Understanding children’s reasoning in multiplication problem-solving

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Abstract:

This article investigates two children’s intuitive thinking in solving multiplication problems from different educational backgrounds. One of the children is in a southern elementary school in the US. He was given the same problems both in first and second grades. The other child was a first grader in a southwest region of China, and she was given the same problems. The findings reveal a variety of intuitive thinking in solving the multiplication problems through addition beyond direct modeling and counting strategies. The authors also discussed how different educational backgrounds in early elementary mathematics education may affect children’s intuitive ideas and reasoning in solving multiplication problems. The study implies the importance of understanding children’s intuitive ideas of multiplication and highlights potential opportunities for developing children’s understanding of multiplicative thinking and algebraic thinking in earlier stages of arithmetic learning.

Keywords: Intuitive thinking of multiplication | problem solving | task-based interview | elementary mathematics education in the US and China

Article:

Introduction

Have you ever wondered when students begin to reason multiplicatively? Children’s development of multiplicative reasoning, in particular, has gained a host of inquiries among teachers and mathematics educators. In this study, we investigate how children with no kind of formal multiplication instruction solved multiplication problems. We inquire about the role of students’ internal sense making or what we label as “intuitive thinking.” For us, intuitive thinking means the ways in which a child reasons multiplicatively before formal instruction. More specifically, we study two children from different countries. For us, gaining insight into children’s intuitive thinking of multiplication across countries and curriculum requirements from other countries is useful in understanding our own in the United States.

Theoretical perspectives

Literature review

Multiplicative thinking is “a capacity to work flexibly with the concepts, strategies, and representations of multiplication (and division) as they occur in a wide range of contexts” (Siemon, Breed, & Virgona, 2005, p. 2). Multiplicative reasoning is vital for understanding fractions, proportions, and functions and developing algebraic thinking. It is a key focus of mathematics instructions in 3-5th grades (NCTM, 2001). Despite multiplication is often formally introduced in second grade in the US (CCSSI, 2010), studies have indicated that young children have considerable knowledge of multiplicative reasoning before they receive formal instruction of multiplication. They can solve multiplication tasks with or without context through a variety of approaches as early as kindergarten (Anghileri, 1989; Bakker, Van Den Heuvel-Panhuizen, & Robitzsch, 2014; Mulligan & Mitchlmore, 1997). Researchers have found that young children can use direct modeling, combined with counting strategies such as counting-on, counting-altogether, and doubling strategy to solve multiplication problems (Anghileri, 1989; Carpenter, Ansell, Franke, Fennema, & Weisbeck, 1993; Downton, 2008; Kouba, 1989; Mulligan, 1992). Bakker et al. (2014) found that even without physical objects, first graders were still able to solve multiplication tasks. Also, contextual problems, especially with pictures of countable objects, and doubling problems are relatively easy for young children as those types of problems have a closer relationship with addition (Bakker et al., 2014).

Thompson and Saldanha (2003), in their study of children’s algebraic thinking, found that knowing addition is not enough in forming a conceptual understanding of multiplication. Instead, to understand multiplication, children need to envision mathematical objects into equal-sized groups and recognize that group size is associated with group numbers (Sullivan, Clarke, Cheeseman, & Mulligan, 2001). Multiplicative reasoning involving recognitions of quantities of different types is different from additive reasoning, where quantities are always the same type (Bakker et al., 2014). It is essential to teach children mathematics through building connections with their informal mathematical knowledge and reasoning.

Children’s informal knowledge comes from their experience and prior knowledge. In the survey conducted by Bakker et al. (2014), individual students in first grade demonstrated different informal knowledge of solving multiplication tasks. Bakker et al. (2014) pointed out that educational background such as the mathematics textbooks used in class at the end of first grade had a substantial effect on students’ pre-instructional knowledge of multiplication. Cultural factors such as parents’ education levels and expectations also have an influence on children’s informal knowledge (Davis-Kean, 2005). Chinese elementary education is rather different from the US in many aspects such as culture, curriculum, ways of teaching and teachers’ knowledge (Cai et al., 2005; Ma, 1999). However, little research has been done to highlight the difference of informal knowledge of young children regarding multiplicative reasoning across the US and China.

The present study

The lead author initially investigated a first grader's intuitive ideas in solving multiplication problems in the US through a task-based interview (Lu, 2013). This article reports a follow-up study examining the growth of his understanding of multiplication after one year. It also describes a replicated study of a first grader in China. Coupling that understanding with a concrete example such as a task-based interview with a child from different educational background adds more insight into what we can do to support the development of multiplicative reasoning in elementary grade students. In this study, we describe how the young children worked through multiplicative reasoning tasks. We are particularly curious about the nature of multiplicative reasoning between them from different countries and the role of intuitive thinking during multiplicative reasoning tasks.

Method

Participants

The study involves two students: David, a 7-year old Asian-American boy in an elementary school in the southern US, and Ying, a 7-year old Chinese girl in an elementary school in southwest China. David was in a gifted and talented program in the American school and Ying was a top student in her class as identified by the school.

Data sources and analyses

The study is a case study based on clinical interviews, in particular, task-based interviews. Task-based interviews study individual's mathematical behavior in the process of solving given problems. It focuses on eliciting students' intuitive thinking and can also be used as a way of assessing their understanding of mathematical ideas. Task-based interviews have been broadly employed for educators to gain an understanding of students' mathematical thinking in problem-solving and provide insights for improving teaching practices (Goldin, 1997). The main task selected for this study states "An old man holds a stick that has 3 branches. On each branch are three cages. In each cage are three birds. How many birds are there together?" (Fang, 2003). The main task is a multiplication contextual problem. It supports sense-making of number relationships and allows flexible ways to solve it. To further deepen understanding of students' thinking, a variation of the main task was also employed in the interviews with the two students. The variation states: An old man holds a stick that has 4 branches. On each branch are four cages. In each cage are four birds. How many birds are there together?" both the task and its variation are intended to "have depth and response flexibility that allow evidence of widely differing subject capabilities to emerge" (Goldin, 2000, p. 540).

The interviews with the two students were conducted separately in the US and China. To probe student intuitive thinking in problem solving, an environment that encourages free thinking is vital (Goldin, 2000). Throughout all the interviews with the two students the researcher (lead author) did not impose any ideas or judgments to the student's thought process; rather, the researcher used why and how questions to elicit the student's thinking. When the student made a mistake, the researcher asked retrospective questions or suggested the student use different words or representations.

The researcher made careful observations and field notes during each interview. The interviews were audiotaped, and transcripts reflect summaries for each minute of the dialogue. The transcripts, field notes, and individual student's work were discussed and used for the data analysis in this study. Since the Chinese girl could not speak or read English, the interview was conducted in Chinese and the transcripts with the Chinese girl were translated from Chinese to English. The translations were presented to a third researcher in the field of mathematics education who is fluent in both English and Chinese for accuracy, simplicity, and validity of the translation.

Case 1: David's approaches

The first interview

When the first interview conducted, David was in January of the second semester of his first grade. The leading researcher presented the main task to him. He used his fingers and counting-on strategy to figure out 3 + 3 + 3 = 9 and then 9 + 9 + 9 = 27. Although he made mistakes in adding 9 + 9 + 9, he finally got the answer correctly after a few tries. In short, he solved the multiplication problem through repeated addition. The lead researcher then gave him the variation problem. He tried to use his fingers again, but he failed. With the suggestion from the lead researcher, he then drew a picture (see Figure 1). He drew four branches, four boxes representing four cages on each branch. Then four short line segments inside and around each box representing four birds. He counted all the line segments altogether. In summary, David used a combination of drawing and counting-altogether to avoid the difficulties of adding multiple larger numbers (Lu, 2013).

Figure 1. David's solution to the variation of the main task in the first interview.

The second interview

David was given the same problems one year later. By the time, he was in the March of second grade and knew that 3 × 2 = 6, which is the same as 3 + 3. The following conversation happened after the researcher presented to him the main task in the second interview.

Researcher: An old man is holding a stick with three branches. There are three cages on each branch.
David: Three branches, three cages, OK.
Researcher: Each cage has three birds.
David: That would be 26…. No. it is 27.
Researcher: How did you get that?
David: Simple! It is just like one branch, 3, and times another 3 is 9, and times another 3. 9 times another 3 is 27.
Researcher: Why is it 3 × 3 × 3?
David: Because it keeps going on 3. One stick and then 3 branches, and 3 cages on each, which is 9 cages, and times 3 more is 27. 9 times 3 is 27, right?
Researcher: Can you draw a picture?

Figure 2 illustrates his method (where he changed the character of the original problem into a turtle as he thought a turtle was a symbol for wisdom).

David: Nine turtles right here (pointing to one branch), because three cages. Nine times those 3 (pointing to the3 branches) is 27.
Researcher: Why 9 times 3?
David: One branch has 9 because there are 3 (pointing to the cages). Three times 3 is in one branch. Nine times 3 (pointing to 3 branches) is 27.

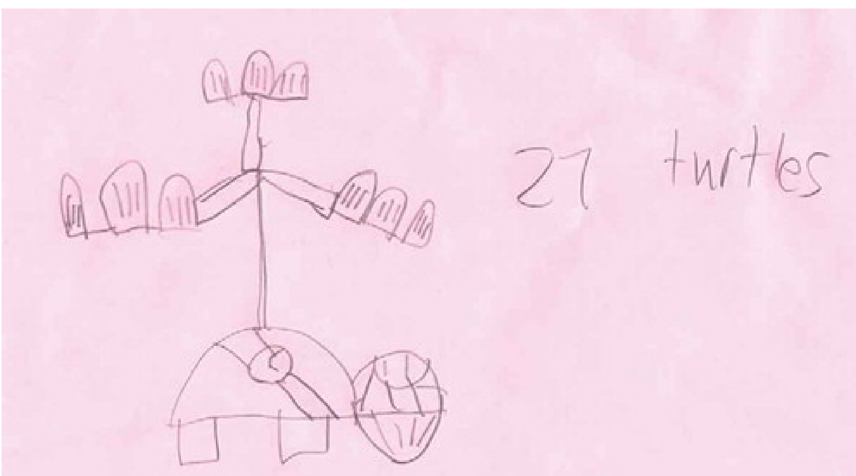

Figure 2. David's transformation of the original problem into a turtle in the second interview.

The lead researcher asked him to solve the problem variation too. He drew a picture (Figure 3. He changed the characters into eagles).

David: So it is 16 × 4
Researcher: Why 16 × 4?
David: 16 × 4, which is…16 times 4, is 64. So the answer is 64.
Researcher: Why?
David: No. it is 62.
Researcher: Why is it 62?

David: Because 16 × 4 is 62.
Researcher: How?
David: Because 4 × 4 is 16 on one branch, and four branches.
Researcher: How did you get 4 × 16 is 62?
David: Because 16 + 16 is 32, 32 × 2 is 62?
Researcher: Are you sure 62?
David: I mean… I knew it (he erased 62 and wrote 64). So 16 + 16 is 32, times 2 is 64.
Researcher: Are you sure that is 64?
David: Yes, yes. I am sure it is 64.

Figure 3. David's solution to the variation of the main task in the second interview.

The conversation above shows David viewed 16 × 4 as four 16s. In fact, he computed two 16s first and then two 32s, which suggests a doubling strategy. The envisioning of three 3s or four 16s in the problems indicates his understanding of quantification and multiplicity in the multiplication problems (Thompson & Saldanha, 2003). David's solutions in the second interview are different from his additive-based solutions in the first interview, indicating his move from additive counting to multiplicative thinking.

Case 2: Ying's approaches

When the interview was conducted, Ying was a 7 year-old Chinese girl in an elementary school in southwest China and she was in the second semester of her first grade. In the beginning of the interview, the leading author gave Ying two addition problems 9 + 5 and 48 + 6. She used a making-10 strategy, 9 + 5 = 9 + 1 + 4 = 10 + 4 = 14;
48 + 6 = 40 + 8 + 2 + 4 = 40 + 10 + 4 = 40 + 14 = 54. The lead researcher then gave her the main task. Ying wrote 3 + 3 + 3 = 9, 9 + 9 = 18, and 18 + 9 = 27 (Figure 4). She explained,

Ying: First see how many birds are on one branch. Three cages on each branch, and each cage has three birds. Add altogether, 3 + 3 + 3 = 9. Then add the number of birds on another branch, 9. 9 + 9 is 18, the number of birds on two branches. We get the numbers of birds on one branch, and the second branch. We now add the number of birds on the third one, 18 + 9 = 27.

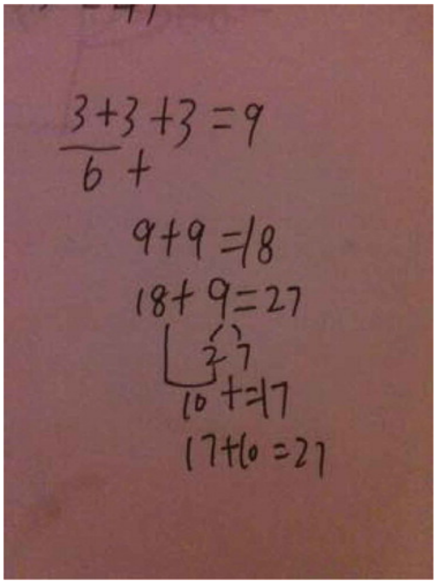

Figure 4. Ying's making-10 strategy in addition.

Ying obviously used the repeated addition approach to solve the multiplication problem, which is similar to David's approach in his first interview. However, Ying used making ten strategy in her calculation (see Figure 4).

The lead researcher then presented the variation of the problem, Ying wrote down 4 + 4 = 8, 8 + 8 = 16, 16 + 16 = 32, and 32 + 32 = 64, and explained,

Ying: Every branch has 4 cages. Every cage has 4 birds. You split 4 (cages) into 2 halves. The first half is 4 + 4 = 8, then add the other half.
Researcher: What do you mean 2 halves? What do you mean by 4 + 4?
Ying: That means 4 birds in one cage. Then add another 4, you get 8 birds.
Researcher: So 4 + 4 means 8 birds in two cages?
Ying: Yes. And then add birds in the other two cages, which gives 8 + 8 = 16. That is how many birds on one branch.

Ying drew a picture below her calculation to illustrate her half-half method (see Figure 5).

Ying: One branch has 16 birds, but you do not need to add for every branch. You can just do 16 + 16 = 32. Because there are 4 branches, split them into half-half, each half has two (branches). This half is 32. The other is 32 too, because they are equal.

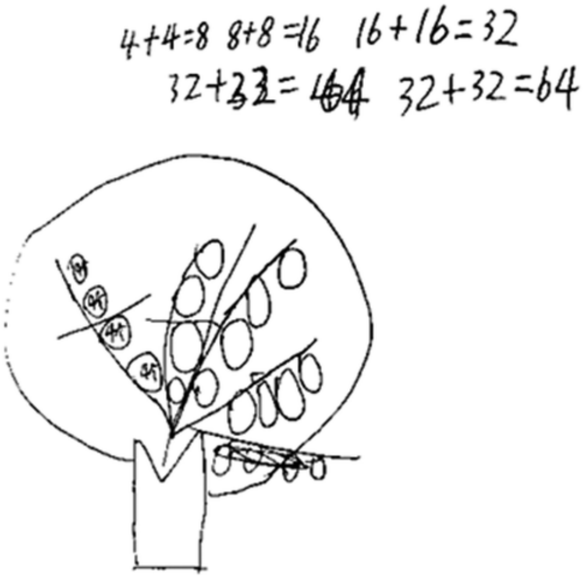

Figure 5. Ying's pictorial representation of her thinking.

To further understand her thinking, the researcher asked what if the number cannot be split into two equal parts; for example, if there are five branches, five cages and five birds in each cage. She drew a picture to illustrate her half-half method and explained (see Figure 6),

Ying: Every cage has 5 birds, 5 + 5 = 10. Ignore this one (refers to the fifth cage on the left side branch in her drawing), you just add these two (she drew a line segment between every two cages). Each has 5 birds, 5 + 5 = 10, you got this half. Then add the other half, 10 + 10 = 20. Now you only have one cage left, then what do you do? You add another 5, 20 + 5 = 25. (She drew a line segment between two branches), like (what we did) before, they are the same (number of birds on each branch). 25 + 25 = 50. Exclude this one (branch), we now have got the half. 50 + 50 = 100. After the 100, only one branch left, which is 5 + 5 = 10, 10 + 10 = 20, 20 + 5 = 25. Then we add the 25. They equal to 125. So, there are total 125 birds.

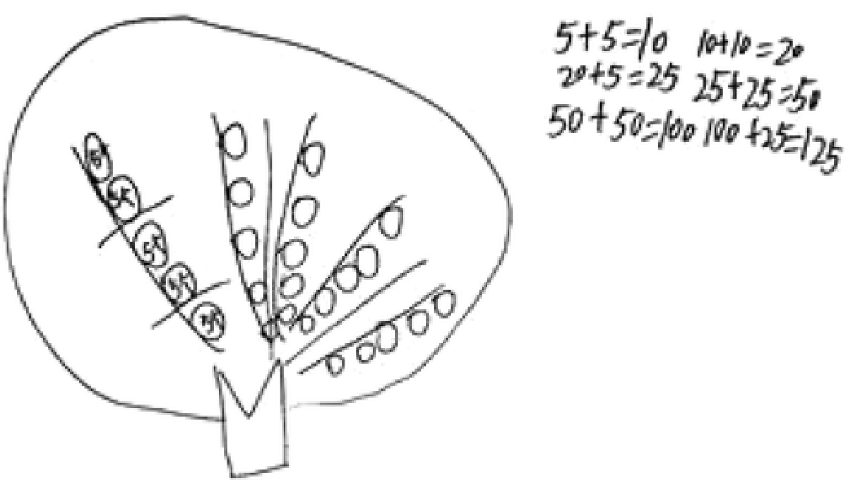

Figure 6. Ying's solution to the "five branches" question.

The researcher continued, asking her "What if there were six instead?" Ying drew another picture (Figure 7) and solved it in a similar way.

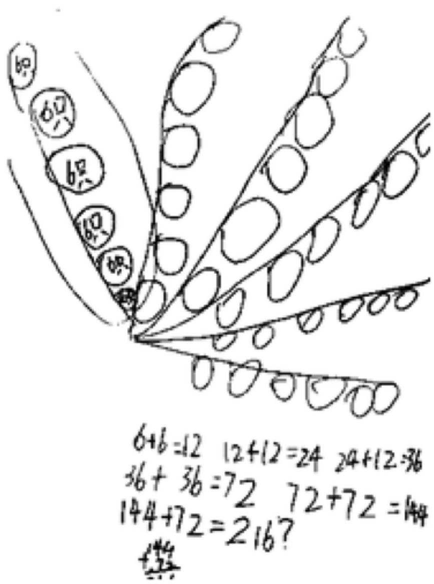

Figure 7. Ying's solution to the "six branches" question.

As we can see, Ying mainly used partitioning and doubling strategies to avoid directly the addition of larger numbers. To calculate the number of birds on each branch, she partitioned the addends in 4 + 4 + 4 + 4 into two groups of (4 + 4), 6 + 6 + 6 + 6 + 6 + 6 into three groups of (6 + 6), then used doubling strategy. She used the same method even when there were an odd number of addends. In the case of 5 cages in one branch, she divided 5 + 5 + 5 + 5 + 5 into two groups of (5 + 5) and one 5.

More subtle is that she not only applied the partitioning and doubling strategy to compute the number of birds on one branch, but also on all the other branches. In the case of 4 branches, after she doubled 8 to get 16 birds on one branch, she then doubled 16 for two branches, and doubled 32 to 64 for 4 branches. In the case of 5 branches, she divided 5 branches into 2, 2, and 1 branches. In the case of 6 branches, she divided branches into three 2 branches. She applied the doubling and partitioning strategies across different scales (cages or branches). The generalization indicates her understanding of the quantitative relationships embedded in the problems. In algebra, those relationships can be illustrated as $4x = 2 (2x)$, $5x = 2 (2x) + x$, and $6x = 3 (2x)$, where x can be either cages or branches. Understanding of quantitative relationships beyond the concrete context is one of the foundations for algebraic thinking (Yackel, 1997).

Discussion and implications

The study investigates the thinking of two children in solving multiplication problems across the US and China. The results indicate that they approached multiplication problems through addition, which is consistent with the existing research findings. However, the two children employed different strategies of addition. Specifically, the study reveals the following differences between the their approaches to the multiplication problems:

First, Ying used making-10 strategy, while David used direct modeling and counting strategies. When numbers were getting larger, David used a combination of drawing and counting-altogether strategy, while Ying exclusively used a combination of partitioning and doubling strategy. David also used the doubling strategy, but it was the second interview when he was in second grade.

Second, Ying tended to use symbolic representations, while David tended to use concrete objects in the first interview. For example, in Ying's drawings, she directly wrote the numbers of birds while David used line segments to represent them (Figure 1).

Third, in her drawings, Ying did not write numbers on all branches and cages, but only on the first branch, which suggests she recognized the quantitative relationships in the problem situation. She could generalize the patterns across different contexts.

In general, David's approaches and strategies are consistent with existing research findings, saying young children like to use direct modeling and counting strategies to solve multiplication problems. The study also confirms that without modeling, young children can also solve multiplication problems through drawings or in an environments that promotes counting strategies. More importantly, this study reveals that young children can go beyond counting and modeling to using comprehension strategies, such as making-ten and a combination of partitioning and doubling. Ying's strategy of using a combination of partitioning and doubling is rarely seen in the existing research on children's intuitive ways of solving multiplication problems. These strategies are especially useful for students to visualize multiplicities of quantification in multiplication problems. This study also shows the importance of patterns and quantitative relationships for algebraic thinking.

Ying's approaches match with the findings of research on Chinese early childhood mathematics education. As Sun and Zhang (2001) noticed that in American elementary mathematics education, whole numbers are often introduced based on one-on-one correspondence, and addition and subtraction rely on counting strategy exclusively. In Chinese curriculum, addition relies heavily on "making-ten" strategy. Sun and Zhang (2001) that Chinese children rarely use manipulatives, but more on logical reasoning to make a connection with previous knowledge. In first-grade math curriculum, numbers are introduced as relationships of two quantities, and subtraction problems primarily are solved through addition facts. The emphasis on quantitative relationships and logical reasoning is beneficial for Chinese children to understand the connections between addition and multiplication. In addition, multiplication is presented as a simple way to calculate addition with repeated numbers in Chinese curriculum. Numerous examples with different contexts are used to help students identify multipliers and multiplicands and revise repeated addition expressions into multiplication expressions. Students are also encouraged to use patterns and derive multiplication results from known facts, for example, using $5 \times 3 + 3$ to find 6×3. Cai et al. (2005) studied Chinese elementary mathematics curriculum and found throughout Chinese curriculum, various contextual examples and tasks are used for students to identify quantitative relationships and generalize patterns from different situations, which contributes to the development of algebraic thinking in earlier grades in China.

Caution should be employed when examine the finding as this study was conducted with only two children through a limited number of interviews in the US and China. In addition, the cultural background may affect their performance; however, it was difficult to determine their cultural context through the task-based interviews with the children. The further larger scale study across different cultural and educational environment is recommended.

Conclusion

NCTM (2001) have advocated conceptual understanding, reasoning, and problem solving for students in all grades in mathematics education. One of most effective ways to achieve the goal is to help teachers develop knowledge and abilities to understand students' internal sense-making and reasoning (Lesh, Hoover, Hole, Kelly, & Post, 2000). The findings from this study shed insights into young children' sintuitive thinking and have implications for helping them conceptually understand multiplication and develop algebraic thinking. Task-based interviews offer a way to probe students' intuitive construction of meaning and deepen their thinking (Goldin, 2000). As educators, we need to take a full advantage of their intuitively mathematical thinking and help them make sense of mathematical concepts, and develop their ideas on the basis of understanding their ways of thinking.

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