

## Prospective elementary teachers use of representation to reason algebraically

By: [Kerri Richardson](#), [Sarah Berenson](#), Katrina Staley

Richardson, K., Berenson, S., & Staley, K. (2009). Prospective elementary teachers use of representation to reason algebraically. *Journal of Mathematical Behavior*, 28(2/3), 188-199.

Made available courtesy of Elsevier: <http://www.elsevier.com>

**\*\*\*Reprinted with permission. No further reproduction is authorized without written permission from Elsevier. This version of the document is not the version of record. Figures and/or pictures may be missing from this format of the document.\*\*\***

### **Abstract:**

We used a teaching experiment to evaluate the preparation of preservice teachers to teach early algebra concepts in the elementary school with the goal of improving their ability to generalize and justify algebraic rules when using pattern-finding tasks. Nearly all of the elementary preservice teachers generalized explicit rules using symbolic notation but had trouble with justifications early in the experiment. The use of isomorphic tasks promoted their ability to justify their generalizations and to understand the relationship of the coefficient and y-intercept to the models constructed with pattern blocks. Based on critical events in the teaching experiment, we developed a scale to map changes in preservice teachers' understanding. Features of the tasks emerged that contributed to this understanding.

**Keywords:** Early algebra, Algebraic thinking, Algebraic reasoning, Patterns, Problem solving, Representations

### **Article:**

#### ***1. Introduction***

As a result of K-12 curriculum reform efforts and state policy changes, early algebra instruction is increasingly popular in the elementary grades (National Mathematics Advisory Panel, 2008; NCTM, 2000). Teaching early algebra concepts in elementary school causes teachers to move from direct instruction with its emphasis on memorization to emphasize student inquiry and higher levels of thinking. As a result, teachers are expected to engage their students in more problem solving and mathematical reasoning tasks. These reforms and redirections suggest changes to mathematics preparation and professional development of elementary teachers (Ball, 2003). Consistent with this need, we look at how teacher educators might promote prospective elementary teachers' understanding of early algebra ideas.

Reasoning. There is a comprehensive body of research on elementary teachers' mathematical understanding across a wide variety of topics (Ball, 1990; Ma, 1999; Schifter, 1998; Simon, 1993). Understanding and knowing mathematics are interchangeable and when students understand mathematics, they are able to see how things work, how things are related to each other, and why they work the way they do (Hiebert et al., 1997, p. 2). This group of authors further describes the dynamic nature of understanding and defines it as seeing relationships and making connections to what we already know. Making sense or understanding, according to Mason (2004) is a process by which an individual perceives a collection of things that share certain similarities and is able to propose a generalization about the collection.

Researchers speak to the processes of mathematical reasoning involved in learning mathematics. This perspective of making connections by seeing relationships and drawing generalizations is increasingly important in national reforms on teaching algebra for understanding.

Deciphering the meaning of algebraic reasoning can be viewed from multiple perspectives. Algebraic reasoning is broadly defined by Carraher and Schliemann (2007) as a *psychological process involved in solving problems that mathematicians can easily express using algebraic notation* (p. 670). Justification of the generalization is another aspect of algebraic reasoning and it plays a vital role in accessing children's thinking about their

generalizations (Maher & Davis, 1990). To accomplish an understanding of children's thinking, teachers ask open-ended questions to access the reasoning and, in some cases, redirect children's reasoning through those questions (Martino & Maher, 1999). With elementary students, justifications are more likely to take the form of a persuasive argument as to their thinking in relation to their generalization rather than a formal, mathematical proof. As Simon and Blume (1996) report, those arguments may take the form of presenting empirical evidence or proof by cases (Maher & Martino, 1996). Generalization and justification are viewed as important components of algebraic reasoning (Blanton & Kaput, 2005; Lannin, 2005).

### ***1.1. Early algebra***

Early algebra describes the algebra concepts taught in Pre-K-5 mathematics while algebra refers to the algebra concepts taught in middle grades and high school. NCTM (2000) defines early algebra as a way of understanding patterns, relations, and functions beginning in kindergarten. As students progress into grades 3–5, expressions and symbols are integrated into the early algebra curriculum to allow for ways to represent and model mathematical problems and situations. NCTM (2000) recommends pattern building and pattern finding activities for elementary schools. The purpose of these activities is to develop algebraic reasoning and problem-solving skills. Noticing or finding patterns helps students make generalizations and/or discoveries that encourage class discussions and promote algebraic reasoning. Students express these generalizations using a variety of contextual representations (Goldin, 1998). Advocates for stronger connections between topics taught in Pre-K-12 mathematics view the potential of incorporating algebraic reasoning across many, if not all, of school topics (for example, Carraher & Schliemann, 2007; Johanning, 2004; Ball, 2003). Goldin (1998) points out that there are a variety of meanings of the term representation; according to his definition these representations are either internal or external to the learner. We use representation here to mean tool systems such as tables, pattern blocks, symbols, and words that involve some actions on the part of the students and serve to mediate learning (Vygotsky, 1978). We focus on situational reasoning and representational tool systems in this study of elementary preservice teachers' understanding.

For those proposing to teach early algebra concepts in elementary school, patterns are seen as useful activities to make the following connections: (1) using symbols to express generalizations, (2) developing meaning of or reasons for those symbolic generalizations, and (3) using other algebraic representations such as graphs and tables (Berenson, Wilson, Mojica, Lamburtus, & Smith, 2007; Bishop, 1997; Kaput, 1999; Lannin, Barker, & Townsend, 2006). The use of concrete materials to build and find patterns is recognized by Kuchemann (1981) as easier for students to understand in terms of generalizing rules with symbols that represent the generalizations. Describing patterns, Smith (2003) focuses on two aspects of patterns: how they change with repetition and how students perceive the pattern as a whole. For this teaching experiment, we select the pattern aspects of change with repetition. It is from the actions of generalization and justification of pattern finding that we view this teaching experiment of elementary preservice teachers' understanding of early algebra concepts.

## **2. Methodology**

### ***2.1. Frameworks***

Our inquiry of elementary preservice teachers was designed to study how pattern-finding tasks promote learning how to generalize and justify rules from an emergent perspective (Cobb, 1995). In order to find out more about preservice teachers' thinking over time, we designed our study as a whole-class teaching experiment (Cobb, 2000; Simon, 2000). The emergent perspective gave us opportunities to study individuals' activities and the interactive activities of the whole class. Not only did the emergent perspective provide us with a way of looking at how the students self-organized, it also provided us with a way of seeing how students contributed to their small groups, the class as a whole, and to our own thinking about the problems we provided. We actively participated as instructors or assistants within the class setting. Their tasks consisted of conducting the experiment over 3 weeks of a mathematics methods course containing 25 preservice teachers. One researcher taught while the others collaboratively observed the class by taking field notes. Additionally, we followed multiple strands of the experiment including the preservice teachers' algebraic reasoning and their use of representations when engaged in pattern finding. After each class, researchers met and examined preservice teachers' learning and algebraic reasoning to analyze events, and plan the next phase of the experiment. Within

this research context, theories relating to preservice teachers' learning were generated and tested.

## 2.2. Subjects

Our study participants were all preservice elementary education majors, taking their first (and only) mathematics methods course. Most were traditional students in their early twenties and all 25 were Caucasian females. Their program involved them for three semesters in a Professional Development School, requiring school participation of 10 h per week each semester. It was possible for the preservice teachers to elect a mathematics concentration; however most of them took one mathematics course and one mathematics methods course to fulfill licensure requirements. The preservice teachers were seniors in their final semester of coursework prior to student teaching, and we conducted the teaching experiment toward the end of the semester during regular class time over a 3-week period.

## 2.3. Intervention and sources of data

In groups of two, preservice teachers completed four pattern-finding tasks and a summary task. We designed the tasks to have similar features and to have similar representations, unlike other research studies that incorporated several different pattern-finding tasks. While the preservice teachers investigated the tasks, laptop computers recorded the preservice teachers' conversations. Transcripts of these small group and whole class conversations, field notes, and artifacts produced by the preservice teachers provided sources of data. The first day, we gave the preservice teachers pattern block squares (see Fig. 1) and asked them to model the perimeter of pattern block trains that were one square, two squares, and three squares long. We suggested a table as a way of organizing the data and we asked the preservice teachers to write a conjecture about the perimeter of a train that was four squares long. The following sets of questions followed: *Does your conjecture work for 5 block trains? Use your conjecture to determine the perimeter of a 10-block train. How could you convince me that your conjecture works? You may have to make another conjecture to find the perimeter of a 100-block train. Try to think of an efficient way of doing this. What is your conjecture? How did you test your conjecture? Explain what the numbers and symbols in your rule mean in relation to your model.* The task presented to the preservice teachers in the second week involved generalizing and justifying a rule for the perimeter of an  $n$ -block triangle train (see Fig. 2). The preservice teachers investigated the  $n$ -block hexagon train's perimeter in the third week (see Fig. 3), and then they developed generalizations and justifications without pattern blocks for the perimeter of an  $n$ -block pentagon train. The last task asked the preservice teachers to find the perimeter of an  $n$ -gon train, using their results from previous tasks.

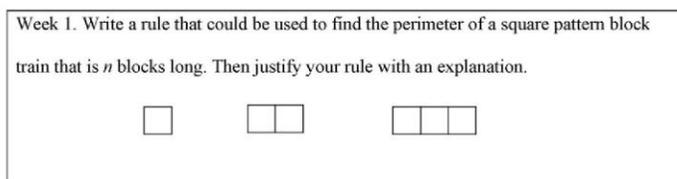


Fig. 1. Task given on week 1 of the teaching experiment.

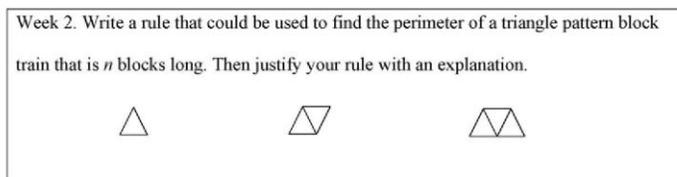


Fig. 2. Task given on week 2 of the teaching experiment.

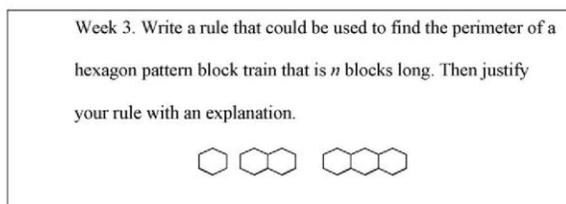


Fig. 3. Task given on week 3 of the teaching experiment.

## 2.4. Analysis

Powell, Francisco, and Maher (2003) developed a model for analyzing critical events' data. They defined a critical event as an element of the data that indicated significant progress from past understanding of a particular idea. Within a teaching experiment, these critical events were evident in the moment and over time. Imbedded within a critical event were central questions, all of which related to preservice teachers' growth of understanding. Our research question aligned with this framework: What critical events provided evidence of preservice teachers' algebraic reasoning including generalization and justification as they engaged in pattern-finding tasks?

## 3. Results: critical events

We examined the data to identify five critical events associated with the tasks. We found some of these events among the preservice teachers' work and conversations in several or all of the tasks. The analysis identified other critical events in only one particular task. Evidence given describing the five critical events was organized by the 3 weeks of the teaching experiment.

Two of the 25 preservice teachers did not generalize an explicit rule, such as  $2n + 2 = P$ , throughout the teaching experiment (Lannin, 2005). While they listened to other preservice teachers' strategies for generalizing explicit rules, and perhaps even noted these generalizations on their papers, they were not successful in abandoning their focus on their last entry into a table of values and the use of additive strategies to focus on the pattern block model. Lannin (2005) refers to this type of reasoning as recursive. For example, Lauren (pseudonyms are used in all cases) explained her recursive generalization of the square block train as: *The – when you start with the first block and to get four you have added three. And the second block to get six you add four, and the third, like it's a progressive thing. Three, four, five, six, seven.* However, 23 preservice teachers generalized an explicit rule and are of more interest to the question of study.

### 3.1. Critical Event 1: generalizing an Explicit Rule (1)

Using either tabular representations or the pattern block representations, the preservice teachers were, in most cases, successful in generalizing a rule that could be used for a block train of any number of blocks. In the case of the square blocks trains, they wrote in one form or another the equivalent of:  $2n+2 = P$ .

#### 3.1.1. Week 1

The first example analyzed was Cassie's work. Although preservice teachers worked in pairs, they each recorded their own work, and in many cases, their written work and their transcript words were different from their partner(s). Week 1 involved the square pattern block trains, and Cassie drew a t-table that consisted of two columns. She labeled the first column "# of blocks" and listed the number of blocks in sequence beginning with 1. The second column, labeled perimeter, named the perimeter associated with the number of blocks in the train beginning with the 1-block train. For example, in column one, she listed 1 and in column two, she listed 4, the next entry in column one listed 2 and column two listed 6. She continued to fill in her table additively, using recursive methods to generalize a rule. Her partner was Jenny and the transcript indicated an example of generalizing an explicit rule with no justification.

Jenny: Once I realize where all the perimeter sides are, yeah, I would think that number of blocks times two plus two.

Cassie: Yeah, because I think that's easier because I wouldn't have to reference all the other ones to write them all down to get [the answer].

Preservice Teacher: Ten plus two is [unintelligible 12]. Well, that would be faster.

Cassie: Okay, so a two hundred block train would be four hundred ...

Jenny: Yes. Four hundred and two.

Cassie: Yea!

Jenny: Yeah, I think I would definitely go with this one. [Jenny points to the generalized formula and they decide not to use the recursive strategy to find the perimeters]

Cassie: So... but, I did that in my head.

Jenny: Yes.  
 Cassie: It's weird that I didn't think of it like that. That I only thought of it the other way [recursively]. But, now that I know that, that's the one I want.  
 Jenny: Which one?  
 Cassie: The number of blocks times two plus two.

Cassie and Jenny's transcript indicated that they realized the power of the explicit formula to get any answer quickly rather than using recursive strategies. When analyzing their written work, though, Cassie offered no justification. Rather than justifying where the notation came from in relation to the pattern blocks or table, her explanation was limited to a description of the actual algebraic notation.

### 3.1.2. Week 2

Fig. 2 illustrates the task given on week 2. We noted that triangle trains were more difficult for eighth grade students to successfully generalize and justify rules than square trains or hexagon trains (Berenson et al., 2007). Thus we modified the activities to begin the first task with the square trains where all exposed sides were either horizontal or vertical. Note in Fig. 2 that the exposed sides of the triangle trains presented themselves as horizontal but not vertical. Also, the triangle train reveals a top or bottom, for a single triangle, thus there is a lack of symmetry between triangles. We acknowledged that this made the triangle train task more difficult than the square train task. On the second week, nearly all of the pairs generalized an explicit rule, but several were unable to offer a justification.

At first, Kim, Betty and Rita generalized a recursive rule using a t-table. Then a discussion about their work on week 1 with Eva, helped them create an explicit rule.

Betty: So we have one, two, three, four ... [Constructs table filling in the number of blocks].  
 Kim: Just add two to it every time .....

Rita: But remember, it's not... that's not what we did last time  
 Kim: But last time we didn't do triangles.  
 Betty: We didn't?  
 Rita: Squares! You're right.  
 Betty: Yeah. So this one's going to be different.  
 Eva:  $x$  times 2 plus 2. So you just add two to the number of blocks.  
 Kim: I think so, yeah.  
 Betty: So wait. Are you getting that the perimeter just goes up one every time?  
 Rita: Mm-hmm.  
 Eva: So, to figure out the perimeter, you just add two to the number of blocks you have?  
 Rita: The triangle plus two.  
 Kim: Or,  $x$  plus 2  
 Betty: Yeah, the triangle,  $x$  plus two, yeah.

Later, another preservice teacher explained to the whole class another way of thinking about the geometric model. This idea, presented to the whole class, provided an alternative way for others to think about their generalizations.

Eva: Okay. Um, it's  $n$ , the number of triangles is  $n$  plus 2 will give you the perimeter and then I came up with  $n+3$ , because you have three sides and then you're taking away one. So it's the same kind of rule, but it's written differently.

Here Eva indicated a 2-block triangle train for her idea of taking away interior sides. Note her generalization was specific to a 2-block train and no other. However, the idea of subtracting the interior sides did circulate around the class. Jo wrote:

$$\begin{aligned}
P &= 3n - 2(n - 1) \text{ and then simplified} \\
&= 3n + -2(2(n + -1)) \\
&= 3n - 2n + 2 \\
&= n + 2
\end{aligned}$$

The second week of the teaching experiment produced discourse and engagement on the part of the preservice teachers, in their diads, among diads, and with whole-class discussion. As noted, the triangle context that was more difficult to perceive visually, seemed to promote spirited discussions. Most of the preservice teachers were able to connect their generalizations of the triangle trains to the generalizations they made about square trains in the first week. The interactions of Rita, Betty, and Kim demonstrated how discourse was used to mediate leaning (Karpov & Haywood, 1998; Salomon & Perkins, 1998; Smith, 2008). Numerical patterns were discovered by the preservice teachers' use of tabular representations to generate the first generalization,  $n + 2 = P$ . The idea of subtracting the interior sides of the triangles was used inconsistently with respect to the preservice teachers' use of symbolic representations and verbal explanations. The focus on the interior of the model instead of the perimeter led others to develop inefficient rules. However, this subtractive approach relied on the pattern block model and helped the preservice teachers to focus on the blocks as an additional mediator of problem solving (Lannin, 2005), and perhaps, paved the way for justifications.

### 3.1.3. Week 3

Hexagon trains were the topic for generalization in the third week of the teaching experiment. It appeared that the isomorphism of the previous problems carried over to the generalizations made about hexagon trains (see Fig. 3). This week the pattern block representation received more attention from the preservice teachers than the tabular representations. Sue, working with Riki, generalized a rule almost immediately while building the model.

- Riki: [Building the pattern block model] I've got five (hexagons).  
Sue: I think it's  $4n+2$ . That's my hypothesis. Are we making one big train? I'm confused.  
Riki: And here's two....  
Sue: I think everybody's getting that (referring to  $4n+2$ ).  
Riki: So with one block ...there's (counts to six pointing to the sides of one hexagon) obviously.  
Sue: And there's one block so it's four times one plus 2 equal 6 because there are 6 sides.  
Riki:  $4n+2$  where  $n=$  ...

Some of the preservice teachers were able to generalize another rule mediated by the pattern block model. They viewed the train in two parts, the inside hexagons and the two hexagons on the outer ends. They reasoned that the inside hexagons each contributed 4 sides to the perimeter and that the two ends contributed a total of 10 sides to the perimeter. Ada wrote:

$P = 10 + 4(n - 2)$ . Ada's partner was Erin, and Ana joined them in the following dialogue.

- Erin: [Considering a 4 block hexagon train] The rule, well, it's confusing, because the bottom two (ends) have five and the middle two have four.  
Ada: Yeah.  
Ana: So whatever the ends are, you're going to have five and the middles are going to have four.  
Erin: So...  
Ana: Like this has five [points to one end of the train] and this has five [points to the other end of the train] but those [points to the inner blocks] have four.

By the third week, the preservice teachers learned to generalize across the tasks, recognizing similarities and difference between the three different tasks. In the first week, the preservice teachers made explicit generalizations that relied on symbolic, numerical and tabular representations. In the second week, an alternative generalization promoted more use of the pattern block model and less dependence on the tabular

representations. The representation of choice in the third week was that of the pattern block model to generalize two different explicit rules. Nearly all of the preservice teachers generalized an explicit rule, but not all were able to provide a justification for those rules.

### 3.2. Critical Event 2: incomplete justification

Another critical event noted in this teaching experiment related to incomplete justification. While the generalizations were valid in terms of a rule, attempts to explain the algebraic symbols of the rules were incomplete with respect to explaining the origins of the coefficient and/or the y-intercept of the rule.

#### 3.2.1. Week 1

The first example analyzed was Bess's work. Although Julie was her partner, the two chose to work independently on the square train task and both the transcripts and written work indicated little exchange of ideas. When Bess engaged in finding the perimeter of the square train task, she drew a table consisting of the number of blocks in one column labeled "# of blocks" (1, 2, 3, 4, 5, ...10, 100) and in the second column labeled "perimeter" (4, 6, 8, 10, 12, ...22, 202). In her written work she noted the following, *2x+ 2 means: In a linear form, the addition of another block only adds 2 extra sides because 2 of those sides are consumed by the area. The 2x is the 2 sides of each block and the 2 added are extra 2 sides gained on the end of the train by the addition of another block.* Bess was not alone in her focus on the last block added to the train as the source of the y-intercept of 2. This idea was persistently present in week 2 among a number of preservice teachers.

#### 3.2.2. Week 2

Week 2 involved finding a rule for the perimeter of an n-block triangle train and the subjects chosen for analysis were Erin and Ada who worked closely together. Their justification concerning the origin of the y-intercept was similar to that of Bess's. They were unable to explain that each interior triangle in the train contributed one side to the perimeter, a top or a bottom side.

Researcher 2: Okay. Have you got a rule?

Ada: We do.

Researcher 2: Okay, what is the rule.

Ada: For every [TRAIN], add two to the number of blocks that you have.

Researcher 2: Okay.

Ada: So, if you had twenty blocks you add two, so it would be twenty-two.

Researcher 2: Okay. Why do you add two?

Ada: Because each one has an extra side is what

Researcher 2: What one has an extra side?

Erin: Each. When you add a triangle to the triangle ...

Researcher 2: Uh-huh.

Erin: The sides become one, so you would just have the two edges (points to the last block added to the train).

Ada: So, you would have two extra sides.

Bess's transcript also offered the same explanation as her written work. She was able to generalize an explicit rule for the square trains,  $2x+2$ , but she had difficulty justifying the rule and this was noticeable in her struggle with explaining the coefficient. When looking at the patterns within the task, her written work also indicated she used only the table as a way to generalize and to justify her patterns.

In an attempt to justify their rules from the pattern block models, some preservice teachers explained only y-intercept but not the coefficient of the number of blocks. In this case, Erin and Ada knew that the n in their rule represented the number of blocks but they did not associate the coefficient of n with a top or a bottom edge because they focused on the last block or part of the model only. While focusing on individual parts is one way for preservice teachers to attempt to justify and generalize, focusing on the whole (in this case the pattern block train model) would allow preservice teachers to focus on the entire model's complexity (Armstrong & Larson,

1995). We noted that the triangle trains were the most difficult to successfully generalize and to justify rules (Berenson et al., 2007). Additionally, there may be a time and experience factor associated with the use of the concrete materials because these faulty justifications were not present by the third week of the teaching experiment.

### **3.3. Critical Event 3: a subtractive approach to generalization**

A third critical event noted in this teaching experiment was related to using a subtractive approach that was first posed by Eva in the second week and described in the first critical event. By using the subtractive approach, the preservice teachers used the inside of the pattern block model to make their generalizations and justifications. While we found no examples for days 1 or 3, Eva's explanation of the subtractive approach to the whole class provoked a lot of trial and error discussion. Joan and Jill provided some examples of this as noted in the following analysis.

#### **3.3.1. Week 2**

Joan and Jill's written work of  $3n - 2(n - 1) = \textit{perimeter}$  appeared to have been copied from Eva's work presented to the whole class on the white board, including the simplification of the equation. They focused on the inside rather than the perimeter of the model and the following excerpt from their transcripts indicated this idea:

- Jill: Yeah. So number of blocks times three and then you need to subtract the number of blocks times two, minus two?
- Joan: Yeah, so I just pulled the two out.
- Jill: Okay. The number ...
- Joan: So for five it's fifteen minus eight gives you seven. Yeah, it works.
- Jill: Yeah. Times two and then subtract two.
- Joan: Yeah.
- Jill: But, I like how you wrote it. Yeah. Is five, yeah, that works. So then, if you distributed that could you pull out that other? No, you couldn't pull out anything, never mind. But you could do ... yeah, you could pull out then.
- Joan: It won't simplify any more than what we had.
- Jill: Yeah. Yeah, because if you simplified it would be.
- Joan: It would be  $3n - 2$ .
- Jill: No, it would be plus two because you have to distribute the negative. So that would be  $n + 2$ .
- Yup.
- Joan: That's what it is.

Joan and Jill spent a considerable amount of time making sense of the formula but finally arrived at a justification that each block has 3 sides but each lost 2 sides when added to the train, except for the 2 end blocks that contribute 2 extra sides to the perimeter. What stood out as a prominent feature, though, was how they attempted the task. By using Ava's subtractive approach, they worked with a sense of inefficiency, which made their generalizations less elegant than that of their peers. While the formula did give the correct answer, it was not accurate in terms of their justification and instead should have been written:  $(3n - 2n) + 2 = P$ . This was particularly evident in their written work because they focused on just one aspect of the model, removing and adding blocks but not extending further to the actual perimeter of the entire set of blocks. Although successful at arriving at a valid justification, their focus was disjointed at times and they had difficulty noticing the model as a whole. It is not clear that Joan had any clear understanding of the meaning of the rule. Preservice teachers who focused on the parts of a problem only, rather than comparing parts to wholes, were less successful at making sense of all of the conditions of a problem (Armstrong & Larson, 1995).

### **3.4. Critical Event 4: generalizes and justifies a reasonable rule**

#### **3.4.1. Week 1**

A few preservice teachers generalized and justified an explicit rule in week 1. During the whole-class

discussion, the explanations provided by other preservice teachers varied from Sue's rule. She generated an explicit rule for perimeter and connected it to the representation of the square pattern block train. Sue drew a table and labeled one column "# of blocks" and the second column "perimeter" as part of her written work. She used a recursive strategy using the model by adding 2 more in the perimeter column for every one in the number of blocks column. And then she justified her rule by using the pattern block representation and writing  $2x+2$  means the top of the number of blocks plus bottom of the number of blocks plus 2 for each side is the perimeter of the train. Some struggled to reach a justification as the following interactions between Star, Nan, and another preservice teacher revealed.

- Star: What does  $2x$  plus 2 mean? I want to know what that means exactly. Okay?
- Researcher 2: Write it down. Where you think that comes from.
- Star:  $x$  equals the number of blocks and  $2x$  equals ...edges. So, two is four so that is one, two, three, four, number of edges (referring to the block representations).
- Nan: I don't know how to explain that.
- Star: How do I explain that (pointing to the  $2x$ )? How do I explain this (pointing to the  $+2$ )?
- Nan: I don't know.
- Star: Like, I know it's the number of these (pointing to the tops and bottoms of the block train) and then plus two is like the number of these (pointing to the vertical edges of the first and last block). Okay.
- Preservice Teacher:  $x$  is the number of blocks...
- Star and Nan: Right
- Preservice Teacher: Say you have a two block train. Two times two is the top and bottom.
- Star: So, that's ...
- Preservice Teacher: The top of the train and the bottom of the train. Plus two is the two end sides.
- Star: That's what I was saying in my head, but I couldn't make it make words.

Star was able to sufficiently understand the justification of her generalization for the square pattern block train in terms of the block representation. While most of the preservice teachers focused on numerical patterns found in the tabular representation in the first week, those who referred back to the train models were most successful in finding a valid justification for their rules.

### 3.4.2. Week 2

Laura drew the triangular model on her paper and then justified her rule. She wrote, In a figure your top and bottom sides equal the number of blocks. In order to figure out the perimeter add 2, which incorporates your 2 ends (one on each end). Laura's explanation is grounded by her work using the model.

- Researcher 2: Okay. This group had that, over here ( $x+2 = P$ ). So, would you explain it the same way?
- Preservice Teacher: I explained it that plus two. I just was justifying the plus two, so I was just saying that plus equals the ends of the train and every time you have a train you're going to add – each triangle represents one – because when they're put together like that there's only one side exposed. And then you have your train, so if you have five triangles you have five sides plus your ends.

In these two examples of the justification of  $n+2 = P$ , once again we saw the importance of the pattern block model as a mediating tool of justification. The preservice teachers' data revealed that deeper meanings were given to the number of blocks, the variable  $n$  or  $x$ , and that the number of blocks in a train determined the perimeter, a beginning notion of covariance.

### 3.4.3. Week 3

By week 3, Joan and Jill abandoned looking for number patterns in the table and looked to the model to write a generalization and justification for the perimeter of a hexagon train. The transcript revealed that as soon as they saw the materials they generalized their rule against the hexagon train.

- Jill: Times four. The number of blocks times four plus two.

Joan: Yep. How long did that take you, thirty seconds?

Jill: Yeah. Four times the number plus two. Four times the number plus two. [Later they explain the rule.]

Jill: Do we have to make a train or can you just think about it?

Researcher 1: Yeah, make a train, otherwise the perimeter won't make sense.

Joan: You have to take a picture of it.

Jill: Well, I can – I already know what it is though.

Researcher 1: You already know what it is?

Jill: By what we said with the triangles and stuff.

Researcher 1: Okay. So can you make a table of values?

Jill: Yeah, oh yeah.

Researcher 1: Okay, but put it down underneath because it's interesting that you saw the formula.

Jill: Mm-hmm.

Jill: Yeah. Because you have four exposed sides

Researcher 1: Mm-hmm.

Jill: ... for each one. So, for each block you have times four.

Researcher 1: Mm-hmm.

Jill: Then two on the end, plus two.

Researcher 1: Okay. Okay.

Joan: Two times four is eight, plus two. Right?

Jill recognized the explicit rule quickly by establishing a connection between the model and the rule to justify the rule while Joan followed in her understanding. Jill noticed the similarities between the task features of the triangle trains and the hexagon trains.

It is apparent that Jill and Joan moved beyond a reliance on the pattern block representation when they were able to discover a rule for the perimeter of an  $n$ -gon pattern block train,  $n(s-2)+2=P$  ( $s$ =number of sides in the polygon); and explained when you connect shapes, you are covering two sides so you multiply the number of blocks' exposed sides, add two end pieces. The reasoning expressed here by Jill and Joan represented a cognitive transition from a reliance on the model to abstract thinking. This happened when the preservice teachers kept the features of the pattern block trains in their minds and no longer relied primarily on a visual, concrete representation of the train. A number of the preservice teachers went through this transformation from concrete to abstract thinking in the third week.

By encouraging preservice teachers to apply their meanings for their symbolic representations, they overcame their initial mistaken strategies and increased their understanding of fundamental algebraic concepts. Preservice teachers can progress in their reasoning if they are provided experiences with similar geometric tasks to encourage making connections to previous knowledge. The importance of combining geometric models in the tasks appeared to be a key to successful justifications.

#### 4. Discussion

Generalization and justification are overarching themes of many studies involving algebraic reasoning (For example, Lannin et al., 2006; Lannin, 2005; Martino & Maher, 1999; Steele & Johanning, 2004; Zazkis & Liljedahl, 2002). Elementary preservice teachers can learn to generalize and justify algebraic rules across a number of related geometric pattern-finding tasks. More specifically, we expect the justification to give meaning to the symbolic rules the preservice teachers develop under the generalization. We propose two conjectures for further study of algebraic reasoning and pattern-finding tasks. The first conjecture is that the design and implementation of the tasks is instrumental in mediating the preservice teachers' learning how to generalize rules and justify those rules for pattern-finding tasks. The second conjecture is that there are observable patterns of growth in reasoning among the preservice teachers that define the critical events of the experiment when engaged in these pattern block train tasks.

**Conjecture 1.** Task design and task implementation were critical to developing algebraic reasoning among preservice teachers. When we provided a first experience with generalizing and justifying rules to the preservice teachers, we selected tasks that enhanced the algebraic understanding of nearly all of the 25 preservice teachers by the end of the 3 weeks. We considered three features of the tasks that we think played a large part in the preservice teachers' success: (1) the linear and geometric nature of the tasks, (2) the use of pattern blocks, and (3) the isomorphism between the three tasks.

**Linear and geometric nature.** The block train tasks promoted the growth of explicit, symbolic generalizations early in the experiment. The linear nature of all the patterns, coupled with the geometric nature of the pattern seemed to contribute to the preservice teachers' success with the task (Lannin, 2005; Steele & Johanning, 2004). This ability to express these explicit generalizations symbolically continued throughout the experiment and took less time to generate each week. We speculated that quadratic and exponential functions would present a problem to the novice in terms of expressing the generalization. Also, we thought it would be more difficult to find and recognize non-linear patterns when represented numerically when compared to patterns that were of a linear nature.

**Pattern block models.** The use of representational tools by preservice teachers to mediate the learning changed significantly over time. In the first week, most focused on numerical data in tables and had difficulty providing a valid justification for their generalizations. Over time, their use of the pattern block model to provide justifications increased. These concrete materials appeared to influence the preservice teachers algebraic thinking more profoundly than the numerical tables of data (Karpov & Haywood, 1998; Simpson, Hoyles, & Noss, 2006). By the third week of the experiment, some preservice teachers moved beyond concrete materials to think abstractly about the hexagon task. Pirie and Kieren (1994) identified this preparatory activity of the growth of mathematical understanding for generalizing as "having an image" (Martin, 2008). This happens when the preservice teachers can keep the features of the pattern block trains in their minds and no longer have to rely on a visual, concrete representation of the train. A number of the preservice teachers went through this transformation from "making an image" to "having an image" by the third week. It is recognized that the use of the concrete materials was initially essential to the preservice teachers learning how to generalize and justify.

**Isomorphic tasks.** The tasks were designed to incorporate features that were isomorphic to one another in terms of repetition across tasks. For example, the two ends of any n-block train each contributed an extra unit to the perimeter when compare with the interior blocks in the train. Hence, the preservice teachers came to attach meaning to the y-intercept [+2] in every rule. There were a number of incidences where the preservice teachers referred to what was done in previous weeks, leading to the conclusion that the similarities between the tasks was a mediating factor in the preservice teachers learning. Simpson et al. (2006) stated that model rules can be similar despite the changes in or configurations in patterns as preservice teachers noticed the similarities between triangle trains and hexagon trains. The unifying ideas of the tasks involved perimeter, inside blocks, outside blocks, and number of n-gon sides. Rather than introducing preservice teachers to algebraic pattern finding by giving three different tasks, we designed the tasks to have similar features and representations.

**Communities of ideas.** The tasks promoted discourse among the small groups of preservice teachers. Ideas flowed among and between the diads. When one group was blocked in their understanding, they turned to a more knowing peer to explain the ideas. The whole class summaries by the preservice teachers promoted questions and alternatives to the ideas presented. In turn, the discourse became another mediator of learning how to generalize and justify patterns using algebraic reasoning (Salomon & Perkins, 1998; Vygotsky, 1986). The tasks presented attainable activities for all of the preservice teachers, and while some had faster and deeper insights, they willingly shared those insights with others who were motivated to bring deeper understanding to the tasks.

These four attributes of the pattern block tasks appear to teach these preservice teachers how to find patterns, generalize rules, and justify those rules. Acknowledging that this was a first experience with this type of algebraic thinking for most of the preservice teachers, we view the teaching experiment as a success. We

suggest that if first pattern-finding tasks are linear, geometric, concrete, and isomorphic they promote classroom discourse and lead to more learning and understanding of foundational ideas in algebra.

**Conjecture 2.** Given that this was a unique mathematical experience for many of these preservice teachers, we noted critical events along the way to provide evidence that the preservice teachers were learning to use patterns to generalize algebraic rules and then learning how to justify those rules. From our observations of the first three tasks we were able to develop a scale specific to the five tasks that indicated different milestones towards learning to reason algebraically (see Table 1).

**Table 1**  
Description of critical events leading to understanding how to generalize and justify explicit rules within the pattern block train tasks.

0	Generalizes a recursive rule with no justification of the coefficient and y-intercept
1	Generalizes an explicit rule with no justification of the coefficient and y-intercept
2	Generalizes an explicit rule with partial or faulty justification of the coefficient and y-intercept
3	Justifies the coefficient and y-intercept and generalizes an explicit rule inconsistently or inefficiently
4	Generalizes an explicit rule and justifies the coefficient and y-intercept

The first critical event (level 1) was generalizing an explicit rule and this event occurred each time the preservice teachers were confronted with a new pattern-finding task. The preservice teachers began with generalizing recursive rules in the first week of the teaching experiment but quickly moved on within that first day to realize the efficiency of explicit generalizations. With two exceptions and without prompting, the preservice teachers expressed their explicit generalizations using algebraic notation. These results were somewhat different from those of Zazkis and Liljedahl (2002) who reported that their elementary preservice teachers had difficulty expressing generalizations of visual numerical patterns using algebraic notation. On the other hand, Steele and Johanning (2004) in their teaching experiment used geometric patterns with 8 seventh graders and noted that some students' explicit generalizations incorporated the use of algebraic notation. Lannin et al. (2006) reported that in their pattern finding teaching experiment with spreadsheets, sixth graders had difficulty moving between recursive and explicit generalizations and incorporating algebraic notation.

The second critical event (level 2) noted was that some of the preservice teachers made faulty or partial justifications for their explicit generalizations. Generally, this type of justification concerned the y-intercept rather than the coefficient of  $n$ . For example in the triangle train, some preservice teachers understood that the two ends of the train explained the y-intercept. A few attributed their last action of adding a triangle as a justification for the y-intercept. Justifications to explain the coefficient of  $n$ , especially with the triangle train task, seemed to be most difficult for some of the preservice teachers.

The persistence of additive reasoning was manifested in the notion that building the triangle block trains should focus on the interior sides that were to be subtracted from the perimeter rather than a focus on the exterior sides that comprised the perimeter (level 3). We did not consider this to be an incorrect approach, although it was not necessarily an efficient way of thinking about the perimeter of the train. As reported, the preservice teacher who symbolized the generalization did it inconsistently even though it resulted in the correct answer. Her symbolic representation of the rule did not match her verbal description but she justified the rule. None of the other preservice teachers who justified this subtractive approach challenged this notation. Inefficiency or inconsistency in generalizations but correctly justifying a pattern was considered a level 3.

By developing this scale that mapped the growth of the preservice teachers' understanding, we determined how ideas evolved among the individual preservice teachers across the three tasks. In addition we assessed the understanding of the class of 25 preservice teachers as they progressed, over time, within each task.

## 5. Implications for teaching

The teacher preparation of elementary teachers includes algebra as a content standard (NCTM, 2000). As is often done in the elementary curriculum, isolating early algebra into a strand of arithmetic is problematic because it promotes little mathematical discourse and ignores the importance of the five NCTM process standards (problem solving, reasoning and proof, communication, connections, and representation). Unfortunately we see third graders taught procedurally to solve equations such as:  $359+x=863$ . Early algebra as

a central component to the elementary curriculum fits well with the process standards and must be an integral part of elementary teacher preparation programs. An important issue to take into consideration, though, is the role of the teacher in early algebra teaching. The two ideas considered here are: (1) task selection and (2) implementing the tasks.

### ***5.1. Task selection***

Selecting appropriate tasks for students is a challenge for teachers introducing early algebra concepts to their classes, and as discussed earlier, it makes sense to select tasks with clear geometric connections (Lannin, 2005). A task that encourages students to base their reasoning on a physical structure or concrete model rather than on numeric thinking can help the students clarify and interpret a problem (Steele & Johanning, 2004). Besides appropriate geometric, linear, and concrete selection of tasks, isomorphic qualities are also of importance. Greer and Harel (1998) speak to isomorphisms as representational acts whereby the student is able to engage in representational thinking at varying levels. While selection of tasks is important, the way in which students are engaged in the task is of equal or even greater importance (Henningesen and Stein, 1997).

### ***5.2. Implementing the task***

Promoting strong mathematical dispositions among students is a goal among all teachers of mathematics. In doing so, a host of factors must be taken into account when implementing problem-solving tasks. An assumption that we make here is that the task is appropriate for the students so that they have ideas about the task that they can share with others. Some of these implementation factors for teachers to consider include knowing when to intervene, when to listen, and when to ask questions. “Teaching as telling” is rightfully criticized as a method not suitable for the development of student autonomy (Kamii, 1989). Interpretations of such critiques are misconstrued and some teachers believe that no intervention whatsoever is necessary on their parts. Chazan and Ball (1995) argue that teachers are an important part of the mathematical discourse in a classroom. They must know when a discussion is leading in a direction that encourages a divisive nature in the environment and when to help those students shift their focus. When students come to an agreement about a problem that leads to an incorrect answer, the teacher must intervene by asking key questions.

Davis (1996) points out the need for attention to the way in which the teacher listens to students. Rather than interpretive or evaluative listening, which tend to take on a focus of seeking only answers, he calls for hermeneutical listening to occur. Hermeneutical listening involves helping students seek relationships among and within tasks. Patterning tasks in particular contain characteristics that lend themselves well to dynamic mathematical discourse in which teachers and students intervene when needed, listen hermeneutically, and ask appropriate questions to orchestrate and extend new directions in the discourse.

## **6. Conclusion**

We view this whole-class teaching experiment as an effort to understand the dynamic interactions, over time, between instructor and researcher, preservice teachers, and, in this case, representations. The experiment provides us with an opportunity to connect our research to our practice as teacher educators through the study of the processes and outcomes of teaching preservice teachers how to generalize and justify rules (Design-Based Research Collective, 2003; Kelly, 2003). This experiment involves a relatively small number of preservice teachers and the results are considered as preliminary, providing a foundation for other studies. Further explorations might include the study of more classes, a focus on different strategies, or to validate the same critical points that emerge with other groups of students. The preservice teachers’ activities within their engagement in the tasks identifies the attributes of the tasks that seem to make a positive difference in the outcomes, especially those involving their representations.

The scale from these activities was based on critical events and provided a way to map the study’s preservice teachers’ development of algebraic reasoning over time by task in a further analysis. These findings showed that all but two of the preservice teachers generalized explicit patterns symbolically in the first week. Justifying these generalizations proved to be a more difficult activity for some of the preservice teachers. A few provided justifications in the first week, more in the second week, and all but two by the third week. The source of

variance among these preservice teachers' in terms of their understanding was attributed to time with tasks (Richardson, Berenson, & Staley, in preparation). This whole-class research was challenging given the multi-dimensional perspectives needed to simultaneously view the interactions and activities of the individual students, those of the class of students, and the teacher. We can say with some certainty that the preservice teachers in our study learned to generalize and justify rules from a series of related tasks. We cannot yet say that they learned how to teach children to generalize and justify.

## References

- Armstrong, B., & Larson, C. N. (1995). Students' use of part-whole and direct comparison strategies for comparing partitioned rectangles. *Journal for Research in Mathematics Education*, 26(1), 2–19.
- Ball, D. L. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.
- Berenson, S., Wilson, P. H., Mojica, G., Lamburtus, A., & Smith, R. (2007). Towards a theory to link mathematical tasks to student understanding. In T. Lamberg, & L. R. Wiest (Eds.), *Proceedings of the 29th annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 205–208). Stateline (Lake Tahoe), NV: University of Nevada, Reno.
- Ball, D. (1990). Breaking with experience in learning to teach mathematics: The role of a preservice methods course. *For the Learning of Mathematics*, 10(2), 10–16.
- Bishop, J. W. (1997). Middle school students' understanding of mathematical patterns and their symbolic representations. Paper presented at the annual meeting of the American Educational Research Association, Chicago, IL.
- Blanton, M. L., & Kaput, J. J. (2005). Characterizing a classroom practice that promotes algebraic reasoning. *Journal for Research in Mathematics Education*, 36(5), 412–446.
- Carraher, D. W., & Schliemann, A. D. (2007). Early algebra and algebraic reasoning. In F. K. Lester (Ed.), *Second handbook of research on mathematics teaching and learning*. Charlotte, NC: National Council of Teachers of Mathematics (Information Age Printing).
- Chazan, D., & Ball, D. (1995). Beyond exhortations not to tell: The teacher's role in discussion intensive mathematics classes. *Craft Paper 95-2*. Michigan State University, MI: National Center for Research on Teacher Learning.
- Cobb, P. (1995). Continuing the conversation: A response to Smith. *Educational Researcher*, 24(7), 25–27.
- Cobb, P. (2000). Conducting teaching experiments in collaboration with teachers. In A. Kelly, & R. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum.
- Davis, B. (1996). *Teaching mathematics: Toward a sound alternative*. New York: Garland.
- Design-Based Research Collective. (2003). Design-based research: An emerging paradigm for educational inquiry. *Educational Researcher*, 32(1), 5–8.
- Goldin, G. (1998). Retrospective: The PME Working Group on representations. *Journal of Mathematical Behavior*, 17(2), 283–301.
- Greer, B., & Harel, G. (1998). The role of isomorphisms in mathematical cognition. *Journal of Mathematical Behavior*, 17, 5–24.
- Henningsen, M., & Stein, M. K. (1997). Mathematical tasks and student cognition: Classroom-based factors that support and inhibit high-level mathematical thinking and reasoning. *Journal for Research in Mathematics Education*, 28(5), 524–549.
- Hiebert, J., Carpenter, T., Fennema, E., Fuson, K., Wearne, D., Murray, H., Olivier, A., & Human, P. (1997). *Making sense: Teaching and learning mathematics with understanding*. Portsmouth, NH: Heinemann.
- Johanning, D. I. (2004). Supporting the development of algebraic thinking in middle school: A closer look at students' informal strategies. *Journal of Mathematical Behavior*, 23, 371–388.
- Kamii, C. (with Joseph, L.). (1989). Autonomy: The goal of education for Piaget. In C. Kamii (Ed.), *Young children continue to reinvent arithmetic (second grade)* (pp. 45–53). New York: Teachers College.
- Kaput, J. J. (1999). Teaching and learning a new algebra. In E. Fennema, & T. Romberg (Eds.), *Mathematics classrooms that promote understanding*. Mahwah, NJ: Lawrence Erlbaum.
- Karpov, Y., & Haywood, H. C. (1998). Two ways to elaborate Vygotsky's concept of mediation. *American Psychologist*, 53(1), 27–36.
- Kelly, A. (2003). Research as design. *Educational Researcher*, 31(1), 3–4.
- Kuchemann, D. (1981). Algebra. In K. Hart (Ed.), *Children's understanding of mathematics* (pp. 11–16).

London: John Murray.

- Lannin, J. (2005). Generalization and justification: The challenge of introducing algebraic reasoning through patterning activities. *Mathematical Thinking and Learning: An International Journal*, 7(3),231–258.
- Lannin, J. K., Barker, D. D., & Townsend, B. E. (2006). Recursive and explicit rules: How can we build student algebraic understanding? *Journal of Mathematical Behavior*, 25,299–317.
- Ma, L. (1999). *Knowing and teaching mathematics: Teachers' understanding of fundamental mathematics in China and the United States*. Mahwah, NJ: Lawrence Erlbaum.
- Maher, C., & Davis, R. (1990). Teachers' learning: Building representations of children's meanings. In R. Davis, C. Maher, & N. Noddings (Eds.), *Constructivist views on the teaching and learning of mathematics*. Reston, VA: NCTM.
- Maher, C., & Martino, A. (1996). The development of the idea of mathematical proof: A five year case study. *Journal for Research in Mathematics Education*, 27(2),194–214.
- Martin, L. C. (2008). Folding back and the dynamical growth of mathematical understanding: Elaborating the Pirie Kieren Theory. *Journal of Mathematical Behavior*, 27(1), 64–85.
- Martino, A., & Maher, C. (1999). Teacher questioning to promote justification and generalization in mathematics. *Journal of Mathematical Behavior*, 18(1), 53–78.
- Mason, J. (2004). Making sense of learning mathematics. Paper presented at the annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education, Delta Chelsea Hotel, Toronto, Ontario, Canada.
- National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics*. Reston, VA: NCTM.
- National Mathematics Advisory Panel. (2008). *Foundations for success: A final report*. Washington, DC: U.S. Office of Education.
- Pirie, S., & Kieren, T. (1994). Growth in mathematical understanding: How can we characterise it and how can we represent it? *Educational Studies in Mathematics*, 26(2–3),165–190.
- Powell, A. B., Francisco, J., & Maher, C. A. (2003). An analytical model for studying the development of learners' mathematical ideas and reasoning using videotape data. *Journal of Mathematical Behavior*, 22(4),405–435 .
- Richardson, Berenson, & Staley. (in preparation). *Developing algebraic reasoning among elementary preservice teachers*.
- Salomon, G., & Perkins, D. (1998). Individual and social aspects of learning. *Review of Research in Education*, 23,1–24.
- Schifter, D. (1998). Learning mathematics for teaching: From teachers' seminar to the classroom. *Journal of Mathematics Teacher Education*, 1(1), 55–87. Simon, M. (1993). Prospective elementary teachers' knowledge of division. *Journal for Research in Mathematics Education*, 24(3),233–254.
- Simon, M. A., & Blume, G. W. (1996). Justification in mathematics classrooms: A study of prospective elementary school students. *Journal of Mathematical Behavior*, 15(1),3–31.
- Simon, M. (2000). Research on the development of mathematics teachers: The teacher development experiment. In A. Kelly, & R. Lesh (Eds.), *Handbook of research design in mathematics and science education*. Mahwah, NJ: Lawrence Erlbaum.
- Simpson, G., Hoyles, C., & Noss, R. (2006). Exploring mathematics through the motion of construction and collaboration. *Journal of Computer Assisted Learning*, 22(2),114–136.
- Smith, E. (2003). Stasis and change: Integrating patterns, functions, and algebra throughout the K-12 curriculum. In J. Kilpatrick, W. G. Martin, & D. Schifter (Eds.), *A research companion to principles and standards for school mathematics*. Reston, VA: National Council of Teachers of Mathematics (change).
- Smith, E. (2008). Representational thinking as a framework for introducing function in the elementary classroom. In J. Kaput, D. Carraher, & M. Blanton (Eds.), *Algebra in the early grades* (pp. 133 –160). New York, NY: Lawrence Erlbaum.
- Steele, D., &Johanning, D. (2004). A schematic-theoretic view of problem solving and development of algebraic thinking. *Educational Studies in Mathematics*, 57(1),65–90.
- Vygotsky, L. S. (1978). *Mind in society: Development of higher psychological processes*. Cambridge, MA: Harvard University.

Vygotsky, L. S. (1986). *Thought and language* (rev. ed.). Cambridge, MA: Harvard University Press.

Zazkis, R., & Liljedahl, P. (2002). Generalization of patterns: The tension between algebraic thinking and algebraic notation. *Educational Studies in Mathematics*, 49(3), 379–402.