathcal{H}/\cong$ represent the quotient set of isomorphism classes of persistent homology $R[x]$-modules. Let $\mathcal{H} \xrightarrow{q_{\cong}} \mathcal{H}/\cong$ be the surjective map induced by the equivalence relation $\cong$. We will let $\mathcal{H}/\cong \xrightarrow{\widetilde{\text{bcode}}} \text{Bcodes}_{\mathbb{Z}^2_{\geq 0}}$ be the induced injective set function defined by

$$\widetilde{\text{bcode}}(\llbracket M \rrbracket) = \text{bcode}|_{\text{Bcodes}_{\mathbb{Z}^2_{\geq 0}}}(M)$$

where $\llbracket M \rrbracket$ is the equivalence class of $M \in \mathcal{H}$. We diagrammatically summarize the situation below:



Proposition E.22. *The quotient map $\mathcal{H}/\cong \xrightarrow{\widetilde{\text{bcode}}} \text{Bcodes}_{\mathbb{Z}^2_{\geq 0}}$ is a set bijection.*

Proof. Fix an arbitrary $(P, \mu_P) \in \text{Bcodes}_{\mathbb{Z}^2_{\geq 0}}$. By **Lemma E.20**, there exists an $M \in \mathcal{H}$ such that $\text{bcode}|_{\text{Bcode}_{\mathbb{Z}^2_{\geq 0}}}(M) = (P, \mu_P)$. This implies that $\widetilde{\text{bcode}}(\llbracket M \rrbracket) = (P, \mu_P)$ and thus $\widetilde{\text{bcode}}$ is surjective. Now suppose

$$\llbracket M \rrbracket, \llbracket N \rrbracket \in \mathcal{H}/\cong$$

and

$$\widetilde{\mathrm{bcode}}(\llbracket M \rrbracket) = \widetilde{\mathrm{bcode}}(\llbracket N \rrbracket).$$

Without loss of generality, assume

$$\widetilde{\mathrm{bcode}}(\llbracket M \rrbracket) = (P, \mu_P) = \widetilde{\mathrm{bcode}}(\llbracket N \rrbracket)$$

$$\text{where}$$

$$P = \left\{[\xi_i, +\infty)\right\}_{i=0}^{\kappa_0} \cup \left\{[\zeta_j, d_j - \zeta_j)\right\}_{j=0}^{\kappa_1}$$

with $\xi_i, \zeta_j, \kappa_0, \kappa_1 \in \mathbb{Z}_{\geq 0}$ and $d_j \leq d_j + 1$. By definition of the set function bcode,

$$M \cong \left(\bigoplus_{i=0}^{\kappa_0} \Sigma^{\xi_i} R[x]\right) \oplus \left(\bigoplus_{j=0}^{\kappa_1} \Sigma^{\zeta_j} R[x] \Big/ \left(x^{d_j}\right)\right) \cong N$$

and hence $\llbracket M \rrbracket = \llbracket N \rrbracket$. Thus $\widetilde{\mathrm{bcode}}$ is injective by arbitrariness of $\llbracket M \rrbracket$ and $\llbracket N \rrbracket$. Therefore $\widetilde{\mathrm{bcode}}$ is a bijection. □

Corollary E.23. *The set function* $\mathcal{H}/\cong \xrightarrow{\left(\mathrm{dgm}|_{\mathbb{Z}^2_{\geq 0}} \circ \widetilde{\mathrm{bcode}}\right)} \mathrm{Dgms}_{\mathbb{Z}^2_{\geq 0}}$ *is a set bijection.*

Proof. Recall **Corollary E.15**. The set function $\left(\mathrm{dgm}|_{\mathbb{Z}^2_{\geq 0}} \circ \widetilde{\mathrm{bcode}}\right)$ is a bijection since set bijections are closed under composition. □

Proposition E.22 and **Corollary E.23** say that persistence barcodes in $\mathrm{Bcodes}_{\mathbb{Z}^2_{\geq 0}}$ and persistence diagrams in $\mathrm{Dgms}_{\mathbb{Z}^2_{\geq 0}}$ are invariants of the isomorphism classes of persistent homology modules of finite type. In the persistence literature, this is typically rephrased by describing the persistence barcode/diagram as a *complete discrete invariant*; see [CZ09, p. 74]. A more geometric explanation for persistence barcode/diagrams being correct summaries of nth persistent homology modules comes from the construction of *persistence triangles*.

Definition E.24. Suppose (Q, μ_Q) is a multiset and define the **total multiplicity of** (Q, μ_Q) by

$$\sharp(Q, \mu_p) \coloneqq \sum_{q \in Q} \mu_Q(q) \leq +\infty.$$

If $(Q, \mu_Q) \in \text{Dgms}$ and $(i, j) \in Q \setminus \Delta$, let the, possibly unbounded, convex set

$$\angle(i,j) \coloneqq ((0,i),(j,0),(i,j-i)) \setminus ((i,j-i),(j,0)) \subseteq \mathbb{R}^2_{\geq 0} \times \bar{\mathbb{R}}^2_{\geq 0}$$

be called a **persistence triangle** and let $\angle Q$ be the **set of Q-persistence triangles**. We will define $\angle(Q, \mu_Q) \coloneqq (\angle Q, \mu_{\angle Q})$ to be the **multiset of Q persistence-triangles** where $\mu_{\angle Q} \colon \angle Q \to \mathbb{Z}_{\geq 0}$ is defined by

$$\mu_{\angle Q}(\angle(i,j)) \coloneqq \sharp\left((\{(i,j)\}, \mu_Q|_{\{(i,j)\}})\right).$$

Lemma E.25 (k-triangle lemma)**.** *Suppose $\mathcal{H}_k(K)$ is a persistent homology module for some $k \geq 0$ and consider $(Q, \mu_q) = \text{dgm}(\mathcal{H}_k(K))$. Assume $(i,j) \in \mathbb{Z}^2_{\geq 0}$ and $0 \leq i < j$. Let $T \subseteq \angle Q$ denote the set of Q-persistence triangles containing the point (i,j), that is, $t \in T$ if and only if $(i,j) \in t$. Then*

$$\beta_k^{i,j} = \sum_{t \in T} \mu_{\angle Q}(t).$$

The proof of the k-triangle lemma can be found in [ELZ02, pp. 520-521]. We close this section with some examples.

APPENDIX F

STABILITY OF PERSISTENCE DIAGRAMS

The persistence barcodes/diagrams introduced in Appendix Appendix E are useful for classification of triangulable spaces that have been replaced with filtered triangulations. In this section we will define the bottleneck distance d_B which provides a metric on persistence barcodes/diagrams. This allows us to apply analytical methods to persistence barcodes/diagrams and estimate the persistent homology of filtered abstract simplicial complexes. Due to time, the scope of this section will not exceed a quick summary of [CSEH07]. In [CSEH07, pp. 105, 107-108], Cohen-Steiner et al. identify $\mathbb{Z}_{\geq 0}$-filtrations of simplicial complexes as a special case of Morse functions on Whitney-stratified spaces; see [GM88, Part 1, c. 1]. Using this framework, Cohen-Steiner et al. generalize persistent homology to *tame* functions and show that persistence barcodes/diagrams are stable with respect to d_B.

Definition F.1. Suppose $f\colon \mathbb{X} \to \mathbb{R}$ is a continuous function on a topological space $\mathbb{X}$ and let $H_n(\mathbb{X})$ denote the nth singular homology module. Limited by time we direct the curious reader to [Rot98, c. 4] for a group theoretic introduction; it is a fairly simple exercise to translate this to the setting of modules. For this section, little more is needed than knowing singular homology provides a functor $\mathbf{Top} \to \mathbf{Mod}_R$ and, by [ES52, pp. 100-101], singular homology and simplicial homology modules are isomorphic up to triangulable spaces. Since $f^{-1}((-\infty, x]) \subseteq f^{-1}((-\infty, y])$ whenever $x \leq y \in \mathbb{R}$, the sublevel sets of f provide an $\mathbb{R}$-filtration of $\mathbb{X}$. In the context of the continuous map $f\colon \mathbb{X} \to \mathbb{R}$, we will say $\mathbb{X}$ is $\mathbb{R}$-filtered.

Lemma F.2. *If $f\colon \mathbb{X} \to \mathbb{R}$ is a continuous map and $x < y \in \mathbb{R}$, then there exists an induced R-module homomorphism $H_n\left(f^{-1}(-\infty, x]\right) \to H_n\left(f^{-1}(-\infty, y]\right)$ for each $n \geq 0$.*

Proof. This is just the evaluation of the singular homology functor having the form $\mathbf{R} \xrightarrow{\leq} \mathbf{Top} \xrightarrow{H_n} \mathbf{Mod}_R$. □

Definition F.3. An n**th homological critical value of** f is a constant $c \in \mathbb{R}$ such that the induced R-module homomorphism

$$H_n(f^{-1}(-\infty, c - \varepsilon]) \to H_n(f^{-1}(-\infty, c + \varepsilon])$$

is not an isomorphism for some $\varepsilon > 0$ and $n \geq 0$. We say f is **tame** if it has at most finitely many homological critical values and $H_n(f^{-1}(-\infty, x])$ are finitely generated for all $n \geq 0$ and $x \in \mathbb{R}$. Clearly, this provides a generalization of nth persistent homology modules of finite type. By way of notation, $F_n^x := H_n(f^{-1}(-\infty, x])$ and $f_n^{x,y}\colon F_x \to F_y$ is the R-module homomorphism induced by functoriality, that is, by the inequality $x \leq y \in \mathbb{R}$. We will also write $F_n^{x,y} := \operatorname{im}(f_n^{x,y})$ and maintain the convention that $F_n^{x,y} = \{0\}$ whenever $x = \pm\infty \in \bar{\mathbb{R}}$ or $y = \pm\infty \in \bar{\mathbb{R}}$. We call $F_n^{x,y}$ the x, y**-persistent** n**th homology module of** f. The n**th persistent homology module** $\mathcal{H}_n(f)$ **of** f is the family $\left\{F_n^x, f_n^{x,y}\right\}_{x<y\in\bar{\mathbb{R}}}$. A persistent homology module is of **finite type** if $f\colon \mathbb{X} \to \mathbb{R}$ is tame. In the case of a filtered simplicial complex, the previous definitions generalize those of **Definition E.8**. For tame functions, we define the x, y**-persistent** n**th Betti number** to be $\beta_n^{x,y} := \operatorname{rk}(F_n^{x,y})$ for all $x \leq y$ in $\bar{\mathbb{R}}$.

In some sense, homological critical points divide the preimage of f into segments in which the homology of the sublevel set is unchanged.

Lemma F.4 (Critical value lemma; [CSEH07]). *If $[x, y] \subseteq \mathbb{R}$ contains no homological critical values for some function $f : \mathbb{X} \to \mathbb{R}$, then $f_n^{x,y}$ is an R-module isomorphism for all $n \geq 0$.*

Proof. Set $m_{-2} := x$, $m_{-1} := y$, and $m_0 := (x + y)/2$. Then $f_n^{x,y} \equiv (f_n^{m_0,y} \circ f_n^{x,m_0})$ by functoriality, that is, by the fact functors preserve compositions. By way of contradiction, suppose $f_n^{x,y}$ is not an isomorphism. It follows that either f_n^{x,m_0} or $f_n^{m_0,y}$ is not an isomorphism since R-module isomorphisms are closed under composition. If f_n^{x,m_0} is not an isomorphism, then set $m_1 := (x + m_0)/2$. Otherwise, we set $m_1 := (m_0 + y)/2$. Continuing inductively, we construct a sequence $\{[m_i, m_j]\}_{i,j \in \mathbb{Z}_{\geq -2}}$. By [Rud76, p. 38, Theorem 2.38]

$$\bigcap_{i,j \in \mathbb{Z}_{\geq -2}} [m_i, m_j] \neq \varnothing.$$

Hence there exists some critical point $c \in \bigcap_{i,j \in \mathbb{Z}_{\geq -2}} [m_i, m_j]$, a contradiction. Therefore $f_n^{x,y}$ is an R-module isomorphism for each $n \geq 0$. □

We will now generalize the definition of persistence barcodes/diagrams to be summaries of the persistent homology of the sublevel sets of tame functions.

Definition F.5. Assume $(c_i)_{i=1}^k$ lists the nth homological critical points of the tame function f such that $c_i < c_{i+1}$ and let $(d_i)_{i=0}^k$ be chosen so that $d_{i-1} < c_i < d_i$. The sequence $(d_i)_{i=0}^k$ is said to be **interleaved**. We will set $d_{-1} = c_0 = -\infty$ and $d_{k+1} = c_{k+1} = +\infty$. The **multiplicity of the persistence interval** $[c_i, c_j)$ is defined by

$$\mu\left([c_i, c_j)\right) := \left(\beta_n^{c_i, c_{j-1}} - \beta_n^{c_i, c_j}\right) - \left(\beta_n^{c_{i-1}, c_{j-1}} - \beta_n^{c_{i-1}, c_j}\right)$$

for $0 \leq i < j \leq k+1$. Notice that if $x, x' \in (c_i, c_{i+1})$ and $y, y' \in (c_{j-1}, c_j)$, then $F_n^{x,y} \cong F_n^{x',y'}$ by the **Lemma F.4**. Hence $\beta_n^{x,y} = \beta_n^{x',y'}$ and thus

$$\mu([x,y)) = \mu([x',y')),$$

that is, μ is well-defined.

This leads us to define $\mathrm{bcode}_n(f) := (P, \mu_P := \mu) \in \mathrm{Bcodes}$ to be the n**th persistence barcode** of the $\mathbb{R}$-filtered topological space $\mathbb{X}$, where

$$P = \left\{ [c_i, c_j) \,\middle|\, 0 \leq i < j \leq k+1 \right\}.$$

The n**th persistence diagram of** f is $\mathrm{dgm}_n(f) := \mathrm{dgm}\left(\mathrm{bcode}_n(f)\right) \in \mathrm{Dgms}$. Letting Quad_x^y denote the infinite quadrant $[-\infty, x] \times [y, \infty]$, we can prove a generalization of the k-triangle lemma; see **Lemma E.25**.

Lemma F.6 (k-Triangle Lemma; [CSEH07]). *Suppose that $f \colon \mathbb{X} \to \mathbb{R}$ is a tame function with critical points $(c_i)_{i=0}^{k+1}$ and interleaved sequence $(d_i)_{i=-1}^{k+1}$. Also, we will set $(Q, \mu_Q) := \mathrm{dgm}_n(f)$ for some integer $n \geq 0$. If $x < y$ are not homological critical points, then*

$$\sharp \left(\left((Q \setminus \Delta) \cap \mathrm{Quad}_x^y, \mu_Q|_{(Q \setminus \Delta) \cap \mathrm{Quad}_x^y} \right) \right) = \beta_n^{x,y}.$$

Proof. Without loss of generality, we may assume that $x = d_i$ and $y = d_{j-1}$ for $0 \leq i < j \leq k+1$ by **Lemma F.4**. Also, we will set

$$\{(c_p, c_q)\}_{0\leq p<q\leq k+1} \coloneqq (Q \setminus \Delta) \cap \mathrm{Quad}_x^y.$$

Observe

$$\begin{aligned}
\sharp\left(\left((Q\setminus\Delta)\cap\mathrm{Quad}_x^y, \mu_Q|_{(Q\setminus\Delta)\cap\mathrm{Quad}_x^y}\right)\right) &= \sum_{0\leq p<q\leq k+1} \mu_Q\left([c_p, c_q)\right) \\
&= \sum_{0\leq p<q\leq k+1} \left(\beta_n^{d_p,d_{q-1}} - \beta_n^{d_p,d_q}\right) - \left(\beta_n^{d_{p-1},d_{q-1}} - \beta_n^{d_{p-1},d_q}\right) \\
&= \sum_{0\leq p<q\leq k+1} \beta_n^{d_{p-1},d_q} - \beta_n^{d_p,d_q} + \beta_n^{d_p,d_{q-1}} - \beta_n^{d_{p-1},d_{q-1}} \\
&= \beta_n^{d_{-1},d_{k+1}} - \beta_n^{d_i,d_{k+1}} + \beta_n^{d_i,d_{j-1}} - \beta_n^{d_{-1},d_{j-1}} \\
&= \beta_n^{d_i,d_{j-1}} \\
&= \beta_n^{x,y}.
\end{aligned}$$

Notice that the last two equalities above are due, respectively, to telescoping sum and the fact $F_n^{x,y} \coloneqq \{0\}$ whenever x or y is infinite. □

Proposition F.7. *Suppose $\mathcal{H}_k(K)$ is a persistent homology module of a $\mathbb{Z}_{\geq 0}$-filtered abstract simplicial complex and consider $(Q, \mu_Q) \coloneqq \mathrm{dgm}\left(\mathcal{H}_k(K)\right)$. Assume $(i, j) \in \mathbb{Z}_{\geq 0}$ and $0 \leq i < j$. Let $T \subseteq \angle Q$ denote the set of Q-persistence triangles containing the point (i, j). It follows that*

$$\sharp\left(\left((Q\setminus\Delta)\cap\mathrm{Quad}_x^y, \mu_Q|_{(Q\setminus\Delta)\cap\mathrm{Quad}_x^y}\right)\right) = \sum_{t\in T} \mu_{\angle Q}(t).$$

Proof. It suffices to show that $(i,j) \in \angle(p,q) \in T$ if and only if $(p,q) \in \text{Quad}_i^j$ for any $\angle(p,q) \in T$. If $(i,j) \in \angle(p,q) \in T$, then $0 \leq p \leq i$ and $0 \leq j \leq q$. This implies that $(p,q) \in \text{Quad}_i^j$. Now assume $(p,q) \in \text{Quad}_i^j$ for some arbitrary $\angle(p,q) \in T$. Consequently, $0 \leq p \leq i$ and $0 \leq j \leq q$. Thus $(i,j) \in \angle(p,q) \in T$. □

The previous proposition and the k-triangle lemma together prove the correctness of the generalization of $\beta_n^{i,j}$ to $\beta_n^{x,y}$.

Definition F.8. Suppose (P, μ_P) and (Q, μ_Q) are multisets of points in euclidean space. We define the **bottleneck distance between** (P, μ_P) **and** (Q, μ_Q) by

$$\mathrm{d}_B\left((P,\mu_P),(Q,\mu_Q)\right) := \inf_{\eta\colon P \xrightarrow{\sim} Q} \left\{ \sup_{x \in X} \{\|x - \eta(x)\|_\infty\} \right\}$$

where the infimum is taken over all multiset bijections between P and Q. It is not a hard exercise to show this is a metric on Dgms.

The main result of [CSEH07] is the stability of persistence diagrams under d_B.

Theorem F.9 (Bottleneck Stability; [CSEH07])**.** *Suppose $\mathbb{X}$ is a triangulable space and assume $f, g\colon \mathbb{X} \to \mathbb{R}$ are tame continuous functions. If $(f - g)$ is bounded, then*

$$\mathrm{d}_B\left(\mathrm{dgm}_n(f), \mathrm{dgm}_n(g)\right) \leq \|f - g\|_\infty$$

where $\|\cdot\|_\infty$ is the supremum norm for bounded real-valued functions and $n \geq 0$ is an arbitrary integer.

For the proof, we direct the reader to [CSEH07, pp. 113-116]. We will use **Theorem F.9** in chapter IV to prove the stability of persistence diagrams that are produced from multiradial filtrations.

APPENDIX G

NOTATION

APPENDIX H

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