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In an era of new standards and emerging accountability systems, an understanding of the supports needed to aid teachers and students in making necessary transitions in mathematics teaching and learning is critical. Given the established research base demonstrating the importance of justification and reasoning in students' mathematics learning and the heightened emphasis on students' abilities to demonstrate the mathematical practices outlined by the Common Core State Standards Mathematics Practice Standards, this study is timely in that it examined mathematical argumentation as it is currently enacted in today's classrooms.

The study investigated students' mathematical argumentation as it is currently practiced in high school classrooms to understand the ways in which the teacher education and professional development communities may better support teachers in this new era of standards and accountability. Five high school Algebra I teachers and their classes comprised the sample. Using a multiple case study design, data in the forms of classroom observations, teacher interviews, and detailed field notes were collected and analyzed. The within- and cross-case analyses revealed a modest number of episodes of mathematical argumentation with a primary focus on using definitions, properties, and procedures to establish students' claims. Further, teachers fostered mathematical argumentation in their classrooms for a variety of reasons, many of which focused on factors affecting learning not explicitly supporting the learning of new mathematical

ideas. Findings suggest that teachers may view argumentation as a means of assessing students' knowledge rather than as a mechanism of learning.

ON THE NATURE OF AND TEACHERS' GOALS FOR STUDENTS'

MATHEMATICAL ARGUMENTATION IN

HIGH SCHOOL CLASSROOMS

by

Tracey H. Howell

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To my daughter, Lauren

APPROVAL PAGE

This dissertation, written by Tracey H. Howell, has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I

INTRODUCTION

Now is a unique time in the history of American education. Advances in the learning sciences have introduced new, productive ways of understanding learning and its relationship to teaching. Progress in psychometrics is leading to the development of new assessment systems that not only more precisely diagnose understandings but also measure content-specific practices. With the unprecedented adoption of common standards by 45 states and surrounding political momentum advancing educational reform, it is a promising time for those in mathematics education interested in advancing school mathematics beyond facts and procedures to include reasoning and sense-making. With the implementation of these new standards and forthcoming assessments aimed toward incorporating these new visions of learning, an understanding of the current state of teaching and learning in mathematics classrooms is critical for considering the types of supports teachers will require in order to adjust their instruction and enable students to meet the new standards.

Thus, the purpose of this study is to investigate one aspect of teaching and learning in today's mathematics classrooms. Specifically, it examines mathematical argumentation as it occurs in high school mathematics classrooms. In this chapter, I first provide a historical review of standards-based reform and the rise of accountability in public education. Next, I argue that this movement has privileged one particular view of student learning and that forthcoming changes in standardized assessment systems establish a need for an expanded understanding of classroom learning. I then delineate the research problem, the purpose of the study, and its significance.

Background

In 1983, the publication of *A Nation at Risk* by the National Commission on Excellence in Education announced:

> We report to the American people that . . . the educational foundations of our society are presently being eroded by a rising tide of mediocrity that threatens our very future as a Nation and a people. What was unimaginable a generation ago has begun to occur--others are matching and surpassing our educational attainments. (p. 9)

Following this publication, concern for the condition of education in the United States and interest in standards for school accountability and improvement rose significantly (Cohen & Hill, 2001; Misco, 2008; Sanders & Horn, 1998). In 1989, President George H. W. Bush convened an Education Summit comprised of the governors of all fifty states to discuss educational issues and to design a course of action to address those problems identified in *A Nation at Risk*. Six National Education Goals were developed, based on input from the governors, the White House, and leading educators (Sanders & Horn, 1998). Implementation of those goals was left to the discretion of the states themselves. By the early 1990s, most states had enacted standards-based reform, and many had attempted “to put schemes in place that would hold schools accountable for students’ performance” (Cohen & Hill, 2001, p. 1). When President Bill Clinton took office in 1993, his administration’s Goals 2000 legislation continued this path by requiring states to develop their own curriculum standards and be accountable for their students' achievement (Ravitch, 2011).

The No Child Left Behind legislation of 2002 brought about a federal mandate requiring individual states to measure students' academic achievement via standardized tests (Amrein-Beardsley, 2008) and a resulting increased focus on accountability and assessment. Along with a heightened emphasis on increasing students' mathematical performance on standardized tests came attempts to link classroom instruction to specific content standards (Means, 2006). Curriculum decisions regarding what students should learn remained under the control of each state (Ravitch, 2011, p. 30).

The recent 2009 Race to the Top initiative (RttT) required participating states to demonstrate that "they are creating or upgrading standards to ensure that all students regardless of race, ethnicity, English proficiency or disability status are being prepared for colleges and careers" (Mullenholz, 2012, para. 4). To this end, "45 states, the District of Columbia, four territories, and the Department of Defense Education Activity have adopted the Common Core State Standards" (Common Core State Standards Initiative [CCSSI], 2012, para. 1). Two national consortia funded largely by the RttT initiative are currently developing new assessments aligned with the CCSS for implementation in 2014-2015. As the Committee on the Foundations of Assessment (CFA, 2001) noted over a decade ago, educational assessment continues to play a prominent role in the decisions surrounding education.

Advances in Understanding Learning

The central problem addressed by the Committee on the Foundation of Assessment (2001) focused on the issue that “the most widely used assessments of academic achievement are based on highly restrictive beliefs about learning and competence not fully in keeping with current knowledge about human cognition and learning” (p. 2). Similarly, Sawyer (2006b) asserted that schools are currently designed around the assumptions that knowledge is a collection of facts and procedures, and teachers’ should transmit those facts and procedures to students. Concepts are to be taught in order of increasing difficulty, and schooling is considered successful when students are able to demonstrate the facts and procedures they have acquired. Traditional testing, from the classroom level to high-stakes assessments, focuses almost exclusively on multiple-choice questions that measure recognition and recall of superficial, lower-level information (Carver, 2006; Darling-Hammond & Adamson, 2010). Sawyer maintained that this traditional vision of schooling is not adequate to prepare our students to participate in our current society.

The last 30 years chronicle advances in the way “knowledge” is viewed and understood. By the 1980s, researchers focused on learning found that children’s abilities to retain information and to generalize to a wide range of contexts improved when they used knowledge in meaningful, real-world social situations (National Commission on Excellence in Education, 1983; Sawyer, 2006b). In the 1980s and 1990s, scientists studying the nature of science began to understand that students come to learn a discipline by participating in the central practices of that discipline (Sawyer, 2006b).

This view of learning moved beyond a transmission and acquisition view of knowledge (Sawyer, 2006b; Sfard, 1998) to embrace ideas of "deep learning" through authentic practices, integration of new and old concepts, participation in dialogue, and self-reflection on understanding (Sawyer, 2006b). As the learning sciences came to see knowledge as embedded in specific social and cultural contexts, it became increasingly difficult to understand learning in terms of isolated individuals (CFA, 2001; Sawyer, 2006b). Thus,

> Contemporary theories of learning and knowing emphasize the way knowledge is represented, organized, and processed in the mind. Emphasis is also given to social dimensions of learning, including social and participatory practices that support knowing and understanding. This body of knowledge strongly implies that assessment practices need to move beyond a focus on component skills and discrete bits of knowledge to encompass more complex aspects of student achievement. (CFA, 2001, p. 3)

The Evolution of Standards for School Mathematics

It is against this background of changes in national education policy and advances in the learning sciences that standards-based reform evolved in mathematics education. In 1989, the National Council of Teachers of Mathematics (NCTM) released *The Curriculum and Evaluation Standards for School Mathematics*, marking a significant change from previous curriculum standards for school mathematics by including ways of knowing mathematics beyond facts and procedures. These standards envisioned school mathematics beyond mastery of computation to one that centered on mathematical reasoning, problem solving, communication, and connections, and urged that, "computational algorithms, the manipulation of expressions, and paper-and-pencil drill must no longer dominate school mathematics" (NCTM, 1991). In promoting a different vision of what it means to know and do mathematics, *Curriculum and Evaluation Standards* highlighted the need for students to experience mathematics through authentic tasks and discourse.

NCTM's release of *Principles and Standards for School Mathematics* (NCTM, 2000) continued to draw upon the advances in understanding learning by explicitly including five standards that describe the mathematical processes that encompass the practices of mathematicians, what Sawyer (2006b) referred to as the "everyday activities of professionals that work in a discipline" (p. 4). These Process Standards address problem solving, reasoning and proof, communication, representation, and connections. Almost concurrently, the National Research Council (NRC, 2001) released their report *Adding It Up* about school mathematics in an attempt to address the lack of reliable information detailing how children learn mathematics in schools. In their effort to define what it means for a student to learn mathematics successfully, they described five strands of mathematical proficiency: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition and noted that the most important observation to make about these strands is their interdependence. These strands of proficiency continued the movement of reconsidering students' mathematical learning beyond previous interpretations of skill and isolated understandings.

As advances in research on learning and standards initiatives in mathematics education continued to develop, the Common Core State Standards for Mathematics (CCSSM; CCSSI, 2010) emerged as the next generation of educational standards. These standards are comprised of grade-specific content expectations that are guided by eight Standards for Mathematical Practice (see Table 1) both of which represent an extension and refinement of previous standards documents. From this review, it is clear that as the collective knowledge of how students learn mathematics has advanced, so to have the mathematical standards upon which school curricula are currently based. Yet the evolution of mathematics standards has not generated the types of learning to which the standards aspire, largely as a result of the aforementioned emergence of accountability systems based on high-stakes assessments.

Table 1

Common Core Standards for Mathematical Practice

CCSSM Practice Standards
1. Make sense of problems and persevere in solving them.
2. Reason abstractly and quantitatively.
3. Construct viable arguments and critique the reasoning of others.
4. Model with mathematics.
5. Use appropriate tools strategically.
6. Attend to precision.
7. Look for and make use of structure.
8. Look for and express regularity in repeated reasoning.

Note. From www.corestandards.org

In *Knowing What Students Know*, the NRC (2001) noted many assessments intended to "ensure that all students have an opportunity to learn mathematics . . . are not well aligned with the curriculum" (p. 4). Even though consistency among standards, policy, and assessments leads to positive learning outcomes for students (Cohen & Hill, 2001), current assessments largely continue to focus almost exclusively on "the recognition and recall of superficial course content" and are not appropriate for evaluating problem solving and reasoning (Carver, 2006, p. 205). Recent progress in psychometrics, however, promises the ability to evaluate a new age of large-scale assessment systems (Mislevy & Zwick, 2012) and a sizeable body of work detailing the substantial advances made in testing technology supports this claim (Darling-Hammond & Adamson, 2010).

With the CCSSM and the surrounding political will, two consortia are developing assessment systems for the CCSS. The Smarter Balanced Assessment Consortium (SBAC) and the Partnership for Assessment of Readiness for College and Careers (PARCC), largely funded with RttT funds, both purport to draw on psychometric advances in designing assessment systems that measure not only proficiency in terms of content but also a wider range of practices encouraged in the Common Core Standards for Mathematical Practice through open-ended performance tasks (PARCC, 2013; SBAC, 2010). SBAC and PARCC seek to measure not only progress in the acquisition of basic skills and concepts but also processes, habits of mind, and dispositions important to a discipline. In particular in their Theory of Action, the SBAC (2010) notes:

> Assessments produce evidence of student performance on challenging tasks that evaluate the Common Core State Standards (CCSS). Instruction and assessments seek to teach and evaluate knowledge and skills that generalize and can transfer to higher education and multiple work domains. They emphasize deep knowledge of core concepts and ideas within and across the disciplines—along with analysis, synthesis, problem solving, communication, and critical thinking—thereby

> requiring a focus on complex performances as well as on specific concepts, facts, and skills. (p. 1)

Changes in what it means to “know” mathematics, as represented by the CCSSM and measured by new assessment systems, underscore a critical need to understand the current state of mathematics teaching and learning in order to underscore ways of supporting teachers in what Elmore (2002) calls “the reciprocity of accountability for capacity” (p. 5). That is, increased expectations of teachers by the education system must be met with increased supports to assist teachers in meeting those expectations. To improve the educational experience of students, effort and resources must also be invested in supporting teachers in improving their knowledge and skills. As these new assessments are implemented nationally and broader understandings of mathematics learning become a recognized and valued part of our students’ curriculum, it is imperative that the research community better understand the supports teachers will need to be able to modify their instruction to aid students in attaining these new educational goals.

Statement of the Research Problem

In an era of new standards and emerging accountability systems, it is crucial to understand the supports needed to aid teachers and students in making necessary transitions in mathematics teaching and learning. Given the importance of classroom discourse in general, and mathematical argumentation specifically, this study seeks to understand the current state of classroom mathematical argumentation to ascertain the types of supports that might be advantageous to teachers. Given the established research base demonstrating the importance of justification and reasoning in students' mathematics learning and the heightened emphasis on students' abilities to demonstrate the mathematical practices outlined by the CCSSM Practice Standards, this study is timely in that it examines mathematical argumentation as it is currently enacted in today's classrooms. It is imperative to build an understanding of the challenges teachers and students face with these new demands and conjecturing the types of support teachers require to fully implement these initiatives.

Significance of the Study

Research on mathematics learning has established that instruction that advances student learning includes an emphasis on justifications and explanations of students' mathematical reasoning through argumentation (e.g., Dixon, Egendoefer, & Clements, 2009; Hoffman, Breyfogle, & Drexler, 2009; Hollebrands, Conner, & Smith, 2010; Kazemi & Stipek, 2008; Lannin, 2005; Levenson, Tirosh, & Tsamir, 2009; Webel, 2010; Yackel, 2002; Yackel & Cobb, 1996) and strategies that press students for their reasoning (e.g., Breyfogle & Herbel-Eisenmann, 2004; Cazden, 2001; Chapin, O'Connor, & Anderson, 2009; Driscoll, 1999; Franke, Kazemi, & Battey, 2007; Smith & Stein, 2011; Stein, 2001). Mathematical argumentation, as a specific type of discourse, promotes students' articulation of their mathematical thinking and subsequent abilities to justify and explain their reasoning in a manner understandable to their peers. Such discourse communities have been shown to advance student learning (e.g., Andriessen, 2006; Krummheuer, 1995; Yackel & Cobb, 1996), and teachers may support students' participation in mathematical argumentation through the creation of social and sociomathematical norms that encourage students' participation in these mathematical communities. Given the CCSS initiative and accompanying assessments, a more thorough understanding of mathematical argumentation in these classrooms is timely and relevant. However, very few empirical studies of high school mathematics classrooms have been conducted and little is known about the quality of classroom discourse at that level (Walshaw & Anthony, 2008). It is vital to better understand the types of assistance teachers will need to help students' rationalize and articulate their mathematical reasoning and advance their learning.

Statement of Purpose

Thus, the intent of this study is to investigate students' mathematical argumentation as it is currently enacted in high school classrooms to understand the ways in which the teacher education and professional development communities may better support teachers in this new era of standards and accountability. Specifically, this study broadly investigates the nature of mathematical argumentation in these classrooms.

CHAPTER II

REVIEW OF THE LITERATURE AND CONCEPTUAL FRAMEWORK

In this chapter, I develop a conceptual framework to support and inform an investigation of the overarching research problem. I begin with the theoretical foundations underlying my study and next review the literature on classroom discourse with a focus on students' mathematical argumentation. Following this review, I argue the importance of mathematical argumentation in developing students' mathematical understanding and provide an overview of teachers' questioning practices that support students' mathematical argumentation. I conclude the chapter with a summary of the salient points from the literature review and refine the research problem by articulating a set of research questions that guided this study.

Theoretical Perspectives

With origins in the work of Vygotsky in the 1920s and 1930s, sociocultural approaches to learning are "based on the concept that human activities take place in a cultural context, are mediated by language and other simple systems, and can be best understood when investigated in their historical development" (John-Steiner & Mahn, 1996, p. 191). From this viewpoint, no conceptual learning occurs without such social interaction (Sfard, 2003) and further, learning cannot be understood outside of the social context (Miller, 2011). As Lave and Wenger (1991) explained:

> Activities, tasks, functions, and understandings do not exist in isolation; they are part of broader systems of relations in which they have meaning. These systems of relations arise out of and are reproduced and developed within social communities, which are in part systems of relations among persons. The person is defined by as well as defines these relations. (p. 53)

From a sociocultural perspective, “learning by a group or individual involves becoming attuned to constraints and affordances of materials and social systems with which they interact” (Greeno, Collins, & Resnick., 1997, p. 17). Classrooms provide an environment allowing for participation in the social practices of inquiry and learning and “participation is always based on situated negotiation and renegotiation of meaning in the world. This implies that understanding and experience are in constant interaction—indeed, are mutually constitutive” (Lave & Wenger, 1991, pp. 51–52).

In a longitudinal study of approximately 1,000 mathematics students, Boaler (2000) postulated the inseparability of knowing and doing and considered realistic problems, argument, discussion, and exploration central to mathematics learning. Following this tradition, a sociocultural perspective is both appropriate and necessary to investigate mathematical argumentation in classrooms as students learn to justify and explain their mathematical reasoning and to better understand the purposes for which the teachers support students in this endeavor.

Discourse in Mathematics Classrooms

Across the literature on discourse in mathematics classrooms, there are two prominent areas of instructional practice that promote student learning involve requiring students to justify and explain their mathematical reasoning through argumentation (Andriessen, 2006; Arzarello & Sabena, 2011; Dixon et al., 2009; Evens & Houssart, 2004; Herbel-Eisenmann, 2009; Hoffman et al., 2009; Hollebrands et al., 2010; Hoyles & Kuchemann, 2002; Kazemi & Stipek, 2008; Krummheuer, 1995; Lannin, 2005; Levenson et al., 2009; Lopez & Allal, 2007; Stephan & Rasmussen, 2002; Toulmin, 1958/2008; Walter & Barros, 2011; Webel, 2010; Yackel, 2002; Yackel & Cobb, 1996) and strategies that press students for their reasoning (Breyfogle & Herbel-Eisenmann, 2004; Cazden, 2001; Chapin et al., 2009; Driscoll, 1999; Franke et al., 2007; Herbel-Eisenmann, 2009; Herbel-Eisenmann & Breyfogle, 2005; Leonard, 2000; McGuiness, 2005; Smith & Stein, 2011; Stein, 2001). In what follows, I provide an overview of the research on mathematical discourse with a focus on students' mathematical argumentation, the role of mathematical argumentation in support of students' mathematical learning, and teachers' practices that support mathematical argumentation.

Supporting Enhanced Mathematics Classroom Discourse

The literature reviewed focuses on enhancing mathematics discourse in classrooms as a key instructional practice to support mathematics learning (Franke et al., 2007; Hufferd-Ackles, Fuson, & Sherin, 2004; Lampert & Cobb, 2003; Smith & Stein, 2011; Stein, 2001). For example, in their seminal work aimed at interpreting students' experiences in mathematics classrooms, Yackel and Cobb (1996) described their view of mathematics learning as "both a process of active individual construction and a process of acculturation into the mathematical practices of a wider society" (p. 460). To assist students' in fully participating in the discourse of the classroom, the authors noted the importance of constructing norms to help facilitate their discussions. They made a distinction between general classroom norms, such as requiring students to justify their answers, and what they called sociomathematical norms. A sociomathematical norm goes beyond simply justifying an answer and encompasses what the classroom community agrees is “an acceptable mathematical explanation and justification” (p. 461). Similarly, understandings of “what counts as mathematically different, mathematically sophisticated, mathematically efficient, and mathematically elegant” (p. 461) in a classroom are also sociomathematical norms. Later in his reflection on the teaching experiments, Cobb (1999) described classroom discourse as “an act of participating in and contributing to the evolution of communal mathematical practices” (p. 32) and noted that the focus on sociomathematical norms helped him and his colleagues to “understand the process by which teachers can foster the development of intellectual autonomy in their classrooms” (p. 8).

Other empirical studies also discussed the ways a focus on enhancing mathematics discourse affects classroom communities. In their investigation of second-grade students’ mathematical explanations and justifications, Dixon et al. (2009) referred to the learning that occurs through interactions between teachers and students “as developing a community of learners engaged in creation of mathematical knowledge” (p. 1067). They also commented on the distinction between social norms and sociomathematical norms, noting that the social norm of raising one’s hand before speaking had a constraining effect on students’ participation in classroom conversations. Once students were given permission to speak without raising their hands, the quality of social interactions increased as students offered more detailed and varying explanations and justifications.

Similarly, Sherin (2002) noted the importance of negotiating social and sociomathematical norms as teachers attempt to balance eliciting student ideas while addressing and moving specific mathematics content forward. In a study of a middle-school teacher's efforts to enhance discourse in her classroom, she focused specifically on those practices that fostered situations where "students are expected to state and explain their ideas and to respond to the ideas of their classmates" (p. 207). Finally, in a year-long classroom study of an urban elementary classroom, Hufferd-Ackles et al. (2004) described the process by which a *math-talk* learning community developed over time. The authors defined a math-talk learning community to be a classroom community where the teacher and students all worked together to understand and extend their mathematical thinking. They found that as the teacher's lessons progressed throughout the year from traditional lecture formats to lessons implementing reform-based practices such as whole-group discourse, students' abilities to reason, justify, and explain their mathematical thinking grew.

This focus on enhancing discourse in classroom communities is recognized as a means of supporting students' mathematical learning. Further, social norms that create an environment where students are comfortable and willing to share their mathematical thinking are essential. The creation of sociomathematical norms defining what is acceptable mathematically is also a necessity for successful classroom discourse about mathematics.

Justifying Mathematical Reasoning through Argumentation

Concept of Argumentation

For the purposes of this study, I take Krummheuer's (1995) interpretation of Toulmin's (1958/2008) concept of argumentation. For him, argumentation is an interaction that "has to do with the intentional explication of the reasoning of a solution during its development or after it" (Krummheuer, 1995). Argumentation is the process by which an argument as a product is constructed. In much of the literature, scholars make distinctions among mathematical explanation, justification, argumentation, and proof. Similar to other researchers investigating mathematical argumentation (Hollebrands et al., 2010; Singletary, Conner, & Smith, 2013; Yopp, 2013), in this study I take any verbal interaction pertaining to one or more person's reasoning for which supporting information is given as mathematical argumentation.

As students participate in mathematical argumentation, they work beyond simply finding an answer and move toward finding a solution, focusing on the reasoning behind that solution, and articulating their reasoning in a manner understandable to other members of the classroom community. These actions are consistent with definitions of argumentation found in the research literature (e.g., Herbel-Eisenmann, 2009; Kazemi & Stipek, 2008; Krummheuer, 1995; Lannin, 2005; Levenson et al., 2009; Webel, 2010) as well as with the sociocultural view that learning occurs through engagement in practice (Lave & Wenger, 1991). For example, Webel (2010) noted that justification of mathematical solutions centers on the ability to convince others and is more than simply knowing an answer is correct. Similarly, in their study of four upper-elementary mathematics classes, Kazemi and Stipek (2008) noted that mathematical argument consists of more than simple procedural descriptions. In his study of 25 sixth-grade students as they studied patterning activities, Lannin (2005) described an argument as something that does more than simply convince but also explains. Argumentation viewed as a social practice involves “a focus on argumentation as a process that brings about joint thinking in a way that favors (re)construction of participants’ perspectives” (Leitao, 2000, p. 336). Further, argumentation viewed as a sociomathematical norm provides a structure to the expectation that students can and will elaborate on their mathematical thinking by emphasizing what is viewed as acceptable mathematical justification and reasoning.

It is important to stress that student participation in this collaborative practice of arguing to learn (Andriessen, 2006) is neither oppositional nor aggressive. In his chapter linking argumentation to learning, Andriessen (2006) proposed that participants in these discussions are actually working together with the expectation of mutual agreement by the end of the argument. In her article on the potential of argument in knowledge building, Leitao (2000) described argumentation as a unique form of discourse that is successful only when participants:

> . . . coordinate their contributions in such a way that enables them to come up with a set of collectively valid statements (accepted by all the participants in the discussion) that will serve as an answer to the disputed question. The compelling need to cooperate constitutes a developmentally relevant experience that gives people the momentum to seek new forms of understanding a phenomenon. (p. 337)

Further, students exposed to collaborative argumentation learn to think carefully about the topic under discussion and conflicting opinions (Andriessen, 2006) as well as to predict opposing arguments and prepare refutations to them (Jonassen & Kim, 2010). Student responsibility and engagement, justification of answers, and challenges to the responses of others are consistent markers of environments that engender mathematics learning and productive argumentation (e.g., Dixon et al., 2009; Goos, 2004; Kazemi & Stipek, 2008; Kuhn & Udell, 2003; Lopez & Allal, 2007; White, 2003; Whitnack & Knipping, 2002; Yackel & Cobb, 1996).

For example, both Zack and Graves (2001) and White (2003) described longitudinal case studies of elementary classrooms where students concentrated on reasoning and sense-making. The fifth graders in Zack and Graves' study kept daily mathematical logs where they recorded the helpful comments of their peers. Similarly, White found second graders in the two classes participating in her study thought critically and carefully as they analyzed the responses of their peers. In another elementary school study, Whitnack and Knipping (2002) observed students' use of multiple representations, such as numbers and drawings, to model their thinking and help to justify their thinking. They found that "collectively, the students may develop new taken-as-shared ways to communicate their ideas, interpretations, and, perhaps, challenges and counterchallenges" (p. 455).

Research conducted in middle school and high school settings report comparable findings. In their study of a middle school mathematics classroom using the Math Talk Learning Community Framework (Hufferd-Ackles et al., 2004), Hoffman et al. (2009)

observed that students took responsibility and articulately justified their thinking, answers, and strategies with their peers whenever the teacher assumed a more peripheral role in the classroom. Similarly, Sherin, Louis, and Mendez (2000) noted that students in their study were willing to share their mathematical thinking, but also reported that students were reluctant to comment about the ideas of their classmates. In a study of ninth grade Calculus students at a thematic school focused on science, Arzarello and Sabena (2011) used students' explorations of graphical activities to explore their argumentation practices. They found students' reasoning evolved from "the truth because of the data to the truth because of theoretical reasons" (p. 204). Teacher's choice of tasks and participation in episodes of argumentation guided this transformation in student thinking and asserted their belief that the teacher's role in the argumentation is essential.

By constructing learning opportunities that allow students to search beyond simply finding a correct answer to a problem, teachers focus the classroom discourse toward deeper mathematical understanding and higher order forms of cognition (Dixon et al., 2009; Jonassen & Kim, 2010). Students' thinking becomes more explicit and this type of "interaction pattern helps students articulate their own thinking to one another and encourages students to make sense of one another's strategies and reasoning" (Herbel-Eisenmann & Breyfogle, 2005, p. 488).

Analyzing Argumentation

Various frameworks for analyzing episodes of mathematical argumentation exist. Some, such as Lannin (2005), offered frameworks for classifying levels of students' justifications and focused on the mathematical content of students' responses. Others, such as Boaler and Brodie (2004), offered questioning frameworks that classify the intent of teacher questions that prompt students' mathematical argumentation. Franke et al.'s (2009) questioning framework not only classified the intent of initial teacher questions, but also went on to analyze the questions that teachers subsequently ask to prompt elaboration on student responses.

However, though these other models exist, for much of the last century scholars interested in studying argumentation have focused on the structure of arguments (Andriessen, 2006). Toulmin (1958/2008) offered a powerful way to examine arguments independent of content and comparable across students and topics. In his book on the uses of argument, Toulmin (1958/2008) contended that certain basic elements compose all arguments, regardless of discipline. To define his three basic components, Toulmin wrote:

> Let it be supposed that we make an assertion, and commit ourselves thereby to the claim which any assertion necessarily involves. If this claim is challenged, we must be able to establish it—that is, make it good, and show that it was justifiable. (p. 90)

Between the *data* (D), which serve as the foundation of the argument, and the *claim* (C) or conclusion to be verified, *warrants* (W) serve as a bridge to explain reasoning. Toulmin provided the following example as an illustration (see Figure 1). He noted there are different types of warrants providing various levels of support for the connection between the data and the claim. Thus, specifying the data, warrant, and claim of an argument might be insufficient, making it necessary to introduce an explicit reference regarding the force of the warrant connecting the data and the claim. He referred to this feature as a *qualifier* (Q) and defined a *rebuttal* (R) to be those instances when the conditions of the warrant do not apply. Additionally, he provided one final component that he referred to as the *backing* (B) of the warrant. The backing consists of additional facts presented specifically for the purpose of supporting the warrant. Figure 2 provides an example of Toulmin's complete model. To support the claim that "Harry is a British subject," Toulmin offered the data that "Harry was born in Bermuda." The warrant clarified the connection between the claim and data by adding the information that "A man born in Bermuda will generally be a British subject" and was backed by legal statutes. The qualifier and rebuttal introduced the potential reasons that the claim might be false.

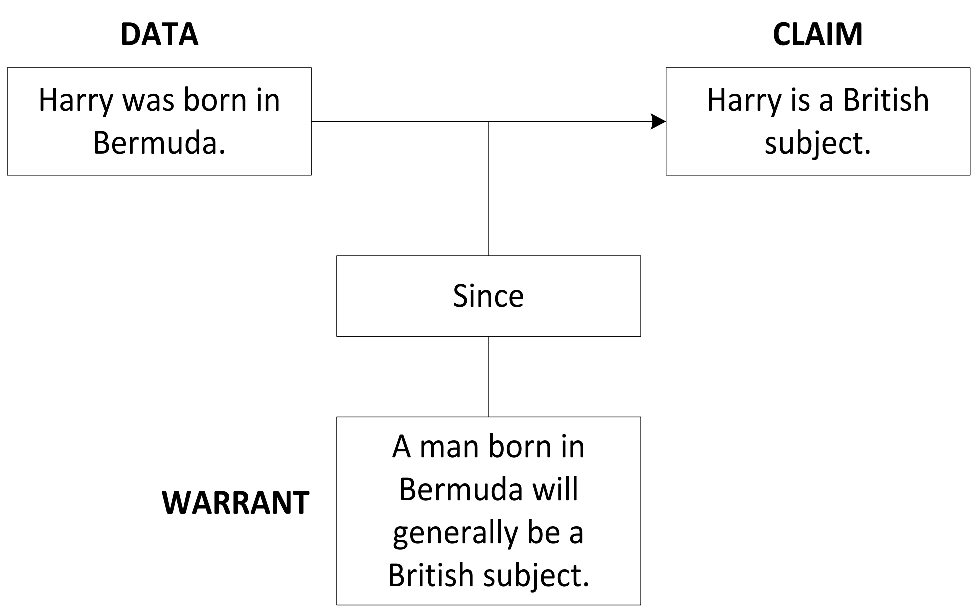


Figure 1. From *The Uses of Argument* (p. 92), by S. Toulmin, 1958/2000, New York, NY: Cambridge University Press.

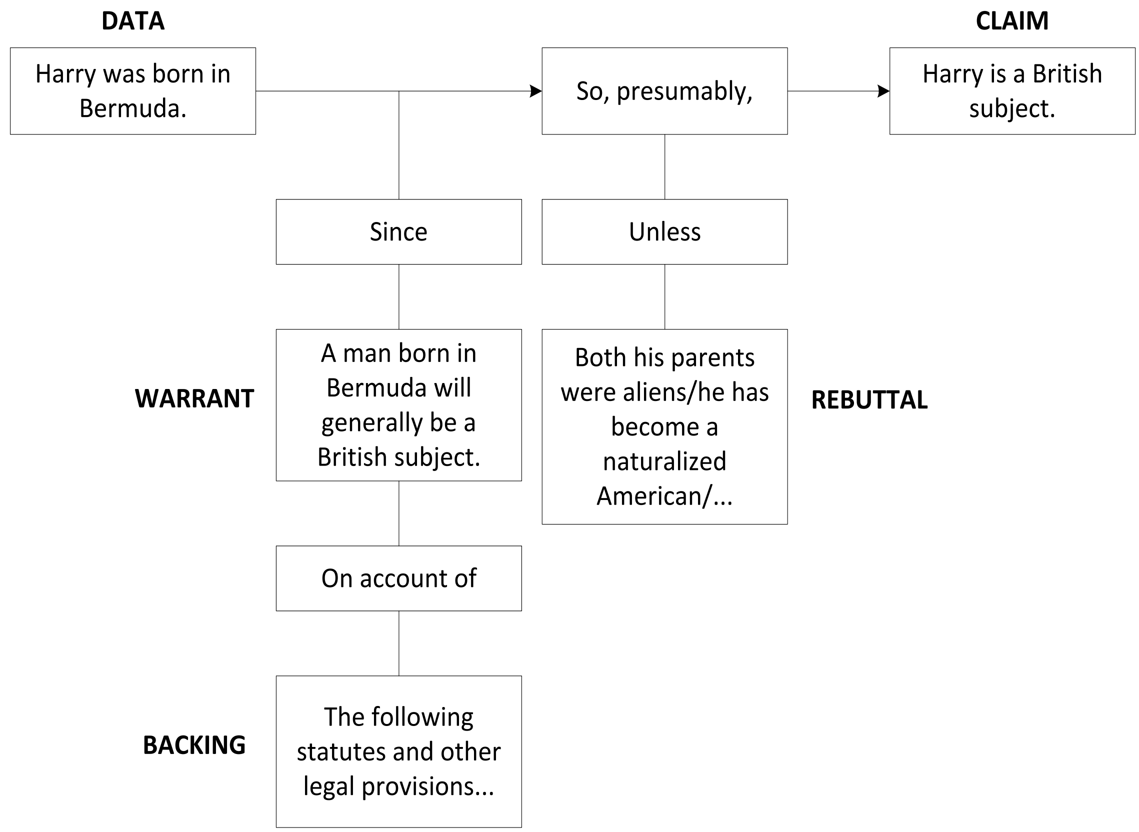


Figure 2. From *The Uses of Argument* (p. 97), by S. Toulmin, 1958/2000, New York, NY: Cambridge University Press.

Scholars studying mathematical argumentation have used Toulmin's model to better understand student arguments. In his book chapter on the ethnography of argumentation, Krummheuer (1995) applied Toulmin's (1958/2008) model to mathematical argumentation. Using data, claims, warrants, and backing, Krummheuer analyzed the mathematical arguments of second-grade students. In Figure 3, he analyzed an argument about the product of four multiplied by four. The number "16" was accepted as the claim based upon the data that there were "8 + 8 = 16 and there are two sets of fours." For the warrant, the student offered, "4 x 4 is like 4 sets of fours" which highlighted his conversion of the multiplication problem into a statement about sets of numbers. A classmate backed the warrant by holding up two fingers on each hand and saying, “Like it’s 2 and 2 make 4.” While making no major changes to Toulmin’s (1958/2008) original model, he drew attention to the fact that what constitutes acceptable backing is constructed by each community of mathematics learners. The examples he provided in his chapter helped to make Toulmin’s structure for arguments more accessible to mathematics education researchers.

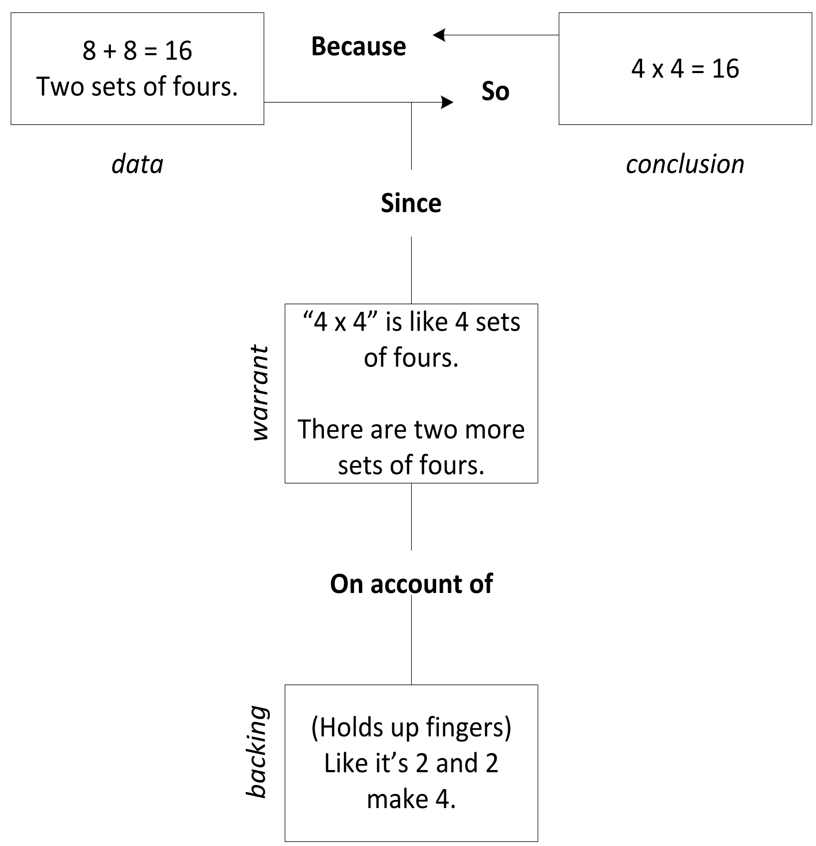


Figure 3. From “The Ethnography of Argumentation,” by G. Krummheuer, in P. Cobb & H. Bauersfeld (Eds.), *The Emergence of Mathematical Meaning: Interaction in Classroom Cultures* (p. 245). Hillsdale, NJ: Lawrence Erlbaum.

Others have used the model to focus specifically on understanding argumentation practices in the classroom. For example, in her study of classrooms from elementary to the university-level, Yackel (2002) investigated the practice of argumentation and how students learned to participate in it. Building from Toulmin (1958/2008) and Krummheuer (1995), Yackel noted:

> The functions that various statements serve in the interactions of the participating individuals are critical to making sense of the argumentation that develops. Thus, what constitutes data, warrants, and backing is not predetermined but is negotiated by the participants as they interact. (p. 424)

By examining cases of argumentation from a given class over time, Yackel observed that this mathematical practice becomes a part of the class's shared repertoire and no longer requires justification. Thus, what the teacher and students require as data, warrants, and backing also evolved.

Other studies using Toulmin's model in a classroom setting include Walter and Barros's (2011) work with university calculus students. They noted the need to analyze the processes through which students offer convincing arguments and commented that detailed analyses of argumentation practices are of "theoretical and practical interest to mathematical educators who are seeking to better understand how students reason to connect prior knowledge with emerging understandings" (p. 324). Similarly, Arzarello and Sabena (2011) used students' explorations of graphical activities in high school calculus, and Hollebrands et al. (2010) drew on the work of college geometry students using technology to solve problems in non-Euclidean geometry to study the ways in which students construct mathematical arguments. Likewise, Stephan and Rasmussen

(2002) chose to study argumentation as it developed in a university differential equations class over the course of a 15-week semester. In each of the studies, the authors noted Toulmin was an effective way to understand argumentation in mathematics classrooms they studied.

These studies provide insight into the process through which students become more skilled at justification and explanation and into the ways that other researchers have investigated mathematical argumentation in classroom settings. Through a focus on the classification of the components of each argument, an analysis of what information is explicitly stated, and by whom, helps to better demonstrate the complexities of mathematical argumentation. This in turn promotes greater understanding of the relationship between argumentation and how students come to understand mathematics.

Argumentation and Learning

In her article proposing the need for analytical procedures to better understand the connection between argumentation and the processes of knowledge building, Leitao (2000) made the following observation:

> It is a matter of quasi-consensus among theorists and researchers that engaging in argumentation sets the scene for the building of new knowledge . . . only rarely however, has the actual process by which transformations in knowledge are accomplished been studied in here-and-now argumentation contexts. (pp. 332–333)

Harel and Sowder (2007) made a similar observation in their *NCTM Handbook* chapter on the learning and teaching of mathematical proofs. They noted that, though such studies were difficult carry out, there is a need for more longitudinal studies to gain a better understanding of the impact of justification and proof on students' mathematical learning throughout their school years. Other researchers also have commented on the need for additional empirical research on the connection between argumentation and learning (e.g., Andriessen, Baker, & Suthers, 2003; Franke et al., 2009; Venville & Dawson, 2010).

Andriessen, Erkens, van de Laak, Peters, and Coirer (2003) presented various ways of understanding the issue of relating argumentation to learning in the book, *Arguing to Learn*. For example, Baker (2003) identified two processes by which argumentation could support new knowledge and understanding. He noted that articulating problem-solving processes during argumentation can enable participants to "elaborate more coherent points of view" (p. 52) and that the process of clarifying, accepting, or rejecting an argument's components can encourage students to reexamine beliefs. In the chapter on argumentation in collaborative writing, Andriessen, Erkens, et al. (2003) contended that argumentation can lead to one or more of the following learning goals:

> (1) shared knowledge, where argumentation leads to both participants better understanding each other; (2) knowledge constitution, where argumentation leads to deeper understanding of a concept; and (3) knowledge transformation, where argumentation leads to a different belief or idea. (p. 83)

They qualified these claims by noting that the relationship between argumentation and learning is indirect. In the concluding chapter of the book, Koschmann (2003) proposed that the difficulty researchers had with demonstrating a clear link between argumentation and learning was due in part to the various ways in which learning is understood. He identified different theories of learning presented in the various chapters of the book and contended that the differing ways in which the researchers understood learning confounded the ability of the field to demonstrate unified results.

Sawyer (2006b) noted that in the 1990s, learning scientists reached a general consensus regarding certain basic facts about learning. Published by the United States National Research Council (see Bransford, Brown, & Cocking, 2000), this consensus stressed “the importance of deeper conceptual understanding . . ., focusing on learning in addition to teaching . . ., creating learning environments . . ., the importance of building on a learner’s prior knowledge . . ., [and] the importance of reflection” (Sawyer, 2006b, p. 2). Andriessen (2006) built upon these ideas, his earlier work, and the work of Baker (2004) to tie argumentation to four specific learning mechanisms: making knowledge explicit, promoting conceptual change, co-elaboration of new knowledge, and increasing articulation.

First, Andriessen (2006) proposed that students who participate in providing explanations and justifying their reasoning benefit by making their knowledge explicit. He noted that “learners who provide explanations, or make explicit the reasoning underlying their problem solving behavior, show the most learning benefits” (p. 445). Thus, the act of participating in mathematical argumentation, which necessitates the statement of students’ reasoning in support of their claims, leads to deeper mathematical understanding.

Next, he noted that deliberating a particular claim may raise doubt related to misconceptions and that supports a transformation of concepts (Andriessen, 2006).

Others also expressed a view of argumentation as a mechanism for supporting and accommodating new ways of thinking. Sawyer (2006a) elaborated on the idea of conceptual change occurring during discourse practices such as argumentation and contended that it is within conversation that knowledge of the group is taken up by the student as individual learning. In their article on arguing to learn, Jonassen and Kim (2010) contended:

> Meaningful leaning requires deep engagement with ideas. Deep engagement is supported by the critical thinking skill of argumentation. Learning to argue represents an important way of thinking that facilitates conceptual change and is essential for problem-solving. (p. 439)

Others have similarly demonstrated the role of argumentation in supporting conceptual change in students' thinking and reasoning (Asterhan & Schwarz, 2007; Nussbaum & Sinatra, 2003; Wiley & Voss, 1999).

Third, Andriessen (2006) also noted the "interactive, interpersonal nature of verbal interaction helps to scaffold individual learning" as students collaboratively build new knowledge through participating in argumentation (p. 445). Over 20 years of educational research has consistently shown that peer collaboration helps students learn in a wide range of subject areas (Sawyer, 2006a). In his chapter on analyzing collaborative discourse, Sawyer (2006a) observed that many researchers, including Forman (1992) and Palincsar (1998), "working within a Vygotskian or sociocultural framework have emphasized how participants build on each other's ideas to jointly construct a new understanding that none of the participants had prior to the encounter" (Sawyer, 2006a, p. 191).

Finally, the obligation to precisely convey thoughts and questions during argumentation strengthens students' abilities to articulate their reasoning (Andriessen, 2006). Sawyer (2006b) wrote:

> Articulating and learning go hand in hand, in a mutually reinforcing feedback loop. In many cases, learners don't actually learn something until they start to articulate it—in other words, while thinking out loud, they learn more rapidly and deeply that studying quietly. (p. 12)

Educational research has shown that when students articulate their knowledge, even, and perhaps especially, while it is still developing, they learn more effectively (Sawyer, 2006b).

In summary, research on learning has established that specific mechanisms support student learning, such as the need to make knowledge explicit, the need to promote conceptual change, the idea that co-elaboration creates new knowledge, and the value of articulating knowledge. Further, though research is not abundant, evidence exists that mathematical argumentation supports each of these learning mechanisms. Further empirical research is needed to support these findings and deepen our understanding of the ways in which mathematical argumentation promotes student learning.

Teachers' Roles in Supporting Students' Participation in Mathematical Argumentation

Teachers play a critical role in establishing and supporting the classroom norms that allow students to move beyond rote answers and to provide more details about their thinking. Teachers and students must both internalize their roles in participating in a community of discourse (Hufferd-Ackles et al., 2004) and the ways in which the teacher encourages and supports students throughout this process is of vital importance. As students first learn to elaborate their answers, they tend to focus more on procedures of the talk rather than reasoning (Breyfogle & Herbel-Eisenmann, 2004). Once norms for discourse are established, fluctuations in the type of talk still occur. For example, in their year-long case study of a third-grade teacher and her students, Hufferd-Ackles et al. (2004) noted that during periods of adjustment, such as those that occur when new mathematical topics were introduced, the teacher "functioned in a more central position and was responsible for more of the discourse" (p. 111). However, once students grasped new vocabulary and representations, they resumed their more significant participation in the mathematical discourse of the classroom. The teacher supported the students through times of transition and thus facilitated the students' attempts to participate in discourse while learning new mathematics content.

Teachers support students as they engage in the practice of mathematical argumentation by establishing social norms delineating the acceptable ways in which it is to be enacted in the classroom. For example, in their overview of the research literature on the teacher's role in classroom discourse, Franke et al. (2007) observed the importance of the teacher's role in structuring and supporting discourse and remarked on the insight mathematical conversations can provide to teachers regarding students' thinking. Based on their synthesis of the literature, they noted teacher practices such as revoicing, the use of specific situations to engage students in whole-class discussions, and press for justifications and explanations all support mathematical conversations. Similarly, both

Smith and Stein (2011) and Chapin et al. (2009) encouraged the use of revoicing as a means of supporting classroom discourse. Noting "deep thinking and powerful reasoning do not always correlate with clear verbal expression" (Chapin et al., 2009, p. 13), the authors presented the technique of revoicing as a way of acknowledging a student's contribution to the discussion while also permitting the teacher to reword the response if necessary.

Finally, teachers support students' participation in mathematical argumentation by making the classroom community a safe environment where students embrace taking mathematical risks (Herbel-Eisenmann, 2009; Stipek et al., 1998). They make a conscious effort to learn how to support their students as they engage with and discuss their solutions to cognitively challenging tasks (Smith & Stein, 2011). By establishing clear expectations of each student's participation along with an understanding of mutual respect, "the teacher plays a central role in establishing the mathematical quality of the classroom environment and in establishing norms for mathematical aspects of students' activity" (Yackel & Cobb, 1996, p. 475).

Teachers' Use of Questions

Historically, the most dominant type of questioning pattern in mathematics classrooms has been the Initiate-Respond-Evaluate (IRE) pattern where the teacher *initiates* a question, a student *responds*, and the teacher *evaluates* that response (Chapin et al., 2009; Franke et al., 2007; Herbel-Eisenmann, 2009; Leonard, 2000; Smith & Stein, 2011; Stein, 2001). Franke et al. (2007) noted that this pattern is a well-documented occurrence in United States classrooms, even in classrooms where teachers are attempting to teach for mathematical understanding. Implementing various questioning formats in the classroom is a necessary and challenging task and alternatives to the IRE-structure are beginning to emerge in mathematics classrooms. As Breyfogle and Herbel-Eisenmann (2004) noted, “changing the questions that [teachers] ask and the ways in which [teachers] facilitate mathematical discussions takes time and conscious effort” (p. 246).

As they attempt to engage students in the discourse practices of the classroom, teachers use questions for a variety of purposes. Questions can assess student learning and also promote learning. They may elicit evidence of student thinking and can support students in making connections between topics. In her book on classroom discourse, Cazden (2001) noted simply that teachers ask many questions. She went on to describe that the best possible types of questions are those that promote student learning while also assessing student understanding. She saw an important distinction between “helping a child somehow get a particular answer and helping that child gain some conceptual understanding from which answers to similar questions can be constructed at a future time” (p. 93). Similarly, in their recommendations for effective lesson planning to promote more productive classroom discussions, Chapin et al. (2009) recommended teachers use questions to engage students in talking about mathematics and to prompt discussion about areas of potential confusion. They pointed out that thoughtful questions require students to analyze their strategies, connect and generalize ideas and relationships, and build on prior knowledge. Likewise, as a component of their book, *5 Practices for Orchestrating Productive Mathematics Discussions*, Smith and Stein (2011) commented that good questions can direct students' attention to aspects of a problem that have been previously unexplored as well as encourage students to articulate their thinking in a manner that is understandable to their peers. They noted expressing mathematical reasoning verbally often supports deeper mathematical understanding.

Other authors focus on the use of teacher questions to better understand student thinking. Following her analysis of the learning outcomes for students at twelve UK schools participating in school-based enrichment projects, McGuiness (2005) concluded that teachers support student learning by going "beyond accepting right or wrong answers and examine processes by posing questions prefaced by words like *explain* and *show*. It is not just a case of 'show me what you know' but rather 'show me how you know it'" (p. 44). Likewise, Herbel-Eisenmann (2009) noted in her book on classroom discourse that questions that focus, as opposed to questions that funnel, help the teacher to better understand the students' thinking. Funneling occurs when a teacher's questions lead students straight through a procedure or directly to a desired outcome (Herbel-Eisenmann & Breyfogle, 2005). Focusing questions require the teacher to put aside how he or she would solve the problem and instead concentrate on the students' responses and guide the discussion based on what is articulated (Herbel-Eisenmann & Breyfogle, 2005).

Some scholars focus on the use of questions to help students make connections across mathematical topics. In his book on fostering algebraic thinking, Driscoll (1999) emphasized the importance of considering both the intention and the context of questions when promoting algebraic habits of mind in students. He advised teachers to "be aware of the variety and breadth of intention behind classroom questions and to seek, over time, patterns of questioning that are balanced across the range of intention" (p. 4). Driscoll also stressed the importance of asking questions that support students' algebraic thinking both "in situations that are patently 'algebraic,' as well as in situations in which the relevance of algebraic thinking [is not] as obvious" (p. 4). He noted that such questions also promote articulation and justification of generalizations students develop.

In summary, though IRE has historically been the most prominent type of questioning pattern in mathematics classrooms, other patterns are emerging in contemporary mathematics classrooms. The literature shows that teachers ask questions for a variety of reasons, such as eliciting evidence of students' reasoning and promoting articulation of students' mathematical thinking. Teachers' use of questions to encourage students to justify and explain their mathematical reasoning is of specific relevance to this study due to the prominent role questions play in prompting mathematical argumentation.

Key Points from the Literature Review

The review of the literature focused on the practice of mathematical argumentation, its role in student learning, and the ways in which and reasons why teachers support argumentation. Mathematical argumentation is a specific type of discourse promoting students' articulation of their mathematical thinking and their abilities to justify and explain their reasoning in a manner understandable to their peers. Toulmin's (1958/2008) model of arguments describes certain basic components typical to all arguments and provides a means to better understand the manner in which students participate in mathematical argumentation in classrooms. Argumentation has been tied to four specific mechanisms that have been shown to support student learning. Specifically, argumentation supports making knowledge explicit, promotes conceptual change, supports co-elaboration of new knowledge, and increases articulation. Teachers support students' participation in mathematical argumentation through the creation of social and sociomathematical norms that promote discourse and through the types of questions they ask.

Yet this review also reveals very few empirical studies from high school mathematics classrooms. From Yackel and Cobb's (1996) work in a second grade classroom to Kazemi and Stipek's (2008) study in fourth- and fifth-grade classrooms, studies of mathematical discourse in elementary classrooms are well represented. Likewise, studies conducted in college and university mathematics classrooms are available, though not abundant. However, the only high school mathematics study analyzed for this review was conducted in a calculus classroom. A search of EBSCO Premier Databases did not return other peer-reviewed articles of studies of classroom discourse of any type in high school mathematics classrooms. To corroborate this observation, I compared my findings to another literature review on classroom discourse compiled by Walshaw and Anthony (2008). They discussed only two studies conducted in high school settings and noted significant gaps in the knowledge base. They noted, "to date, we do not know as much about the quality of classroom discourse at the high school (secondary) level as we do about the elementary (primary) level" (p. 542). Andriessen (2006) concurred, noting specifically the small number of research studies investigating argumentation. Given the paucity of studies conducted in high school mathematics classrooms to investigate students' mathematical argumentation, empirical investigations are needed to support claims that mathematical argumentation positively affects student learning in mathematics at the high school level.

Research Questions

The research literature highlights various practices that promote student learning in mathematics. Of specific interest to this study, mathematical argumentation has been linked to a deeper, more meaningful understanding of mathematics. Yet what largely remains to be shown are the ways in which this practice manifests in high school mathematics classrooms. Specifically, how it is enacted and for what reasons it is supported is of interest in order to better understand the relationship between teaching and learning. Thus, I conclude this chapter with a refinement of my broad research problem into the following two research questions:

1. What is the nature of mathematical argumentation in high school mathematics classrooms?
2. For what goals do teachers foster mathematical argumentation in high school mathematics classrooms?

CHAPTER III

METHODOLOGY

In this chapter, I discuss the theoretical foundations underlying my study and the choice of a qualitative research design to address my research problem. Then, I describe the sampling procedures with descriptions of the participating teachers and the schools in which they teach. Next, I provide details of the data collection procedures for classroom observations and teacher interviews. I describe the methods of analysis, followed by a discussion of both validity and ethical issues.

Paradigmatic Perspectives

I chose to approach this issue through an interpretivistic lens. As described by Schram (2006), I actively engaged in the interpretation process as I observed these teachers' classrooms and collected data in the forms of field notes, interviews, and video recordings. Central to this approach was the idea that "what people know and believe to be true of the world is constructed—or is made up—as people interact with one another over time in specified social settings" (LeCompte & Schensul, 1999, p. 48) and that I was "interested not only in the physical events and behaviors that are taking place, but also in how the participants in [my] study make sense of these, and how their understanding influences their behavior" (Maxwell, 2005, p. 22). By engaging with my participants through in-depth observations and interviews, I focused and refined my interpretations. I acknowledge that what I chose to notice, what I chose to ignore, and what I failed to see all impacted my perception of what happened in these classrooms and my interpretations of their meaning (Denzin & Lincoln, 2005; Schram, 2006). Given the complex cultural, political, and social environments in which these teachers work, an interpretivistic paradigm was an appropriate lens through which to present a “multiply-voiced” story of their classrooms (LeCompte & Schensul, 1999).

Research Study Design

To examine the nature of mathematical argumentation emerging in mathematics classrooms, this study followed a case study research design (Gerring, 2007; Merriam, 1998; Miles & Huberman, 1994; Stake, 1995; Yin, 2003). A case can be described as a phenomenon of interest occurring within a bounded context (Gerring, 2007; Merriam, 1998; Miles & Huberman, 1994; Stake, 1995; Yin, 2003) and serves as the unit of analysis for inquiry. A case study research design may be composed of one or several cases (Gerring, 2007; Merriam, 1998; Miles & Huberman, 1994; Stake, 1995; Yin, 2003). Multiple-case studies, defined as those studies incorporating two or more cases, often yield a more compelling and robust interpretation of findings.

Thus, this study used a multiple-case study design to maximize the likelihood of observing and understanding mathematical argumentation and the purposes for which it is fostered in these mathematics classrooms (Stake, 1995). It explored in depth both the nature of and the teachers’ supports for episodes of mathematical argumentation in an attempt to gain greater understanding of their purpose and meaning to those involved (Merriam, 1998). The cases were “bounded by time and activity” (Creswell, 2003, p.

15); that is, they were “bounded” in the sense that they are focused on the teachers of these Algebra I classes during my weeks of observation.

Case study research designs are characterized as particularistic, descriptive, and heuristic (Merriam, 1998). *Particularistic* means that “case studies focus on a particular situation, event, program, or phenomenon” (Merriam, 1998, p. 29). The study was particularistic in that it focused specifically on the phenomena of episodes of mathematical argumentation in classrooms. Merriam (1998) contended that the value of the case in particularistic studies resides in what it can reveal about the phenomenon of interest. As Stake (1995) explained, “We take a particular case and come to know it well, not primarily as to how it is different from others but what it is, what it does” (p. 8).

The study was *descriptive* in that it presented a thorough description of the phenomenon of interest. Specifically, the study gave a complete depiction of teacher supports for mathematical argumentation will illustrating the complexity of these classroom settings. The end product of the study is “thick description” (Merriam, 1998) of the nature of mathematical argumentation and the purposes for which it is foster in these classrooms that was used to understand and explain my research questions.

Finally, the study was *heuristic* in that it will “illuminate the reader’s understanding of the phenomena under study” (Merriam, 1998, p. 30). Moustakas (2001) described the focus of heuristic studies as the “recreation of the lived experience, that is, full and complete depictions of the experience from the frame of reference of the experiencing person” (p. 264). To that end, the study confirmed and extended the knowledge base from the literature regarding mathematical argumentation.

Sampling Procedures

In 2006, a cohort of mathematics educators, community activists, and school system personnel initiated a project that partnered a state university system and a school district to recruit and retain qualified mathematics teachers in the ten traditionally lowest performing high schools on state-mandated standardized testing in a school district in a southeastern state. By stabilizing the rate of teacher attrition and supplementing teachers' knowledge both in terms of mathematics content and pedagogy, the project intended to increase student mathematical learning and thus raise student test scores. The main components of the project over its first five years were differential teacher pay incentives, year-long professional development seminars, intensive professional development opportunities following a summer institute model, mentoring, and classroom technology purchases. The primary focus of the project was on raising students' scores on state achievement tests. The project awarded performance bonuses of up to $4,000 a year to project teachers based upon their value-added scores as determined by the EVAAS model (Sanders & Wright, 2008).

This study extended the research of the project and therefore was situated within a subset of the ten project schools. The school district is one of the largest school systems in the state and one of the fifty largest districts in the country. With a diverse student population spread out over urban, suburban, and rural areas, the school system has more than 70,000 students. Sixty percent of the student population in the district is non-white, and approximately 55% of the students receive free or reduced-price lunches. Eight of the ten project schools are classified as urban, with the remaining two considered suburban.

Of the 80 teachers participating in the project during its fifth year, 38 of them had been a part of the project for all five years. A small subset of seven teachers had been the most successful at supporting student learning as measured by end-of-course tests and had received the $4,000 performance bonus each year for high value-added scores. Of those seven teachers, two retired at the end of the fifth school year, and five teachers remained for possible selection in this study during the sixth year of the project. These five participants included one African-American male, two African-American females, and two white females, teaching in three different project schools across the district: Mountainside, Lakeside, and Hillside (pseudonyms). Denae and Kendra both taught at Lakeside while Abby and Leslie taught at Mountainside and Hillside, respectively. Will transferred from Hillside to Lakeside during the sixth year of the project. I emailed these teachers at the beginning of the school year in August 2011 to request their permission to visit their Algebra I classrooms for a week of data collection. I asked specifically to observe Algebra I classes since Algebra I was the only high school mathematics course tested with a statewide standardized assessment in 2011-2012. All five teachers agreed to participate in the study. I visited the classrooms of the five teachers during the 2011-2012 school year to collect data prior to the ending of the larger project and prior to the implementation of the Common Core State Standards.

The three high schools at which the five teachers taught ranged in size from approximately 1,000 total students to 1,400 total students for the 2011-2012 school year.

They reported school attendance rates between 93-95% and graduation rates between 77-85%. For that school year, Hillside, the smallest of the three schools, had approximately 200 students enrolled in Algebra I and an average class size of 21 students. In comparison, Mountainside and Riverside had Algebra I enrollments of approximately 300 students each and average class sizes of 18 and 13, respectively. Hillside reported 77% student proficiency on the state end-of-course test while Mountainside and Lakeside both reported proficiency levels between 53-59% (see Table 2). Over 70% of the students enrolled in Algebra I at each of the schools were classified as economically disadvantaged. Additional information by demographic subgroup is presented in Table 2. (www.ncpublicschools.org/accountability/reporting/leaperformancearchive/, www.ncreportcards.org).

This purposeful sampling assisted me as I attempted to understand the nature and purpose of mathematical argumentation in these classrooms. By selecting study participants from a sample of only those teachers most successful under the current accountability system, I increased the likelihood of observing classrooms that support students' learning in our "traditional vision of schooling" (Sawyer, 2006b, p. 2). As Stake (1995) noted, while variety of participants can be useful in maximizing a researcher's opportunity to learn. By including multiple cases in the study, I also aimed to maximize the variability among these classroom communities while ensuring that the practices under investigation were present. This strategy of deliberately selecting from these participants provided information that could not be as easily obtained from other project teachers (Maxwell, 2005; Merriam, 2002).

Table 2

Algebra I Enrollment and Proficiency by Demographic Subgroups

School subgroups	Number of students enrolled in Algebra I	Percentage of total number of students enrolled in Algebra I	Number proficient	Percent proficient
Hillside: Overall	201	100%	154	76.6%
Hillside: Black	93	46%	69	74.2%
Hillside: Hispanic	28	14%	20	71.4%
Hillside: White	59	29%	47	79.7%
Hillside: Economically Disadvantaged	145	72%	107	73.8%
Hillside: Limited English Proficiency	19	9%	15	78.9%
Mountainside: Overall	332	100%	178	53.6%
Mountainside: Black	177	53%	78	44.1%
Mountainside: Hispanic	50	15%	27	54.0%
Mountainside: White	74	22%	53	71.6%
Mountainside: Economically Disadvantaged	266	80%	133	50.0%
Mountainside: Limited English Proficiency	45	14%	24	53.3%
Riverside: Overall	308	100%	183	59.4%
Riverside: Black	197	64%	109	55.3%
Riverside: Hispanic	60	19%	42	70.0%
Riverside: White	18	6%	8	44.4%
Riverside: Economically Disadvantaged	280	91%	169	60.4%
Riverside: Limited English Proficiency	55	18%	34	61.8%

Methods of Data Collection

Following qualitative research tradition, data for this dissertation study consisted of classroom observations and a participant interview (see Appendix A). During the 2011-2012 school year, I conducted classroom observations with each of the five participant teachers, observing the same Algebra I class of each teacher for five consecutive ninety-minute sessions, totaling 7.5 hours of observations per teacher (Howell, 2012). As a non-participant observer (Creswell, 2003), I took field notes and video and audio recordings of the activities that occurred in the classroom and gained firsthand knowledge of what was occurring. As noted by Maxwell (2005), observation "provides a direct and powerful way of learning about people's behavior and the context in which this occurs" (p. 94). Though the focus of the field notes was mainly on documenting instances of mathematical argumentation, I also remained open to other practices demonstrated to support student learning in the research literature (see Appendix B).

After the observations, I conducted an interview with each teacher to uncover historical information (Creswell, 2003) regarding their familiarity with and use of various strategies for supporting more productive mathematical discourse in their classrooms. I was also interested in their beliefs about teaching and the reasons behind their choices of various activities (see Appendix C). The interviews were conducted at each teacher's convenience and took approximately 60 minutes to complete. As Maxwell (2005) noted, "interviewing is often an efficient and valid way of understanding someone's perspective" (p. 94). Following the interview questions on mathematical argumentation, I used segments of the observation recordings to ask stimulated recall questions (Bloom, 1954) regarding the teacher's practice in support of mathematical argumentation. Together, the interview and classroom observations helped provide the rich rigor called for by Tracy (2010). It is important to note that all information reported in this dissertation study comes from this study and not the larger project. I attempted to bracket my previous experiences with the teachers to "take a fresh perspective toward the phenomenon under examination" (Creswell, 2003). Any characteristics or comments attributed to the teachers came either from my observations of their classroom or from the post-observation interview.

Methods Data Analysis

The data analysis for this study consisted of both within-case and cross-case analyses (Merriam, 1998; Miles & Huberman, 1994). Beginning with a within-case analysis of each teacher's observations, I completed detailed descriptions and an analysis of the mathematical argumentation occurring in each classroom. I then conducted a cross-case analysis in an attempt to understand patterns that transcend the five cases.

Within-Case Analysis

The purpose of within-case analysis is to learn as much as possible about each individual case that comprises a multiple-case study. Merriam (1998) described within-case analysis as the initial stage of analysis in multiple-case studies where "each case is first treated as a comprehensive case in and of itself" (p. 194). To identify and analyze the instances of argumentation that occurred in these classrooms, I used my field notes and video recordings of the classes. I began with a careful reading of my field notes, looking for periods of class time where any discussions about mathematics occurred between the teacher and students and making note of the time when these discussions took place. For this initial review of the field notes, my conception of a "mathematical discussion" was broad and included all episodes of talk between teacher and students around specific problems or topics. This also allowed me to identify those blocks of time that focused on activities other than discussion. Once these periods of discussion were identified, I began my analysis of the video recordings which took place in several passes.

On my first pass through the classroom videos, my intent was to identify potential episodes of argumentation. Toulmin (1958/2008) described all arguments as beginning with a claim or assertion that one seeks to establish and data or facts which support the claim. Further, he asserted that, "a bare conclusion, without any data produced in its support, is no argument" (p. 98). Thus, I noted all instances where a student made a claim and discussion about that claim, the mathematical topic, or problem continued beyond the statement of the claim. This yielded 53 potential episodes of argumentation (see Table 3).

The second pass of analysis involved transcribing the potential episodes of argumentation and including as many context details as available from the video recordings and my field notes. During this period of analysis, I removed 19 of the episodes. Reasons for removal included the discovery that the discussions were not mathematically focused and that claims were present without data. At the end of this second analysis, 34 potential episodes of argumentation remained (see Table 3).

During the third pass of analysis, I created models for each episode of argumentation, clearly identifying claims, data, warrants, and, when present, backing. Though most episodes lacking data in support of the claim had previously been removed, this analysis yielded the removal of 15 additional episodes for other reasons, such as not being mathematical in nature, leaving a total of 19 episodes of argumentation (see Table 3).

Table 3

Episodes of Argumentation

Teacher	Abby	Denae	Kendra	Leslie	Will	Total
First Pass	5	11	12	6	19	53
Second Pass	5	7	10	6	6	34
Final Count	2	3	6	3	5	19

In all episodes, claims and data were explicit and attributable to specific individuals. Color coding on the models identifies to whom each component is credited. Warrants, however, were frequently implicit and had to be inferred. In these cases, the warrants appear on the models in dashed boxes and are shaded to match the individual to whom I attributed the warrant. Only one instance of implicit backing occurred in the 19 episodes, and there were no examples of qualifiers or rebuttals. In addition to identifying the argument components, I also highlighted each teacher question prompting the students to supply a claim, datum, or warrant. To establish inter-rater reliability, I trained three other mathematics education researchers on my operationalized definitions of data, claims, warrants, and backing and had each of them code the instances of classroom argumentation. Inter-rater reliability between me and each one of the others was 80%. A fourth mathematics education researcher assisted with the resolution of discrepancies.

A specific example of my modification of Toulmin's model follows (see Figure 4). This example was chosen for its representation of multiple aspects of my interpretation and use of Toulmin's model. The transcript from which the two pieces of the model were created is given at the bottom of the figure. From the transcript, I first constructed the model on the left of the figure under the heading "Raw Data." For this version of the model, I used the exact wording of the teacher and students. In a slight variation of Toulmin's model, I present the claim directly above the data that support it to emphasize the manner in which the students' claims rely upon the information presented as data. The colors of the boxes denote the speaker or speakers. In this particular episode of argumentation, multiple students offered the claim, as indicated by the blue shading. One student alone offered the data and this is indicated by the green shading. Though not shown in this example, in episodes where another student also contributed to the argument, those boxes are shaded in purple. All statements and questions attributed to the teachers were coded in red in these models. Teacher questions prompting student response are included as an additional modification of Toulmin's model. This use of shading is consistent throughout the analysis.

My interpretation of the teacher and student comments is shown in the version of the model on the right of the figure, under the heading "Interpreted Data." Based on the mathematical example prompting the episode of argumentation and my field notes that provided additional context, I revised the raw data model to more clearly represent the mathematics of the argument and to create a model that could stand alone without additional explanation of context. Note that in this example, the warrant was not explicitly stated and thus was not a part of the raw data model. However, based on my interpretation of the episode, I included the implicit warrant in the interpreted data model. I attributed the warrant to the student who offered the data as denoted by the green shading and used a dashed border to indicate that warrant was implicit.

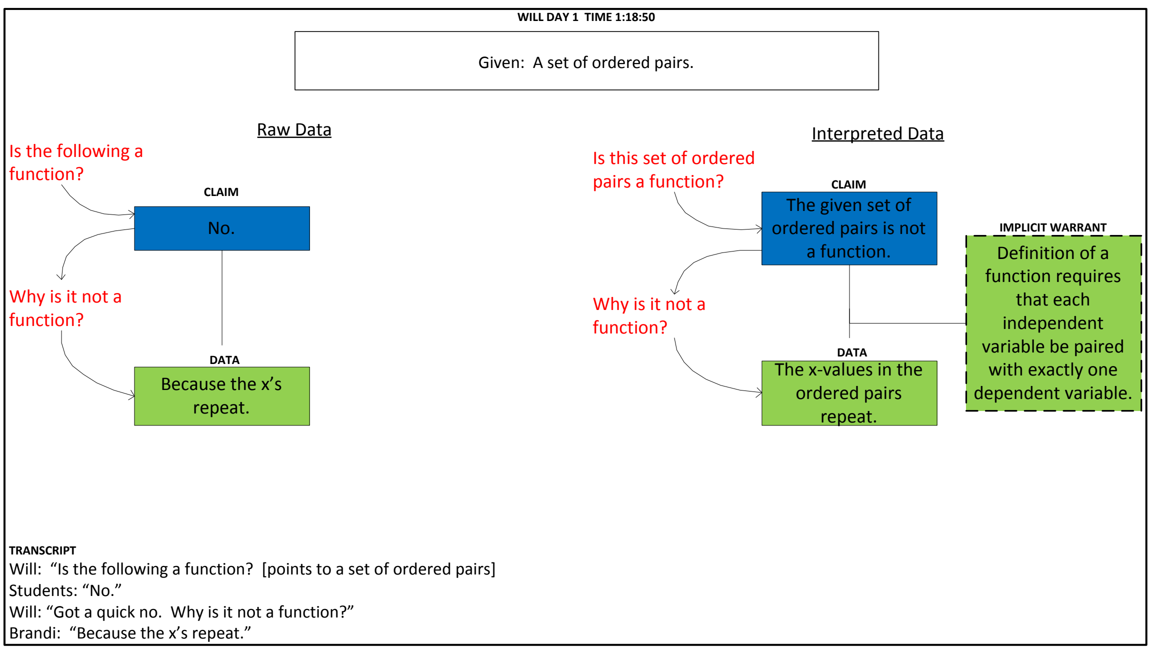


Figure 4. Will Argument 1.

Cross-Case Analysis

Following my analysis of each case, I conducted a cross-case analysis to bring together the findings from the five individual cases. As Merriam (1998) noted, the purpose of cross-case analysis is to abstract across the cases. Miles and Huberman

(1994) described cross-case analysis as an attempt to determine the “relevance or applicability of our findings to other similar settings” (p. 173). Cross-case analysis helped to strengthen my understanding of the commonalities and differences between the mathematical argumentation occurring in these five classrooms and the purposes for which teachers fostered such discussions.

The cross-case analysis drew on methods from both case-oriented and variable-oriented approaches (Miles & Huberman, 1994). In a case-oriented approach, the referent for the analysis is the case, and the aim of investigation is to understand the phenomena in question in relation to the case. In this study, such an approach implied developing an understanding of each teacher’s argumentation practices and then searching for commonalities and instructive differences across the five teachers. A variable-oriented approach takes the focus of inquiry as primary with an aim to describe the variability in the phenomena. This approach examined the argumentation practices across the cases and investigated relations between them. Miles and Huberman (1994) noted that a combination of these two approaches is “necessary for careful description and explanation” (p. 91). Thus in this study, I used both approaches to investigate my research questions.

Validity

As Maxwell (2005) noted, validity relates the conclusions of one’s inquiry with the reality in which the phenomenon rests. With qualitative research, “the understanding of reality is really the researcher’s interpretation of participants’ interpretations or understandings of the phenomenon of interest” (Merriam, 2002, p. 25). Thus, the best course of action for researchers is to identify validity threats and address how these threats will be minimized (Creswell & Plano Clark, 2011; Maxwell, 2005). For this study, I noted the following potential threats to the validity.

First, the purposeful sample might have been comprised of teachers who were successful in terms of supporting students in better performance as measured by tests but whose classroom practices did not enhance mathematical argumentation. Based upon my review of the mathematics education literature, it was reasonable to believe that there is a strong relationship between student learning as measured by tests and the practices that occur with the classroom. If the anticipated instructional strategies had not been observed, this would have demonstrated that existing measures of student learning do not capture the extent to which argumentation is a part of mathematical learning.

Second, as I attempted to provide the rich, thick description originally called for by Geertz (as cited by Denzin & Lincoln, 2003), I was also aware that what I chose to notice, what I chose to ignore, and what I failed to see all impacted my perception of what was happening in these classrooms and my interpretations of their meaning (Schram, 2006). I attempted to minimize this threat by video recording during my classroom observations while also taking field notes. I then triangulated the data (Creswell, 2003) by comparing my observations from each of the two sources and my interview transcriptions.

Third, as I analyzed the episodes of argumentation that occurred in these classrooms, I had a preconceived understanding of what I valued as mathematical argumentation. From the research literature, I perceived that arguments with multiple participants, conceivably without the teacher's mediation, about aspects of mathematics that had not been previously discussed or that pushed students to think more deeply about a topic held the most potential for increasing students' mathematical learning and understanding. As my analysis revealed arguments with structures differing from this perceived ideal, it was necessary for me to reexamine my basic definition of mathematical argumentation so that I could better understand the types of arguments that were present in these classrooms.

Finally with interviews, information is filtered by the participant and can be biased by the researcher's presence (Creswell, 2003, p. 186). Given my position as the project manager of the larger grant, my presence in the classroom during observations had the potential to cause the teachers and students to alter their classroom routines. Observing five consecutive classes of the teacher helped me to better determine classroom patterns that consistently occur. Further, the information provided by the teachers during the interviews was compared against what I observed in the classroom as a check of internal validity (Merriam, 2002).

Potential Ethical Issues

The most important ethical consideration for the study concerned the real and perceived power that I had over my study participants. As the manager of the larger project, study participants had come to know me through that role since I was frequently in their schools and classrooms. Further, they were aware of my affiliation with the university and with the fact that I frequently attended meetings with school system administrators. My hope was that the relationships that I had built with the teachers over the course of the larger project helped to dissipate issues of power and any concerns that they might have had regarding my use of study findings.

I am aware of the procedural ethics of this study. Tracy (2010) noted that "procedural ethics encompasses the importance of accuracy and avoiding fabrication, fraud, omission, and contrivance" (p. 847). With this in mind, I was careful to make certain that I did not make assumptions throughout the interviews and observations about the teachers' meanings based on my prior knowledge of their teaching practices. As previously noted, I was careful to ensure that all findings reported in this dissertation study were gleaned from my observations of the teachers' classrooms for this study and our post-observation interviews. Further, I note my commitment to represent these teachers in a respectful manner from a strengths-based perspective.

Finally, Merriam (2002) referenced ethical dilemmas concerning the dissemination of research findings, and I recognize that this is a concern for this study. Realizing that my research findings could potentially be used in a manner in which I do not intend, it is imperative that I protect the identities of my participants. I have made every effort to ensure that the participating teachers cannot be inadvertently recognized through any information or descriptions that I included in my study findings.

CHAPTER IV

WITHIN-CASE ANALYSIS

In this chapter, I describe each of the participating teachers in greater detail, giving information about their experience, background, and beliefs on teaching as well as descriptions of their Algebra I classes. All teacher and student names are pseudonyms and the cases are presented in alphabetically order by teacher pseudonym. Next, I provide details of all instances of mathematical argumentation that occurred during my observations of their classes. The cases are presented in chronological order unless otherwise noted. I provide my interpretation of each of these and describe their relevance to this study. Then, I discuss patterns relating to those instances of mathematical argumentation. Finally, the chapter concludes with a summary of the analysis.

Abby

Description of and Characteristics of Abby

Originally from out of state, Abby moved to North Carolina upon completion of her undergraduate degree 12 years ago to teach in a school system approximately 60 miles away from Mountainside High School. After teaching there for two years, she transferred to Mountainside where she has taught for the last ten years. Commuting for over two hours each day, Abby has a fierce determination to help her students succeed and tutors both before and after school during the week and often on Saturdays as well. She experiences great frustration when dealing with colleagues that do not display what she perceives as the same levels of dedication to their students. This feeling of dissatisfaction along with the stress of teaching at a school with a falling performance level in mathematics over the past several years has Abby seriously considering resigning from Mountainside. In our post-observation interview, Abby noted that she was "so tired of dealing with the stress."

In the classroom observations, Abby had a remarkable rapport with her students, and they obviously liked and respected her. She noted in our interview that she "talks to [the students] in the same way they talk to [her]." While an outsider may misunderstand her sarcastic demeanor when teaching, it was clear that her students were very comfortable conversing with her, participating in the class, and answering her questions. She explained to me that she does not have a list of rules posted in her classroom, because she feels that mutual respect is all that is necessary. I observed that she was aware of almost everything that was occurring in her classroom, even when the students were working with partners or in groups. In her words, "I want them to have the ability to do different things, but yet I'm still in control, because otherwise it's just not going to work."

Abby described herself as "student-centered" and noted that she "tries to make her classes as interesting for her students as possible" by having them work with partners or in groups on a daily basis. She allowed the students to choose their own seats and partners, saying, "You know who you can work with!," and then held them accountable for what they accomplished during the class. She described this as "an important experience for them and an opportunity for them to learn how to become productive members of a group."

At the same time, Abby was determined to prepare her students mathematically for more than just a standardized test. In our post-observation interview, she noted that she teaches all topics completely by hand before showing them how the same problems can be solved using technology such as a graphing calculator. When her students expressed frustration that she did not "show them the easy way" first, she replied that their goal should be to move beyond Algebra I. She stressed that understanding the mathematics behind the calculator commands is ultimately more important to them as students. In her opinion, "they make a better connection between the methods when I hold the calculator off until the end of the unit." She also worked to strengthen their understanding of mathematics by answering their questions with questions of her own. She explained, "If they ask me a question, I won't answer it. I question them all the way to the end." She described herself as very persistent about pushing her students to figure things out for themselves and believed that they would understand and retain the information better than if she "[fed] them answers."

Description of Abby's Class

Abby's second block class of sixteen students came in each day and started their warm-up activity without any prompting from Abby. The thirteen ninth grade and three tenth grade students stopped by the front table to pick up the assignment on their way to their desks and then chose a seat and began work while Abby attended to hall duty. Throughout my week with them, I observed that the students worked well together in group and partner settings, and took this as evidence that Abby made these activities a regular part of the classroom routine. The students seemed willing to participate in whole class discussions as well and responded to Abby's questions with little prompting. I observed that her calm, straightforward attitude with her students allowed her to address any discipline issues without disrupting the flow of the lesson. While the atmosphere in the class was somewhat relaxed, the students seemed to understand, as Abby put it, "When it's time to work, it's time to work."

Description of Observations

During my observations, the class covered a progression of lessons that built from finding the greatest common factor (GCF) of integers through factoring polynomials with up to four terms. Using the students' prior knowledge of finding the largest factor common to two or more integers, the first day's lesson built upon that understanding to find the GCF of terms containing both integers and variables. The warm-up exercises that Abby assigned allowed the students to work together in groups to review the properties of exponents and thus helped to remind them of the properties they would need to use to as they factored the GCF from polynomials. I observed that Abby was very careful to explain that the day's lesson was the first "little" step toward more complex problems. She and the students worked together through several examples before they completed examples of their own.

The second day of class began in a similar fashion as the students completed warm-up exercises reviewing GCF problems from the previous day. As they worked, Abby circulated around the room to answer questions and address issues that might hamper the students' abilities to proceed with the next section on factoring by grouping. Following the warm-up exercises, Abby carefully talked through factoring a polynomial by grouping. She then asked the students to factor the next example and monitored their progress as they worked. When she confirmed that everyone had the correct answer, she put the work for the problem up on the board and asked for additional questions. The rest of the class proceeded in the same manner with progressively more difficult examples.

For the third day of my observations, I noted that the class began in the same manner as the previous two days. However, after completing a warm-up activity covering examples of factoring by grouping, Abby instructed the students to find a partner as she passed out game boards she had made for Slap Jack, a game with which the students were already familiar as evidenced by their eagerness to get started. Abby had prepared a flipchart consisting of questions that reviewed content from their previous units, and they spent approximately 30 minutes playing. I observed the students to be very engaged throughout the entire game. When the game ended, the students quickly transitioned back to their seats to start the next part of their factoring lesson. The remainder of the class was spent on factoring trinomials by a procedural method, with Abby modeling the first problem and the students then working subsequent problems as Abby circulated around the room

The fourth day's lesson combined the factoring of trinomials with the factoring of a GCF from a polynomial and the warm-up for the day consisted of five trinomials to factor. For the first time during my observations, three of the students were off-task and disrupting the classroom atmosphere. Abby gave them a serious look and a warning to "chill." When one student folded his warm-up sheet into a paper airplane, Abby turned to him and announced that she would be taking off 10 points for each crease. The students all laughed, but then everyone got back to work. From there, the class proceeded as in previous days, with Abby working the first example and carefully showing the procedure before allowing the students to work through additional examples on their own.

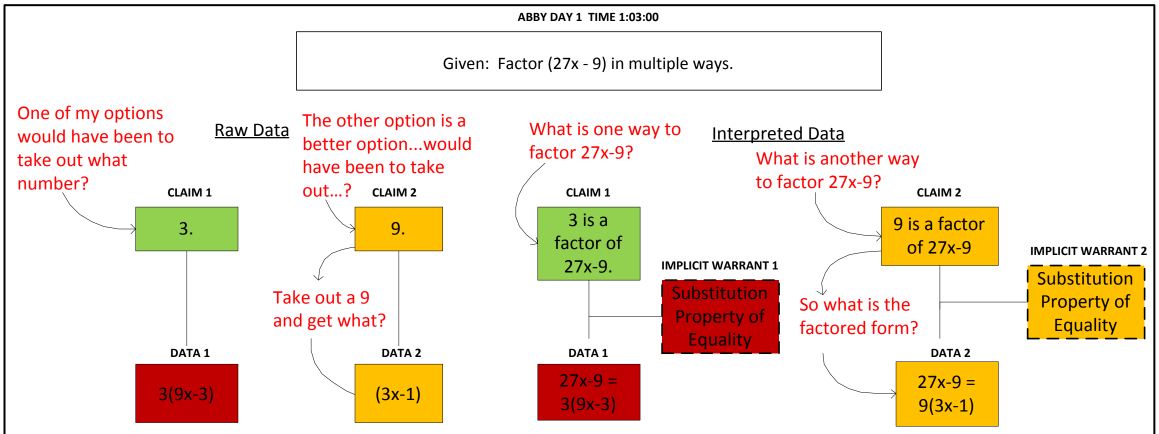
The fifth day's warm-up activity consisted of a review of ten problems from topics they had already covered in the course. In pairs with clickers, the students completed each problem and locked in their answer. Abby was able to immediately see how many students got the correct answer to each question and then worked through problems that multiple students missed. They then moved on to factoring binomials and followed the same format as they had in previous days.

Examples of Mathematical Argumentation in Abby's Class

Two instances of mathematical argumentation occurred in Abby's class during my week of observations and both episodes occurred on the first day.

Abby: Argument 1. While working on the new notes for the day, Abby used a flipchart titled "Factoring Polynomials: GCFs and Grouping." After working several examples of factoring out the GCF from binomials, the students asked Abby if they could practice a few more examples. She made up two problems and put them on the board. The students worked on the problems as she monitored their progressing. Abby noticed that the problems she made up both had more than one possible way to factor. She moved to the board to discuss the first binomial, $27x – 9$, and asked, "You have two options for both of them. Stan, on the first one, one of my options would have been to take out what number?" Stan offered that a possible factor for the binomial was 3. Abby nodded in confirmation and added, "One option was to take out a 3, and then you would have had $3(9x - 3)$." She continued, "That's one option. The other option is a better option . . . [It] would have been to take out . . .?" Troy stated that 9 was also a factor of the binomial, and Abby encouraged him to continue stating, "Take out a 9 and get . . .?" Troy explained that factoring out a 9 would leave the binomial $3x - 1$.

The model of this episode of argumentation is depicted in Figure 5. Abby prompted the argument by requesting that the students "factor" the binomial $27x - 9$. Because she did not explicitly request that the students factor out the GCF, two claims were possible. Abby made note of this as she monitored the students' progress, as evidenced by the fact that she prompted two students for their claims. In response to Abby's question, "One of my options would have been to take out what number?", Stan offered the first claim of 3. I interpreted this exchange to mean "What is one way to factor $27x - 9$?" and "3 is a factor of the binominal", respectively. Abby did not prompt for Stan's data and instead offered it herself by saying, "And you would have had $3(9x - 3)$" She then prompted for the next claim with her statement, "The other option is a better option…would have been to take out . . .?" and Troy offered the claim of 9. I similarly interpreted this exchange in the same manner as the first, with Abby's question to be a request for another way to factor the binomial and Troy's response to mean "9 is a factor of the binomial." For this second claim, Abby prompted Troy to provide his data by urging him to continue with the question, "Take out a 9 and get . . .?" which I interpreted to mean "What is the factored form?" Troy offered $3x - 1$ which, when combined with his original factor of 9, represented the factored form of the binomial $9(3x - 1)$. In both instances, the claims and data were accepted without a request for an explicit warrant.



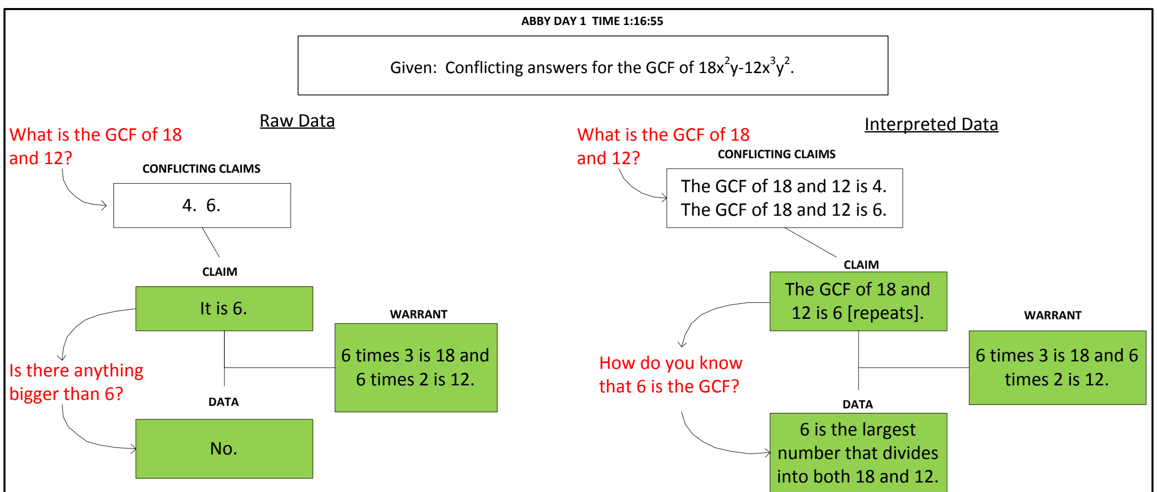
TRANSCRIPT
Abby: "Alright, are you ready? Some of you are. You have two options for both of them. Stan, on the first one, one of my options would have been to take out what number?"
Stan: "3."
Abby: "One option was to take out a 3, and you would have had 3(9x-3) [*several students say this answer out loud along with Abby*]. That's one option. The other option is a better option...would have been to take out..."
Boy: "1!"
Abby: "No, better than 1."
Troy: "9."
Abby: [*nods*] "Take out a 9 and get..."
Troy: "(3x-1)" [*a couple of other students also offer this answer*]
Abby: "Either way will work."

Figure 5. Abby Argument 1.

Multiple correct claims made this episode of argumentation unique. Because Abby made up this example at the students' request and initially asked how it could be factored, she allowed the possibility of factoring out either a 3 or a 9. As she circulated around the room and realized that the students had not interpreted her request to "factor" as a request to "factor out the GCF," she allowed the students to continue working without rephrasing her directions. Though Abby had intended for the students to factor out the GCF, she recognized that some students' use of a common factor and others' use of the GCF provided an instructional opportunity for students to see the utility of the GCF. As they began to discuss the problem together, Abby was careful to sequence her questions to specific students based upon the answers she had seen on their papers as she monitored their work. Allowing the discussion of the problem to emerge in this manner provided the students the opportunity to see multiple ways of interpreting Abby's initial directions and to consider the benefits and drawbacks of each. By not pressing for explicit warrants for either claim, Abby assumed that the process of replacing a binomial with its equivalent factored form was understood by all the students.

Abby: Argument 2. The second episode of argumentation occurred as Abby and the students moved on to examples containing variables as a part of the GCF. Abby pointed out differences between factoring out numbers and factoring out variables and worked through several examples with the students. When they started to factor $18x^2y - 12x^3y^2$, two students, Omar and James, offered conflicting ideas about the GCF of the coefficients 18 and 12. Abby observed, "He says 4 [pointing to Omar] and he says 6 [pointing to James]" and then waited for responses from the students. James reiterated that 6 was the GCF and Abby asked, "Is there anything bigger than 6?" James responded, "No, 6 times 3 is 18 and 6 times 2 is 12."

Figure 6 depicts the model used to analyze this episode of argumentation. Abby prompted this argument by requesting the GCF of the coefficients 18 and 12. Omar and James offered conflicting claims as they simultaneously answered "4" and "6." I interpreted their replies to mean "the GCF of 18 and 12 is 4" and "the GCF of 18 and 12 is 6," respectively. Abby encouraged discussion by pointing to each student and repeating their answers and then waiting to see what the students would say. James then confidently repeated, "It is 6." Abby prompted James for his data, by asking, "Is there anything bigger than 6?" I interpreted this question to mean, "How do you know that 6 is the GCF?" James responded with a statement offering both his data and warrant as he said, "No, 6 times 3 is 18 and 6 times 2 is 12." I understood this answer to provide his data that 6 is the largest number that divides into both 18 and 12 as well as his justification for that data in the form of the two multiplication facts.



TRANSCRIPT

Abby: "I'm going to do one more before I cut you loose… You do the two numbers; if you need to type it in, you type it in. I need the GCF of the two coefficients, 18 and 12, what's the GCF of 18 and 12?"
Omar: "4."
James: "6."
Abby: "He say 4 [*points*] and he says 6 [*points*].
James: "It is 6."
Abby: "Is there anything bigger than 6?"
James: "No, 6 times 3 is 18 and 6 times 2 is 12."
Abby: "Ok."

Figure 6. Abby Argument 2.

This episode of argumentation was unique due to the conflicting claims offered by Omar and James, and the fact that Abby did not indicate which student was correct. By allowing the students to resolve the issue, James became the mathematical authority in the room as he asserted that his answer was correct and also provided the data and warrant to support his claim. By explicitly offering the warrant connecting his claim and data, James' understanding of the meaning of GCF was accessible to his classmates as well.

Observations from Mathematical Argumentation that Occurred in Abby's Class

I observed several similarities when examining the two arguments that occurred in Abby's class. First, both arguments concerned mathematics with which the students were already familiar. The second argument dealt with finding the GCF of integer coefficients of the terms in a binomial and required only arithmetic skills. The first argument dealt with the mathematics of that day's lesson, factoring common factors from the terms of a polynomial, and also provided an opportunity for students to explore the benefit of factoring out the GCF as opposed to another factor.

Second, both arguments followed the same structure. All prompts for claims were prompts for answers to problems and both arguments were between Abby and multiple students. When Abby prompted for data, she used phrases other than simply "Why?" or "How?" For example, in the first argument, Abby used Troy's claim of 9 in her prompt for data as she inquired, "Take out a 9 and get . . .? She followed the same pattern in the second argument as she took James' claim of 6 and pushed, "Is there anything bigger than 6?" Both instances of data offered by students relied on a previously-learned property. It is also significant to note that Abby did not prompt for warrants. In the first episode, Abby offered the data for first claim, left implicit the warrant that the

Substitution Property of Equality allowed her to rewrite $27x - 9$ as $3(9x - 3)$, and proceeded to discuss the second half of the argument. While Troy offered the data for the second claim, Abby still did not prompt for an explicit statement of the Substitution Property of Equality as the warrant. In the second argument, James offered the explicit warrant along with the data with no additional prompting from Abby.

Finally, Abby allowed the arguments to proceed as discussions of possibilities. For instance in the first argument, Abby allowed the students to present two options for factoring the given binomial and then led the discussion into considering the reasons why one option might be preferred over the other. In the second episode, students offered conflicting answers to Abby's question. Instead of indicating which student's answer was correct, she instead allowed the students to come to that conclusion on their own. James reasserted his claim and then justified it by providing both data and warrant in its support.

Summary of Mathematical Argumentation in Abby's Class

From my observations of her class and the post-observation interview, it is evident that Abby is dedicated teacher focused on helping her students learn mathematics. To this end, she carefully sequenced her instruction to build from her students' prior knowledge and followed a pattern throughout the week of reviewing the previous day's work and then introducing a new conceptual topic that logically followed the previous day's topic. Also, she attempted to engage the students in her lessons by incorporating partner work, group work, and games into her instruction. The atmosphere in her classroom was relaxed, yet the students were attentive and quick to follow her directions.

Abby stressed in our post-observation interview that she focuses on preparing her students for their mathematics classes beyond Algebra I. She pointed out, “If I ask them something and I think that they guessed, I’ll ask them ‘why?’” and acknowledged that this frustrated the students. I take this as evidence of the potential for more substantial episodes of mathematical argumentation in Abby’s class, since she already has a focus on questioning students to determine their level of understanding and a commitment to insuring they are prepared for future mathematics courses.

Denae

Description of and Characteristics of Denae

Denae studied and obtained a Bachelor’s degree in engineering and did not intend to become a teacher. Though both her mother and sister are teachers, Denae reported that she joined the mathematics faculty at Lakeside High School with feelings of hesitation and uncertainty. Eleven years later however, Denae declares that she now “loves teaching”. Interestingly, though she has been very successful with other courses, she taught Algebra I for the first time during the 2011-2012 school year.

Denae’s patience with and understanding of her students was evident immediately upon entering her classroom on my first day of observations. It appeared that she had a personal connection with each of her students as she took several minutes each day to talk about non-mathematical issues important to them. In our post-observation interview, she explained that she felt this interest in their lives helped to strengthen their connection to her and helped to create a positive learning environment. In her words, Denae explained, “If taking five minutes to talk to them about something non-math-related gets them to work for me for the next 50 minutes, then it is worth the time.” She also described how she starts each semester by telling her students that “we take care of each other in here.” She does not allow them to disparage or interrupt each other. She compares classroom discussions to family members talking back and forth; they do not have to raise their hands, but they know not to talk when others are talking. Since this Algebra I class had only eight students on the roster and at most only six students in attendance during the 5 days of observations, Denae encouraged them to work together throughout most of the class period. She felt strongly that students learn from helping other students.

Though Denae created a safe place for students to discuss mathematics, she described how difficult she found it to get the students to explain their answers. She believed this was in direct consequence of the fact that the students in this Algebra I class had failed the course in previous semesters. Denae explained in our interview, “At this point, they are just so glad to get to the right answer, because for some of them, this is the second or third time taking the class. Just getting them to a correct answer was a big step.” Since they did not wish to discuss problems beyond the answer, Denae reported that getting the students to explain their reasoning was a challenge she struggled to overcome.

To help the students feel a greater sense of accomplishment and success, Denae reported that she modified her teaching in this Algebra I class in comparison to the larger class she taught later in the day. I observed that she avoided showing multiple methods for solving a problem and focused on being consistent from one problem to the next. When I questioned her about this in our post-observation interview, she noted that whenever possible, she taught the students to solve problems on their calculators. While she acknowledged that this might not provide the students with as strong of a mathematical background as those of the students in her other Algebra I class, her immediate concern was to help them pass this course which has become such an obstacle for them. She noted, "I just want them to know that they can be successful and that they can do this. I think if they realize that they can get some problems right, it will help to build their confidence." I observed that she stressed highlighting key terms and double-checking their answers with every problem they did.

Description of Denae's Class

Denae's second block Algebra I class was unique in several ways. There were only eight students on the roster, the maximum number of students present on any one day of observation was six, and only four students came to class regularly. Though classified as freshmen, the students were older than traditional ninth graders. When only those four students were present, I observed them to be cooperative and willing to work with Denae. Even when struggling with the mathematical content, they remained upbeat, asked questions of Denae and each other, and worked together to complete difficult problems. On the days when other students were present, however, there was a shift in the atmosphere of the classroom and Denae struggled to keep them working and engaged.

Description of Observations

On the first two days of my observations with Denae's class, she and the students focused on systems of equations problems. She allowed the students to talk for a few minutes about non-mathematical topics and occasionally participated in the conversations herself. On both days, Denae distributed worksheets to guide their work and began the lessons with a discussion of what the students knew about systems of equations. They then spent the class period going through the problems on the worksheets, sometimes working the problems together with Denae and other times on their own. Even as they attempted to work independently, the students would ask questions of Denae and each other.

The third day of my observations had a modified schedule with a shorter class period. Denae used the first half of the period to have the students complete a worksheet reviewing topics previously covered in class. As the students work, Denae walked around to check their progress and answer their questions. The students also talked quietly together about the problems. The remainder of the class was spent analyzing three story problems together with Denae, deciding how to solve them, and then using their calculators to get the answers.

The fourth day of my observations proceeded in a similar manner as the first two days, with the topic of quadratic equations and multiple problems worked both with Denae and independently. However, the fifth day followed a different routine. As the students came in, Denae had a brightly colored flipchart displayed on the board with the first of several multiple-choice questions ready for them to start. She had placed little whiteboards, markers, and erasers at the students' desks prior to their arrival. Denae started the class with a discussion about how to recognize specific types of problems, such as quadratics, systems of equations, and predictions. She then pointed to the problem on the board, saying, "Ready to attack? Go!" She then let them work together on the problem and redirected any questions addressed to her to one of the other students. When all of them had reached an answer, Denae flipped to a new problem, announced "Everybody can do this one!" and led a quick discussion about the type of problem before having the students work it. The remainder of the class was spent following the same pattern, with Denae cheerfully encouraging them on every problem, and the students persistently working to identify the type of problem and find its solution.

Examples of Mathematical Argumentation in Denae's Class

Three instances of mathematical argumentation occurred during my observations of Denae's class. Two of those arguments occurred on the first day as the four students worked on problems dealing with systems of equations.

Denae: Argument 1. The students spent the first part of the class working on solving systems of equations using their calculators and then moved to a new worksheet of systems word problems. Denae told them that they were going to work through the problems together. The first problem had to do with numbers of dogs and cats and served as the basis for both of the first two instances of argumentation. The task described a total number of animals to be 51 with the condition that there were twice as many cats as dogs, and the students were to determine the exact numbers of cats and dogs. Before having the students attempt the problem, Denae drew a table on the board with 2 columns labeled "dogs" and "cats" and filled in the number 2 to represent two dogs. She then stated, "It says, 'There are twice as many cats as dogs,' so if there are two dogs, how many cats [do we have]?" Portia responded quickly with the answer of 4. When Denae prompted for her to explain how she knew this, Portia answered, "Because there are two dogs."

The model of this episode of argumentation is depicted in Figure 7. Denae prompted the argument with her question requesting the number of cats given two dogs. Portia offered the claim, "Four," and Denae pressed her to explain how she found this answer. Portia replied with the data, "Because there are 2 dogs." The warrant expressing the connection between the number of dogs and number of cats had been stated by Denae as a part of her prompt.

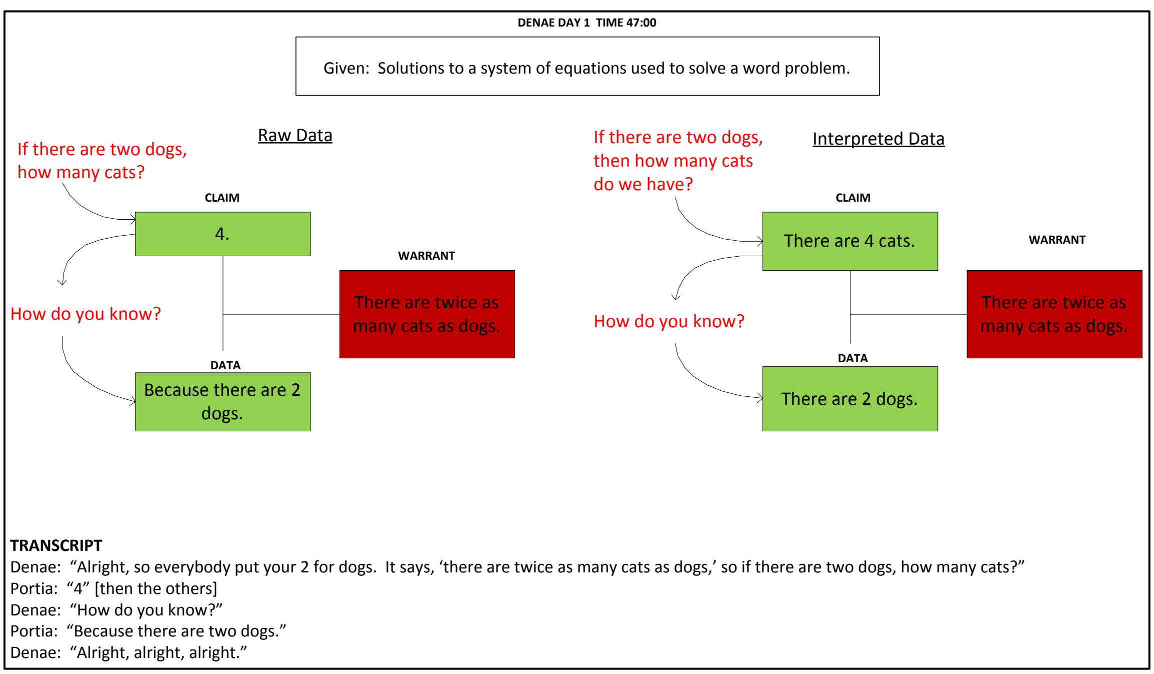


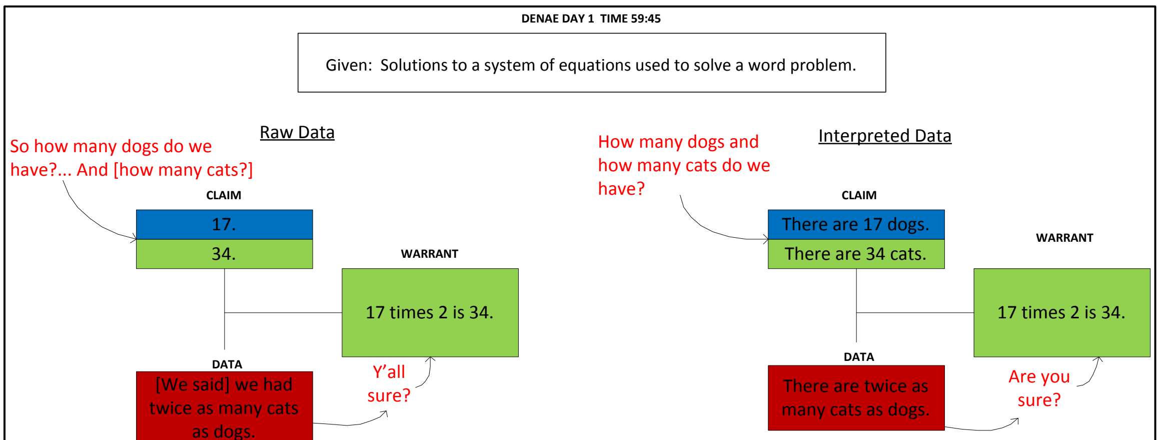
Figure 7. Denae Argument 1.

Here, Denae was attempting to insure that the students understood the wording of the problem and the relationship their numerical answers would have in the context of this problem. She highlighted the phrase “there are twice as many cats as dogs” and repeated it for emphasis. Thus, the explicit warrant linking Portia’s claim and data was offered by Denae herself prior to Portia’s responses.

Denae: Argument 2. After the first argument helped the students understand the context of the word problem, Denae encouraged them to continue the problem. They reached the point where they had the two equations for the system about cats and dogs: $x + y = 51$ and $y = 2x$. Denae instructed that the two equations needed to be in standard form and the students tried to figure out what they needed to do with the second equation. They then spent a few minutes putting the equations in the calculator. When the students all had answers, Denae asked, “So how many dogs do we have?” which led several students to answer “17.” She then continued, “And how many cats?” Destiny replied quickly, “34.” Nodding, Denae added, “And didn’t we say that we had twice as many cats as dogs? Is that twice as many? Are you sure?” Destiny responded, “It is. Seventeen times 2 is 34.”

Figure 8 depicts the model used to analyze this episode of argumentation. Denae prompted the argument with two questions, one asking for the number of dogs and the other the number of cats. Multiple students offered the claim that there were 17 dogs while Destiny gave the claim that there were 34 cats. As she attempted to encourage the students, Denae offered the data, “And didn’t we say that there were twice as many cats as dogs?" and prompted to make sure the students agreed by asking, "Are you sure?" Destiny confirmed the claim with the warrant, "Seventeen times 2 is 34."

Here, Denae continued to stress the importance of the phrase "twice as many cats as dogs" and engaged all four of the students in the discussion. Not only did she push the students to identify which numerical answer applied to the number of cats and which to the number of dogs, but she also urged them to explicitly state their warrant in the form of the multiplicative justification that "17 times 2 is 34."



TRANSCRIPT
Denae: "So how many dogs do we have?"
Students: "17."
Denae: "So y'all got 17 and who?"
Destiny: "34."
Denae: "What does the 17 represent?"
Destiny: "Dogs."
Denae: "The x, the dogs. And what does the 34 represent?"
Destiny: "Cats."
Denae: "And didn't we say that we had twice as many cats as dogs? Is that twice as many? [*pointing at the board*]"
Students: "Uh-huh."
Denae: "Y'all sure? I don't know if y'all really know?"
Destiny: "It is. 'Cause…"
Denae: "'Cause what?"
Destiny: "Because 17 times 2 is 34."

Figure 8. Denae Argument 2.

Denae: Argument 3. A third example of argumentation occurred on the fourth day of my observations. The students were unfocused that day, and Denae had a difficult time keeping them on task. They discussed parabolas and worked to find the vertex of several problems. Denae then asked about the vertex of $y = 3x^2 - 24x + 40$ and gave them multiple-choice options for the answer. Portia responded, “I got ‘B’,” which led Denae to inquire, “How did you get it?” Portia responded simply, “I put it in the calculator.”

The model of this episode of argumentation is depicted in Figure 9. Denae prompted this argument by requesting the multiple-choice solution to the problem requiring that the students find the vertex of a parabola. I took Portia’s claim, “I got ‘B’” to mean that the correct answer to the problem was the ordered pair in answer choice B. Denae pressed for more information by asking, “How did you get it?” Portia offered as data, “I put it in the calculator.” I interpreted this to mean that “the calculator gave that ordered pair as the minimum of the function.” Denae did not ask her to elaborate further.

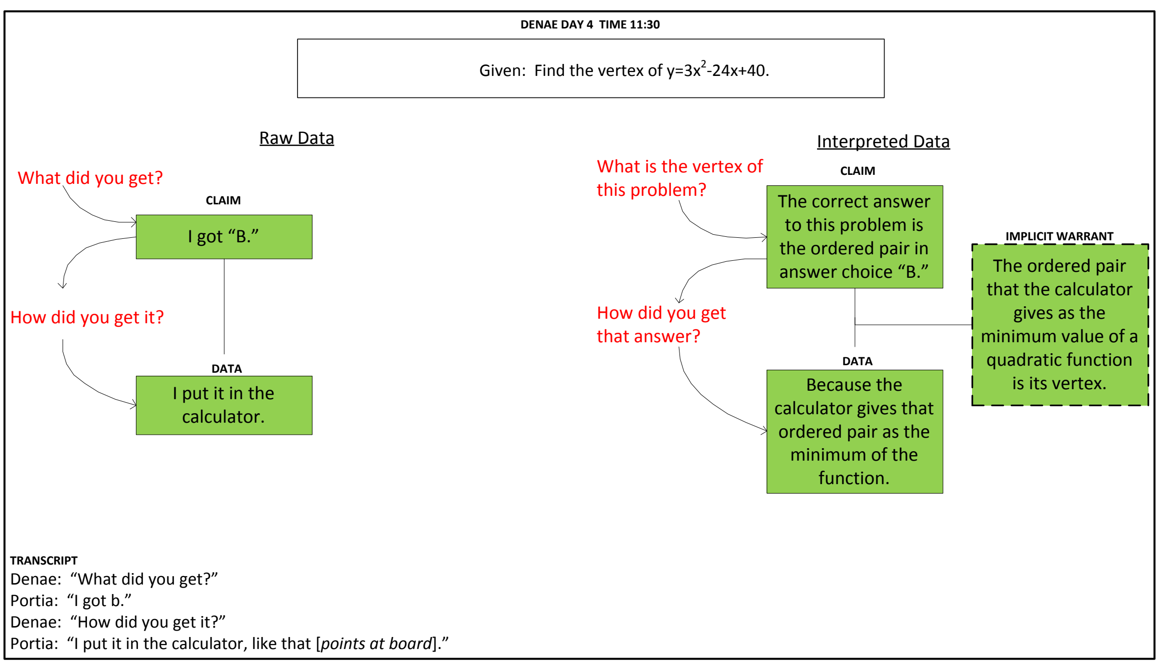


Figure 9. Denae Argument 3.

Here, Denae had taught the students a procedure for finding the vertex of a parabola using their calculators and accepted Portia's claim and data without asking for further reasoning. By allowing the warrant to remain implicit in this case, Denae assumed that all of the students understood the connection between the calculator's answer and the correct multiple-choice answer. It is unclear if the students understood the connection between the minimum value of a quadratic function given by the calculator and the vertex of a parabola.

Observations from Mathematical Argumentation that Occurred in Denae's Class

I observed several similarities among the mathematical arguments that occurred in Denae's class. First, the arguments concerned mathematics that the students had already experienced in Denae's classroom. Two of the three arguments concerned the mathematics of the lesson as the students worked on systems of equations for the first time and the other argument was based on problems the students were working as a review.

Second, the arguments followed a similar structure. It was especially interesting to note that only one argument was prompted by a request for a solution to a problem while the remaining two arguments were prompted by requests for the students' interpretations about the meanings of problems and their answers. Given the students' reluctance to discuss beyond a problem's solution and Denae's self-described challenge to help them overcome that hesitancy, the fact that two instances of argumentation occurred around students' interpretations is noteworthy. There was a blend of teacher-student and teacher-class arguments during my observations with multiple participants in one of the three arguments. Denae consistently prompted for data by asking, “How do you know?” and the data offered by the students was always a repetition of fact previously stated in the class or a procedure. Likewise, the two explicitly stated warrants were also repetitions of information previously stated.

Finally, Denae prompted and supported arguments for reasons other than promoting deeper mathematical argumentation. Two of the arguments were used to reinforce how to interpret answers to systems of equations. The third argument involving finding the vertex of a parabola was used to reinforce the calculator procedure for solving the problem. They all emphasized procedures that Denae wanted the students to follow when attempting to solve problems in testing situations.

Summary of Mathematical Argumentation in Denae’s Class

Denae cares deeply about the success of her students and feels that “successful completion” of the course for them is synonymous with passing the high-stakes standardized test at the end of the course. Her instructional routine was aimed at helping them be more confident about their mathematical abilities. Only one of the three arguments contained an explicit warrant given by a student and in that instance, the warrant was not algebraic in nature since it referred to a basic multiplication fact.

When Denae asked “How do you know?” to get the students to provide data, she expected the students to repeat information previously stated. Her consistency with the arguments’ structures paralleled her stated desire to show the students one method to solve each type of problem in her efforts to help them successfully pass the end-of-course standardized exam. As she noted in the post-observation interview, “I’ve got to get them to find those key words and then those key words trigger exactly what to do [next]." For example, she used the first two arguments to reinforce the importance of finding and highlighting the phrase "there are twice as many cats as dogs" which was a test-taking skill she wanted the students to use. Thus in Denae's class, mathematical argumentation is used as a means of reinforcing patterns and procedures in preparation for the final test.

Kendra

Description of and Characteristics of Kendra

A teacher at Lakeside High and a colleague of both Will and Denae, Kendra has spent her entire 13 year teaching career in this school where she feels she gives something back to the community. She feels a strong attachment to both her students and fellow math teachers. In our post-observation interview, Kendra explained, "I can't imagine [myself] anywhere else!" Kendra describes herself as a motivator and tries to give her students multiple opportunities to be successful. She explained to me in our post-observation interview that she tells her students that she has two jobs: "First, to get them to pass the [end-of-course] test and second, to make sure that they are prepared for their next mathematics course." She described herself as "very, very patient" and noted that she spends time trying to help the students overcome their fear of words. She works with them to highlight key terms and encourages them not to skip problems that appear intimidating. In our interview, she explained, "We spend a great deal of time trying to get over the fear of words, because it has been my experience in the past that when they see a long word problem, they just guess or skip it."

Kendra notes that she had a difficult time getting the students in this class to talk about math. While she encouraged both partner and group work on a daily basis, management of this class was challenging and keeping the students on task demanding. Kendra often repeated the terminology that her students used when answering her questions and then followed up with the correct phrasing. In our interview, she explained, “Sometimes the kids come up with ways to say things that I wouldn’t necessarily think of, but may help someone else.” When Kendra tried to get them to explain how they reached their answers, the students in this class were very resistant. Kendra believes that many of the students were not confident speaking up in such a large class and that others did not understand how to respond when she asks them to explain their thinking. She commented, “Some of them are not willing, are not confident, …just passing Algebra I will be such a confidence booster for them.”

Description of Kendra’s Class

The large Algebra I class I observed was a constant challenge for Kendra. With approximately 20 students, discipline issues required Kendra to modify her plans several times during the week. Six students seemed to dominate the class with disruptive behaviors and Kendra had to call for an administrator twice during my observations to have a student removed from the class. On the third day with approximately 20 minutes left in the class, the students’ inattention and refusal to participate in the lesson led Kendra to stop teaching and give a graded “drill assignment” in an attempt to get the students focused. Kendra’s obvious displeasure with their behavior along with a new seating arrangement led to improved behavior for the remainder of my observations.

Despite Kendra's best efforts, the majority of the students were not willing to speak up when she asked questions.

Description of Observations

On the first day of my observations in Kendra's class, the students came in to find six problems reviewing functions, midpoint, and slope on the board for their warm-up. They started the work promptly and worked independently on the problems for 10 minutes. Kendra then moved to the board to work through each of the problems. She did this by asking for volunteers and then wrote their responses on the board. As she wrote, she indicated important points that she wanted them to remember, such as key words or slope-intercept form for lines. Following the warm-up, Kendra quietly said, "Let's change gears," and started a discussion about exponential functions by asking that they "tell [her] what you know." The discussion led into 45 minutes of group work on a packet of worksheets about exponential functions. With approximately 20 minutes left in the class, they transitioned back to their seats for a whole-class review of 13 problems similar to ones that would appear on their test.

The pattern for the second day's activities was much the same. The first 30 minutes were spent with the students individually completing warm-up problems on functions, midpoint, and slope and then handing the problems in to Kendra for a grade. The students then returned to their same groups to complete their work on exponential functions and make a poster to display in the classroom. Once the posters were completed, they spent the remainder of the class completing a unit test.

On the third day of my observations, class was delayed due to a fire drill, and the students were noticeably less attentive when Kendra started class. Her intentions were to spend the day reviewing for an upcoming benchmark using a released standardized test and clickers. For approximately 40 minutes, Kendra put up different multiple-choice problems for the students to work. Whenever several students missed a problem, she stopped and worked through it before continuing. During this time, she had to send two students out of the classroom and call an administrator to address an issue with another student. When it became obvious that the students were completely unfocused and that continuing would be unproductive, Kendra had the students take out a piece of paper. She informed them that she was taking this work up began calling out questions for them to answer, all the while refusing to raise her voice.

On the fourth and fifth days of my observations, Kendra and the students moved on to a new chapter on matrices, after spending the first three days on review material. The students came in on both days and worked five warm-up problems and then discussed them with Kendra. Kendra then introduced definitions, properties, and procedures involving matrices and led the students through various examples that could be solved using a matrix. On both days, Kendra had the students complete an impromptu survey on a topic such as favorite color or favorite type of car. She then showed the students how the information could be organized in a table and represented by a matrix. The students were very engaged at those points in the lesson. On the fourth day, they worked together until the bell rang for class to dismiss and on the fifth day, the last 15 minutes of class were spent completing a quiz.

Examples of Mathematical Argumentation in Kendra's Class

Six episodes of mathematical argumentation occurred during my observations of Kendra's class. The first three instances of argumentation happened on the first day of class as Kendra and the students discussed the answers to the warm-up problems.

Kendra: Argument 1. Pointing to the first problem which contained a set of ordered pairs and the directions to determine if it represented a function, Kendra nodded to Tasha and prompted the first argument by saying, "Ok, I need for you to explain everything about #1." Tasha replied, "You only have to look at the x-values to tell whether it's a function or not." Kendra then prompted Tasha to "keep going." Tasha responded by explaining, "When you look at them, the numbers, they shouldn't repeat the same numbers. So then if they don't, then yes; if they do; then no."

Figure 10 depicts the model used to analyze this episode of argumentation. Kendra prompted the argument by instructing Tasha to "explain everything about #1." Tasha offered the well-articulated claim that "you have to look at the x-values to tell whether it's a function or not." Kendra's encouragement for her to keep going led Tasha to offer both data and warrant for this argument. When Tasha stated her data, "When you look at them, the numbers, they shouldn't repeat the same numbers," I took this to refer to the x-values of the ordered pairs in the set. I interpreted her statement to mean "When you look at the x-values, they should not repeat." Likewise with the warrant, I interpreted her reference to "they" to mean the x-values of the ordered pairs.

With this episode, Tasha was very eager to participate in the discussion and thus required little prompting from Kendra. Her claim, data, and explicit warrant were communicated clearly and used the same terminology Kendra had used previously with the class. Her warrant was based on the definition of function and clarified her reasoning for examining the x-values of the ordered pairs.

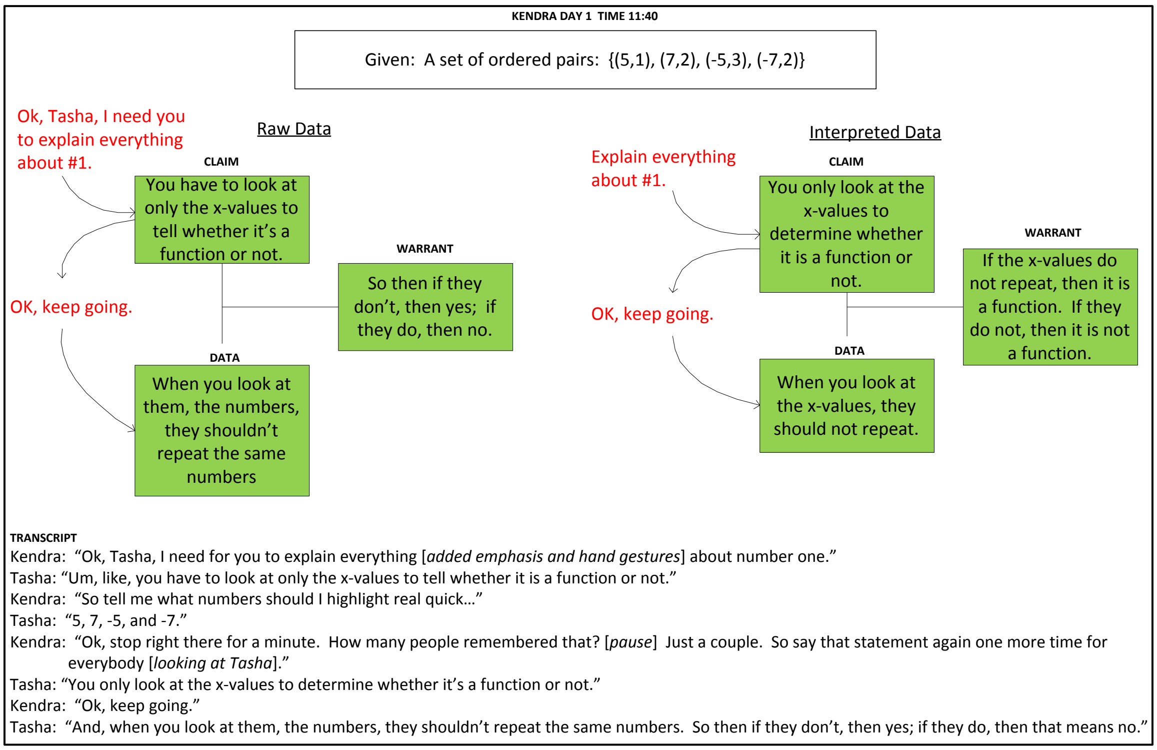
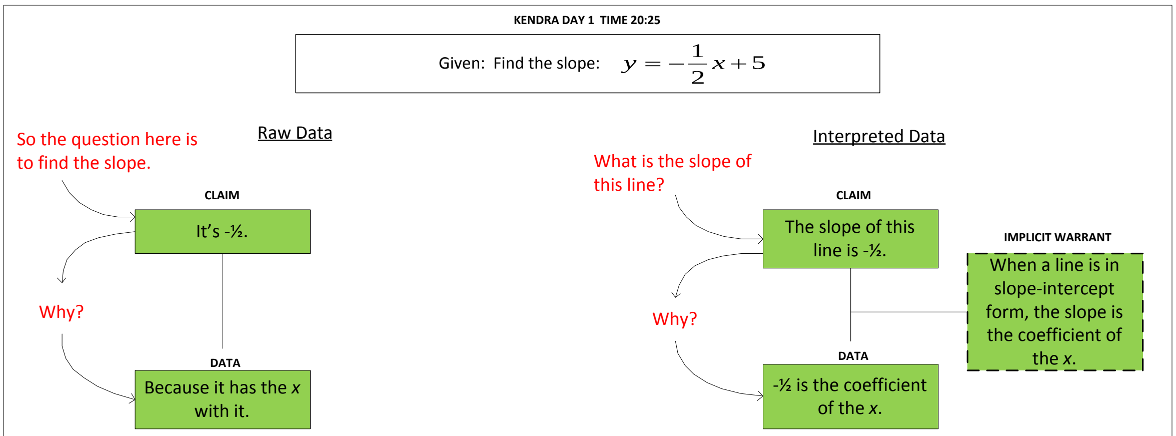


Figure 10. Kendra Argument 1.

Kendra: Argument 2. The second argument occurred around an equation of a line and the directions to find its slope. Kendra prompted this argument by repeating the directions and addressing one of the students, "Selena, what have you got for me?" Selena responded, "It's -1/2" and Kendra urged her to continue by asking, "Why?" Selena answered, "Because it has the *x* with it."

The model of this episode of argumentation is depicted in Figure 11. Kendra prompted the argument by repeating the directions to find the slope of the line and then calling on Selena. Selena offered the claim, "It's -1/2" which I interpreted to mean, "The slope of this line is -1/2." Kendra prompted for her data with the question, "Why?" When Selena replied, "Because it has the x with it," I took this to be a reference to the fact that -1/2 was the coefficient of the *x* in the equation. Kendra did not ask Selena for additional information.



TRANSCRIPT
Kendra: "So the question here is find the slope. Does it have to be a straight line to find the slope?"
Students: "No."
Kendra: "What if I'm given a graph? Then I can find it, right? But here, I was just given an equation. So…somebody that hasn't talked to me yet today."
Kendra: "Selena, what have you got for me?"
Selena: "It's -1/2 ."
Kendra: "It's -1/2 ? Why?"
Selena: "Because it has the x with it."

Figure 11. Kendra Argument 2.

Figure 11. Kendra Argument 2.

This argument, based upon an equation of a line, was very straightforward and built from the students' knowledge of the slope-intercept form of a line. Kendra called on Selena to give the claim and data and did not prompt her to explain her reasoning as to why the coefficient of the *x* represented the slope of the line. By not pressing for an explicit warrant, it is unclear whether the other students understood the connection.

Kendra: Argument 3. The third argument also started with an example asking the students to find the slope of a line. However, this problem was not presented in slope-intercept form and thus caused confusion among some students. The context of the argument revolved around attempting to identify the form of the line as it was given. As Tasha bounced in her seat and waved her hand in the air, Kendra nodded to her and said, “Go ahead.” Tasha stated that the equation was in standard form. Kendra asked her to explain, “What makes it standard form?” Tasha first replied, “Because you taught us that last semester,” but elaborated when Kendra pushed for more information and said, “Because Ax + By = C.”

The model of this episode of argumentation is depicted in Figure 12. As Kendra called on Tasha to discuss this problem, Tasha offered the claim that the equation of the line was in standard form. When Kendra asked Tasha, “How do you know?” to elicit her data, Tasha instead offered the warrant “Because you taught us that last semester,” which I interpreted more formally as “The definition was taught previously.” Kendra continued to press with the question, “What makes it standard form?” Tasha then offered the data that the symbolic representation of a line in standard form is $Ax + By = C$.

This episode of argumentation was interesting due to the order in which the claim, data, and warrant were presented. Tasha’s datum that supported her claim that the equation was in standard form was not immediately given when Kendra prompted for more information. Instead, Tasha first offered her explanation for how she recognized the line by saying, “Because you taught us that last semester.” When Kendra did not accept her response as sufficient information, she then offered the symbolic representation for a line in standard form as her data.

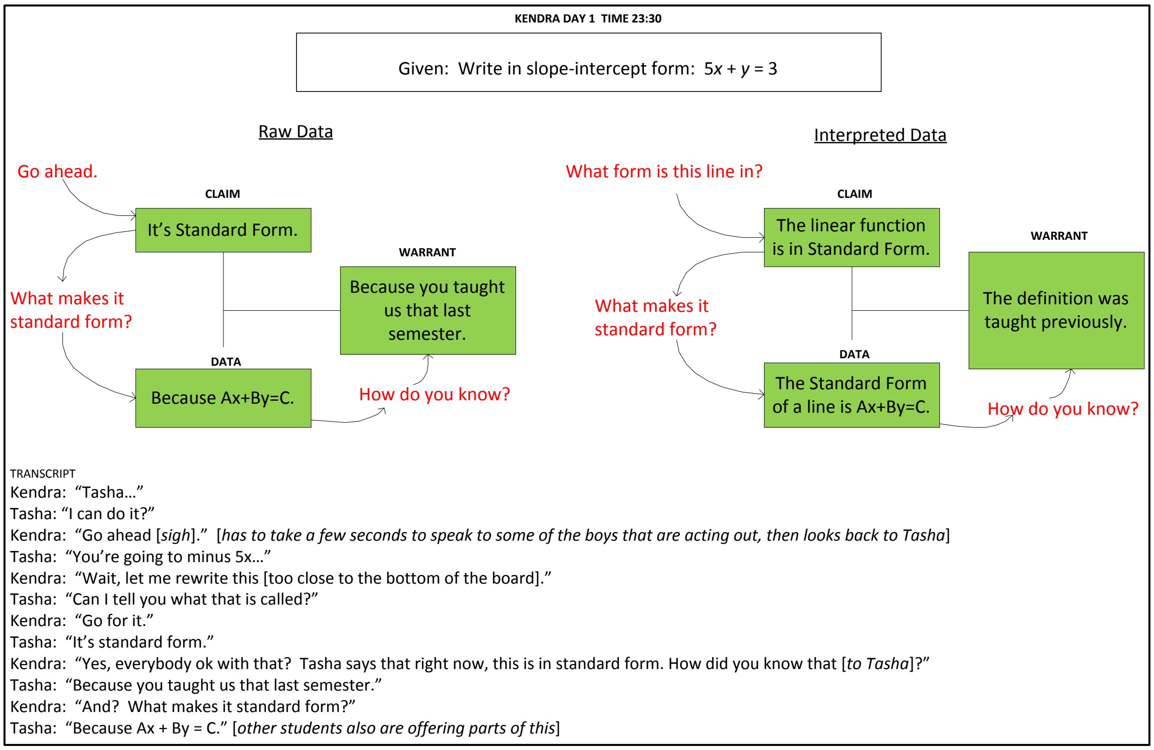


Figure 12. Kendra Argument 3.

Kendra: Argument 4. The next two examples of argumentation occurred on the second and third days of my observations, respectively, and both were prompted by examples that the students were asked to complete. On the second day, one of the warm-up problems required the students to determine if a set of ordered pairs represented a function. Kendra addressed the class and asked, "Ok, #1, the answer is . . .? Several students responded with an answer of "no." When Kendra prompted for more information, Tasha offered, "Because the 0's repeat."

Figure 13 depicts the model used to analyze this episode of argumentation. Kendra prompted this argument with her request for the answer to the first problem. Multiple students offered the claim, “No,” which I understood to mean the given set of ordered pairs did not represent a function. Kendra pressed for the students’ data by asking, “Because . . .?” Tasha then replied, “Because the 0’s repeat.” Since two of the ordered pairs in the set had an x-value of zero, I interpreted this more formally as, “There are two ordered pairs with the same x-value; the x-values repeat.” Kendra did not ask for additional information and the warrant based upon the definition of function was left unstated.

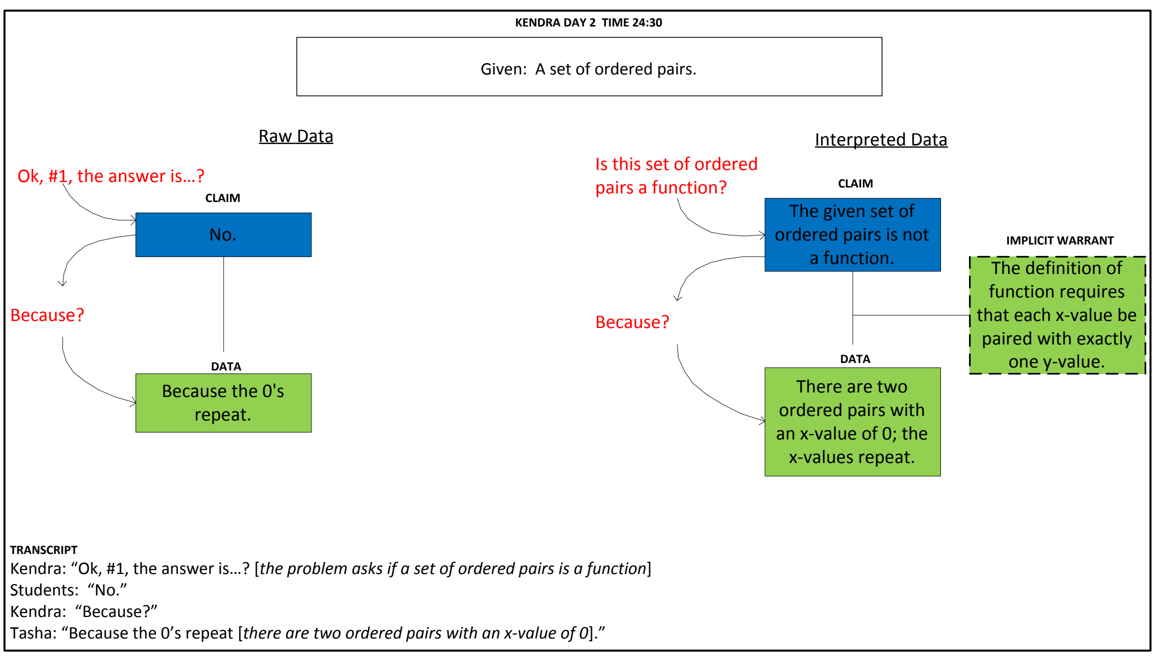


Figure 13. Kendra Argument 4.

This was the only argument that occurred in Kendra’s class that had multiple participants. Several students offered the claim that the set of ordered pairs did not represent a function, though only Tasha spoke up to provide the data. Perhaps because this example was almost identical to one completed the day before, Kendra did not prompt Tasha to explain the connection between her data that “0’s repeat” and conclusion that the ordered pairs did not represent a function.

Kendra: Argument 5. On the third day during the students’ work with the clickers as a review for an upcoming benchmark, Kendra asked the students to identify the type of problem represented by a word problem. Mark raised his hand and answered, “It’s a midpoint problem.” Kendra asked, “How do you know that?” leading Mark to reply, “Because it says halfway.”

The model of this episode is depicted in Figure 14. Kendra prompted the argument by asking the students to identify the type of problem this word problem represented. Mark offered the claim that “It’s a midpoint problem” which I understood to mean, “This problem requires the use of the Midpoint Formula.” When Kendra asked Mark, “How do you know?” and he replied, “Because it says halfway,” I took this to refer to the fact that the problem asked for a point halfway between two houses.

While the goal of the problem itself was to find the midpoint between two points, this argument focused on the students’ abilities to recognize specific types of problems. Kendra promoted this strategy to aid the students on tests and used this instance of argumentation to reinforce that strategy. When Mark offered the data that the problem contained the word “halfway,” Kendra did not prompt for him to explain further and left the warrant based upon the definition of midpoint unstated.

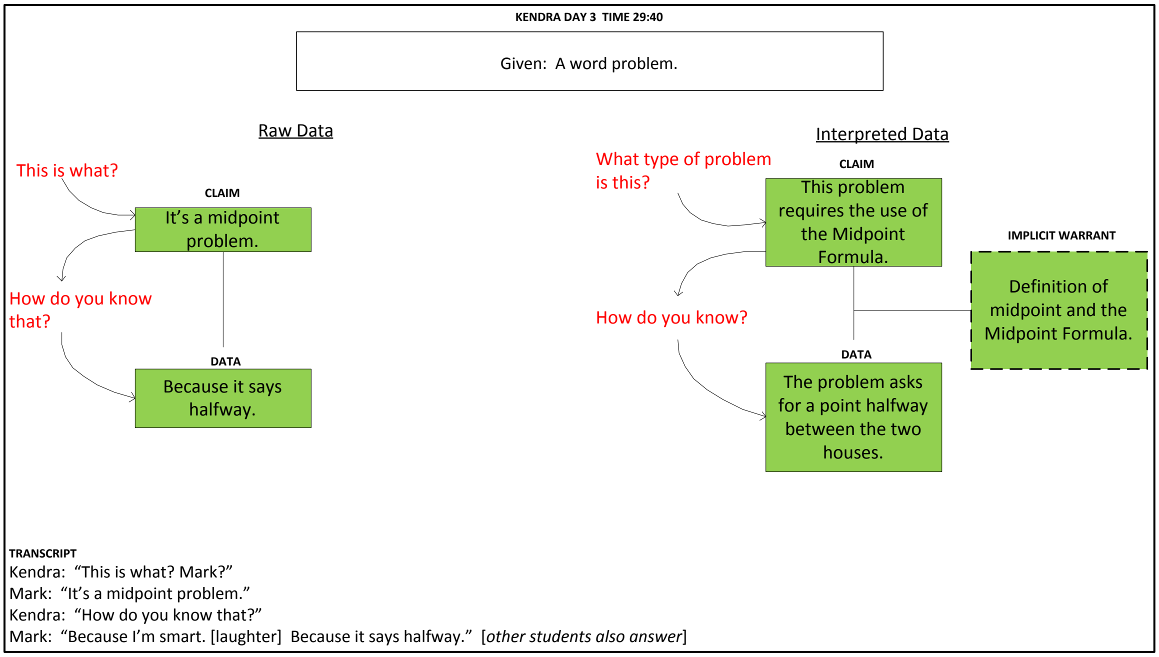


Figure 14. Kendra Argument 5.

Kendra: Argument 6. The final episode of argumentation that I observed in Kendra's class occurred on the fourth day. Kendra and the students began discussing the definition of and properties of matrices. She started by writing an example of a matrix on the board and defining what is meant by the "rows" and "columns" of a matrix. She then paused and asked the class, "What is the smallest matrix we can have?" For several minutes, the students discussed this among themselves as Kendra quietly waited. Finally, Abraham offered, "One by one." Kendra asked him, "Why one by one? What does that mean?" Abraham replied only one row and one column.

Figure 15 depicts the model used to analyze this episode of argumentation. Kendra prompted this argument with her question, "What is the smallest matrix we can have?" I interpreted Abraham's response, "One by one," to mean "The smallest matrix we can have is one by one." Kendra prompted for his data with her questions, "Why one by one? What does that mean?" Abraham offered, "Only one row and one column" which I understood to mean, "A matrix must have at least one row and one column." (Note that the possibility of an empty matrix was beyond the scope of this course.) Kendra did not prompt for an explanation as to why this must be so and left the warrant based upon the definition of a matrix unstated.

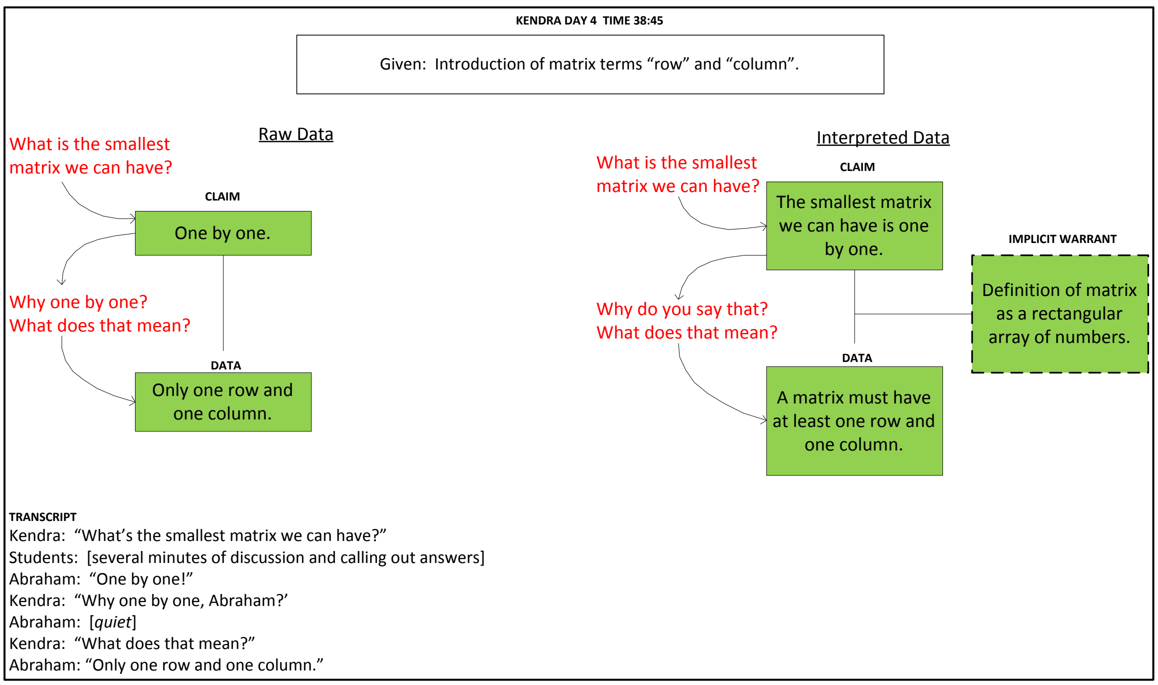


Figure 15. Kendra Argument 6.

This argument was unique among all of the arguments I observed for this study. Prompted by a discussion of new definitions and properties, the argument was not based upon the students' answer to or interpretation of a given problem. Instead, it pushed the students to broaden their mathematical thinking and offer insight about the characteristics of matrices that they had not previously discussed. However, since Kendra did not press Abraham for his reasoning, it is unclear how many students in the class understood his answer.

Observations from Mathematical Argumentation that Occurred in Kendra's Class

Several similarities emerged from the analysis of episodes of argumentation that occurred in Kendra's class. First, five of the six arguments concerned mathematics that the students had already experienced in Kendra's class. Further, the five arguments all occurred during periods of review and involved the definitions and properties of functions, lines, and midpoint.

Second, all of the arguments in Kendra's class followed the same structure. Five of the six arguments were based upon problems that the students had been asked to solve. Of those, three had claims that were the answers to those problems, and two had claims that indicated how to start the problems. Kendra prompted for data differently in every argument as she attempted to get the students provide additional information, yet all data were based upon definitions and properties. Four of the six warrants remained implicit, and in the two arguments with explicit warrants, one of those was prompted by Kendra and the other offered by the student with the data. All four of the implicit warrants were based on definitions or properties.

Finally, Kendra prompted and supported arguments for reasons other than promoting deeper mathematical understanding. Specifically, she used arguments to reinforce definitions and properties as well as for classroom management purposes. For example, both the first and the fourth episodes of argumentation reinforced the definition of a function and procedures for determining whether a given example represents a function. Likewise, the second and third arguments reinforced student understanding of different ways to write equations of lines and how to identify the slope of the line from those different forms. In terms of classroom management, Kendra used the instances of argumentation as an attempt to engage off-task students in the lesson. She also used them as a means to redirect outspoken students to contribute to the class discussion. For example, Kendra allowed Tasha to participate in three of the six arguments as she attempted to keep her from continually interrupting the lesson.

Summary of Mathematical Argumentation in Kendra's Class

From my observations of her class and the post-observation interview, it was apparent that this class represented a challenge for Kendra to keep on task and motivate to contribute. Her patience and persistence led her to continually attempt to engage the students in conversations about the problems they worked. Her description of herself as "a motivator" was apparent as she repeatedly gave the students opportunities to be successful through the use of almost daily review problems.

While five of the six arguments involved mathematics with which the students were already familiar and served as a means of reinforcing definitions and procedures, the sixth argument that occurred while discussing matrices was unique. As Kendra asked the students to determine the smallest possible size of a matrix based upon only the definitions of a matrix, rows, and columns, she presented the students with an opportunity to contribute to the discussion with comments that had not previously been discussed in class. For that argument, both the claim and data represented mathematical thinking that was new and original for the students.

Leslie

Description of and Characteristics of Leslie

Of the five teachers invited to participate in the study, Leslie was the first to respond to my email request to observe her Algebra I classroom for a week. An experienced teacher who has a strong sense of loyalty to her school and students, Leslie has been teaching there for 25 years. Leslie graduated from Hillside High School, where her father taught for 35 years. When she had the opportunity to return as a teacher, she described the experience as "coming home."

In our post-observation interview, Leslie expressed feelings of responsibility to and concern for her students, many of whom were repeating the Algebra I course. She explained how she tries to anticipate where the students will struggle with the content and what questions they will ask. She noted that she refuses to teach them "tricks" and instead focuses on being very detailed in her explanations. In her words, Leslie said, "I try to be very specific in the way that I explain how to do problems. I want them to understand…not just a trick that might work now and then not later." Leslie asks many questions to draw her students into discussions about mathematics and encourages them to engage in mathematical argumentation to justify the answers that they give. She believes students' hesitations in offering explanations or even answers comes from the fact that they have never experienced success in their mathematics courses. Leslie also spends time teaching her students how to participate in both partner and group work. She described these activities as useful ways of keeping the students engaged with the material and helping them to feel more comfortable about their mathematical abilities.

In the classroom observations that serve as data for the present study, I observed Leslie to be a calm and positive teacher who was genuinely respected by her students. She worked hard, in her words, to “make them [students] believe that I believe they can be successful.” Her caring and supportive demeanor helped create a classroom atmosphere where her students appeared to feel safe to answer and ask questions.

Description of Leslie’s Class

Leslie’s first block Algebra I students came in each day and took their calculators from the table, quickly found their seats, and got started on their warm-up assignment. From this, I inferred that her students were familiar with her expectations for their conduct, and I observed only a few minor classroom management issues during my week with them. Approximately half of this group of 15 students had failed to pass Algebra I in previous semesters. All of the students took a Foundations of Algebra course in the spring semester immediately prior to this one. Designated as an elective mathematics credit, this introductory course was designed to study concepts such as number and quantity, functions, and data and statistics to better prepare students for success in Algebra I.

Though the students started off each day with their desks in traditional rows, they often moved throughout the room to work with partners or in groups. Based on what appeared to be normalized routines in her classroom, I inferred that Leslie has taught them how to participate in such activities, and they clearly knew her expectations during these periods. It was also evident that the students were accustomed to talking with their classmates and helping each other during practice and review activities. In almost all cases where the students talked together during practice and review activities, I observed their discussions to be about the problems on which they were working. However, across the five days of observation, when Leslie attempted to get the students to participate in whole class discussions, only two of the students consistently were willing to speak.

Description of Observations

During my observations, Leslie and the students worked on lessons pertaining to the rules of exponents for three of the days. On the first day, the students reviewed with Leslie what they remembered about the rules of exponents as they apply to the real numbers. Then, they began a graphic organizer which extended the properties of exponents to expressions containing variables. Leslie gave the students several minutes to assemble the graphic organizers using colored paper, scissors, and tape. Once the organizers were constructed, Leslie carefully read through the first property, the Zero Property of Exponents. She discussed the common misconception held by students that $x^0 = 0$ and showed the progression $2^4, 2^3, 2^2, 2^1, 2^0$ to help explain why this assumption is incorrect. Leslie then helped the students complete several examples requiring the use of this property. They continued this pattern of instruction with three additional properties; Leslie introduced each property and then the students worked with Leslie to complete several examples exemplifying that property. The students spent the last ten minutes of class working on a combination of problems. Some students chose to work individually while others chose to work quietly with their neighbors.

All of the second day was spent in a similar fashion. After reviewing questions about the homework assigned on the first four properties of exponents, Leslie introduced the three remaining properties and led the students through multiple examples of each of them. Once all the properties had been discussed in detail, Leslie gave the students a worksheet with examples of the same properties with different wordings and containing different variables. The students spent the remainder of the class attempting to match the new examples to the ones in their graphic organizer.

By the third day, they had completed numerous examples of each property of exponents. Leslie divided them into teams, and they played a mathematical version of the game Connect Four with problems requiring the students to simplify expressions with exponents. On the fourth day of my observations, the class reviewed for 45 minutes and then took a quiz on the rules of exponents. Following the quiz, they began a chapter on polynomials. The remainder of that day, along with the fifth day, were spent on the topics of addition, subtraction, and multiplication of polynomials. As she did with the properties of exponents, Leslie introduced each of the topics, pointed out areas of common mistakes, and worked through examples while soliciting help from the students. She then gave them a worksheet focusing on perimeter and area problems from Geometry, which required the addition and multiplication of polynomials. The students worked alone and in pairs to complete the problems. As they worked, Leslie circulated around the room and answered any questions that they asked. They then came back together for a whole class discussion about the problems, their answers, and any remaining areas of uncertainty.

Examples of Mathematical Argumentation in Leslie's Class

Three instances of mathematical argumentation occurred during the five consecutive observations. While two occurred on the first day and one on the fourth, all three episodes dealt with properties of exponents.

Leslie: Argument 1. On the first day of my observations, in an attempt to help the students understand the Zero Power Rule of Exponents, Leslie showed a sequence of powers on the board: 2^4, 2^3, 2^2, 2^1, and 2^0 and asked the students if they could evaluate those powers. After discussing 2^2, 2^3, and 2^4, an argument occurred when Leslie prompted, "Does anyone know 2^5?" After a pause of a few seconds, Keisha replied that the answer was 32. Leslie then nodded encouragingly to Keisha and asked, "Uh-huh, how did you get that?" Keisha justified her answer by indicating that she had done the calculation in her head.

Figure 16 depicts the model used to analyze this instance of argumentation. Leslie prompted this argument with her question, "Does anybody know 2^5?" Keisha offered the claim of 32 and Leslie encouraged her to provide more information by asking, "How did you get that?" Keisha's data in support of her claim was the comment that she "did it in [her] head" which I more formally interpreted to mean that she did the calculation mentally. Leslie did not prompt for additional details and left the warrant based upon the definition of an exponent unstated.

In this argument, Leslie pressed Keisha to provide her process to finding a solution. By nodding encouragingly and indicating that Keisha's initial answer of 32 was indeed correct, her demeanor was supportive and nonthreatening. Based on my interview with Leslie, I believe that her attempt to draw Keisha into a mathematical argument suggested confidence that Keisha would be willing to attempt to articulate her thinking. Keisha's response, though acceptable to and understood by Leslie, was not necessarily sufficient for her classmates to understand how she came to the answer of 32. Because Keisha did not provide more details and make a clear connection between the process she did in her head and the answer she gave, the argument was left with an implicit warrant implying that everyone understood Keisha's meaning.

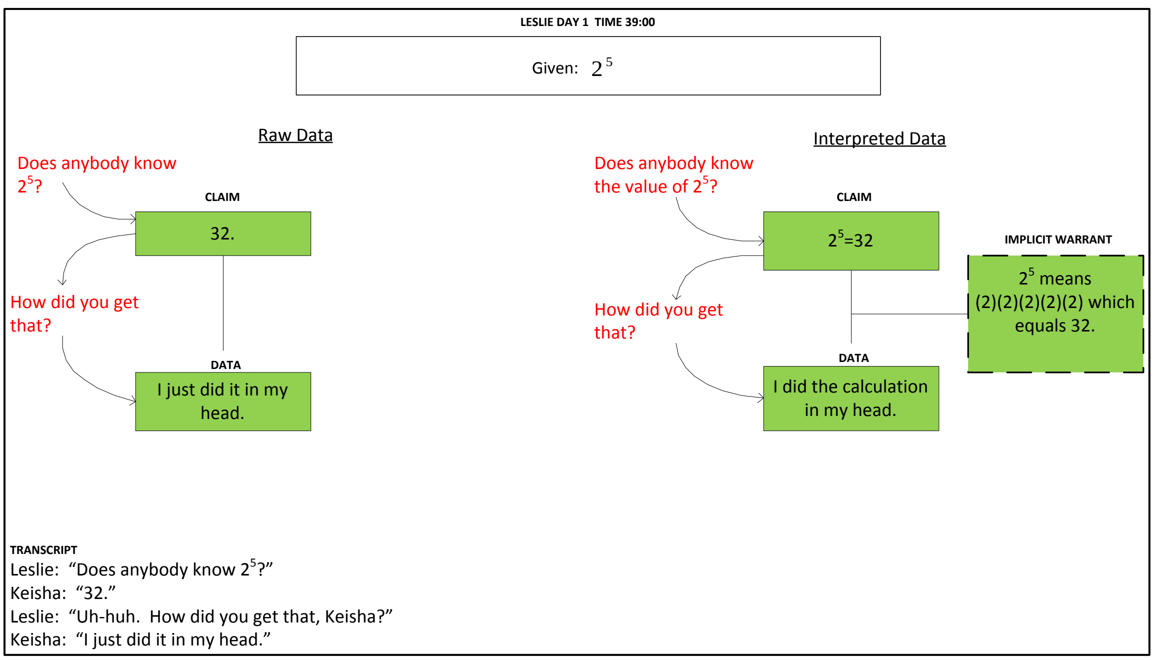


Figure 16. Leslie Argument 1.

Leslie: Argument 2. Later in that same class, after working various examples containing variables with exponents, Leslie presented the students with the expression $-3x^{-3}y^2$ and asked that they simplify it. Terry offered the answer $\frac{y^2}{3x^3}$ which led to murmurs of confusion from several students in the class. Without confirming whether or not Terry's answer was correct, Leslie asked the other students if they agreed with his response. Noticing the manner in which John shook his head that he did not agree, Leslie directed her question to him. John then made the statement that the answer was not correct, leading Leslie to prompt for why he believed the answer was different from the one given by Terry. John supported his answer by offering that the coefficient -3 should remain in the numerator with the variable *y*. Leslie then revoiced his statement and pushed John to explain further. John offered the observation that "it didn't have a negative exponent" as his explanation for leaving the -3 in the numerator of the answer.

The model of this episode of argumentation is depicted in Figure 17. I interpreted Leslie's question, "Are we good with this?" to mean "Do we agree that this answer is not correct?" based upon the way in which she initially responded to the incorrect answer with the statement, "Ok. Anybody have another option?" I understood John's claim, "I don't think so" to mean that he believed the answer given by his classmate was incorrect. Leslie prompted him to provide the data behind his claim, which led him to respond, "I would keep the -3 with the y^2." In this case, Leslie requested an explicit warrant from John as she repeated his response and then asked, "Why?" John offered his reasoning by stating, "Because it didn't have a negative exponent." I interpreted his response to mean, "The -3 does not have a negative exponent and thus should stay in the numerator" based upon the Property of Negative Exponents.

In this episode, Leslie directed her questions to Josh after noting his challenge to Terry's answer, as evidenced by him shaking his head in disagreement. Leslie then pressed John to justify why he disagreed with the answer given by his classmate without indicating whether or not Terry's answer was correct. After revoicing John's statement about the position of the -3 in the answer and marking its importance to the other members of the class, she urged him to explain further. By prompting him to provide the explicit warrant of this argument, she gave him the opportunity to describe how his reasoning differed from that of his classmate.

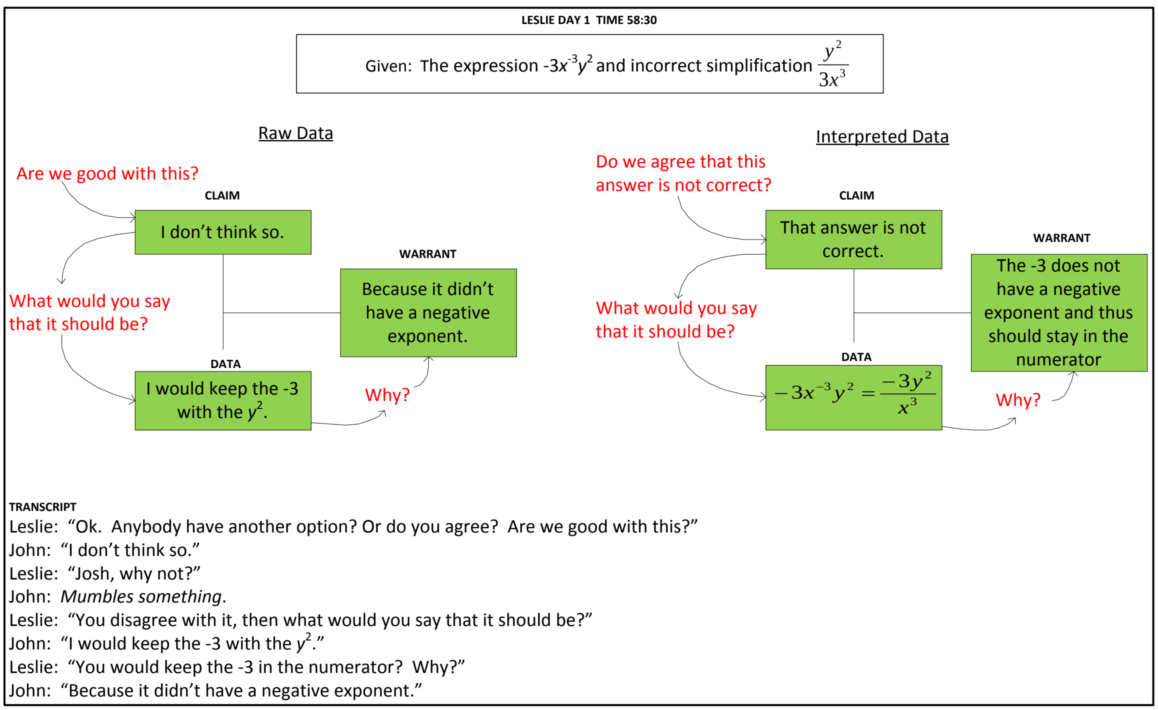


Figure 17. Leslie Argument 2.

Leslie: Argument 3. The third episode of argumentation occurred on the fourth day after the students had completed their quiz on the rules of exponents. They began a lesson on polynomials, and Leslie put several examples on the board and asked the students if they could "combine like terms." A brief discussion about the meaning of "like terms" followed, with Amal offering that like terms must contain the same variables and Leslie confirming that, "Yes, they must have the same variable and same exponent." As the students collectively helped Leslie simplify several polynomials, it was apparent that they were consistently listing the terms of the polynomials in descending order. When they reached the third problem, $7 + 2b^2 - b - 4 - b^2 + 5b$, and the students offered the answer beginning with the term b^2, Leslie took the opportunity to connect the problem back to the properties of exponents. She pointed to the constant term in the answer, 3, and asked how it fit into the descending order of exponents on the variable *b*. As the students hesitated to reply, she wrote a *b* beside the 3 and pointed to the exponent of the *b* as she asked, "What would the exponent have to be for this term to still be 3?" At this point, John offered that the exponent must be 0. Leslie prompted for more information which led John to explain that, "$b^0 = 1$ and then 1 times 3 is 3." Leslie nodded encouragingly and confirmed, "Exactly!" before moving on to the next example.

Figure 18 depicts the model used to analyze this episode of argumentation. Leslie prompted this argument by pointing to the variable *b* she had written beside the constant 3 and asking, "What would the exponent have to be for this term to still be 3?" I interpreted this to mean symbolically, "What does *x* need to be for $3b^x = 3$?" John responded with a claim that the exponent should be 0, and Leslie prompted for more information with the question, "Why 0?" John offered data to support his claim with the statement, "Because b^0 is 1 and then 1 times 3 is 3." The implicit warrant based upon the Zero Property of Exponents was left unstated.

Here, Leslie asked the students to consider the final term in the polynomial, 3, and if it could be written as 3 times *b* to some power without altering the value of the polynomial. As with Keisha in an earlier exchange, I believe that Leslie was confident that John would be able to articulate his reasoning and explain his thinking to his classmates. Perhaps due to the fact that they had just completed the unit on the rules of exponents, Leslie accepted John's data without pressing for further explanation and thus left this warrant implicit.

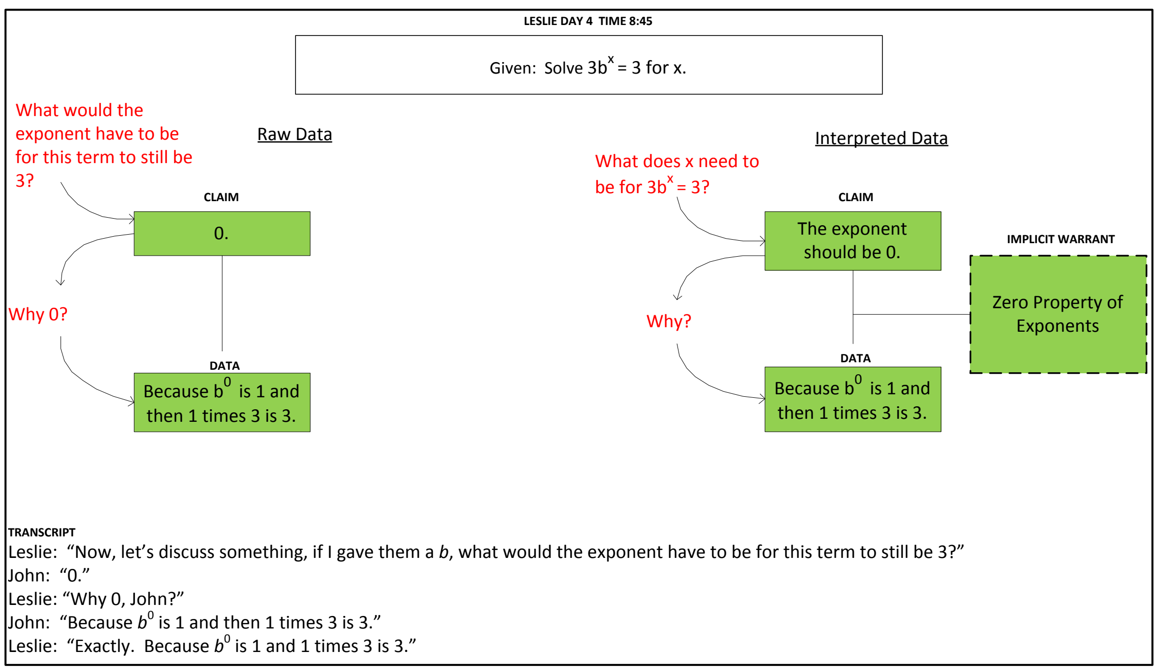


Figure 18. Leslie Argument 3.

Observations from Mathematical Argumentation that Occurred in Leslie's Class

I observed several similarities when examining the arguments that occurred in Leslie's class. First, all three arguments concerned mathematics that the students had already experienced in their curriculum. Two of the problems concerned simple examples of the rules of exponents. In the first argument, the problem dealt with a whole number raised to a positive power, which is a basic pre-Algebra skill. In the third argument, students intuitively solved an equation of the form $3b^x = 3$. Though more complex than the first example, this solution required merely the recall of the Zero Property of Exponents. Of the three problems that Leslie used to prompt episodes of mathematical argumentation, only the second problem containing multiple variables, negative coefficients, and negative exponents required skills learned in Algebra I.

Second, all three arguments followed the same structure. Leslie prompted for claims and data with various questions such as, “Does anybody know?” and “Are we good with this?” while exhibiting a calm, supportive demeanor. All of the arguments were between Leslie and only one student. The first argument occurred between Leslie and Keisha and the remaining two arguments between Leslie and John. As Leslie noted in our post-observation interview, she did not randomly call on students since she did not wish to “put them on the spot” and cause them to feel unsuccessful. She carefully selected the students who she would ask for elaboration about their reasoning based on their willingness to participate as indicated by their raised hands or nodded responses. All data offered by students relied upon a definition or property. In the first argument, the data relied upon the basic definition of exponents and in the second and third arguments, the data relied upon the Property of Negative Exponents and the Property of Zero Exponents, respectively. Leslie did not push for explicit warrants across the episodes. In the first argument, Leslie accepted Keisha’s claim and data without asking for further clarification, perhaps due to the fact that the problem required only pre-Algebraic skills based upon the definition of exponent. Likewise, the warrant for the third example pertained to a property of exponents previously covered that week and it was also left unstated. The one instance of an explicit warrant occurred when John simply offered it along with the data in the second argument. Allowing arguments to conclude unstated warrants assumed that the reasoning between the claim and data was clear to all members of the class.

Finally, Leslie prompted and supported arguments in her classroom for reasons other than promoting deeper mathematical understanding. Leslie used arguments as a means of building student's confidence in their mathematical abilities and only called on students who volunteered or indicated that they were willing to participate. Further, she only pursued arguments with those students who she felt, based upon her prior knowledge of the students, would be able to answer correctly. It is also evident that Leslie relied upon mathematical argumentation to reinforce her students' understanding of definitions, as in the first argument, and properties, as in the second and third arguments.

Summary of Mathematical Argumentation in Leslie's Class

From the post interview, Leslie's concern for her students and desire to have them experience success in mathematics mediated the choices she made in the classroom. Believing that most of the students had little confidence about their mathematical abilities, Leslie explained that she stressed to them that "you won't be the first person who's ever given a wrong answer in this classroom!" as she encouraged them to participate. She noted that "some of these kids have never experienced any success in mathematics and . . . they are [not] going to talk about it, let alone try to make somebody think they know what they're doing."

As she attempted to make her classroom safe and non-threatening, she addressed questions which promoted mathematical argumentation only to those students she felt would be able to correctly answer her. Leslie carefully monitored her students' progress as they worked on problems independently and was deliberate in her choice of students to call on for answers. I believe this was a result in part of Leslie not wanting her students to feel intimidated or to feel as if they had failed in front of their peers if they gave a wrong answer. Her phrases that often preceded these episodes such as "Does anyone feel comfortable . . .?" and "Do we have someone in that group that feels secure about her answer . . .?" lend support to this interpretation of her actions. Admirable and understandable in her intent, this practice of carefully selecting students to press for more information allowed mathematical argumentation to occur in the classroom in a positive, supportive manner that reinforced both the students' prior mathematical knowledge and the properties and definitions of their current lesson.

Will

Description of and Characteristics of Will

Will is a veteran teacher with 25 years of experience teaching mathematics at four different high schools. He spent 15 of those years at Hillside High School where he and Leslie were colleagues. During the 2011-2012 school year, he transferred to Lakeside High School to accept a position that allows him to teach for half of the school day and to fulfill administrative duties for the remainder of the day. Will was assigned a large ESL inclusion class as one of his two Algebra I classes. My observations of this Algebra I class and post-observation interview with him occurred within a month of his move to Lakeside.

During the post-observation interview, Will explained to me that he has the philosophy of wanting students to understand mathematics as opposed to just getting the answers and, in his words, he “[tries] to teach them how to think.” He worries that students do not see the real-life applications of the mathematics they learn in school. He noted, “I think that they often don’t see the big picture. I start with that and then move back to the ‘pieces.’” He begins his lessons with realistic situations that students might not usually connect to their mathematics lessons. For example, to introduce a lesson on direct variation, he talked with the students about the relationship between a child’s age and shoe size. In our interview, he emphasized the importance of the relevance of the topics he teaches and described his pleasure when his students are able to relate mathematics to their daily lives.

From my observations of Will’s classroom, he asked numerous questions throughout his lessons and selected different students to answer. His instructional routine appeared to have students work through examples with him or to have them work independently before sharing their solutions with the whole class. Will explained during our interview that he believes that students working together requires a certain amount of maturity, and that it is his responsibility to know the students’ strengths and weaknesses before assigning them to work together. Since he had only been working with these students for a few weeks at the time of my observations, I concluded that he felt a lecture format was the most appropriate method of instruction at that time. I also took this to mean that he planned to incorporate more partner and group work into his lessons once he felt as though he knew the students better.

Of all the teachers I observed, Will expected and maintained the highest levels of courtesy and respect among his students. Throughout the week, Will consistently reminded the students that any unkind phrases or rude behaviors would not be permitted. When I asked him about this during our interview, he pointed to all of the motivational posters around his classroom and explained that as a teacher, he "expects the same level of respect from his students as they would show a pastor or their grandmother." Further, he believes that establishing such relationships and boundaries promotes trust between students and teacher.

Description of Will's Class

Will's second block Algebra I class of 18 students came in each day to a classroom with assigned seats and desks in traditional rows. An ESL and inclusion class, these students had been together throughout the previous fall semester with another Algebra teacher who was fluent in four languages. I noticed several times throughout my week of observations that students would quietly translate Will's comments into a different language for a classmate seated nearby. Overall, they seemed willing to participate in the lessons, and there were few discipline issues throughout my week with them. When Will addressed questions to the class as a whole, the students were eager to answer and often several of them spoke at the same time. The most prominent behavioral issue that Will continually corrected dealt with the students' manners towards one another and Will's insistence on politeness.

Description of Observations

On the first day of my observations, the students spent approximately 30 minutes completing test revisions before Will began the day's lesson. As an introduction to linear functions, he began with a graph depicting the speed of a boy's walk through his neighborhood from his home to a friend's. After discussing the graph with the students, Will linked their discussion to the topics of position and time followed by dependent and independent variables. He then passed out a packet of worksheets for them to use throughout the remainder of the lesson and used the document camera to go over basic definitions and examples of relations, functions, domain, and range.

On the second day of my observations, Will and the students returned to the packet of worksheet. An hour of the class was spent with Will introducing definitions and properties, showing the students how to work specific examples, and having the students work similar problems independently. For the problems that the students worked independently, Will called on students for their answers. With 30 minutes left in the class, Will distributed a new worksheet and began a discussion about direct variation and proportions. He used an example about a child and her shoe size to try to connect the topic to the students' lives. They worked through several examples together until the bell rang to dismiss class.

All of the third day and half of the fourth day followed the same pattern as Will and the students worked to complete the packet on linear functions and the worksheet on direct variation. Will remained at the document camera at his desk throughout the time as he worked through examples or filled in answers from problems the students completed independently. The students then took a short quiz that Will collected for a grade. Following the quiz, Will instructed the students to get out a piece of paper for new notes and moved on to the next section on rate of change. For the rest of the fourth day and all of the fifth day, Will and the students worked in a similar fashion to discuss properties related to rate of change and to complete numerous examples, both together as a class and independently.

Examples of Mathematical Argumentation in Will's Class

Five episodes of mathematical argumentation occurred during my week of observations in Will's class.

Will: Argument 1. On the first day of my observations, after the test corrections were completed, Will and the students discussed the definitions of functions, relations, domain, and range. Will pointed out important aspects of the notation with which students would need to be familiar and worked through several examples with them demonstrating how to identify functions given certain information. With just a few minutes left in the class, Will had the students consider a set of ordered pairs on their own. He then pointed to the set of ordered pairs and asked, "Is the following a function?" Several students quickly responded, "No." Will observed, "That was a quick 'no.' Why is it not a function?" Brandi answered that "the x's in the ordered pairs repeat."

The model of this episode of argumentation is depicted in Figure 19. Will prompted this argument with his question, "Is the following a function?" which led several students to offer the claim, "No." Will then pressed for more information by asking, "Why is it not a function?" At that point, Brandi spoke up with the data that "the x's repeat." I interpreted this to mean, "The x-values in the ordered pairs repeat." Will did not prompt for her reasoning and the warrant based up the definition of function was left implicit.

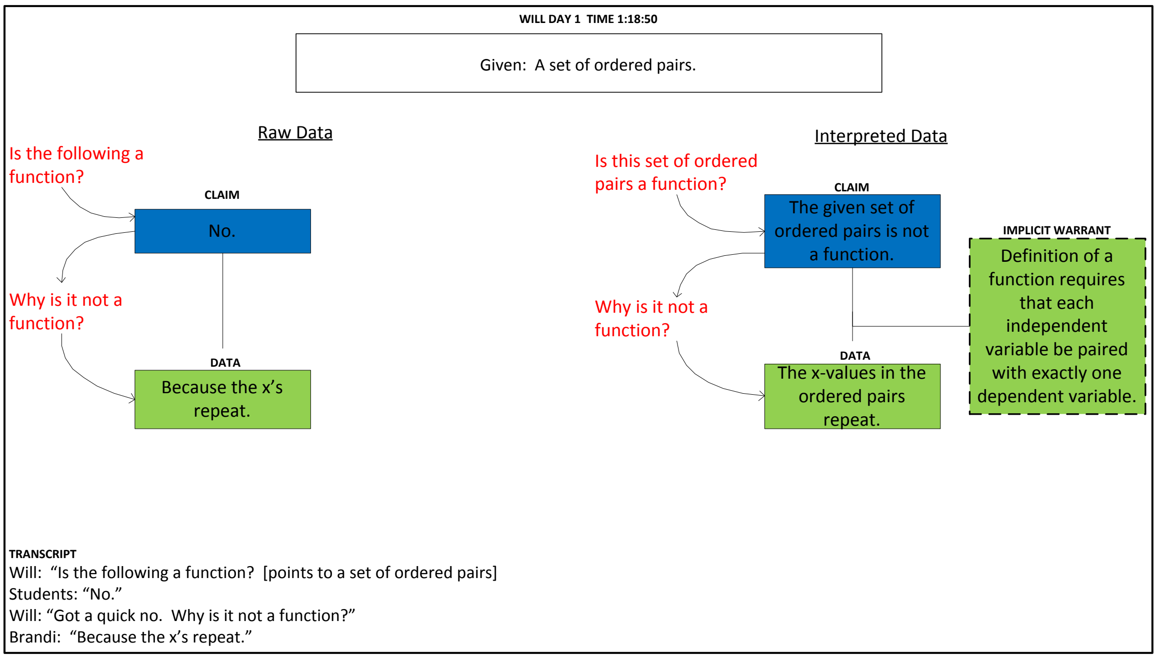


Figure 19. Will Argument 1.

This argument is representative of four of the five arguments that occurred in Will's class during my week of observations. Based on a problem given to the students to judge their level of understanding of the definition of function, this argument was used to reinforce the procedure that Will wanted the students to follow to determine whether or not a set of ordered pairs represents a function. The episode allowed Will to assess the students' abilities to repeat that procedure as well as their memorization of the definition of function.

Will: Argument 2. The second episode of argumentation occurred on the second day of my observations. The class was still working on methods for identifying functions. The problem prompting this argument was presented as a table and contained the names of US Open winners and years. The intent of the problem was to determine if the information in the table represented a function. Will asked, “Do any of the years repeat?” and multiple students answered, “No.” After looking around the classroom, Will expressed concern that not all the students understood the reason for that answer and asked, “Marri, why did you write ‘no’?” Marri responded, “Because none of the years repeat.”

Figure 20 depicts the model used to analyze this episode of argumentation. It should be noted that the original problem asked the students to determine if the table of values represented a function and the answer to that question was “yes.” In his attempt to reinforce the manner in which he wanted the students to approach the problem, Will asked a different question, “Do any of the years repeat?” Since the years represented the independent variable in this problem, the answer to Will’s question was “no.” As he pressed Marri to give her justification for answering, “No,” she replied, “Because none of the years repeat.” Will did not ask for an explicit warrant based upon the definition of function and it remained unclear to me whether or not the students thought the table represented a function.

This instance of argumentation differed from the first episode in that Will called on a specific student for the data. Otherwise, it followed the same structure, with a problem requiring a “yes” or “no” response being used to determine the students’ ability to identify a function from a table of values. This argument was somewhat confusing due to the fact that the answer to the problem on the worksheet, "Is this a function?" was in fact "yes" and yet the discussion revolved around the students' answer of "no" to Will's separate question, "Do the years repeat?"

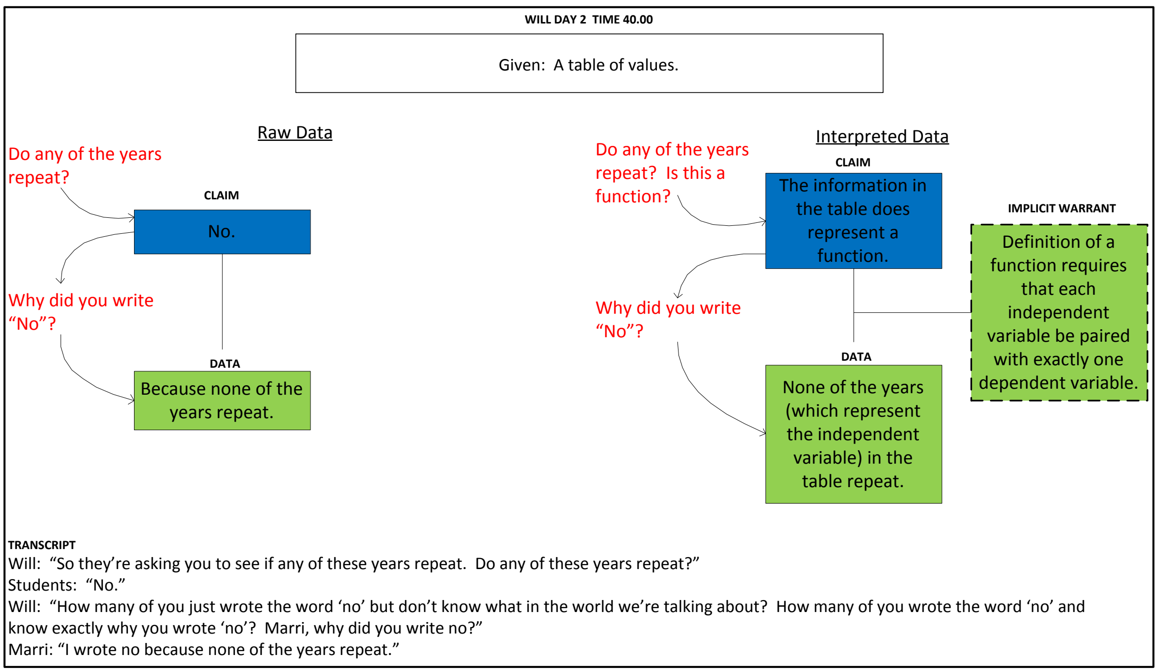


Figure 20. Will Argument 2.

Will: Arguments 3 and 4. The remaining two episodes that followed this pattern both occurred on the fourth day of my observations. Will and the class returned to studying direct variation problems, a topic that they had already briefly covered. They discussed the definition of direct variation and worked through several examples together before Will instructed them to work independently on two problems. The directions were for the students to determine if the given equation represented a direct variation. After allowing time for the students to work, Will called on two different students for their response and then pressed them for a reason by asking, “Why?”

Figure 21 depicts the model used to analyze the first of these two episodes. In this episode, Will prompted, “I need another person to raise their hand and tell me. Jillian?” Jillian responded with the claim, “No,” which I took to mean that the first equation did not represent a direct variation. Will asked for her data with the questions, “Can you tell me why?” and she replied, “Because there’s a number behind the x.” I interpreted this more formally as, “There is a constant term added to the term containing x.” The warrant based upon the definition of a direct variation was left implicit.

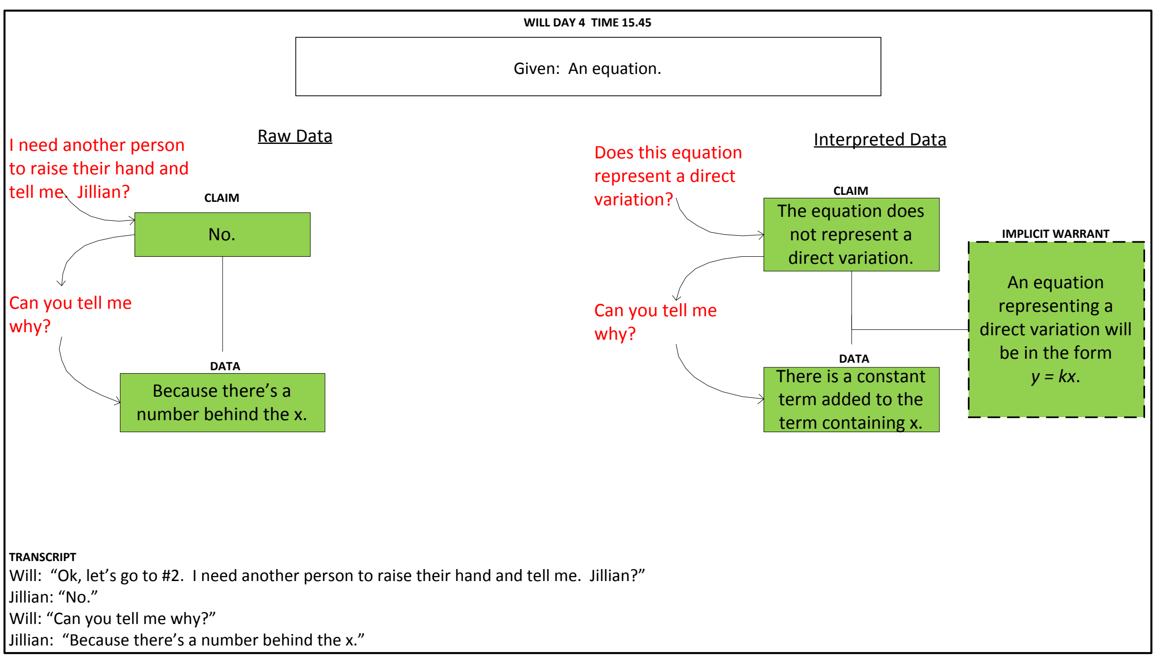


Figure 21. Will Argument 3.

Figure 22 depicts the model used to analyze the second of these two episodes. As in the previous argument, Will called up a specific student, John, to answer whether or not the equation represented a direct variation. John offered the claim, "Yes," and Will prompted for more information with the question, "Why?" I took John's data, "Because it's by itself" to mean "the equation does not have a constant term added to the term with x." Again, the warrant based up the definition of a direct variation was left unstated.

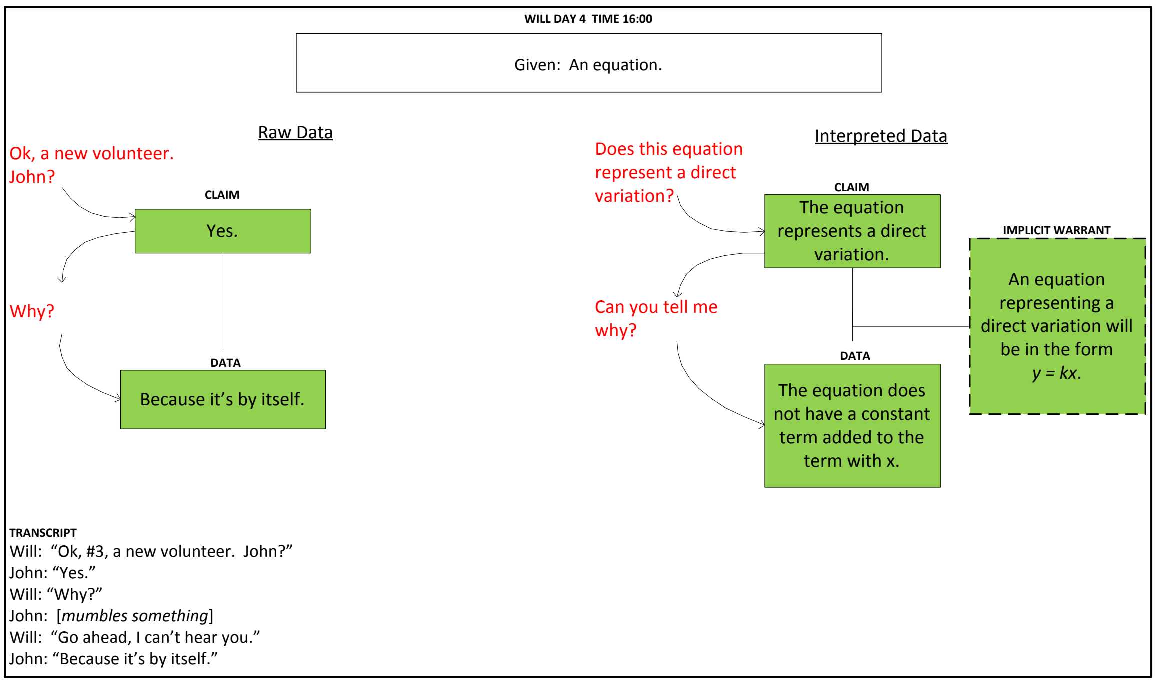


Figure 22. Will Argument 4.

Both of these episodes followed the same structure, with a problem requiring a "yes or no" response being used to determine the students' ability to identify a direct variation. Will's prompts for data were used to have the students repeat the evidence they used to determine their answer. This evidence was based directly upon the definition of direct variation, though Will did not ask either student to explicitly state their reasoning.

Will: Argument 5. One additional instance of mathematical argumentation that was somewhat different from the other four episodes occurred on the third day of class. Students were considering various types of examples to determine if they represented functions. This example involved the graph of an equation. Will prompted the argument by saying, "Ok, let's look at this. Vertical line test. What do they want for us to do?" He then drew a vertical line through the graph and turned to the class, asking, "So what do I need to do? Stop or keep going?" Several students responded, "Stop!" and Will encouraged them to continue by asking, "Why?" Marri replied, "Because it's not a function," but was unable to offer more details. Will then called on Trey and asked that he help Marri. Trey offered, "Because the number repeats."

The model of this episode of argumentation is depicted in Figure 24. Will prompted this argument as he demonstrated how to apply the Vertical Line Test (VLT) to a graph of an equation. When he asked whether he should stop or keep going, I interpreted this to mean, "The vertical line has intersected the graph twice, so do I stop or keep going?" Several students offered the claim, "Stop" which I understood to mean that they felt the VLT was complete, because it had intersected the graph more than once. Will pressed for data by asking, "Why?" and Marri responded, "Because it is not a function." When Will then asked, "And why is it not a function?,"Marri was unsure of how to respond. Will called on one of her classmates to assist and Trey offered the warrant, "Because the number repeats." I interpreted this to mean, "There are two points with the same x-value."

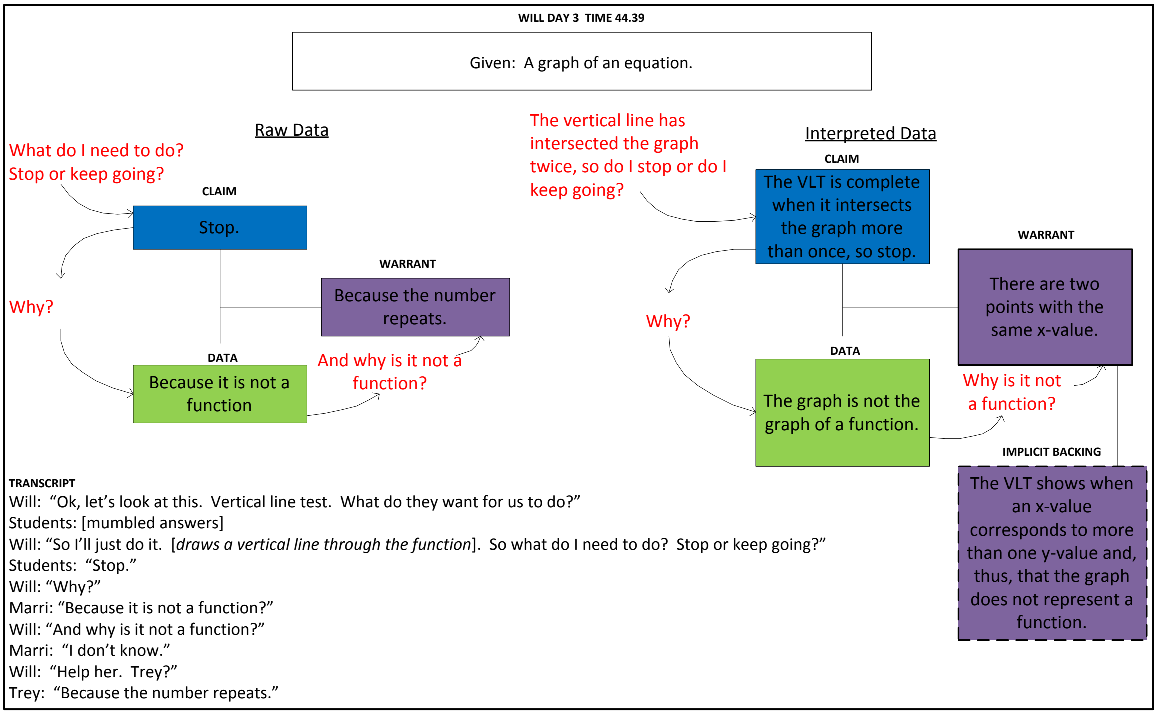


Figure 23. Will Argument 5.

This episode had two unique characteristics. First, though it started out just as the other instance of argumentation with a request to classify the example as a function or not, it proceeded beyond the simple statement of claim and data. In this example, Will prompted for the connection between the claim and data, which led Trey to offer an explicit warrant. The second unique characteristic of this argument stems from the statement of the explicit warrant. In this instance, backing for the warrant is needed to explain the connection between the vertical line intersecting the graph twice and the failure of the graph to represent a function.

Observations from Mathematical Argumentation that Occurred in Will's Class

Several similarities are evident when examining the arguments that occurred in Will's classroom during my observations. First, all five of the arguments dealt with the mathematics of that day's lesson. Three of the arguments reinforced the definition of a function and the procedures for recognizing them. Similarly, the remaining two arguments involved the definition of a direct variation and the procedures for identifying them.

Second, all of the arguments followed a similar structure. All prompts for claims were prompts for "yes/no" answers to exercises the students had just completed. All five of the exercises were of the form, "Is this a function?" or "Is this a direct variation?" All prompts for data were the simple question, "Why?" and all data were based on the definitions and properties of either functions or direct variations. In four of the five arguments, Will did not prompt for an explicit warrant and in all of those cases, the warrant was based upon a definition. Also of interest in term of the arguments' structure, in the three arguments where Will did not specifically call on a student to offer the claim, multiple students offered a choral response. When he prompted for more information, one student spoke up to offer the data.

Finally, Will prompted and supported arguments for reasons other than promoting deeper mathematical understanding. Specifically, he used arguments as a means of reinforcing students' understanding of definitions and memorization of procedures and to assess students' comprehension of those definitions and procedures. For example, in the three arguments where the goal of the problem was to determine if the given information represented a function, the arguments addressed a procedure, such as examining the list of independent variables or doing a vertical line test. Likewise, the two arguments involving the ability to identify a direct variation focused on the students' abilities to determine if the example matched the pattern given in the definition. Since Will did not prompt for explicit warrants in four of the five arguments, it was implied that the students were able to make the connections between the examples, procedures, and definitions.

Summary of Mathematical Argumentation in Will's Class

From my observations of his class and the post-observation interview, it is clear that Will is a dedicated teacher who does what he thinks is best for his students. From his focus on manners to his real-world examples, Will approaches teaching with the intent of preparing students for life beyond school. In our interview, he expressed his desire to encourage students "to become genuinely interested in math" and to "think outside the box." Though these two wishes were not readily apparent during my observations, I believe Will's recent move between schools played a role in this since he was still getting to know these students. Will encouraged the students to participate in the lessons, as evidenced by the many questions he asked. Though the five episodes of argumentation were used as opportunities to reinforce the students' understanding of definitions and procedures that would be helpful on standardized tests, they did represent instances where multiple students shared their thoughts with the rest of the class.

CHAPTER V

CROSS-CASE ANALYSIS

In the previous chapter, I described the episodes of argumentation that occurred in five teachers' classrooms, my analysis and interpretation of each episode, and my summarization of the mathematical argumentation in each classroom. I observed similar argument structures in these classrooms, where the teachers facilitated arguments by requesting answers to specific problems and students participated by providing those answers and the reasoning behind them. In this chapter, I look across the five cases to answer the research questions presented in Chapter II (and repeated here):

1. What is the nature of mathematical argumentation in these classrooms?
2. For what goals do teachers foster mathematical argumentation in these classrooms?

To this end, I first examine the nature of mathematical argumentation in these classrooms and discuss the common characteristics of the arguments detailed in Chapter IV. Then, I discuss three primary goals for teachers' uses of mathematical argumentation in their classrooms. Throughout the chapter, examples used to illustrate these findings are selected for clarity and are intended to be representative of the cases.

Question 1: What is the Nature of Mathematical Argumentation in these Classrooms?

To address my first research question, I describe five characteristics that help portray the nature of the mathematical argumentation that occurred in these classrooms. I begin with a discussion of the frequency of arguments that occurred in each classroom during my five days of observation, an analysis of the prompts that the teachers used to encourage those arguments, and evidence of what were acceptable data in the arguments. Next, I discuss the presence of explicit and implicit warrants and the potential effects on student learning and offer evidence of the potential to move beyond the IRE structure of classroom discourse. Then, I examine the mathematical content of the arguments and the participants in the arguments and conclude by summarizing these findings in relation to the first research question.

Frequency of Arguments and Description of Prompts and Data

Frequency. The frequency of mathematical argumentation occurring in these classrooms was minimal, as shown in Table 4. While the teachers overwhelmingly asked many questions throughout the observations and discussions of mathematical content occurred, my use of Toulmin (1958/2008) for analysis specifically required the presence of both a claim and data to designate the discussion an argument. While the number of occurrences of mathematical argumentation was modest, it is of note that all five teachers demonstrated the ability to engage and support students in argumentation. From their interviews, it should be recognized that mathematical argumentation was not a focus of instruction for these teachers.

Table 4

Frequency of Mathematical Argumentation per Teacher's Classroom

Teacher	Abby	Denae	Kendra	Leslie	Will	Total
Episodes	2	3	6	3	5	19

Prompts. To prompt students to supply data in support of their claims, the teachers used the simple question, "Why?" in 10 of the 19 arguments. In 5 of the 19 arguments, the teachers asked "How do you know?" to encourage students to continue beyond the statement of the claim. In both arguments that occurred in Abby's class, as well as one in Denae's class and one in Kendra's, the prompts used to elicit data were encouraging phrases, such as "Keep going!" or questions of other types, like "Is there anything bigger?" Table 5 shows the breakdown for each teacher.

Table 5

Descriptions of Teachers' Prompts for Data

Teacher	Why?	How do you know?	Encouraging phrases/Other questions
Abby (2)	0	0	2
Denae (3)	0	2	1
Kendra (6)	3	2	1
Leslie (3)	2	1	0
Will (5)	5	0	0
TOTALS (19)	**10**	**5**	**4**

Data. Regardless of the prompt, the data for all nineteen arguments were based upon mathematical definitions, properties, or procedures (see Table 6). For example, both of the arguments that occurred in Abby's classroom involved the Substitution Property of Equality as the students factored polynomials. In Denae's class, all three claims were supported by data based upon procedures such as multiplication or calculator use. Five of the six arguments in Kendra's class were established by definitions of functions, midpoints, and matrices, while the remaining argument used a procedure to justify the claim. Leslie accepted a combination of definitions and properties, while all arguments in Will's class were justified by definitions.

Table 6

Types of Accepted Data

Teacher	Definition	Property	Procedure
Abby (2)	0	2	0
Denae (3)	0	0	3
Kendra (6)	5	0	1
Leslie (3)	1	2	0
Will (5)	5	0	0
TOTALS (19)	**11**	**4**	**4**

In all the cases of argumentation, the teachers accepted definitions, properties, and procedures that had been previously stated either in that lesson or earlier as suitable justification of the students' claims. Though it cannot be concluded that these are the only forms of data the teachers would accept, the types of problems used to prompt these arguments directly promoted data of these kinds. Further, the teachers' acceptance and endorsement of definitions, properties, and procedures suggests that the teachers valued the students' abilities to repeat information verbatim, and I take this as evidence of the teachers' conceptions of what it means to learn mathematics in these classrooms.

Implicit Warrants versus Explicit Warrants

Toulmin (1958) described a warrant as the reasoning that connects the claim and data necessary to each argument. In approximately one-third of the arguments I observed (7 out of 19), the warrants were explicitly stated. In two instances, students offered the warrant at the same time they stated the data and thus required no additional prompting from the teacher. When Abby pressed James to justify his claim regarding the GCF of 12 and 18 in Argument 2, James responded with both the data that 6 is the largest number that divides into 12 and 18 and with the multiplication facts supporting this information (see Figure 6). Similarly, in Kendra's Argument 1, Tasha offered the data based upon the definition of function and proceeded to explain how she used that information to determine if the given set of ordered pairs represented a function (see Figure 10). In the other five instances where explicit warrants were stated, the teachers pressed the students for the connection between the claim and data by asking either "Why?" or "How do you know?", using the same types of prompts as those with which they elicited data. In the remaining eleven arguments, the warrants were not explicitly stated and were left to be implied by the students in the class (see Table 7).

As with the data, all nineteen of the warrants, both implicit and explicit, were based upon definitions, properties, or procedures and are indicative of what the teachers wanted the students to learn. Across these instances, there is no noticeable pattern related as to why some warrants were explicitly stated and others not. I perceived no discrepancies in the difficulty of examples prompting the discussions or differences in the behavior of the students, as evidenced by the number of hands raised and vocal responses. This suggests that the explicit statement of warrants was not a focus of the teachers and that the level of explanation they expected from their students was inconsistent.

Table 7

Analysis of Implicit Warrants versus Explicit Warrants

	Implicit Warrants		**Explicit Warrants**	
	Count	**Type**	**Count**	**Prompt/Type**
Abby (2)	1	1 Property	1	Prompt: How do you know? 1 Property
Denae (3)	1	1 Procedure	2*	Prompt: How do you know? 2 Procedures
Kendra (6)	4	1 Procedure 3 Definitions	2	Prompt: How do you know? 2 Definitions
Leslie (3)	2	1 Property 1 Definition	1	Prompt: Why? 1 Property
Will (5)	4	4 Definitions	1	Prompt: Why? 1 Definition
TOTALS	**12**		**7**	

**Note*: One of these explicit warrants was given by the teacher.

Toulmin (1958) described basic arguments where the warrants were trivial. As an example, given the claim that “Harry’s hair is not black,” and the data that “It is in fact red," Toulmin noted the triviality of the warrant, "If anything is red, it will not also be black" (pp. 90–91). I propose that in the episodes of mathematical argumentation where the teachers allowed the warrants to remain implicit, this was due to the fact that the warrant based upon a definition, property, or procedure was obvious to the teacher. Whether it was equally apparent to all students in the class is unclear. Recall from Chapter II that educational research shows that student learning increases and deepens as students elaborate upon and explain their reasoning through mathematical argumentation (Andriessen, 2006; Bransford et al., 2000). Thus, though all 19 of the warrants were based upon explanations of the definitions, properties, and procedures that the teachers wanted the students to learn, the explicit statements of these warrants were not a priority for the teachers and opportunities to strengthen students' mathematical understanding were missed.

Evidence of the Potential to Move beyond the IRE Structure

As reviewed in Chapter II, the most dominant type of questioning pattern in mathematics classrooms has been the Initiate-Respond-Evaluate (IRE) pattern where the teacher *initiates* a question, a student *responds*, and the teacher *evaluates* that response (Chapin et al., 2009; Franke et al., 2007; Herbel-Eisenmann, 2009; Leonard, 2000; Smith & Stein, 2011; Stein, 2001). Franke et al. (2007) noted that this pattern is a well-document occurrence in United States classrooms, even in classrooms where teachers were attempting to teach for mathematical understanding. Throughout the observations, I would characterize the dominant discourse pattern for all five classrooms as IRE. During their lessons, the teachers asked many questions where they accepted an answer from a student without elaboration, evaluated that answer for correctness, and continued with the lesson.

However, all 19 instances of mathematical argumentation documented by this study represent episodes demonstrated the potential to move beyond the IRE discourse pattern. Rather than simply evaluating claims in these cases, the teachers prompted students to provide data in support of their answers. For example, consider Will’s Argument 1, where he prompted students to determine if a given set of ordered pairs represented a function (see Figure 19). When the students offered the claim that the ordered pairs did not represent a function, rather than confirming the accuracy of their answer and moving on to another problem, Will pressed the students to provide additional information about the claim. In each of the 19 episodes of argumentation, the presence of data suggests that the teachers have emerging expertise in moving mathematical discussions beyond the IRE pattern.

Further, the arguments that contain explicit warrants provide additional indications of the teachers’ developing abilities to engage students in mathematical discussions where the students themselves elaborate on the data justifying their claims in the form of explicit warrants. For example, consider Leslie’s Argument 2, the argument where Leslie engaged John in a discussion about an incorrect answer given by a classmate (see Figure 17). Following John’s claim that the answer on the board was not correct, Leslie prompted John to explain not only what he believed the correct answer to be, but also his reasoning for this belief. This explicit statement expressing the connection between his data and claim not only encouraged John to articulate his thinking, but also helped to ensure that the other students in the class had the opportunity to understand John's reasoning. All five of the teachers demonstrated at least one episode of mathematical argumentation that reached this level of discussion, suggesting the potential to move beyond the IRE structure (see Table 8).

Table 8

Analysis of Argument Components

Teacher	Abby	Denae	Kendra	Leslie	Will
Claims with No Data	Many IRE-Type Examples in Each Classroom				
Basic Arguments with Claims and Data	2	3	6	3	5
Arguments with Claims, Data, and Explicit Warrants	1	2	2	1	1

Evidence of Mathematics Content

The teachers prompted 18 of the 19 episodes of mathematical argumentation by presenting specific problems to the students (see Table 9). Thirteen of these arguments focused on the answer to the given examples, such as Abby's Argument 1 where she asked the students to factor a binomial and Will's Argument 3 where he asked the students to determine if a given line represented a direct variation (see Figures 5 & 21). In these 13 episodes, student claims consisted of the specific answers to the given examples and their data provided insight into how they arrived at those answers. The remaining five episodes of the 18 investigated some aspect of the problem, such as understanding the context as in Denae's Argument 1 or identifying the type of problem as in Kendra's Argument 3 (see Figures 7 & 12). These five episodes suggest that the teachers have taught the students to approach new problems by carefully analyzing those problems before attempting to solve them.

Table 9

Prompts for Episodes of Mathematical Argumentation

Teacher	**Answer to given problem**	**Analysis of given problem**	**Other question**
Abby (2)	2	0	0
Denae (3)	2	1	0
Kendra (6)	3	2	1
Leslie (3)	1	2	0
Will (5)	5	0	0
TOTALS (19)	**13**	**5**	**1**

Further, all 18 of these episodes built upon mathematical knowledge that the students had previously experienced (see Table 10). For some, it was mathematics from a previous class, such as in Leslie's Argument 3, when she engaged John in a discussion based upon the Zero Property of Exponents that they had discussed earlier in the week (see Figure 18). For others, it was the mathematics discussed earlier in that day's lesson, such as each of Will's arguments based on the definitions of functions and direct variations. The fact that these arguments were built upon mathematical facts to which the students had already been exposed shows the importance teachers placed on continuous repetition and review of mathematical topics. The students' contributions to these arguments relied upon their abilities to recall information that they had seen either days or mere minutes before and did not require in-depth mathematical thinking.

Table 10

Analysis of Mathematical Content Knowledge

Teacher	**Mathematics of the Day's Lesson**	**Mathematics from a Prior Lesson**	**Mathematics Yet to be Discussed**
Abby (2)	2	0	0
Denae (3)	2	1	0
Kendra (6)	0	5	1
Leslie (3)	2	1	0
Will (5)	5	0	0
TOTALS (19)	**11**	**7**	**1**

Only one episode of argumentation was prompted by a spontaneous question posed by the teacher in the course of the lesson for the purpose of eliciting new mathematical information from the students. In Argument 6, Kendra and her students had just begun a lesson on the properties of matrices. After introducing the concept of a matrix and the vocabulary terms "row" and "column," Kendra asked the class, "What is the smallest matrix we can have?" (see Figure 15). The answer to this question relied upon the students' understanding of the new terms, yet required them to think beyond what had previously been discussed. Abraham offered the claim that the smallest matrix must be one by one and provided additional information at Kendra's prompting. Though the argument was simple in nature, it allowed Abraham to construct new mathematical knowledge as opposed to repeating information to which he had already been exposed.

To make and justify this claim required Abraham to understand the newly-introduced topics about matrices and extend that understanding to a new situation.

Participation in Mathematical Argumentation

In each episode of mathematical argumentation, participation was shared between the teacher and students. The teacher controlled the format of the discussions and the students volunteered to respond. In some classrooms, the students were expected to speak out with a response, such as in Abby's class or Denae's class, while in others, such as Leslie's, the teacher called on the student who indicated an inclination to participate. In almost all cases, some indication of willingness by the students precipitated their participation in the episodes of argumentation. Though occasionally one of the teachers would request a student to offer a claim without volunteering, there seemed to be an overall consensus among the teachers that calling on students who were unsure of their answers would be detrimental to their self-confidence about their mathematical abilities. Twelve of the 19 arguments were between a teacher and only one student (see Table 11).

In all the episodes of mathematical argumentation, the teachers mediated the discussions and guided the students to give the claim first, then the data, and possibly the warrant. Even in the cases where multiple students spoke up to offer a claim, they directed their responses to the teacher and not to each other. Since none of the arguments occurred without the teachers' support and guidance, I took this as evidence of the teachers' uncertainty of the students' abilities to articulate their mathematical thinking and, as will be elaborated in response to the second research question, their belief that their role was to protect students from perceived failure.

Table 11

Participants in Episodes of Mathematical Argumentation

Teacher	Argument between teacher and one student	Argument between teacher and multiple students
Abby (2)	0	2
Denae (3)	2	1
Kendra (6)	5	1
Leslie (3)	3	0
Will (5)	2	3
TOTALS	**12**	**7**

Summary

Though mathematical argumentation was not prevalent in these classrooms, some episodes of argumentation did occur. Teachers prompted for data and explicit warrants using questions such as, “Why?” and “How do you know?” All data and warrants relied on definitions, properties, and procedures. The majority of warrants were implicit and the majority of arguments occurred between a teacher and one student. Eighteen of the 19 arguments were prompted by a problem introduced by the teacher that required the use of mathematical knowledge to which the students had already been introduced. The teachers mediated all arguments that occurred in their classrooms. Together, these findings suggest that the teachers recognized the need for students to move beyond merely offering the answers to problems. However, due to the classroom environments, I inferred that the teachers also felt compelled to maintain control of the discussions and to mediate all of them. Students that did not indicate a willingness to participate were not asked to do so. Teachers indicated that they felt the need to help students acquire a positive self-image about their mathematical abilities and tried to protect them from offering incorrect answers that might be perceived as failure. Further, the content of the arguments and the form of the data and warrants indicated a strong focus on the students' abilities to repeat information verbatim as well as to implement specific procedures when working certain types of problems.

Question 2: For What Goals Do Teachers Foster Mathematical Argumentation in These Classrooms?

To address the second research question, I explored the teachers' reasons for incorporating mathematical argumentation into their classroom discussions. Though the concept of argumentation was familiar to all of the teachers in terms of having students "justify" and "explain" their answers and reasoning, the link between mathematical learning and argumentation for the teachers was unclear. Thus, mathematical argumentation became a mechanism for various purposes and accomplished different goals. I begin with a discussion of the use of mathematical argumentation as a means of assessment by the teachers. Then, I investigate the social and affective goals accomplished by the use of mathematical argumentation in these classrooms, including nurturing, motivating, managing, and character development. Finally, I describe the use of mathematical argumentation as a means of promoting mathematical learning and examine its use in helping students understand context, explore possibilities, and create new mathematical understandings and conclude this section by summarizing these findings in relation to the second research question.

Goal for Mathematical Argumentation: Assessment of Learning

Overwhelmingly, the purpose of mathematical argumentation in these classrooms was to assess and reinforce student understanding of definitions, properties, and procedures. This is most obvious with Will, since all five of the arguments that occurred in his classroom followed a very similar pattern. Two of the five arguments in Will's class were based upon problems asking the students to determine if a given set of ordered pairs or table represented a function, while two other arguments involved a comparable question regarding equations representing direct variations. I inferred that the purpose of the problems was to assess the students' understanding of these mathematical concepts in terms of their abilities to analyze specific characteristics of each problem. As data in each of these arguments, a student offered the procedure he or she followed to determine the answer. For example, in the first argument, Brandi inspected the x-values in the ordered pairs and offered, "Because the x-values repeat" (see Figure 19). Likewise, in the third argument, Jillian decided that the given equation was not a direct variation "because there's a number behind the x" (see Figure 21). The warrants in these arguments were unstated and based upon the definitions of functions and direct variations, respectively.

The use of arguments to assess and reinforce definitions, properties, and procedures was evident with the other teachers as well. As Leslie asked John to elaborate on the correct position of the coefficient in the answer of Argument 2, he offered the procedure that he followed to get his answer as data and the Property of Negative Exponents as warrant (see Figure 17). Portia's data offered to Denae in Argument 3 and her unspoken warrant were based upon the procedure for using the calculator to find the vertex of a quadratic function (see Figure 9). When Abby pressed James to find the GCF of 12 and 18, he grounded his responses on the definition of GCF and basic multiplication facts (see Figure 6). With Selena's data and implicit warrant to Kendra regarding the slope of a line from a given equation, she based her answer on the procedure she had been taught and the slope-intercept form of a linear equation (see Figure 11). Without exception, the data and warrants in all 19 of the arguments dealt with definitions, properties, and procedures, and I infer this to be a reflection of the teachers' desire to assess the students' abilities to replicate solutions as the teachers had modeled and suggest that mastery of the content was equated with students successfully emulating the teachers' procedures.

Further, 18 of the 19 arguments dealt with mathematics the students had already experienced. Only the last argument in Kendra' class dealing with matrices moved into mathematical concepts that were new for the students (see Figure 15). Thus, the use of mathematical argumentation in these classrooms was clearly not for the purpose of advancing students' mathematical learning as argued by Andriessen (2006). Instead, arguments were used as a mechanism to review material previously taught and assess the retention level of the students. They were also used as a means of having students continuously repeat procedures that the teachers had instructed them to follow for certain types of problems. The connection between mathematical argumentation and students' mathematical learning was lost in the practices common to these Algebra I classrooms.

Goal for Mathematical Argumentation: Social and Affective Intentions

Mathematical argumentation in these classrooms served various social and affective intentions of the teachers as well. As the teachers worked to establish a positive classroom atmosphere, mathematical argumentation supported these other domains that affect learning by engaging the students in the classroom discussions and helping the teachers accomplish many important, yet non-mathematical, goals.

Nurturing. Given the large number of students in these Algebra I classes who had previously been unsuccessful in the course, the teachers all assumed nurturing personas in the classroom. This was most obvious with Leslie, as her caring and supportive demeanor encouraged students to participate in the lessons. In an attempt to protect the students from perceived failure, she was very selective on whom she would request to answer questions during class and explained that she never called on a student who did not indicate that he or she was willing to contribute to the discussion. Further as reported in her interview, Leslie only pressed for data and warrants if she felt certain that the student would be able to answer correctly. She expressed concern about the students' level of self-confidence about their mathematical abilities and felt that incorrect answers given in front of their peers would exacerbate the issue. In the cases where she did press students for their reasoning, she gave affirming responses such as nodding her head in agreement or verbally encourage before asking for additional information.

This nurturing attitude was also evident in Denae's classroom as she spent time each day talking with her students about their lives outside of school creating, in her words, "a connection with each of her students." In her interview, she explained that this connection "makes them willing to talk about math" and in turn, as they talk about math, they "try a little bit harder, [act] a bit more confident." Similarly, in Kendra's classroom, she worked to help the students "get over their fear of words." She explained in her interview that she believed "most of [her students] are not comfortable with [explaining] at all and may not even know how to do it." Meant to support the students as they worked through a course that was difficult for them, mathematical argumentation provided an opportunity for the students to experience articulating their mathematical thinking in a safe, positive environment.

Managing. During times when the students were off-task and engaging in activities apart from the mathematics lesson, the teachers used classroom discussions involving mathematical argumentation to help the students reengage in the lesson. This was most often observed in Kendra's large inclusion class where several students continually attempted to disrupt the lessons. In her interview, she described the class as a "challenge" and explained that "some [students] in the class were very serious, but others were not." To help prevent disruptions, Kendra kept the students' attention focused on her as she called on different students for answers to questions and then asked for more information about their reasoning. She also used mathematical argumentation to help outspoken students engage in the lesson in a more productive manner.

Similarly, Will used mathematical argumentation to engage students in the lesson and keep the lesson progressing at the pace he desired. By calling on students to offer their answers and reasoning, he kept the students' attention focused on him and the problems he wanted to discuss. While mathematical argumentation used in this manner did not necessarily promote growth in the students' mathematical thinking, it did serve the purpose of focusing the students on the day's lesson and engaging them in the discussions about various mathematical topics.

Motivating. Also related to the students' perceived low self-esteem in mathematics, the teachers often encouraged and motivated the students throughout their lessons. This was most evident with Denae as she continually cheered for her students as they worked through difficult problems with phrases such as, "Y'all got this!" and "Make me proud!" Positive and energetic during reviews of old material and discussions of new topics, Denae radiated assurance her students could and would be successful. In her words, "I just want them to know that they can be successful and that they can do this!" She was very patient with her students and determined that they not see her frustrated, "no matter how many times [she has] to repeat the same information."

Similarly, in their interviews, Leslie expressed the significance of her students believing that she felt they could be successful, and Abby stressed the importance of her students realizing that they are preparing for mathematics beyond Algebra I. The teachers believed that as the students learned to speak up in front of their peers and articulate their thinking about the mathematics they were studying, the students gained confidence in themselves. Further, each time they successfully answered a question correctly and offered their reasoning behind that answer, the teachers felt the occurrence helped to bolster the students' views of themselves as mathematics learners.

Character development. Finally, the teachers used mathematical argumentation as an opportunity to enculturate their students in what they considered to be appropriate manners for general social interaction, such as how to talk together and how to be polite. This was most evident in Will's classroom as he continually paused to correct students' language to each other and to him. In his interview, he expressed that a "certain level of respect" was necessary to establish relationships between students and teacher. When the students participated in mathematical argumentation in Will's class, they often raised their hands to contribute, seldom interrupted one another, and offered to help when a classmate was unsure of an answer.

This intent was also evident in Denae's classroom as she emphasized that the students "take care of each other" and not disparage or interrupt one other. Likewise, Abby noted that mutual respect between teacher and students was an important aspect of the climate of her classroom. When used in this manner, mathematical argumentation helped to instill norms of behavior useful to the students outside of the mathematics classroom.

Goal for Mathematical Argumentation: Supporting Mathematical Learning

Several episodes of mathematical argumentation that occurred showed the potential for supporting students in deeper mathematical understanding. Arguments that examined problem context and explored different answer possibilities pushed the students to think beyond the mere repetition of previously discussed information. To a greater extent, the one argument of the 19 that required the students to build upon definitions and properties to conjecture new mathematical ideas supported the students in new mathematical learning.

Context. In several episodes of argumentation, the focus of the discussion was not the solution to a given problem but on contextualizing aspects of the problem. Two of the arguments in Denae's class provided examples of this use of argumentation. In Argument 1, Denae and her students took time to discuss the wording of the problem about the numbers of cats and dogs (see Figure 7). Before attempting to solve the problem, they engaged in an argument that I inferred was meant to ascertain the students' understanding of the connection between the two variables in the problem. The results of this discussion extended over into the second argument where the students solved the system of equations involved with the word problem and then needed to translate the numerical answers to the system into the context of the problem (see Figure 8). As Denae explained in her interview, "I really have to make them look at the problems carefully . . . I've got to get them to find those key words."

Kendra used mathematical argumentation to assist her students in determining how to proceed to solve specific examples by carefully examining the context of those examples. In Argument 3, the discussion revolved around identifying the form of the linear equation presented in the problem, which then aided the students in deciding how to approach the problem (see Figure 12). Likewise, in Argument 5, the claim and data presented dealt with identifying the type of problem the students were asked to solve (see Figure 14). Kendra used the arguments to help the students focus on key words and problem structures that would assist the students in determining how to proceed based on previous examples they had already worked. In her interview, Kendra noted, "[I tell] them several times what they are going to see the most of [on the exam] and make sure that they are confident with that information."

The episodes of mathematical argumentation that focused on the context of problems helped strengthen the students' mathematical understanding by pushing them to connect key words and "types" of problems to methods of solving those problems. They also aided the students in discerning an overall understanding of each problem that supported them in interpreting their answers. This ability to analyze problems and their underlying mathematical structure is an important skill that will be of use to the students as they move to subsequent mathematics courses.

Exploring possibilities. In Abby's classroom, arguments were not as straightforward as those in the other classrooms. Abby allowed the arguments in her class to have multiple sources of data and warrants and to proceed as discussions of possibilities. In Argument 1, she accepted claims representing both a common factor and the GCF of the binomial the students had been asked to factor (see Figure 5). Since the directions for this problem had not explicitly requested the GCF, Abby used the opportunity for the students to compare the outcomes of each factoring approach. I inferred that this was a conscious choice on Abby's part to proceed by accepting multiple answers, since she apparently realized the issue with the directions while the students were working as evidenced by the manner in which she looked toward the camera and shook her head. In Argument 2, Omar and James disagreed as to the correct GCF of the given binomial (see Figure 6). Abby did not offer an opinion as to which answer was correct and instead waited for the students to resolve the issue. This led to James confidently offering both data and warrants to justify his claim.

Episodes of mathematical argumentation such as these provided an opportunity for students to experience the process of doing mathematics in an authentic manner. While the arguments did not involve solving problems unlike those they had previously seen, they differed from the clear, step-by-step presentation of claims, data, and warrants of many of the other arguments. Allowing the students to experience this more natural flow of mathematical problem solving helped to expand their mathematical thinking by offering examples of successfully solving problems via methods that differed in some ways from the methods previously demonstrated by the teacher.

Possibilities of supporting new mathematical learning. One episode of mathematical argumentation clearly showed the potential to support students in new mathematical learning. In Argument 6, Kendra asked the students a theoretical question based upon the new definitions and properties of matrices, rows, and columns (see Figure 15). This question, "What is the smallest matrix we can have?", required no mathematical computation and was not based upon any problem that the class had previously discussed (recall that the possibility of an empty matrix was beyond the scope of this course). The answer to Kendra's question relied upon the students' abilities to understand the new definitions and properties and to synthesize that information in a new way. The use of mathematical argumentation in this way was the closest representation in this study to the descriptions of mathematical argumentation used to support new mathematical learning in the research literature. I take this as evidence that the use of mathematical argumentation to promote authentic student learning is possible in high school mathematics classrooms.

Summary

Teachers fostered mathematical argumentation in their classrooms for a variety of reasons. Overwhelmingly, when argumentation occurred, teachers used it as a means of assessing and reinforcing definitions, properties, and procedures. They also supported argumentation to accomplish social and affective teaching goals, such as nurturing and motivating students, managing behavior, and promoting appropriate manners for social interactions. Several episodes of mathematical argumentation supported mathematical learning by encouraging students to investigate problem context, to explore various answer possibilities, and to create new mathematical learning. Together these findings suggest that these teachers fostered mathematical argumentation to support teaching goals not directly related to new mathematics learning. Given the current climate in education requiring students to successfully pass standardized end-of-year assessments and the pressure on teachers to help students achieve this goal, it is not surprising that the majority of the arguments in these classrooms revolved around the teachers' attempts to assess and reinforce student understanding of definitions, properties, and procedures. This pressure on teachers and students also offers a possible reason for the teachers' use of mathematical argumentation in ways that helped to create classroom atmospheres perceived as nurturing and motivating. The episodes of mathematical argumentation that supported students' in analyzing problem context, exploring possibilities, and synthesizing new information offer evidence of the teachers' abilities to use mathematical argumentation in ways that support students' new mathematical learning.

CHAPTER VI

FINDINGS AND IMPLICATIONS

In the previous two chapters, I described the within- and cross-case analyses of the episodes of mathematical argumentation that occurred in five Algebra I classrooms to answer the two research questions. In this chapter, I situate those findings in the context of the mathematics education research literature and the current political context affecting schools. I then discuss implications of this study, describe its limitations, and conclude with a set of final thoughts concerning potentially productive ways of supporting teachers in an era of new standards and assessments.

Relating Findings to Research Base

This study found that minimal mathematical argumentation occurred in the five Algebra I classrooms. While numerous questions were asked throughout each lesson by all of the teachers, only 19 instances of mathematical argumentation transpired during the observations, and the overwhelming majority of teacher questions led to an IRE-patterned interaction. This finding concurs with evidence documented in the research literature that the most dominant discourse pattern in mathematics classrooms is the IRE pattern (Franke et al., 2007; Herbel-Eisenmann, 2009; Leonard, 2000; Stein, 2001; Wood, Williams, & McNeal, 2006). For example, Franke et al. (2007) found that within elementary classrooms, teachers struggle to follow up on students' answers with additional questions that will help make student thinking explicit. Though teachers in their study asked numerous questions, they found only sequences of specific questions supported students in clearly justifying and explaining their mathematical reasoning. Similarly, in her teacher-research study, Leonard (2000) found that even when the teacher was attempting to promote discourse through questioning, the fact the teacher posed all of the questions led to more teacher talk, less student talk, and a reemergence of the IRE pattern.

In the observed episodes of argumentation, the teachers recognized a need for students to move beyond simply offering answers to questions with no elaboration about their reasoning and acknowledged the importance of prompting for additional information. These findings also support the findings of other researchers (e.g., Hoffman et al., 2009; Kazemi & Stipek, 2008; Lopez & Allal, 2007; O'Donnell, 2009; Sherin et al., 2000; Webel, 2010; Yackel & Cobb, 1996) regarding the importance of students justifying and explaining their mathematical reasoning. In particular, in their seminal work, Yackel and Cobb (1996) noted how "the teacher's requests for different solutions initiate a change in the setting from solving the problem to comparing solutions" (p. 464). They considered the students' abilities to articulate answers in a manner understandable by their peers to be an important component of their mathematical learning.

This study also found that the majority of teacher prompts for data and all teacher prompts for warrants took the forms "Why?" or "How do you know?" In the research literature, questions of this type have been acknowledged as supporting higher-level reasoning. For example, Smith and Stein (2011), building on the work of Boaler and Brodie (2004) and Driscoll (1999), offered a framework for classifying teachers' questions. In their classification, they highlighted the categories of Exploring Mathematical Meanings and/or Relationships, Probing/Getting Students to Explain Their Reasoning, and Generating Discussion as questions types that can be especially useful to teachers attempting to promote discourse that supports mathematics learning in their classrooms. They emphasized that these types of questions “scaffold thinking to enable students to think harder and more deeply about the ideas at hand” (Smith & Stein, 2011, p. 62).

This study showed a heavy reliance on the repetition of specific definitions, properties, and procedures as the data and warrants of arguments. For the teachers in this study, students’ abilities to repeat information verbatim and implement specific strategies when asked to solve certain types of problems remained a priority. These finding are similar to those of Watson (2002), who in his study of low-achieving students in high school found teaching in these classrooms “often involves simplification of the mathematics until it becomes a sequence of small, smooth steps which can be easily traversed” (p. 462) and that students were often only responsible for offering the repetition of a low-level fact or an arithmetical answer. As Haberman (1991) noted in his seminal work on the pedagogy of poverty, one of the four syllogisms that often undergirds the act of teaching in urban schools is the belief that “directive pedagogy” is required to ensure student acquire basic skills (p. 291). Likewise, Ruthven (2002) showed that in situations where teachers and students were working under the pressures of accountability, strategies adopted in the classroom focused on “immediate objectives of mastering specified mathematical material” (p. 189) and insufficient attention was given to overall components of students' mathematical learning that were not directly assessed. These findings are in accord with Franke et al.'s (2007) claim that teachers funneled and guided the students through procedures, while the teacher did most of the mathematics. Questions were often unrelated to supporting mathematical thinking and instead corresponded to "strategies the teacher thought would enable students to solve the problem" (p. 390). Similarly in this study, even though students' responses were prompted by question types that have the potential to support higher-level thinking, the recall and repetition of information that was elicited does not meet the criteria to be considered higher-level thinking (Resnick, 1987).

The teachers in this study used mathematical argumentation to support three main goals: to assess, to promote social and affective purposes, and to support new mathematical learning. Researchers such as Cazden (2001) and Franke et al. (2009) discussed the use of questioning techniques that enable the teacher to assess students' understanding and misconceptions and noted that those practices play an important role in promoting classroom discourse. However, the findings of this study indicated the questions prompting the episodes of argumentation did not exclusively serve the purpose of assessing the level of students' understanding, but also assessed their abilities to repeat definitions, properties, and procedures.

The literature describes the necessity of establishing safe classroom environments where teachers support students as they participate in challenging mathematical activities and discourse (Herbel-Eisenmann, 2009; Smith & Stein, 2011; Stipek et al., 1998; Yackel & Cobb, 1996). For example, Yackel and Cobb (1996), Dixon et al. (2009), and Sherin

(2002) all reported the importance of teachers establishing social and sociomathematical norms that provided structure for classroom discourse and expected students to articulate their mathematical reasoning. Stipek et al. (1998) studied 24 fourth- through sixth-grade classrooms and concluded teachers' instructional practices positively influenced students' motivation and self-esteem. However, it is important to note the distinction between social and affective goals that create an atmosphere conducive to mathematical argumentation and the use of mathematical argumentation itself to fulfill social and affective goals of teachers as was documented in this study.

Finally, learning scientists have reached consensus on specific mechanisms that support student learning, and their research has demonstrated how argumentation supports those mechanisms (e.g., Andriessen, 2006; Andriessen, Erkens, et al., 2003; Baker, 2003; Bransford et al., 2000; Sawyer, 2006b). Andriessen, Baker, and colleagues (2003) found that argumentation led to better understanding between participants, deeper understanding of mathematical topics, and transformation of students' mathematical ideas. Baker (2003) also found that argumentation supported new knowledge and understanding as articulation led to reexamination of beliefs. Later, Andriessen (2006) linked argumentation to the learning mechanisms identified by Bransford et al. (2000) of making knowledge explicit, promoting conceptual change, co-elaboration of new knowledge, and increasing articulation. The instances of mathematical argumentation observed in this study where students explored the context of problems, investigated possibilities, and attempted to go beyond what had previously been discussed in class all show the potential of these teachers to support argumentation that promotes deeper mathematical understanding.

Thus, this study contributes to the knowledge based about argumentation as it is currently enacted in high school Algebra I classrooms. It adds further evidence of the impact of standardized testing and accountability measures on what teachers teach and how they engage students in learning mathematics. It provides additional insights into supports needed by teachers and students as we transition into an era of new standards and assessments by offering a view of the nature of mathematical argumentation as it is currently enacted and teachers' goals for supporting it.

Discussion

This study took place at the end of the NCLB era and the beginning of the RttT initiatives and thus in school climates with increased emphasis on linking classroom instruction to specific content standards and student standardized test performance (Means, 2006). Most standards-based assessments "are based on highly-restrictive beliefs about learning and competence" (CFA, 2001, p.2), "focus almost exclusively on the recognition and recall of superficial course content" (Carver, 2006, p. 205), and are not appropriate for evaluating problem solving and reasoning. If, as Sawyer (2006b) asserted, our current education system promotes the belief that knowledge is a collection of facts and procedures and that schooling is considered successful when students are tested and able to demonstrate the facts and procedures they have acquired, then deeper mathematical understanding, such as that supported by mathematical argumentation, is not a goal of, or a necessity for, student "success."

One explanation for the disparity between the vision of argumentation as it is described in the literature and the realities of how it is enacted in classrooms might be that teachers experience a tension between goals of pressing their students to justify and explain their mathematical thinking through argumentation and the very real requirement that the students perform well on the end-of-course standardized test. With financial incentives given for positive student performance and the possibility of repercussions such as transferal to another school for negative student performance, current accountability models have affected almost all instructional practices occurring in classrooms. This results in teachers maintaining control of classroom discussions, mediating all episodes of mathematical argumentation in an attempt to keep the discussions moving in the directions that they wanted. In the majority of cases, that "direction" is toward a recall and repetition of how to work specific types of mathematical problems and not an understanding of deeper mathematical structures.

Another explanation may be that teachers are unclear about the value of mathematical argumentation as a mechanism for supporting student learning. On the one hand, it is possible that teachers are aware of the need to press students to justify and explain their reasoning through mathematical argumentation, but are unaware of the ultimate purposes of promoting learning and deeper understanding behind that need. On the other hand, teachers may be cognizant of the potential of mathematical argumentation as a learning mechanism but feel unable to regularly implement such practices due to the pressures of accountability that do not emphasize students' abilities to reason mathematically.

In trying to incorporate the ideas of justifying and explaining through mathematical argumentation into the high-stakes atmosphere of accountability that dominates contemporary mathematics classrooms, teachers' uses of mathematical argumentation runs the risk of becoming a procedural exercise, far removed from the concept of mathematical argumentation described by reformers. Mathematical argumentation becomes a means of assessing and reinforcing students' understanding of definitions, properties, and procedures. Though it may also serve the social and affective goals of the teachers, these goals are not content-specific. Such goals are likely a direct consequence of the high-stakes atmosphere and its negative impact on students' self-esteem about mathematics and their success.

In sum, teachers are being asked to practice their craft in the tensions of two competing views of learning. One view is that knowledge is a collection of facts and procedures, teachers' responsibilities are to transmit those facts and procedures to students, and schooling is successful when students are able to demonstrate these facts and procedures on standardized tests (Sawyer, 2006b). The other view is that learning and knowing are embedded in specific social and cultural contexts (CFA, 2001; Sawyer, 2006b), practices such as mathematical argumentation serve as mechanisms to promote deeper student understanding (Andriessen, 2006), and assessments of student learning need to encompass additional aspects of student achievement (CFA, 2001). In many cases, the value and emphasis placed upon students' performance on standardized tests greatly outweighs the minimal support given to ideas promoting a wider view of learning. Thus, it is not surprising that teachers often yield to the pressures of accountability and focus little of their efforts on practices such as mathematical argumentation as they are not seen as directly impacting student success as measured by standardized tests.

Implications

Teachers

The findings of this study suggest that teachers have the ability to support students in the practice of mathematical argumentation and demonstrated that they have the emerging expertise needed to move beyond the traditional IRE pattern still predominant in mathematics classrooms. Purposeful planning to include opportunities for students' explanations of their mathematical reasoning into other classroom activities, such as the introduction of new mathematical concepts, may present occasions for teachers to expand the context of student arguments. Likewise, establishing classroom norms that guide the manner in which students respond to one another might encourage students' willingness to agree or disagree with the comments of their peers and promote episodes of mathematical argumentation that are not mediated by the teacher. Also, consideration of various question formats for use after the statement of data may yield richer warrants and more insightful arguments, thus increasing the potential for deeper student understanding of mathematics.

Teacher Educators and Professional Developers

A clear message of the study is that teachers may not fully understand the learning benefits of supporting students in mathematical argumentation. In their work with teachers, teacher educators and professional developers should acknowledge different teaching goals, but underscore the importance of argumentation as a mechanism for moving learning forward. To assist teachers in distinguishing between assessment and learning perspectives, representations of practice that specifically highlight mathematical argumentation, professional learning tasks that decompose practice into components such as discourse facilitation and questioning strategies, and experiences that allow for opportunities to engage in approximations of practice (Grossman et al., 2009) may support teachers in better understanding ways in which mathematical argumentation can be implemented in their classrooms.

Further, teachers would benefit from opportunities to see episodes of mathematical argumentation as it is envisioned in the research literature as a true mechanism for enhancing student learning. Observations and interviews from the study indicated reluctance on the part of the teachers to push students with questions that they may be unable to answer due to the perception that this is in the best interests of the students. Professional development dedicated to providing examples of argumentation where students are allowed to not know the correct answer, to be frustrated during the process, and to find resolution would aid teachers by providing a perspective where students are allowed to struggle with mathematics with positive results. This could assist them in reexamining their beliefs about how they engage with their students during their lessons.

Researchers

Another implication of this study is that the connection between mathematical argumentation and deeper student learning is not clearly understood. It was evident that in most cases, the episodes of argumentation were viewed as sufficient as long as the students answered the questions “Why?” or “How do you know?” The mathematical thinking underlying the students’ responses were rarely considered. It is evident from this study that the teachers acknowledged the need to have students justify and explain their answers, but did not fully understand the reasoning as to why they should be doing so. This led to the use of the practice of argumentation in ways that researchers did not intend. Thus, additional research is needed to understand the mechanisms by which teachers learn and change their practice. Moreover, researchers working in the areas of classroom discourse and argumentation should take seriously the ways their results are taken up in practitioner communities and more fully attend to connecting their research findings with the realities of practice.

Administrators and Policymakers

A final implication of this study is that teachers, and thus their students, are highly affected by the pressures of accountability. It was evident that both positive and negative consequences connected with student proficiency affected the teachers’ instructional decisions. Financial incentives, meant to support increased student test scores, may contribute to practices that are known to be conducive to only lower level thinking skills and rote memorization (Sawyer, 2006b). Performance is not synonymous with learning, and “increased test scores” may in fact work against learning. It is imperative that administrators and policymakers recognize and deeply consider the difference between achievement and learning when creating and enacting policy.

Limitations

One limitation of this study was the small number of cases of participating teachers and episodes of mathematical argumentation. However, given that the point of case study is not to generalize but to better understand the phenomena of interest, starting with a small number of teachers was intended to allow for in-depth analysis. A second limitation of this study concerns the fact that ideas that are taken-as-shared by a classroom community cannot be determined within one week of observations. It remains unclear whether the large number of implicit warrants in the episodes of mathematical argumentation was a result of ideas that were already accepted as known by the students and teachers. A way to mitigate this limitation in a future study might be to begin with a different methodology allowing for longer periods of observation with each teacher. Finally, the use of Toulmin's (1958/2008) model to analyze the arguments might also be seen as a limitation of this study. With the focus on the structure of the arguments, the mathematical content was not evaluated. Also, varying the modifications to the model can lead to different interpretations of frequency and understandings of whether components are explicit or implicit (Singletary et al., 2013; Yopp, 2013). This limitation could be addressed through the use of an additional framework designed to examine the quality of the mathematics involved in the arguments.

Final Thoughts

The move toward full implementation of the Common Core State Standards and their accompanying assessments makes this is an exciting time in mathematics education. Not only are the Mathematical Practice Standards of the Common Core State Standards reflective of the student mathematical dispositions that mathematics education scholars have championed for years (NCTM, 2000; NRC, 2001), they are also recognized as components of student learning that are as equally important as the content standards. To this end, the consortia designing the new assessments for the CCSS have indicated their commitment to including these practices into their assessments (SBAC, 2010, PARCC, 2013). For the first time ever, assessments that are designed to measure students' mathematical knowledge will move beyond views of knowledge as facts. Though teachers will still be held accountable for their students' performance on these tests, the conflict between the design of the tests and what is known about student learning will be less pronounced and will hopefully permit teachers greater freedoms to promote practices such as mathematical argumentation to support students' mathematical learning.

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APPENDIX A

RESEARCH CROSSWALK

What do I need to know?	***Why do I need to know this?***	***What kind of data will answer the question?***	***Where can I find the data?***
What is the nature of mathematical argumentation in this classroom?	To assess the how students justify and explain their mathematical reasoning	Classroom Observations Teacher Interviews	From the participating teachers, their students, and their classrooms
For what goals does the teacher support students' mathematical argumentation?	To understand the reasons teachers promote mathematical argumentation	Classroom Observations Teacher Interviews	From the participating teachers, their students, and their classrooms

Note: Adapted from *Qualitative Research Design: An Interactive Approach* (2nd ed.), by J. A. Maxwell, 2005, Thousand Oakes, CA: Sage Publications.

APPENDIX B

OBSERVATION PROTOCOL

Date:	School:	Time in Classroom: ____ to ____
Teacher:	Subject:	Number of Students:

CLASSROOM MATH TALK COMMUNICATING, JUSTIFYING, AND REASONING "Look-Fors"
Classroom Environment:
❖ Evident that the norm of listening to and speaking with one another has been established
❖ Arrangement of room allows for student participation and engagement
The teacher is:
❖ Providing opportunities for students to talk with one another about math
❖ Providing opportunities for students to explain their problem solving approaches
The students are:
❖ Participating in meaningful dialogue with each other and/or the teacher
❖ Willing to explain their thinking
❖ Questioning the reasoning/answers of their peers
❖ Making connections between what they are learning and real life

APPENDIX C

INTERVIEW PROTOCOL

***Additive Impact* Teacher: ________________________**

Script: As you know, we are currently in the sixth year of our Additive Impact project. While we have a great deal of quantitative data and survey data from all of the participating teachers, I am interested in finding out more about the classroom practices of individual teachers. I know that you do excellent work with your students and have excellent VAD scores, so now I am hoping that you can share with me more details about what you actually do in your classroom. Please know that this will be completely anonymous and that there are no "right" or "wrong" answers. Also, please do not feel that you need to tailor your answers to be what you think I would want to hear. What you are doing is obviously working for you and your students and that is what I want for you to describe for me! I'll be video recording with the audio recorder as back-up.

1. Please tell me how long you have been teaching at this school and why you chose to teach here.

2. How would you describe your approach to teaching math?
   (*Probe: How would you describe your teaching style? Could you describe for me a typical math class that you might have?)*

3. Can you tell me the different ways in which you have your students talk with each other and you throughout a lesson?
   a. Anything else?
   b. I heard you say: topic 1, topic 2, etc. Anything else?
   c. Now, let's talk a bit more about topic 1. Can you describe for me an instance when this occurred in your classroom?
   d. Now, let's talk a bit more about topic 2. Can you describe for me an instance when this occurred in your classroom?
   e. Continue until all topics are covered.

4. Can you please tell me all the ways that your students _____________________ (Ex. use mathematical argumentation) in your classroom? If this is not a part of your lessons, then please just say so.
   (*Example Probe: By mathematical argumentation, I mean practices like justifying and defending answers, proving, convincing others that their reasoning is correct*.)

a. Anything else?
b. I heard you say: topic 1, topic 2, etc. Anything else?
c. Now, let’s talk a bit more about topic 1. Can you describe for me an instance when this occurred in your classroom?
d. Now, let’s talk a bit more about topic 2. Can you describe for me an instance when this occurred in your classroom?
e. Continue until all topics are covered.

5. Can you please tell me three of your classroom practices that you believe have the most impact on your students’ EOC performance?

6. Can you please tell me three of your classroom practices that you believe have the most impact on your students’ mathematical learning?
(*Probe: Are these the same practices as in #6?*)

7. Do you feel that classroom discourse/mathematical argumentation play a role in your students’ mathematical learning?

8. Do you focus on EOC scores, mathematical learning, or both? (Remember, it doesn’t have to be what you think I want to hear!) Can you tell me why?

9. What are your plans once *The Additive Impact* ends?

10. Thank you for allowing me to observe your class earlier in the quarter. I’d like for us now to take a look at a few clips from those videos and talk a bit about them.
    a. Clip 1: ______________________________ Video ____ at time marker ____
    b. Clip 2: ______________________________ Video ____ at time marker ____
    c. Clip 3: ______________________________ Video ____ at time marker ____
    d. Clip 4: ______________________________ Video ____ at time marker ____
    e. Clip 5: ______________________________ Video ____ at time marker ____