The goal of this short note is to observe that the singular part of the second bounded cohomology group of boundedly simple groups constructed in [3] is trivial. Recall that a group $G$ is called m-boundedly simple if every element of $G$ can be represented as a product of at most $m$ conjugates of $g$ or $g^{-1}$ for any $g \in G$.

We recall that bounded cohomology $H^b_2(G)$ of a group $G$ (we will be considering only cohomology with coefficients in the additive group of reals $\mathbb{R}$ with trivial action, so in our notations for cohomology the coefficient module will be omitted) is defined using the complex

$$
\cdots \leftarrow C^{n+1}_b(G) \xleftarrow{\delta^n_b} C^n_b(G) \leftarrow \cdots \leftarrow C^2_b(G) \xleftarrow{\delta^1_b} C^1_b(G) \xrightarrow{\delta^0_b = 0} \mathbb{R} \xrightarrow{\delta^{-1}_b = 0} 0
$$

of bounded cochains $f: G \times \cdots \times G \to \mathbb{R}$, and $\delta^n_b = \delta^n|_{C^n_b(G)}$ is the bounded differential operator. Since $H^b_0(G) = \mathbb{R}$ and $H^b_1(G) = 0$ for any group $G$, investigation of bounded cohomology starts in dimension 2. One observes that $H^b_2(G)$ contains a subspace $H^b_{2,2}(G)$ (called the singular part of the second bounded cohomology group), which has a simple algebraic description in terms of quasicharacters and pseudocharacters, and the quotient space $H^b_2(G)/H^b_{2,2}(G)$ is canonically isomorphic to the bounded part of the ordinary cohomology group $H^2(G)$. See [2] for background and available results on bounded cohomology of groups.

A function $F: G \to \mathbb{R}$ is called a quasicharacter if there exists a constant $C_F \geq 0$ such that

$$
|F(xy) - F(x) - F(y)| \leq C_F \quad \text{for all } x, y \in G.
$$

A function $f: G \to \mathbb{R}$ is called a pseudocharacter if $f$ is a quasicharacter and in addition

$$
f(g^n) = nf(g) \quad \text{for all } g \in G \text{ and } n \in \mathbb{Z}.
$$

We use the following notation: $X(G) =$ the space of additive characters $G \to \mathbb{R}$; $QX(G) =$ the space of quasicharacters; $PX(G) =$ the space of pseudocharacters; $B(G) =$ the space of bounded functions. Then

$$
H^b_{2,2}(G) \cong QX(G)/(X(G) \oplus B(G)) \cong PX(G)/X(G)
$$

as vector spaces (cf. [2, Proposition 3.2 and Theorem 3.5]). Special interest in $H^b_{2,2}$ is motivated in part by its connections with other structural properties of groups such as commutator length [1] and bounded generation [2].

Theorem 1 If $G$ is a boundedly simple group, then $H^b_{2,2}(G) = 0$. 
Proof. In view of (1) it suffices to show that the group \( G \) does not have any nontrivial pseudocharacters. First, we observe that every pseudocharacter is constant on conjugacy classes. Indeed, suppose that \( f \in PX(G) \) and \( |f(gxg^{-1}) - f(x)| = a > 0 \) for some \( x, g \in G \). Then on the one hand
\[
|f(gx^n g^{-1}) - f(x^n)| = |f(gx^n g^{-1}) - f(x^n) - f(g) - f(g^{-1})| \leq 2C_f
\]
is bounded independent of \( n \), on the other hand
\[
|f(gx^n g^{-1}) - f(x^n)| = n|f(gxg^{-1}) - f(x)| = na \to \infty \quad \text{as } n \to \infty,
\]
whence a contradiction.

Suppose that \( G \) is \( m \)-boundedly simple. Then every element \( x \) of \( G \) can be written in the form
\[
x = g_1 \cdots g_k
\]
where \( k \leq m \) and every \( g_i \) is a conjugate of either \( g \) or \( g^{-1} \) for some fixed \( g \in G \), whence \( |f(g_i)| = |f(g)| \) for all \( i = 1, \ldots, k \). Then
\[
|f(x)| = |f(g_1 \cdots g_k) - f(g_1) - \cdots - f(g_k) + f(g_1) + \cdots + f(g_k)| \\
\leq |f(g_1 \cdots g_k) - f(g_1) - \cdots - f(g_k)| + |f(g_1)| + \cdots + |f(g_k)| \\
\leq (m-1)C_f + m|f(g)|
\]
which implies that \( f \) is bounded on \( G \), hence must be trivial. \( \Box \)

References


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