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This study focused on the practices of selecting and sequencing student strategies for whole-class discussions with teachers working toward a vision of teaching that is responsive to students' mathematical thinking. In responsive teaching, students' thinking about mathematical ideas is central, and instruction is shifted from how teachers think about mathematics to students' thinking about mathematics. Whole-class discussions are the time in many lessons when teachers are explicitly focused on eliciting and building on students' thinking. In these discussions, teachers typically showcase one or more student strategies and then facilitate a conversation that explores the mathematics in individual strategies or across strategies. Thus, purposefully selecting and sequencing the strategies to be used in whole-class discussions is important because the strategies provide the foundation for the discussions and what students have the opportunity to learn.

Selecting and sequencing strategies are two of the five practices identified in Smith and Stein's (2018) seminal work on orchestrating productive whole-class discussions, but I argue that these two practices have been underappreciated and under-researched, and implementation remains challenging for many teachers. Existing research identifies lists of general criteria for selecting and sequencing, but these criteria often lack specificity and a coherent structure thereby restricting their usefulness for teachers. My dissertation builds on this limited research by investigating the decision making of two groups of teachers engaged in the practices of selecting and sequencing. Specifically, I explored the decision making of three teachers who had demonstrated expertise in

responsive teaching as they selected and sequenced strategies in their classrooms. I also explored the decision making of 30 teachers with varying levels of expertise in responsive teaching as they selected and sequenced strategies during professional development activities.

Findings led to the creation of a three-level framework which showcases teachers' criteria for selecting strategies for whole-class discussions. The levels progressively include more specificity, starting with the main criteria that apply to all mathematical content and ending with detailed criteria linked to specific mathematical content, which in this case is fractions. The framework for selecting strategies for whole-class discussions provides benefits to both researchers and practitioners. For researchers, the framework provides specificity and structure to criteria already identified in the literature, while also incorporating new criteria and elaborating on existing criteria. For practitioners, the framework criteria can inform teachers' decision making and help them become more purposeful when selecting strategies, thereby providing students with more opportunities to learn in whole-class discussions. Further, the leveled structure of the framework makes guidance for the purposeful selection of strategies accessible and useful to teachers at any phase of their development in being responsive to students' mathematical thinking. The framework currently includes only the practice of selecting strategies as there were insufficient patterns in teachers' decision making about sequencing strategies to identify commonly used criteria, but ideas for future research on sequencing strategies are provided.

EXPLORING THE PRACTICES OF SELECTING AND SEQUENCING STRATEGIES FOR WHOLE-CLASS DISCUSSIONS

by

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APPROVAL PAGE

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CHAPTER I

INTRODUCTION

My vision of teaching and learning as building on students' mathematical thinking has been supported by professional development (PD) and policy documents, (Daro et al., 2011; National Council of Teachers of Mathematics [NCTM], 2014; National Research Council [NRC], 2001) and is driven by a robust research-base that indicates the importance of mathematics instruction that builds on students' thinking (see, e.g., Cai, 2017). Teaching and learning with a focus on students' mathematical thinking has been referenced in many ways in the literature, but I draw specifically from the work on *responsive teaching*,[1] a type of instruction that shifts the focus from teachers' ideas to building on students' thinking (Jacobs & Empson, 2016; Robertson, et al., 2016).

Research on mathematics teaching that is focused on students' thinking, such as responsive teaching, has shown that both teacher and student learning are enhanced. Specifically, teachers have learned about the ways students think about mathematics, students' sense-making has been valued, and student achievement has increased

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[1] I use the term responsive teaching to refer to teaching that is responsive to students' mathematical thinking, which is critical if students are to learn mathematics with understanding. However, there are other important ways teachers also need to be responsive to students. In particular, culturally responsive teaching highlights the need for teachers to be responsive to students' thinking and to students' multiple identities, which reflect their cultural, linguistic, and community-based resources (see, e.g., Gay, 2002; Ladson-Billings, 1995). While culturally responsive teaching and other types of responsive teaching are important and complementary, this dissertation foregrounds teaching that is responsive to students' mathematical thinking.

(Carpenter et al., 1989; NRC, 2001; Wilson & Berne, 1999). One typical way that teachers are able to learn about their students' thinking is by providing opportunities for students to articulate their mathematical ideas to everyone in the classroom during whole-class discussions.

Whole-class discussions have been identified as a core practice of mathematics teaching in which the ideas of students are central (Jacobs & Spangler, 2017). In these discussions, teachers typically showcase one or more strategies and then facilitate a conversation by posing questions based on the mathematical ideas within and across strategies. Smith and Stein (2018) highlighted the complexity of orchestrating productive whole-class discussions in their seminal book identifying five practices in which teachers must purposefully engage: (a) anticipating strategies, (b) monitoring strategies, (c) selecting strategies, (d) sequencing strategies, and (e) connecting strategies in whole-class discussions. Although the five practices have been well-received in research and in practice, I argue that two of the practices—selecting and sequencing strategies—are insufficiently researched and deserve more attention. I define *selecting strategies* as the teachers' decision making about which student strategies will be shared with the rest of the class and *sequencing strategies* as the teachers' decision making about the order the selected student strategies will be presented (Cirillo, 2013).

Currently, the literature documents the importance of selecting and sequencing strategies for whole-class discussions, but there is little specificity or guidance for what teachers should be considering when engaging in these practices. General criteria for selecting strategies have been suggested, including selecting strategies with important mathematical details or those that reflect common misconceptions, but less attention has been given to how to select the mathematical details and why certain details might be particularly desirable (Cirillo, 2013; Meikle, 2016; Smith & Stein, 2018). Similarly, there are general suggestions for sequencing strategies, such as sequencing strategies from least to most sophisticated to reach a mathematical goal (Kazemi & Hintz, 2014; Smith & Stein, 2018), but there is limited variety in the suggestions provided.

These general criteria not only lack specificity but also a coherent structure. As such, the disjointed lists of criteria often do not provide sufficient assistance for teachers learning to engage purposefully in the practices of selecting and sequencing, yet this purposeful engagement is critical for orchestrating productive whole-class discussions. Purposefully selected and sequenced strategies, or parts of strategies, are important for whole-class discussions because they allow for certain mathematical ideas to be discussed based on the mathematics in the strategies. Additionally, purposefully selected and sequenced strategies can affect the mathematical identities of the authors of strategies that are shared with the class.

This study was designed to illuminate the selecting and sequencing criteria of teachers who were working toward a vision of responsive teaching. The specific research question guiding this study was:

> *How do elementary school teachers, who are working toward being responsive to students' thinking, select and sequence strategies for fraction story problems for whole-class discussions?*

Positionality

I first became interested in the ways teachers select and sequence strategies for whole-class discussions while working with teachers in PD, which was part of a larger research project, *Responsive Teaching in Elementary Mathematics* (RTEM). RTEM is a multi-institutional design study in which 92 teachers participated in three years of PD focused on students' fraction thinking. In this PD, teachers frequently participated in activities in which they reviewed strategies to select and sequence strategies they would potentially share in a whole-class discussion. During these activities, teachers, even in their third year of PD, often seemed uncertain of which strategies to select and sequence for the discussion. Teachers were unsure of what mathematical details to highlight in students' strategies or how to select and sequence strategies to reach a particular lesson goal. Teachers' struggles set the stage for why I decided to explore teachers' selecting and sequencing of student strategies in whole-class discussions using the context of the RTEM project. My role on this project facilitated my data collection and analysis because not only did I have insider knowledge about the PD but the teachers also felt comfortable working with me, analyzing their practice, and having their practice captured on video (and audio).

My Study

I explored teachers' selection and sequencing of strategies through the use of two data sets. In one data set, I worked with teachers who had demonstrated expertise in being responsive to students' mathematical thinking. I observed selecting and sequencing practices in their classrooms and interviewed them about their decision making. In the second data set, I explored teachers with a range of expertise in responsive teaching as they engaged in PD activities in which they selected strategies for whole-class discussions. The teachers in both data sets were drawn from the RTEM project, and the focus was on fraction instruction in the upper elementary school grades. It was important to focus on a particular mathematical content area because the mathematical details of strategies play an important role in teachers' questioning when they are being responsive to students' thinking. Thus, my dissertation not only elevates the importance of being purposeful when selecting and sequencing strategies for whole-class discussions, but also provides insight into the specific types of mathematical details teachers choose to showcase in the area of fractions.

In shedding light on the different ways teachers select and sequence strategies for whole-class discussions, I contribute to the existing research literature and provide a resource for practitioners. First, my study was designed to further the field's understanding of the practices of selecting and sequencing strategies for whole-class discussions by identifying teachers' criteria and then developing a framework to organize the criteria into a coherent structure. Second, my study was designed to support teachers and professional developers (who are working to support teachers) in the purposeful enactment of the practices of selecting and sequencing strategies for whole-class discussions.

In the chapters that follow, I review the literature that guided this study and the methods used to collect and analyze data. I then provide two findings chapters that present the framework, with the first chapter highlighting the criteria that cut across all mathematical content and the second narrowing the focus to criteria specific to fraction content. In the final chapter, I summarize the contributions of the framework for researchers and practitioners and share some thoughts about future directions for research on selecting and sequencing strategies for whole-class discussions.

CHAPTER II

REVIEW OF THE LITERATURE

In responsive teaching, students are provided opportunities to develop their mathematical understanding, and teachers use their knowledge of how students make sense of mathematical ideas to support and extend students' understanding (Jacobs & Ambrose, 2008; Jacobs & Empson, 2016). Therefore, teachers' responsibilities are to facilitate instruction from students' individual understanding by watching, listening, and responding. I specifically adopt the three characterizing features of responsive teaching identified by Richards and Robertson (2016): (a) attending to the substance of students' ideas, (b) recognizing the important mathematical connections within those ideas, and (c) taking up and pursuing those ideas. In other words, teachers' instructional decisions are made and adjusted based on the ideas and strategies students share in the classroom.

The conceptual framework of responsive teaching guides my study in two ways. First, I draw on the work on mathematical talk because students' ideas and strategies can only be the foundation of instruction if they are visible, and this visibility happens through mathematical talk, or when conversations are held in the classroom. Second, I draw on the work on students' mathematical thinking because understanding the mathematical content of students' ideas is essential for productive pursuit of those ideas. Therefore, my literature review is structured around these two main ideas of mathematical talk and students' mathematical thinking, followed by a section outlining how my study builds on this work.

Mathematical Talk

Mathematical talk, often referred to as discourse, has been emphasized as an essential component in helping students learn mathematics (Chapin & O'Conner, 2007; Cobb et al., 1997; Lampert, 2001; NCTM, 2014). Mathematical talk can provide opportunities for students to learn because when students talk, they have to organize and clarify their mathematical ideas to present to others in the classroom. Further, as students talk about their own ideas, they become curious about what others think and strive to make sense of others' ideas. In this way, students also learn by reflecting on discourse that is focused on the strategies, ideas, and conjectures of others in the class (Chapin & O'Conner, 2007).

Mathematical talk between the teacher and students or among students can happen throughout a mathematics lesson, but it is elevated in whole-class discussions, which are the focus of this study. These discussions typically occur at the conclusion of a particular lesson set-up—launch, explore, and discuss (Hirsch et al., 1995). The *launch phase* occurs when teachers pose a task to the class. The launch includes the set-up of the task and any directions teachers might give about how to complete the task, resources the students can use to complete the task, and any expected products from the students. During this phase, teachers may have the opportunity to gain an idea of students' initial knowledge and motivate students to complete the task. In the *explore phase*, students are provided with time to complete the task, typically in a way that makes sense to individual students. During this phase, teachers often circulate the classroom interacting with individual or small groups of students. In the *discuss phase*, teachers engage students in a whole-class discussion centered around student strategies generated during the explore phase.

In these whole-class discussions, teachers and students work together to collectively make sense of mathematical ideas embedded in the strategies. Specifically, teachers showcase one or more strategies and then facilitate a conversation by posing questions based on the mathematical ideas within and across strategies. Students are generally expected to listen, question, and validate or reject ideas created by the interactions of the students and teacher in the classroom. I draw on the definition of whole-class discussions in the work of Grossman and colleagues in the Core Practice Consortium:

> In a whole-class discussion, the teacher and all of the students work on specific content together, using one another's ideas as resources. The purposes of a discussion are to build collective knowledge and capability in relation to specific instructional goals and to allow students to practice listening, speaking, and interpreting. In instructionally productive discussions, the teacher and a wide range of students contribute orally, listen actively, and respond to and learn from others' contributions. (Grossman et al., 2014, Introduction, 10:07–11:10)

According to this definition, whole-class discussions are a core practice in mathematics classrooms that allow for teachers and students to engage with, make sense of, and critique the thinking of others.

Despite the increased attention on whole-class discussions, implementing them has remained a challenge for teachers. My study contributes to our understanding of these discussions and how to implement them by focusing on two teaching practices—selecting and sequencing strategies—that are central to this work and are often under-represented in research and PD. The following sections first document the benefits of whole-class discussions and then explore the enactment of these discussions, with special attention to the practices of selecting and sequencing strategies, which are critical for achieving these benefits.

Benefits of Whole-Class Discussions

The importance of whole-class discussions in mathematics classrooms is supported by policy and consensus documents (NCTM, 2014; NRC, 2001), documents for practitioners (see, e.g., Hurtado et al., 2017), and research (for a summary, see Jacobs & Spangler, 2017). The use of whole-class discussions in mathematics classrooms can lead to a range of benefits for teachers and students.

Teachers who engage their classes in whole-class discussions can benefit because they have the opportunity to gain an understanding of students' mathematical understandings (Stein et al., 2008). By eliciting students' thinking and listening to how students interact with each other's ideas, teachers can more easily respond to how students are making sense of mathematical ideas by adjusting their instruction (Jacobs & Spangler, 2017; Lampert, 2001; Schoenfeld, 2011). Adjusting instruction based on students' sensemaking has been studied in a variety of ways, including identifying characteristics of powerful instances of student thinking on which teachers can build to advance student thinking along particular pathways (Leatham et al., 2015).

Students can benefit because during whole-class discussions, students are generally positioned as owners of their thinking (Boaler & Staples, 2008; Gresalfi, 2009; Jacobs & Spangler, 2017). When they share their ideas, they have opportunities to clarify understandings, construct convincing arguments, and develop a language for expressing mathematical ideas (NRC, 2001). When listening to other students' ideas and strategies, they have opportunities to adopt new, more advanced problem-solving strategies or simply expand their current understanding (Chapin & O'Conner, 2007; Webb et al., 2014). Finally, whole-class discussions have been connected to higher achievement levels for students, when combined with instruction that foregrounds students' ideas (Boaler & Staples, 2008; Chapin & O'Conner, 2007). For example, in their longitudinal study of three high schools, Boaler and Staples (2008) found that participating in classrooms that included small and whole-class discussions was one of the characteristics linked with higher achievement levels for all students.

Enactment of Whole-Class Discussions

Despite the widespread appreciation for the benefits of whole-class discussions, teachers have consistently found their enactment to be challenging (Franke et al, 2009; Sleep, 2012; Tyminski et al., 2013). Smith and Stein highlighted the complexity of orchestrating productive whole-class discussions in their seminal work identifying five practices in which teachers must purposefully engage: (a) anticipating strategies, (b) monitoring strategies, (c) selecting strategies, (d) sequencing strategies, and (e) connecting strategies in whole-class discussions. The five practices were introduced in a 2008 research article and then transformed into a ground-breaking book for practitioners, that is now in its second edition (Smith & Stein, 2011, 2018; Stein et al., 2008). This team has also published grade-band specific books that describe in detail what the practices look like as teachers engage in them (Smith et al., 2019; Smith et al., 2020; Smith & Sherin, 2019). The 5 practices work has had widespread influence in multiple spheres. Numerous researchers have built on this work (see, for example, Cirillo, 2013; Meikle, 2016) and it is widely cited in publications written for practitioners (see, for example, Hurtado et al., 2017; NCTM, 2014).

The following sections expand on the five practices with special attention to the third and fourth practices—selecting strategies and sequencing strategies—that are the focus of this study. All of these practices are important, and there are varying levels of research on each practice, but I argue that selecting and sequencing, in particular, are under-researched and under-appreciated. Note that most of the practices occur prior to the whole-class discussion, reinforcing the idea that purposeful work needs to occur not only during but also prior to the discussion for it to be productive. I begin this exploration of Smith and Stein's work with a sixth practice—setting goals—that was added after publication of the original *5 Practices* book (Smith & Stein, 2011, 2018).

Setting goals. Prior to enacting the five practices for orchestrating productive whole-class discussions, Smith and Stein (2018) suggest identifying a lesson goal, which they refer to as "practice 0" (p. 17). This practice involves specifying a goal that clearly indicates what mathematics students are to learn as a result of the lesson. The identified goal should be specific as it serves as the foundation from which the other practices are enacted (see also, Meikle, 2016).

Anticipating. The first of the 5 practices identified for orchestrating productive whole-class discussions is *anticipating* likely student strategies for specific mathematical tasks. To anticipate student strategies, teachers typically not only complete the tasks themselves, but also generate multiple strategies that students might use when engaging with the tasks. Teachers can draw on their prior teaching experiences as well as their knowledge of research on students' mathematical thinking to anticipate student strategies that reflect a range of sophistication. Anticipating student strategies is important for orchestrating whole-class discussions because it allows teachers to focus on the thinking of the students rather than their own approaches to completing tasks.

Monitoring. The second practice for orchestrating productive whole-class discussions is *monitoring* student strategies during the explore phase of a mathematics lesson. Monitoring is what teachers do when circulating the classroom once students have started working on the task. This practice supports teachers in accomplishing many goals, such as assessing how many students are working on the task and their frustration levels as well as providing customized learning opportunities for students through one-on-one teacher-student conversations (Jacobs & Empson, 2016). Monitoring also supports teachers in selecting and sequencing strategies that will be productive to share in the whole-class discussion. Specifically, as teachers circulate, they become familiar with the mathematical details in student strategies so that they can more easily consider the learning potential of sharing particular strategies. Note that anticipating student strategies can enhance the quality of monitoring, and thus the selection and sequencing of student strategies, by improving teachers' abilities to make sense of student strategies.

Selecting. The third practice for orchestrating productive whole-class discussions is *selecting* one or more strategies to present during the whole-class discussion. Purposefully selected strategies make it more likely to have a discussion where mathematical meaning and positive mathematical identities are constructed by everyone in the classroom. Selecting can occur during monitoring or after monitoring—outside of class time, if the launch and explore phases occur during one day and the discuss phase occurs on a different day.

Researchers have shared numerous general criteria for purposefully selecting strategies. Criteria identified have included selecting strategies that include both correct and incorrect answers to create classroom climates that view mistakes as sites for learning (Smith & Stein, 2018); selecting strategies that showcase mathematics linked to pre-determined lesson goals (in Practice 0) (Cirillo, 2013; Kazemi & Hintz, 2014; Meikle, 2016); selecting strategies with common misconceptions so that they can be addressed (Smith & Stein, 2018); selecting strategies to boost the confidence of the authors of strategies shared (Kersaint, 2017); and selecting strategies to ensure all students regularly have an opportunity to share their ideas with the class (Stein et al., 2008). Criteria such as the ones above provide a starting point for teachers working to improve their selecting expertise, but the information is disjointed and lacks specificity, which can make it challenging for teachers to know how to enact the selecting practice. My study was designed to build on this earlier work to add both structure and specificity.

Sequencing. The fourth practice for orchestrating productive whole-class discussions is *sequencing* strategies, which refers to teachers determining in what order strategies will be shared during whole-class discussions. Similar to selecting, sequencing can occur during monitoring or after monitoring (outside of class time) when the discussion occurs on a different day.

Researchers have shared a variety of ideas for purposefully sequencing strategies. Suggestions have included sequencing strategies to make the lesson coherent by supporting the goals identified for the entire lesson or sequencing strategies to make the discussion accessible to all students by ensuring that a typical or easy strategy is shared first (Smith & Stein, 2018). Stein and colleagues (2008) recognized the complexity of sequencing of student work for whole-class discussions, and they specifically called for more research on this practice. My study was designed, in part, to address this call.

Connecting. The fifth practice for orchestrating productive whole-class discussions is *connecting* strategies during the discuss phase of lessons. When teachers are connecting student strategies, they make visible students' mathematical thinking—they help students articulate and notice key mathematical ideas embedded in student strategies and make mathematical comparisons among multiple strategies, including judgements regarding the accuracy and efficiency of strategies.

Some researchers have focused on the different types of connecting that are possible during discussions. For example, Kazemi and Hintz (2014) offered five types of *targeted sharing*, in which teachers purposefully select and sequence strategies so that the discussion can be structured to target—or accomplish—specific mathematical goals (see Table 1). However, most of the research has focused on the types of talk moves teachers use to enact this connecting practice (NCTM, 2014; Sleep, 2012). Some of the most commonly researched talk moves include pressing students to elaborate, justify, or defend their ideas to push students to think more deeply or to highlight mathematical ideas (Kazemi & Stipek, 2001; Sahin & Kulm, 2008; Staples, 2007); revoicing—or repeating—students' ideas to clarify the mathematics or position students competently (Franke et al., 2007; O'Connor & Michaels, 1993); or orienting students toward each other's ideas (Lampert et al., 2013). These talk moves link to the purposeful selection and sequencing of strategies because the types of talk moves teachers can enact are dependent on the mathematical details embedded in the strategies.

Table 1

Types of Targeted Sharing (Kazemi & Hintz, 2014)

Types of Targeted Sharing	Mathematical Goal
Compare and Connect	To compare similarities and differences
Why? Let's Justify	To justify why a strategy works
What's Best and Why?	To determine the most efficient strategy for a particular problem
Define and Clarify	To define and discuss particular mathematical models, tools, vocabulary, or notation
Troubleshoot and Revise	To reason through what does and does not make sense in an incomplete strategy or a strategy with an incorrect answer

Students' Mathematical Thinking

The previous sections explored mathematical talk as the first component of responsive teaching that drives my study, and the following sections address the second component, which is students' mathematical thinking. Specifically, only when teachers attend to the mathematical substance of students' ideas, can they be responsive to students by recognizing key ideas and pursuing them (Richards & Robertson, 2016). These key ideas are discipline based (i.e., mathematically specific), and teachers must consider students' thinking about these ideas. They can draw from a large body of research on students' mathematical thinking that identifies patterns in students' thinking and development (Daro et al., 2011; NRC, 2001).

Research has shown that students and adults often think differently about mathematics content, including about story problems, which are a focus of this study because they play a critical role in helping students make sense of key mathematical ideas (Carpenter et al., 1993; Fuson et al., 1997; NRC, 2001). To convey patterns in how students think about solving story problems, researchers have developed frameworks and trajectories of students' mathematical thinking, and they have found that when teachers have access to information about students' mathematical thinking, instruction looks different and can be linked to student achievement gains (Carpenter et al., 1989; Clements et al., 2011; Sztajn & Wilson, 2019; Wilson & Berne, 1999).

There are many approaches to studying the teaching and learning of mathematics content from a students' thinking perspective, but I draw specifically from Cognitively Guided Instruction (CGI), a longstanding research and PD project focused on understanding students' mathematical thinking and its role in instruction. CGI researchers were one of the first groups to identify relationships between students' views of story problems and their strategies (Carpenter et al., 1989; Carpenter & Moser, 1984; Hiebert et al., 1982), and I draw on their work in the content area of fractions, which is the focus of this study.

Developing fractional number sense is an essential concept in students' education. A deep understanding of fractions allows students to make sound decisions and reasonable judgments when problem solving in real life (Lamon, 2007). Developing this understanding of fractions does not always come easy for students, but researchers are finding ways to utilize the knowledge students bring to the classroom as a foundation for building their fraction understanding. This "bottom-up approach" (Lamon, 2007, p. 642) is consistent with responsive teaching in that it supports teachers in eliciting students' intuitive knowledge and helps them make sense of students' thinking about fractions. Research on how students think about fractions has been influential in the development of a variety of frameworks, trajectories, and taxonomies of students' fraction strategies (Charles & Nason, 2000; Confrey et al., 2009; Empson & Levi, 2011; Lamon, 1996; Pothier & Sawada, 1983). These tools can help teachers know what problems to pose and how to make sense of students' strategies and the understandings reflected in those strategies.

In the following sections, I draw from the CGI work to provide an overview of two of these frameworks—fraction problem type and strategy frameworks—that were central to my study (Carpenter et al., 2015; Empson & Levi, 2011). These particular frameworks helped me during analysis to make sense of the problems and strategies discussed by teachers when they were selecting and sequencing strategies in my study. They also played an important role in the PD that these teachers attended.

Types of Fraction Story Problems

CGI distinguishes types of fraction story problems by how students view their mathematical structures—differences which are often reflected in students' strategies. I begin describing these problem types by focusing on addition and subtraction story problems, followed by multiplication and division story problems.

Addition and subtraction story problems. When solving addition and subtraction story problems, students pay special attention to the action being described, and whether it is a joining or separating action. Another distinguishing feature of these story problems is location of the unknown. The unknown can be at the end of the story (result unknown), in the middle (change unknown), or at the beginning (start unknown). Figure 1 provides examples of six story problem types showcasing both the action being described and the location of the unknown. Figure 1 illustrates the problem types with fraction story problems, but these distinctions apply to both whole-number and fraction story problems.

Figure 1

Sample Addition and Subtraction Story Problem Types for Fractions

Join Result Unknown	Join Change Unknown	Join Start Unknown
(2 + 1 1/4 = unknown) Robin has 2 cups of sugar for making cupcakes. Sam gave her 1 1/4 cups of sugar. How much sugar does Robin have now?	***(2 + unknown = 3 1/4)*** Robin has 2 cups of sugar for making cupcakes. How many more cups of sugar does she need to get to have 3 1/4 cups of sugar to make the cupcakes?	***(unknown + 1 1/4 = 3 1/4)*** Robin has some sugar. Sam gave her 1 1/4 cups of sugar. Now Robin has 3 1/4 cups of sugar. How much sugar did Robin start with?
Separate Result Unknown ***(3 1/4 – 1 1/4 = unknown)*** Robin has 3 1/4 cups of sugar. She gives Sam 1 1/4 cups of sugar. How much sugar does Robin have left?	**Separate Change Unknown** ***(3 1/4 – unknown = 2)*** Robin has 3 1/4 cups of sugar. She gives some to Sam. Now she has 2 cups of sugar. How much sugar did she give to Sam?	**Separate Start Unknown** ***(unknown – 1 1/4 = 2)*** Robin has some sugar. She gave 1 1/4 cups of sugar to Sam. Now she has 2 cups of sugar. How much sugar did Robin have to start with?

Multiplication and division story problems. When solving multiplication and division story problems, students again attend to where the unknown is located. These grouping problems have three components—the number of groups, the amount per group, and the total—and the problem types are defined by the location of the unknown. In multiplication story problem types the unknown is the total. In measurement division story problems, the number of groups is unknown, and in partitive division story problems, the amount per group is unknown (see Figure 2). Note that the problem types again capture students' perspectives because, whereas adults typically view the two division problems similarly, students often see them differently and subsequently solve them differently.

Figure 2

Sample Multiple Groups Multiplication and Division Story Problem Types for Fractions

Multiplication	**Measurement Division**	**Partitive Division (Equal Sharing)**
There are 4 children who will each get 2 1/2 brownie. How many brownies do I need?	I have 10 brownies and each child can have 2 1/2 brownies. How many children can get brownies?	Four children want to share 10 brownies so that everyone gets exactly the same amount. How much brownie can each child get?

These problem-type distinctions apply to both whole-number and fraction story problems, but for fraction story problems, Empson and Levi (2011) have made an additional distinction based on students' solution patterns. They have found that students can more easily solve these problems when the number of groups is a whole number than when it is a fraction. Thus, they distinguish between multiple groups problems (the number of groups is a whole number) and partial groups problems (the number of groups is a fraction). These distinctions result in six fraction problem types: multiple groups multiplication, multiple groups measurement division, multiple groups partitive division, partial groups multiplication, partial groups measurement division, and partial groups partitive division.

A special focus of this study is on multiple groups partitive division story problem types, which are often referenced as equal sharing story problems, as they are in this dissertation. Equal sharing problems are foundational for helping students understand fraction concepts and operations. They can provide a bridge between whole-number and fraction work because the most basic version of these problems has fractions only appearing in the answer (as in the last example in Figure 2). Because no fractions are stated in the problem stem, students have access to these problems even before they have learned fraction language and notation.

Strategies for Fraction Story Problems

Researchers have identified three holistic categories to describe advancement in students' understandings as evidenced in student strategies for the above story problem types. These categories include direct modeling strategies, counting strategies, and relational thinking strategies. *Direct modeling strategies* include the use of pictures, tally marks, or concrete objects to explicitly represent the problem situation and all of the known quantities in the problem. *Counting/adding strategies* are abstractions of direct modeling strategies. Rather than including a concrete representation of all quantities in the problem, quantities are abstracted and operated on repeatedly by skip counting, adding, or some combination of the two. *Relational thinking strategies* are less directly tied to the problem situation than direct modeling and counting/adding strategies, and they are typically more efficient. Students create and operate on strategic groupings or decompositions of problem quantities, and they are often based on students' knowledge of number facts and relationships as well as their implicit use of properties of operations. These strategy categories apply to both whole-number and fraction story problems, and students naturally progress through the categories as their understandings increase with experience. However, teachers' strategic selection of problems and strategic use of follow-up questions can enhance this development.

Exploring Selecting and Sequencing in My Study

Teaching that is responsive to students' mathematical thinking—with special emphasis on mathematical talk and students' mathematical thinking—plays a key role in my study of selecting and sequencing strategies for whole-class discussions. I build on Smith and Stein's (2018) seminal work in which they identified five practices to orchestrate productive whole-class discussions. Despite the widespread acceptance and use of the five practices, I argue that two practices—selecting and sequencing—have been underappreciated and under-researched. Researchers have identified a disjointed set of general criteria to guide teachers' selecting and sequencing of strategies, yet teachers' implementation of them remains a challenge. My dissertation builds on this limited research base with the goal of providing more structure and specificity by examining the two practices of selecting and sequencing in the content area of fractions. The specific research question guiding this study is:

- *How do elementary school teachers, who are working toward being responsive to students' thinking, select and sequence strategies for fraction story problems for whole-class discussions?*

CHAPTER III

METHODS

I investigated teachers' selecting and sequencing of student strategies using two sets of data—one that focused on teachers who had demonstrated expertise in teaching that is responsive to students' mathematical thinking and one that included teachers with a range of expertise in this type of teaching. All teachers were engaged in a PD that addressed these practices as part of an emphasis on teaching that is responsive to students' thinking and, in particular, students' fraction thinking. I analyzed these data to develop a framework for selecting and sequencing strategies for whole-class discussions.

This study took place in the context of a larger project—Responsive Teaching in Elementary Mathematics (RTEM)—which focused on the teaching and learning of fractions in the upper elementary school grades. I first provide a brief description of the RTEM project context because my participants were drawn from this context. I will then address each data set separately to describe the participants and data sources and finally conclude with a single discussion of the analysis that cuts across both data sets.

RTEM Project Context

In the larger RTEM project, 92 grades 3–5 teachers participated in up to three years of PD. Participants were divided into three cohorts of about 30 teachers each and completed 8.5 workshop days per year (4.5 days during the summer and 4 days during the academic year). The overall purpose of the PD was to develop teachers' instructional expertise in building on students' thinking to advance students' mathematical understandings. The PD was guided by research-based knowledge of students' thinking about fractions and how this knowledge can be used in instruction. Heavy emphasis was placed on the use of fraction story problems as a tool for helping students make sense of fraction quantities and operations. This emphasis included use of a variety of story problem types, with special attention to the foundational role of equal sharing problems (Empson & Levi, 2011). During workshops, teachers: (a) learned about research on students' thinking, largely through frameworks reflecting problem types and strategies students use when working with whole numbers and fractions; (b) analyzed student strategies in video and in written work; (c) practiced follow-up questioning and problem selection for individual students and groups of students; and (d) adapted existing curricular resources to enhance student learning. In addition, between workshops, teachers were asked to pose story problems and experiment with pedagogical strategies in their classrooms and then reflect on those experiences with colleagues at their schools.

Teachers were drawn from one of three school districts in the southern region of the United States. All three district administrations had endorsed the PD and instruction that was responsive to students' mathematical thinking, but the districts reflected variations in instructional histories, philosophies, and resources—all of which could influence teachers' responsiveness to students' thinking.

- District A had a long history of supporting its teachers in learning about students' thinking to inform instruction. Almost all teachers in the district had been offered PD to learn about students' thinking with whole numbers or fractions as well as other resources that aligned with this focus. Sample resources included school-based mathematics facilitators and district-created documents that unpacked curriculum standards in ways that showcased and aligned with students' thinking.

- District B also had a long history of supporting its teachers in learning about students' thinking to inform instruction. This district had similar resources to District A; however, during the time of data collection, this district was in the midst of shifting priorities given a new administration, and less emphasis was being placed on instruction that builds on students' ideas.
- District C had only recently begun to embrace the vision of instruction that builds on students' thinking, and thus resources were still emerging.

The districts also differed in the student populations they served. The percentage of students who qualified for free and reduced-cost lunch was 33% for District A, 47% for District B, and 9% for District C. The percentage of students who were classified as Limited English Proficient was 61% for District A, 71% for District B, and 40% for District C. The race and ethnicity classifications for students in the state-level database also indicated variations: District A (48% White, 45% Hispanic, 2% Black, and 5% Other); District B (46% Hispanic, 36% White, 3% Black, and 15% Other); and District C (68% White, 12% Hispanic, 10% Black, and 10% Other)

Data Set 1: Teacher Cases

In this data set, I explored selecting and sequencing strategies with teachers who had demonstrated some level of expertise in responsive teaching. I investigated three teachers for two whole-class discussions that occurred across three mathematics lessons—I observed their instruction and probed their selecting and sequencing decision making in interviews.

Participants

Three teachers were drawn from a cohort of teachers who had recently completed three years of RTEM PD. I purposefully selected teachers who had completed the PD and had demonstrated some expertise in responsive teaching. Determination of this expertise was based on conversations with the PD facilitators and research team, as well as examination of field notes from the PD and other data collected from the larger RTEM project, including classroom observations.

The teachers were all experienced, female teachers, and their teaching assignments spanned the range of grades (3–5) addressed in the PD. The third-grade teacher, Ms. Henry[2], had 12 years of teaching experience and taught in District A. The fourth-grade teacher, Ms. Wilbern, had 20 years of teaching experience and taught in District C. The fifth-grade teacher, Ms. Dustman, had 7 years of teaching experience, taught in District A, and was considered a leader for her school, especially in relation to mathematics. None of the teachers had completed any prior PD focused on students' mathematical thinking.

Field notes from the broader RTEM project indicated that all three teachers regularly posed story problems, encouraged students to solve problems using their own strategies, and facilitated whole-class discussions in which students' strategies were shared and discussed with the class. As such, their fraction instruction included multiple

[2] Teachers' real names were used with their permission.

opportunities to solve story problems so that students could make sense of important fraction concepts. After extensive work with fraction story problems, some fraction equation work was also incorporated. All three teachers shared that they greatly valued fraction instruction—they addressed fractions at the beginning of the school year and wove fraction instruction throughout the year rather than restricting it to a stand-alone unit.

Data Sources

Data for each teacher case included four interviews and three observations. The first interview, which occurred prior to any observations, was a general interview that explored how the teachers broadly thought about classroom discussions and the ways in which they typically selected and sequenced strategies. The remaining three interviews were follow-up interviews to observations of mathematics lessons. The observations provided an opportunity to see the general instructional context in which teachers' whole-class discussions took place, and the follow-up interviews provided an opportunity for teachers to articulate their decision making around selecting and sequencing strategies for the whole-class discussions.

I conducted all the observations and follow-up interviews during a two-week period in the spring, and they were connected to two common lesson scenarios: a 1-day scenario and a 2-day scenario. I investigated each teacher's 1-day scenario prior to her 2-day scenario. See Table 2 for a summary of the data sources linked to each scenario.

Table 2

Overview of Data Sources for the Teacher Cases

Lesson Scenario	**Day 1**				**Day 2**		
1-day scenario	1-day scenario observation	→	1-day scenario reflection interview		NA		
2-day scenario	2-day scenario pre-selection observation	→	2-day scenario selection interview	→	2-day scenario post-selection observation	→	2-day scenario reflection interview

Note: The general interview occurred prior to the 1- and 2- day scenarios.

In the *1-day scenario*, teachers posed a fraction story problem of their choosing, and then students solved the problem and engaged in a whole-class discussion all in the same class period. Teachers' selecting and sequencing of student strategies to be shared during the whole-class discussion occurred in-the-moment, generally while the teacher was circulating when students were solving the problem. The 1-day scenario included one observation (*1-day scenario observation*) and one follow-up interview (*1-day scenario reflection interview*) in which I explored the teachers' decisions for selecting and sequencing strategies as well as their reflection on how effective the selected strategies and their sequencing were in the whole-class discussion.

In the *2-day scenario*, teachers posed a fraction story problem of their choosing and had students solve it during one class period, but saved the discussion for a different class period, the following day. In this case, teachers' selecting and sequencing of student strategies to be shared during the whole-class discussion occurred after school between the two lessons, giving the teachers more time for reviewing, selecting, and sequencing strategies. The 2-day scenario included two observations and two interviews over two days. On the first day, I conducted a *2-day scenario pre-selection observation* of the mathematics lesson. After school, teachers participated in the *2-day scenario selection interview* to select and sequence strategies for the whole-class discussion, to take place the following day. On the second day, I conducted a *2-day scenario post-selection observation* of the whole-class discussion that used the strategies identified during the 2-day selection interview. After school, teachers participated in the *2-day scenario reflection interview*, in which I explored teachers' thoughts on how effective the selected strategies and their sequencing were in the whole-class discussion.

I observed each of the teachers in both lesson scenarios because I considered the possibility that teachers' decision making for selecting and sequencing strategies was different in the two scenarios given the different time constraints. If so, I wanted to explore both types of decision making. Note that it was reasonable for me to ask these teachers to engage in both scenarios because all three reported that they regularly used both scenarios in their instruction.

Classroom observations. Classroom observations were video- and audio-recorded. Data were collected using a video camera, a lapel microphone for the teacher, and three individual microphones set up around the classroom. The individual microphones were placed in locations that would best capture the voices of the students and were occasionally moved if the students changed locations during the lesson. The focus of the camera was either on the teacher or on where the students were looking. Specifically, if the teacher moved about the classroom or talked with individual students, the camera followed. If the teacher focused the attention of the class toward a whiteboard, poster, or student, the camera captured the teacher's and students' focus. The classroom observations included three problem types, and Figure 3 summarizes the problems posed in each scenario for each teacher.

Figure 3

Overview of Problems Posed in the Teacher Cases

Teacher	**Problem Posed (*Problem Type*)**	
	1-Day Scenario	**2-Day Scenario**
Ms. Henry (grade 3)	6 friends want to share 8 pizzas. How much pizza will each friend get so that they all get an equal amount? *(Equal Sharing)*	8 friends went to IHOP to celebrate National Pancake Day. They ordered 5 pancakes to share equally. How much pancake will each friend get so that they all get the same amount? (*Equal Sharing*)
Ms. Wilbern (grade 4)	Twelve kids share 8 sandwiches. How much does each child get if the sandwiches are shared equally? (*Equal Sharing*)	The zookeeper has 28 sticks of wood to feed 16 rhinoceroses. How much wood can each rhinoceros get, if the wood is shared equally with the rhinoceroses? (*Equal Sharing*)
Ms. Dustman (grade 5)	Sean has a lawn-mowing business. The gas tank on his lawnmower holds 2 1/2 gallons. It takes about 5/8 gallon to mow each lawn. If his tank is full, does he have enough gas to mow 3 yards? *(Multiple groups multiplication)*	The bakery had ___ chocolate chip cookies left over at the end of the day. They took up ___ of the tray. How many cookies fit on a whole tray? (14, 1/2) (15, 3/4) (12, 2/3) (*Partial groups partitive division*)

Interviews. The 12 interviews (four per teacher) were video- or audio-recorded and then transcribed. In all but the general interview, teachers brought written work from the classroom lesson to the interview so that the conversation could be specific to the details in strategies. During the interviews, I kept track of the main ideas teachers shared and probed these ideas to capture as much of the teachers' perspectives as possible. I strove to keep my own perspectives out of the conversations so that teachers could provide authentic accounts of their practices, rather than explaining what they thought I wanted to hear.

Each interview began with an overview of the purpose of the interview: understanding the decision-making underlying teachers' selecting and sequencing of strategies. The 2-day scenario selection interview had additional directions because teachers were selecting and sequencing strategies *during* the interview (vs. other interviews in which they were reflecting on their earlier selecting and sequencing). These additional directions encouraged teachers to think aloud while selecting and sequencing strategies:

> As you select and sequence pieces of student work, I want you to think out loud all of the things going through your mind. Think of it as voicing your inner speech. It is important for me to try and hear all of the thoughts going through your mind.

During the think-aloud portion of this interview, teachers were prompted if they went too long without speaking (e.g., "Keep talking." or "What are you thinking?").

All four interviews were semi-structured and designed to explore teachers' decision making around selecting and sequencing strategies. Based on pilot work, three broad categories and eight focal areas were identified to provide a common structure across the interviews. The three broad categories were selecting strategies, sequencing strategies, and reflecting on the effectiveness of the selection and sequence of strategies. This last category addressing effectiveness was not the focus of my study, but I thought that in reflecting on the effectiveness of discussions, teachers might reveal additional information related to the selecting and sequencing of strategies. The eight focal areas will be described below, in connection with each of the broad categories. See Figure 4 for sample questions and Appendices A–D for the full interviews. Note that I also collected information about the context of the lessons observed, including information about the story problem posed, connections to prior instruction, and the typicality of the whole-class discussion I observed.

Selecting strategies. This category of interview questions addressed three focal areas. Questions in the first focal area (*process*) tracked the process teachers used while selecting strategies and, when possible (2-day scenario selection interview), included the teacher walking me through their decision making as they engaged in selecting strategies. Questions in the second focal area (*reasons*) investigated teachers' reasons for why particular strategies, or sets of strategies, were selected and why others were not. Questions in the third focal area (*strategy details*) explored the teachers' perspectives on the mathematical details in the strategies selected. I paid special attention to when teachers linked strategy details to research-based frameworks on students' strategies that were used in the PD (Empson & Levi, 2011).

Sequencing strategies. This category of interview questions also addressed three focal areas. Questions in the first focal area (*process*) explored the process the teachers used for sequencing the strategies. Questions in the second focal area (*reasons*) investigated the teachers' reasons for their sequencing decisions. Questions in the third focal area (*timing*) explored when the sequencing occurred (i.e., during circulating, between lessons, or during the discussion).

Reflecting on the effectiveness of the selection and sequence of strategies. This category of interview questions addressed two focal areas. Questions in the first focal area (*discussion goals*) explored the teachers' goals for using selected strategies in whole-class discussions. Questions in the second focal area (*indicators of effectiveness*) explored teachers' ideas about what constituted a successful conversation focused on student strategies in a whole-class discussion and the indicators they sought as evidence of success. The purpose of these questions was for teachers to elaborate on the identified criteria described for selecting and sequencing strategies and their effectiveness in the implementation of the whole-class discussion. By considering the student engagement and mathematical quality of the discussion, teachers were able to reflect on whether the criteria they used for selecting and sequencing strategies resulted in an effective discussion.

Figure 4

Common Interview Structure for the Four Interviews of the Teacher Cases

Broad Categories	**Focal Areas**	**Sample Interview Questions**	
		General Interview	**1- or 2-Day Scenario Interviews**
Selecting strategies	*Process*	Describe the process you typically use to select student work for whole-class discussions.	Walk me through the process of how you selected the student work used for the discussion.
	Reasons	Tell me about some of the guiding principles you use for selecting student work.	Summarize why you selected these pieces of student work.
	Strategy details	What makes selecting student work easy or difficult?	Summarize the pieces of student work you selected.
Sequencing strategies	*Process*	Describe the process you typically use to sequence student work for whole-class discussions.	Walk me through the process of how you sequenced the student work for whole-class discussions.
	Reasons	How do you decide how to sequence student work for whole-class discussions?	Tell me about how you decided on the sequence of the strategies used during this discussion?
	Timing	Do you sequence all of the pieces that will be shared before the discussion starts or do you sequence during the discussion?	When did you decide how to sequence the selected pieces of student work?
Reflecting on the effectiveness of the selection and sequence of strategies	*Discussion goals*	Why do you have whole class discussions?	Did you have any specific goals for this discussion?
	Indicators of effectiveness	How do you know if the student work you selected worked the way you wanted?	Did you feel the discussion worked the way you wanted? What is your evidence?

Data Set 2: PD Conversations

In this data set, I explored selecting and sequencing strategies with teachers in the RTEM PD who, collectively, reflected a range of expertise with responsive teaching. This variety increased the likelihood of identifying a wider range of reasons teachers use when selecting and sequencing strategies for whole-class discussions. I investigated the conversations among teachers during the RTEM PD workshops, and I focused on PD activities in which groups of teachers were asked to review a collection of student strategies and collaboratively select strategies that they would showcase during a whole-class discussion. Teachers were not explicitly asked to sequence the strategies, but some chose to do so.

Participants

A total of 30 teachers completed one or more of the PD activities that were the focus of this data set. The teachers included 28 classroom teachers (13 grade 3 teachers, 10 grade 4 teachers, and 5 grade 5 teachers), one mathematics coach, and one resource teacher. Consistent with the teaching population at the elementary school level, most of the participants were female (26 females and 4 males). The teachers were generally experienced, with a mean of 11 years of teaching experience, and experience that ranged from 2–26 years. Nine participants were drawn from District A, ten participants were from District B, and eleven participants were from District C. In addition, 7 of the 9 teachers from District A had received prior PD on students' thinking about whole numbers in the primary grades. All teachers were enrolled in the RTEM PD at the time of data collection, and 18 were in year 2 of PD and 12 were in year 3 of the PD.

Data Sources

I investigated a total of 17 video-recorded conversations among small groups of teachers (see Table 3). Their conversations focused on four PD activities that occurred during year 2 or year 3 of the PD. (Activities that focused on selecting and sequencing strategies for whole-class discussion did not occur in year 1, due to the complexity of these practices). The PD included numerous activities in which teachers looked at strategies, but these four PD activities were chosen because teachers were asked to select strategies for whole-class discussions and the conversations were video- (rather than just audio) recorded. Each of the four PD activities had a unique story problem, and these four problems reflected a range of mathematical structures and operations, including three problems types (see Table 3).

Table 3

Overview of Data Sources for the PD Conversations

PD Activity Name	Year of PD	Number of Video-Recorded Small-Group Conversations	Number of Teachers[b]	Fraction Story Problem (*Problem Type*)
Snow Cones	2	3	6	Omar has a snow cone machine. It takes 2/3 cup of ice to make a snow cone. How many snow cones can Omar make with 4 cups of ice? (*Multiple Groups Measurement Division*)
Bananas	3	3	9	The zookeeper has 8 bananas to feed the 6 monkeys. If she wants to use up all the bananas and give the same amount to each monkey, how much should she give each monkey? (*Equal Sharing*)
Hot Chocolate	2, 3[a]	7	15	You have 4 2/3 cups of hot chocolate powder. Each serving requires 2/3 cup of hot chocolate powder. How many servings can you make? (*Multiple Groups Measurement Division*)
Butter	3	4	11	You're baking cookies for a bake sale. You already have 7/8 of a stick of butter. How much more butter do you need to buy so that you have a total of 3 1/8 sticks of butter? (*Join Change Unknown*)

[a] The Hot Chocolate activity was used in both year 2 and year 3 of the PD.
[b] 11 participants engaged in more than one of the video-recorded small-group conversations.

In each PD activity, small groups of teachers (2–4 teachers) were given a common, facilitator-selected, set of student strategies and asked to collaboratively select 2–4 strategies that could be used in a potential whole-class discussion. The small groups were also asked to create a poster with their selected strategies and then generate and record sample questions that could be posed in the discussion. Two activities had additional directions. Directions for the Hot Chocolate activity asked teachers to explicitly link the mathematical details in their selected strategies to the research-based strategy framework presented in the PD (Empson & Levi, 2011), and directions for the Butter activity asked teachers to articulate a specific instructional goal for their selected strategies.

Unlike in the interviews for the teacher cases, teachers were generally not explicitly asked to state their selecting rationales. Instead, teachers collaboratively decided on their selections and thus rationales naturally emerged in their conversations. One exception was the Butter activity, which was followed by a brief interview that included these three questions:

- How did you decide on these strategies?
- Were there a few strategies that you eliminated right away? If so, what was your reasoning?
- What kinds of things would you look for in the discussion of these pieces of student work to let you know if your selection of student work was successful or not?

Data Analysis

The purpose of the analysis was to identify and provide a structure for the range of criteria teachers considered when selecting and sequencing strategies for whole-class discussions. Overall, I used a constant-comparative analysis (Glaser, 1965) to identify teachers' criteria and so that I could organize these criteria into a framework for selecting and sequencing strategies for whole-class discussions. The analysis plan consisted of four phases, each described below. The first analysis phase will include a separate description for the teacher cases and the PD conversations. The remaining three analysis phases will discuss the data sets together because they were explored together using the same analysis plan. Coding the two data sets in an integrated and iterative manner, rather than sequentially, was desirable because both data sets addressed the same ideas, but with different depth, with teachers who had different levels of expertise, and with different fraction content.

Phase 1: Completing Foundational Tasks

Phase 1 of the analysis involved several tasks to familiarize myself with the data to begin making sense of the teachers' selecting and sequencing of strategies, which more formally occurred in Phase 2.

Teacher cases. To begin making sense of the teacher case data, I organized the data by collecting, sorting, and labeling all data and transcribing the 12 interviews from the three teachers. I characterized the instructional contexts of the three teachers by creating a memo for each teacher based on their general interviews. These memos provided an initial lens for understanding each teacher's mathematics instruction so that the upcoming selecting and sequencing decision making could be considered in context.

I also characterized the strategies from the three teachers' classroom observations to provide a sense of the landscape of strategies from which teachers were selecting and sequencing. A total of six classroom sets of strategies were characterized because each teacher had two classroom sets of strategies—one for the 1-day scenario and one for the

2-day scenario (see Figure 5). Note that I categorized the entire classroom set of strategies, not just the strategies the teachers chose to use during the discussion.

For each strategy, I tracked (a) whether the strategy included a correct answer, incorrect answer, or no answer and (b) the holistic strategy category—direct modeling, counting/adding, or relational thinking—as identified in the research shared in the PD (Empson & Levi, 2011). Note that in addition to the holistic strategy categories, researchers have identified strategy sub-categories linked to specific problem types. Given the special role that equal sharing problems played in this study, I also tracked some of the strategy sub-categories linked to this problem type.

Direct modeling strategies for equal sharing problems are divided into two sub-categories that are distinguished by whether students initially consider the number of sharers when partitioning items. A strategy is *non-anticipatory direct modeling* if there is no coordination between the partitioning of items being shared and the number of sharers when students start the strategy. In other words, the strategy includes direct modeling of all the items and the problem situation, but students are unsure of how to partition items initially so they may have to use multiple types of partitioning to generate a number of pieces that can be evenly distributed to the number of sharers. For example, students may use a trial and error strategy for partitioning, or they may automatically begin partitioning in halves without consideration for the number of pieces that will be generated. A strategy that is *emergent-anticipatory direct modeling* has some coordination between the partitioning of items and the number of sharers. In other words, the strategy includes direct modeling of all of the items and the problem situation, but students initially partition items by the number of sharers (or a factor of the number of sharers). For example, for problems in which 6 children are sharing 4 items, students might partition all items into sixths or thirds.

Relational thinking strategies for equal sharing problems are also referenced as anticipatory strategies. An *anticipatory strategy* involves mental coordination between the partitioning of items and the number of sharers. Unlike the direct modeling strategies in which all problem quantities are represented, students are mentally able to solve the problem using their understanding of number relationships. Specifically, they view an equal sharing problem as a fraction, *a/b*, and understand that the value of the fraction can be determined by dividing the numerator (the number of items) by the denominator (the number of sharers).

Figure 5

Characterization of the Class Sets of Strategies for the Teacher Cases

Problem **(*Number of strategies*)**	**Answer**		**Strategy**	
Ms. Henry 1-Day Scenario (*N*=22) 6 friends want to share 8 pizzas. How much pizza will each friend get so that they all get an equal amount?	64% 9% 27%	Correct Incorrect No answer	5% 95%	Direct Modeling (non-anticipatory) Direct Modeling (emergent-anticipatory)
Ms. Henry 2-Day Scenario (*N*=17) 8 friends went to IHOP to celebrate National Pancake Day. They ordered 5 pancakes to share equally. How much pancake will each friend get so that they all get the same amount?	65% 29% 6%	Correct Incorrect No answer	94% 6%	Direct Modeling (emergent-anticipatory) Invalid
Ms. Wilbern 1-Day Scenario (*N*=23) Twelve kids share 8 sandwiches. How much does each child get if the sandwiches are shared equally?	74% 26%	Correct Incorrect	35% 48% 13% 4%	Direct Modeling (non-anticipatory) Direct Modeling (emergent-anticipatory) Relational Thinking (anticipatory) Invalid
Ms. Wilbern 2-Day Scenario (*N*=24) The zookeeper has 28 sticks of wood to feed 16 rhinoceroses. How much wood can each rhinoceros get, if the wood is shared equally with the rhinoceroses?	42% 46% 12%	Correct Incorrect No answer	38% 50% 12%	Direct Modeling (non-anticipatory) Direct Modeling (emergent-anticipatory) Relational Thinking (anticipatory)
Ms. Dustman 1-Day Scenario (*N*=16) Sean has a lawn-mowing business. The gas tank on his lawnmower holds 2 1/2 gallons. It takes about 5/8 gallon to mow each lawn. If his tank is full, does he have enough to mow 3 yards?	100%	Correct	19% 44% 31% 6%	Direct Modeling Counting/Adding Relational Thinking Invalid
Ms. Dustman 2-Day Scenario (*N*=14) The bakery had ___ chocolate chip cookies left over at the end of the day. They took up ___ of the tray. How many cookies fit on a whole tray? (14, 1/2) (15, 3/4) (12, 2/3)	64% 22% 14%	Correct Incorrect No answer	29% 21% 29% 21%	Direct Modeling Counting/Adding Relational Thinking Invalid

PD conversations. To begin making sense of the PD conversations, I characterized the common set of strategies that was selected by the PD facilitator for each of the four activities, again to provide a sense of the landscape of strategies from which teachers were selecting and sequencing. The four sets of strategies were categorized using the same two criteria from the teacher cases: correctness and holistic strategy categories (see Figure 6).

Figure 6

Characterization of the Sets of Strategies for the PD Conversations

Problem **(*Number of strategies*)**	**Answer**		**Strategy**	
Snow cones (*N*=13) Omar has a snow cone machine. It takes 2/3 cup of ice to make a snow cone. How many snow cones can Omar make with 4 cups of ice?	100%	Correct	46% 54%	Direct Modeling Counting/Adding
Bananas (*N*=6) The zookeeper has 8 bananas to feed the 6 monkeys. If she wants to use up all the bananas and give the same amount to each monkey, how much should she give each monkey?	83% 17%	Correct Incorrect	33% 67%	Direct Modeling (non-anticipatory) Direct Modeling (emergent-anticipatory)
Hot Chocolate (*N*=11) You have 4 2/3 cups of hot chocolate powder. Each serving requires 2/3 cup of hot chocolate powder. How many servings can you make?	73% 27%	Correct Incorrect	36% 36% 28%	Direct Modeling Counting/Adding Relational Thinking
Butter (*N*=10) You're baking cookies for a bake sale. You already have 7/8 of a stick of butter. How much more butter do you need to buy so that you have a total of 3 1/8 sticks of butter?	90% 10%	Correct Incorrect	30% 50% 10% 10%	Direct Modeling Counting/Adding Relational Thinking Invalid

Phase 2: Identifying Selecting and Sequencing Criteria

Phase 2 involved use of a constant-comparative method to make sense of all of the data and identify the range of criteria teachers considered when selecting and sequencing strategies for whole-class discussions. I also drew from the literature linked to responsive teaching to guide my analysis. Specifically, the literature on mathematical talk provided a starting place for identifying criteria for selecting and sequencing strategies.

The literature on students' mathematical thinking helped me understand the students' strategies and appreciate the powerful mathematics embedded in those strategies so that I could make sense of the teachers' decision making around selecting and sequencing.

To begin this phase of analysis, I immersed myself in the 2-day scenario selection interviews of the three teacher cases because this interview captured teachers' criteria—in the moment—as they were selecting and sequencing strategies. Additionally, in this interview I was able to ask more follow-up questions about the practices of selecting and sequencing because these practices were slowed down given that they occurred when students were not present (vs. in the 1-day scenario when selecting and sequencing occurred during the mathematics lesson). Slowing down the practices also allowed teachers to be more reflective about their decision-making because they had more time to think and articulate their criteria. After reviewing the 2-day scenario selection interviews, I incorporated other interviews from the teacher cases and the PD conversations. The analysis was iterative, regularly moving between data sets to identify criteria. This fluidity was particularly important because the different data sources included teachers who reflected varying expertise in responsive teaching and were making decisions about problems that represented five problem types.

To begin identifying criteria, I first used descriptive coding (Miles et al., 2014) to capture the main idea of each rationale teachers gave for selecting and sequencing particular strategies. The purpose of identifying the main ideas was to begin generating the range of criteria teachers used to select and sequence strategies for whole-class discussions. Looking across all data sources, I generated a comprehensive list of criteria for selecting and sequencing. To be included on the list, the idea had to have been stated more than once by an individual teacher or by more than one teacher. The comprehensive list included not only criteria as found in the literature but also other criteria that arose in the data.

Counter-examples. After criteria from both data sets were compiled into a comprehensive list of criteria, I reviewed all data with the criteria in mind and looked for potential counter-examples. The purpose of identifying counter-examples was to solidify the existence of the criteria and present any critical ideas that ran counter to the identified criteria as a way to add credibility to my findings (Miles et al., 2014). I did not find that teachers contradicted the criteria, but not all criteria were used every time teachers selected and sequenced strategies, as teachers prioritized particular criteria differently across lessons. Thus, I determined that organizing the criteria into a framework would need to illuminate potential criteria for selecting and sequencing strategies rather than outlining criteria that teachers should use as a checklist.

Phase 3: Building a Framework

Phase 3 of analysis involved organizing the comprehensive list of criteria to create a framework that conveyed the landscape of teachers' selecting and sequencing strategies for whole-class discussions. To generate the framework, I grouped (and regrouped) criteria to make visible a structure that was conceptually coherent (vs. based on frequency, for example). The intent was that the framework would be useful for both researchers and teachers, and thus the framework was purposefully designed with special attention to the grain-size. I searched for a grain size that would be precise enough for researchers but also usable by teachers in the midst of teaching. This goal influenced both the number of categories included in the framework and the language used to describe the categories. In creating the framework, I built on the literature on students' mathematical thinking, and in particular the work of the CGI project, because these researchers have a long history of creating frameworks that are usable by both teachers and researchers (Carpenter et al., 1989; Carpenter et al., 2015; Empson & Levi, 2011).

During this phase, I also recognized that I did not have sufficient data to include sequencing strategies for whole-class discussions in the framework. I had less data on teachers' decision making for sequencing strategies than for selecting strategies because the PD activities did not require teachers to sequence the strategies they selected, and even the teacher cases chose to discuss selecting much more frequently than sequencing. Further, teachers' sequencing decisions were all at least superficially similar, thereby not lending themselves to categorization. I return to this issue in the final chapter. As a result, the framework developed in this phase and the remainder of this dissertation focus solely on selecting strategies for whole-class discussions.

Phase 4: Confirming and Revising the Selecting Strategies Framework

I conducted two activities to confirm and revise the framework for selecting strategies for whole-class discussions. First, I revisited the case examples from the 5 practices books (Smith et al., 2019; Smith & Stein, 2011, 2018), and I tried to apply my framework criteria to the teachers' decision making when they were selecting strategies for whole class discussion. In general, my framework mapped well onto those teachers' decision making.

Second, I conducted three member-checking interviews (Miles et al., 2014), one with each of the teacher cases. The purpose of member checking was to add credibility to the framework and to ensure I broadly captured the essence of the teachers' decision-making. In the interviews, I systematically solicited feedback about the framework to verify that the criteria identified had face validity and broadly reflected the teachers' criteria for selecting strategies. Additionally, I gave the teachers an opportunity to question criteria, add criteria, and change phrasing of how criteria were communicated. Overall, the criteria described in the framework resonated with the teachers, but they did provide some different ways to communicate information about the selecting criteria.

The purpose of developing a framework was to identify and put structure to the range of ways teachers select strategies. The next two chapters will describe the framework, supported by examples of teachers' decision making put forth for selecting strategies for whole-class discussions. In the first findings chapter, the framework will be introduced, and criteria that apply across multiple content areas will be showcased. In the second findings chapter, the content area of fractions becomes central, and the framework will be extended to address criteria connected to this specific content area.

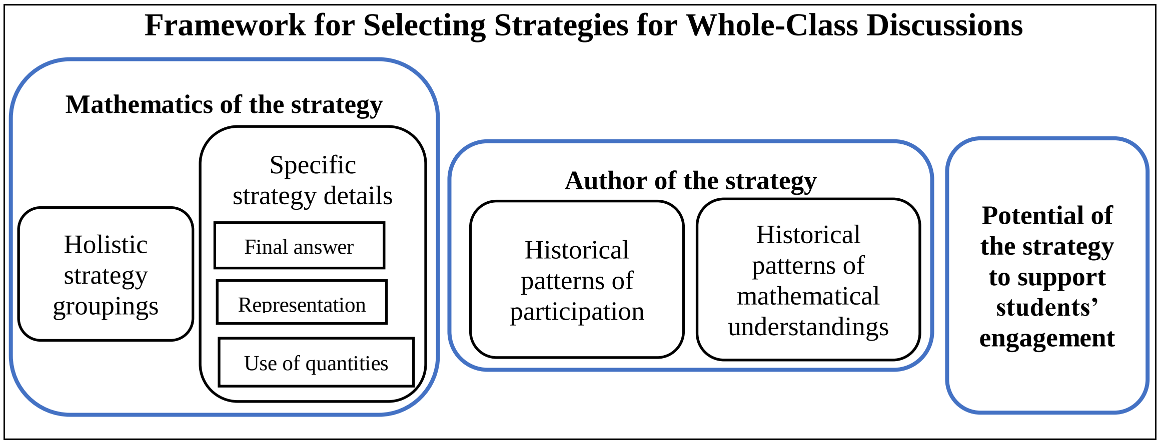
CHAPTER IV

FINDINGS: FRAMEWORK FOR SELECTING STRATEGIES FOR WHOLE-CLASS DICUSSIONS

Based on the analysis of teacher cases and PD conversations, I created a framework for selecting strategies for whole-class discussions. In this chapter, I will provide an overview of the framework, highlighting teachers' three main criteria for selecting strategies: (a) the mathematics of the strategy, (b) the author of the strategy, and (c) the potential of the strategy to support students' engagement (see Figure 7). Data from one of the teacher cases will be used to illustrate these criteria (and some sub-criteria). The criteria described in this chapter are sufficiently broad to cut across multiple content areas. However, there exists an additional level of the framework that is content specific, and in the second findings chapter, I will focus on this content-specific level. Specifically, I will focus exclusively on the criteria of the mathematics of the strategy and address this content-specific level by highlighting how teachers took into consideration specific strategy details linked to the content area of fractions. Data from both the teacher cases and PD conversations will illustrate these ideas.

Figure 7

Framework for Selecting Strategies for Whole-Class Discussions



Overview of Criteria for Selecting Strategies

Teachers based their strategy selections for whole-class discussions on one or more of three main criteria. One main criterion was the *mathematics of the strategy* which refers to teachers' consideration of the mathematical features of strategies, and they considered these features in two ways. Sometimes teachers grouped strategies holistically to capture different levels of understandings, and they often selected strategies to reflect the most common strategies generated by students in the class. Other times, teachers' selections emphasized specific strategy details that were of interest, such as the final answer, the representation of the strategy, or the use of quantities. (Additional strategy details will be introduced in the next chapter.) Teachers used these two types of sub-criteria for the mathematics of the strategy to determine what mathematics would be made visible to the class in the discussion and to highlight particular mathematical ideas.

A second main criterion teachers considered when selecting strategies was the *author of the strategy* which refers to teachers' consideration of who generated the strategy. They used what they knew about the strategy author—their insider knowledge—to assist in their decision-making. Specifically, teachers considered the author's historical patterns of participation and mathematical understandings. The author's historical patterns of participation refer to how recently or how regularly the author has shared strategies during whole-class discussions. Historical patterns of mathematical understandings refer to the teachers' consideration of how the understandings underlying the author's current strategy were advancing compared to understandings the author had displayed in previous lessons. Teachers used these two types of historical knowledge to learn more about students' understandings and to strategically position authors in positive ways in whole-class discussions. For example, teachers might have selected a strategy because the author had typically participated minimally in discussions, and they wanted to encourage increased participation as well as elevate the author's mathematical status in the classroom.

Finally, a third main criterion for selecting strategies, *potential of the strategy to support students' engagement*, refers to teachers' consideration of whether other students in the class will be able to engage with the mathematics of the selected strategies in the discussion, so that they can learn from those strategies. Specifically, to support the class in engaging with the selected strategies, teachers attended to the visibility of the mathematics in the strategy, including the neatness of the written representation.

The rest of this chapter will illustrate the three main criteria using the decision-making underlying the selection of strategies for two whole-class discussions by one of the teacher cases (Ms. Henry). Specifically, I will illustrate the framework using data from Ms. Henry's interviews and observations spanning her 1- and 2-day scenarios. The next section provides an overview of Ms. Henry's 1- and 2-day scenarios, and then the following sections illustrate the framework criteria using Ms. Henry's data. See Figure 8 for a summary of which data sources will be used to illustrate each criterion.

Ms. Henry's case provides a coherent way to illustrate the framework for selecting strategies for whole-class discussions, but note that the framework was developed using data from all teacher cases and PD conversations. My goal is to share Ms. Henry's rationales for selecting strategies to highlight particular mathematical ideas, to create regular participation opportunities and learn more about the advancing understandings of the students whose strategies are shared, and to maximize the opportunity for the class to learn.

Figure 8

Overview of Ms. Henry's Data Sources Used to Illustrate Framework Criteria

FRAMEWORK		MS. HENRY'S DATA SOURCES	
Criteria	**Sub-criteria**	**1-day scenario**	**2-day scenario**
Mathematics of the strategy	Holistic strategy groupings	X	X
	Specific strategy details	X	X
Author of the strategy	Historical patterns of participation	X	
	Historical patterns of mathematical understandings	X	X
Potential of the strategy to support students' engagement		X	

Background Information for Ms. Henry's Whole-Class Discussion

Ms. Henry is a third-grade teacher with 12 years of teaching experience, and she works in a school district in the southern region of the United States. She has 22 students in her classroom, and all students were present for the 1-day scenario while only 17 students were present for the 2-day scenario. In both lessons, she followed the Launch-Explore-Discuss format, and prior to the launch of the story problem, students reviewed class expectations. Expectations for students included solving independently, showing their thinking on paper and stating it verbally, listening, and posing mathematical questions to other students. Students were also asked to record a goal for each mathematics lesson (a task required by the district) and rate themselves on their confidence with the goal for the lesson.

In each of her observed lessons, Ms. Henry launched the story problem by reading the problem to the class and discussing it. After the launch, the explore phase of the lesson began. Students worked individually to solve the problem, and Ms. Henry walked around the classroom visiting almost every student and documenting their strategies on her anticipation guide—a list of possible student strategies for the problem, which was generated by Ms. Henry prior to the lesson. In the 1-day scenario, she also used this circulating time to select strategies for the whole-class discussion and ask strategy authors if they would be willing to share with the group. The discuss phase followed the explore phase and took place either in the same lesson (1-day scenario) or in the lesson the following day (2-day scenario). During the discussion, students—one at a time—displayed their strategies using the document camera and the class asked them questions. When students were describing their strategies, Ms. Henry recreated the strategy on a piece of poster paper at the front of the classroom so that mathematical connections could more easily be made across strategies by both students and teachers.

This chapter showcases the multiple decisions Ms. Henry shared for her selection of strategies in these two scenarios. Her case is not presented as an exemplar of how to select strategies because there is never a single best selection. Instead, Ms. Henry's case is meant to illustrate the decision making captured by the framework. The following sections use Ms. Henry's decision-making to illustrate each of the framework's criteria after two sections that provide details about the observed lesson in her two scenarios—information needed to understand the evidence presented.

1-Day Scenario

Ms. Henry's 1-day scenario involved one mathematics lesson in which she posed this story problem: *6 friends want to share 8 pizzas. How much pizza will each friend get so that they all get an equal amount?* For this lesson, students wrote the goal of comparing fractions by reasoning about their size, and then rated themselves on how confident they felt with the goal at the beginning of the lesson (and again at the end). The more specific goal Ms. Henry had for her class was to reason about the relationship between thirds and sixths (e.g., 1/3 = 2/6) because today was the first opportunity students had to partition into 6ths (which was a likely strategy given that 6 friends were sharing pizzas).

When circulating during the lesson, Ms. Henry selected four strategies to share during her whole-class discussion. Figure 9 displays the four strategies in the order they were shared in the discussion, and Appendix E provides a description of each. All four of the strategies selected were valid strategies, and all four strategy authors verbally shared the correct answer during the whole-class discussion. However, only two written strategies included a correct answer (strategies of Carlos[3] and Sammy). One did not include a written answer at all (strategy of Jeff) and one included an ambiguous answer that nonetheless showed valid reasoning ("8/1/6" for 8 one-sixths in Leo's strategy).

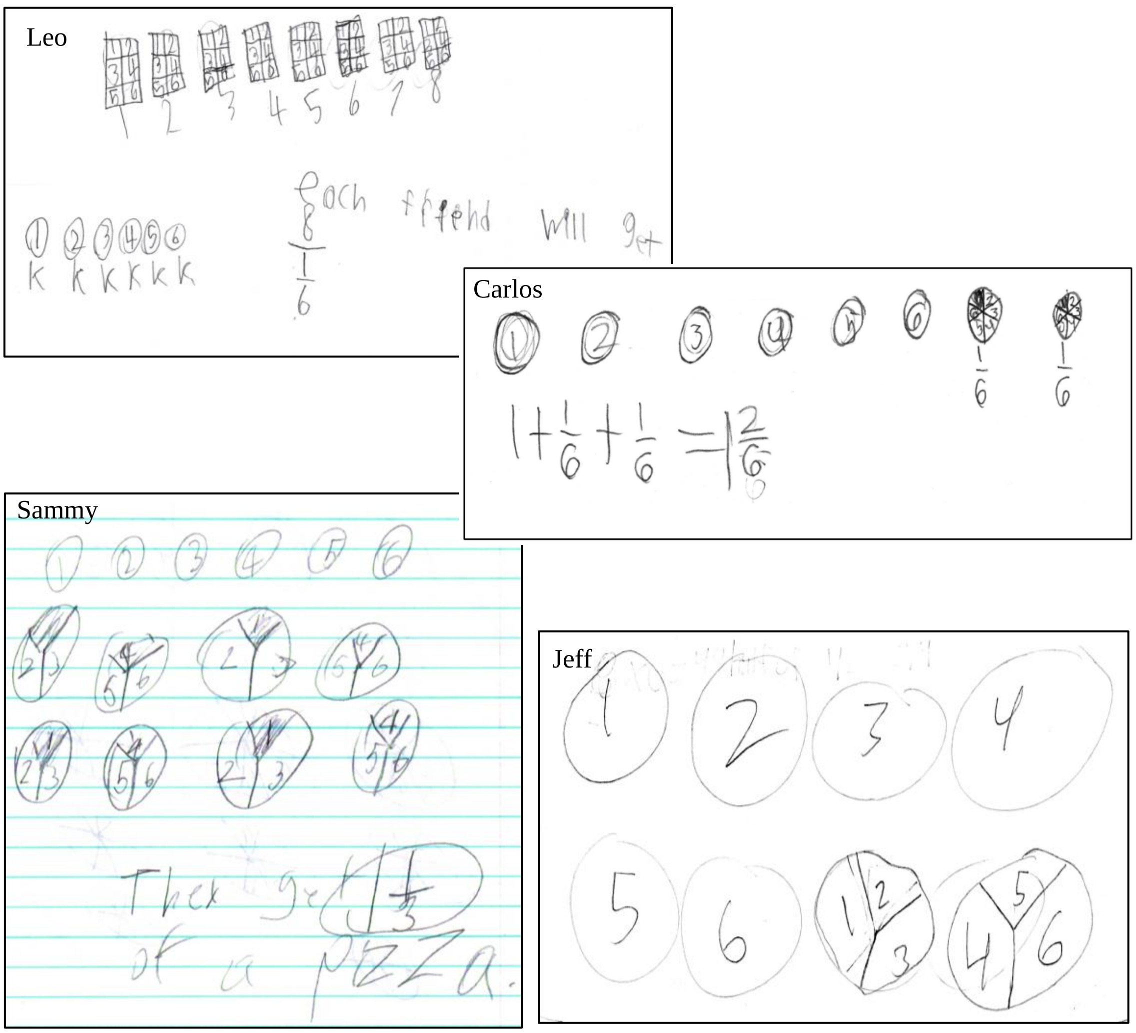
[3] All student names are pseudonyms.

All four strategies included drawings that showed how the students partitioned and distributed the pizzas. The students' partitioning involved sixths or thirds, both of which were related to the number sharers (6) in the problem. Specifically, two of the students partitioned pizzas into sixths—Leo partitioned all eight pizzas into sixths whereas Carlos distributed one whole pizza to each friend and then partitioned only the final two pizzas into sixths. The other two students partitioned pizzas into thirds—Sammy partitioned all eight pizzas into thirds whereas Jeff distributed one whole pizza to each friend and then partitioned only the final two pizzas into thirds. All four strategies used the numbers 1-6 to indicate distribution of whole pizzas (or pieces of pizzas) to each of the 6 students.

Figure 9

Four Strategies Shared in Ms. Henry's 1-Day Scenario Discussion

6 friends want to share 8 pizzas. How much pizza will each friend get so that they all get an equal amount?



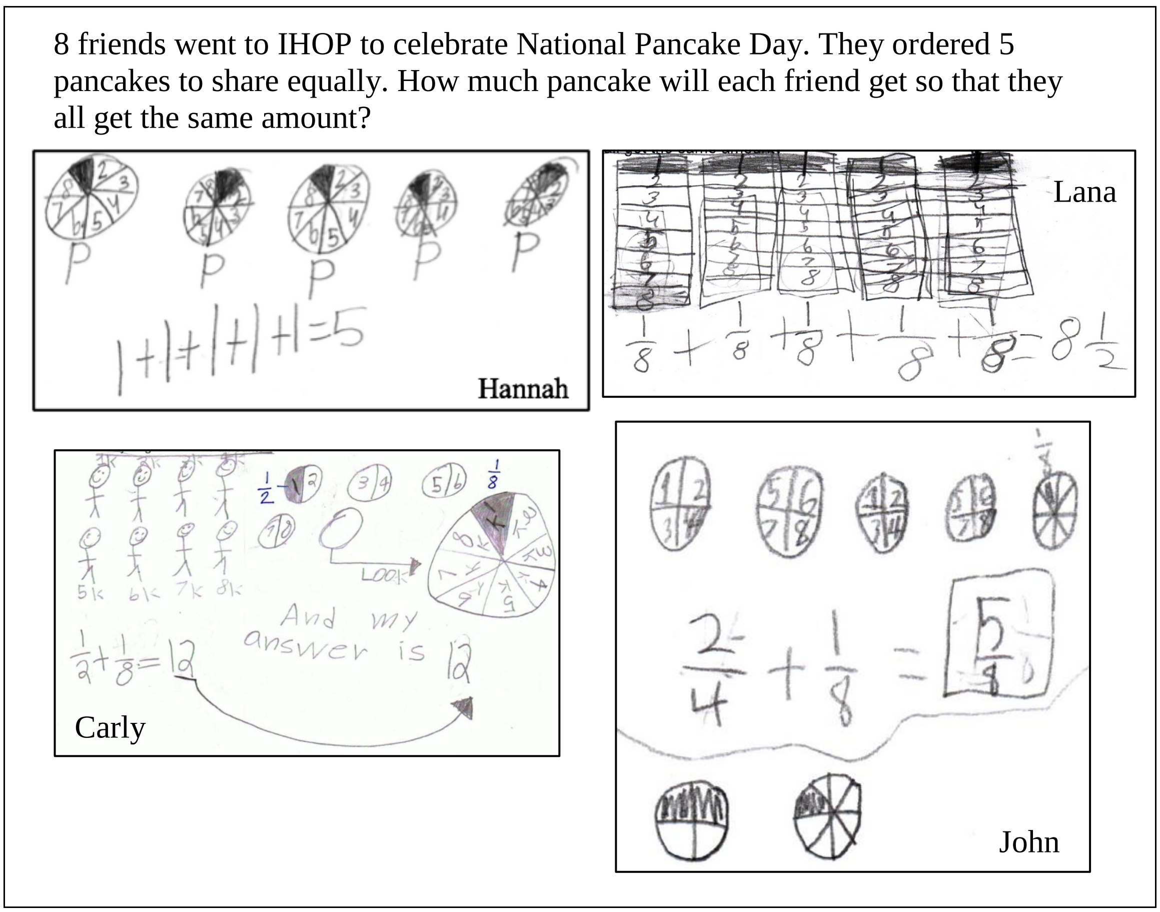
2-Day Scenario

Ms. Henry's 2-day scenario involved two mathematics lessons—one in which the students solved a story problem and one in which students engaged in a whole-class discussion about their strategies. The focus for both lessons was this story problem: *8 friends went to IHOP to celebrate National Pancake Day. They ordered 5 pancakes to share equally. How much pancake will each friend get so that they all get the same amount?* For this lesson, students again wrote the goal of comparing fractions by reasoning about their size and then rated themselves on how confident they felt with the goal at the beginning of the lesson (and again at the end). The more specific goal Ms. Henry had for her class was to reason about the relationship between halves, fourths, and eighths (e.g., 1/2 = 2/4 = 4/8) because today students would likely have the opportunity to combine different fractions using equivalent amounts given the 8 sharers.

In between the two lessons, Ms. Henry again selected four strategies to share during her whole-class discussion. Figure 10 displays the four strategies in the order they were shared in the discussion, and Appendix F provides a description of each. All four of the strategies selected were valid strategies, but only one included a correct written answer (strategy of John). The other three strategies had incorrect answers (strategies of Hannah, Lana, and Carly). All four strategies included drawings that showed how the students partitioned and distributed the pancakes. The partitioning involved halves, fourths, or eighths, all of which were related to the number of sharers (8) in the problem. Specifically, two of the students partitioned all of the pizzas into eighths—Hannah and Lana. The other two students partitioned the final pizza into eighths after using another partition on the first four pancakes—Carly partitioned the first four pancakes into halves whereas John partitioned the first four pancakes into fourths. All four strategies used the numbers 1-8 to indicate distribution of pieces of pancakes to each of the 8 friends.

Figure 10

Four Strategies Shared in Ms. Henry's 2-Day Scenario Discussion



Ms. Henry's Decision Making Linked to the Mathematics of the Strategy

One criterion teachers considered for selecting strategies for whole-class discussions is the mathematics of the strategy—a criterion many might expect because the discussion is occurring in a *mathematics* lesson. Consideration of this criterion came in two forms (or sub-criteria), holistic strategy groupings and specific strategy details, which will each be described using Ms. Henry's 1- and 2-day scenarios.

Holistic Strategy Groupings

Ms. Henry's decision-making included holistic consideration of the strategies her class generated. She often began her decision making by holistically grouping the strategies, and these groupings were initially linked to her anticipation guide that she created prior to the lesson. For these equal-sharing problems, her anticipation guide was sectioned into the three categories of strategies shared in the RTEM PD: non-anticipatory direct modeling strategies, emergent-anticipatory direct modeling strategies, and anticipatory strategies (Empson & Levi, 2011). Within each category, she anticipated a range of possible strategies by drawing the specific strategies.

Holistic strategy groupings allowed Ms. Henry to make sense of the landscape of students' understandings reflected in her class set of strategies before she began considering the specific strategy details she might want to showcase in the whole-class discussion. Her holistic groupings began in her anticipation guide but also were also customized for the performance of her class.

In the 1-day scenario, Ms. Henry selected strategies as she was circulating the class. She described the use of her anticipation guide of expected student strategies to assist her with grouping and eventually selecting strategies:

> It gave me a starting place to start looking for things when I knew the possibilities of what my kids might do. If not, it's just like throwing a dart board and saying like, I guess we'll talk about this one today.

When circulating, Ms. Henry visited 16 of her 22 students, and she noted that one strategy that was a non-anticipatory direct modeling strategy, and fifteen strategies were emergent-anticipatory direct modeling strategies. Further, all the strategies were specific strategies that Ms. Henry had anticipated in those two strategy categories. Ms. Henry selected four strategies from the emergent-anticipatory direct modeling strategies, thereby choosing to showcase the most common strategies used by her class.

In the 2-day scenario, Ms. Henry selected strategies after school and between the lesson focused on problem solving and the lesson involving the whole-class discussion. Because this selection process occurred outside of instructional time, she was able to consider all of the strategies generated, not just ones she saw while circulating. She began her decision-making process for selecting strategies by again using her anticipation guide to holistically group the 17 strategies from her students (as 5 of the 22 students were absent that day). Similar to the 1-day scenario, almost all (94%) of the strategies generated by the class were emergent-anticipatory direct modeling strategies. In this case, Ms. Henry further sorted based on three *specific* emergent-anticipatory direct modeling strategies that she had anticipated. She described the process she used as follows:

> What I'm going to start out doing is thinking about these 3 [anticipated] strategies and I'm kind of going to sort them into three piles, three or more depending on what we see, but three strategies that I anticipated that we would see…I sorted them by the 3 different strategies that I thought I would see and then I had a 4th pile, actually I had 2 piles at the opposite ends of the spectrum. One that knew every strategy and needs her thinking

> pushed further in a different way and then Kasey was in her own pile because she was my only student who didn't solve it correctly or didn't partition in a way that would work.

Ms. Henry's holistic sort resulted in five groups. The first four groups were all emergent-anticipatory direct modeling strategies that varied based on the size of the partitions. The first group consisted of 10 valid strategies, including the selected strategies of Hannah and Lana. These strategies included drawings of all the pizzas, and each pizza was partitioned into eighths. The second group consisted of 3 valid strategies, including the selected strategy of Carly. These strategies included a drawing of all the pizzas, and the first four pizzas were partitioned into halves and the remaining pizza was partitioned into eighths. The third group consisted of 2 valid strategies, including the selected strategy of John. These strategies included a drawing of all the pizzas, and the first four pizzas were partitioned into fourths and the remaining pizza was partitioned into eighths. The fourth group consisted of the work of one student who had multiple valid emergent-anticipatory strategies, including the three strategies described in the first, second, and third groups. The fifth group consisted of one strategy that was an invalid strategy by the child mentioned in the quote (Kasey), who tried multiple partitions but was unable to find a partition size that worked for her.

When selecting her four strategies for the whole-class discussion for the 2-day scenario, Ms. Henry selected two strategies from the first group and one strategy from both the second and third groups, thereby reflecting some of the range of strategies in her class.

Specific Strategy Details

Specific strategy details are another sub-criterion of the mathematics of the strategy. Specific strategy details refer to particular pieces—or details—of strategies that are mathematically significant and might be helpful to address in a whole-class discussion. There are three main categories of mathematical details that teachers considered when selecting strategies for whole-class discussions: the final answer, the representation of the strategy, and the use of the quantities. Each detail category will be described below using examples from either Ms. Henry's 1- or 2-day scenario.

Final answer. One detail category teachers often considered when selecting strategies for whole-class discussions is the final answer. Sometimes their focus was on the correctness of the answer. For example, in the 2-day scenario, Ms. Henry selected Hannah's strategy because of her incorrect final answer of "5." Hannah had a valid strategy of partitioning all the pancakes into eighths but had called each piece "1" instead of "1/8" and thus added them together to answer 5. Ms. Henry shared that she selected Hannah's strategy because "she just cut them all into 8ths… but she doesn't know what to call her final answer."

In addition to selecting strategies to showcase incorrect final answers (with valid strategies), teachers sometimes focused on how the final answers were written. Student answers have traditionally included fractions that are written as proper fractions, improper fractions, or mixed numbers. However, student answers can also be written in non-traditional formats, such as fractions written as words, expressions that include fractions that have not been combined (e.g., each child receives 1/2 + 1/6 brownie), or other invented notation. To illustrate Ms. Henry's consideration of non-traditional formats for final answers, I draw from Ms. Henry's selection of Leo's strategy in the 1-day scenario. He had a valid strategy of partitioning all the pancakes into eighths, but his final answer was written as "8/1/6." Ms. Henry pointed to the non-traditional format of his answer when explaining her decision making:

> What caught my eye was the way he wrote his answer at the end, that's what I wanted to get to… 8 one-sixth and I knew what he was trying to say. And I was kind of surprised he didn't just write 8/6. I wanted to make sure that he knew… I also thought that will give some kiddos an opportunity to ask a question because I knew they were going to wonder about that too.

In this quote, Ms. Henry discussed how she selected Leo's strategy as a way to make visible what quantities Leo was referencing with the non-traditional answer, since that information was not clear as written.

Representation of the strategy. A second detail teachers often considered when selecting strategies for whole-class discussions is the representation of the strategy. Sometimes teaches focused on the extent of the drawings present in the strategy. For example, all of strategies Ms. Henry selected for whole-class discussions (in both scenarios) included a drawing of all the items (pizzas or pancakes) rather than only the items that were partitioned. Sometimes she specifically identified this strategy detail as her rationale, such as when selecting Jeff's strategy to share in the 1-day scenario. In this lesson, Ms. Henry wanted to share a strategy in which one whole was distributed and then the two remaining pizzas were partitioned into thirds. She noted that three students used this strategy, but their representations were different—Jeff's strategy represented all eight pizzas (6 whole pizzas and 2 pizzas partitioned into thirds) whereas Mya's and Jacqulyn's strategies only represented the final two pizzas that were partitioned into thirds. Ms. Henry selected Jeff's strategy because she wanted a drawing of all of the pizzas for the whole-class discussion:

> I knew that Mya or Jacqulyn both didn't draw out every single pizza and that's why I didn't choose them…if I threw up Jacqulyn's or Mya's thinking, I'm going to lose some kids cause they're like, she didn't have to draw them all. Even though I want them to eventually get there, not all of my kids—I think—are ready to see that. So that's why I went with Jeff on that one.

One of the reasons Ms. Henry selected Jeff's strategy was to capitalize on the familiarity of this specific strategy detail of drawing all of the pizzas, because most of her students represented all of the items, so she argued "I knew it at least looked to the rest of my class, it looked like the other strategies at least from the pizza standpoint."

Use of quantities. A third detail teachers often considered when selecting strategies for whole-class discussions is the use of quantities, or the ways that quantities from story problems were manipulated during the problem-solving process. For example, teachers sometimes focused on the possibility of distributing wholes items, such as in the 1-day scenario, when Ms. Henry selected Leo's and Carlos's strategies as a pair so that students could compare how whole pizzas may or may not be distributed. Leo's strategy involved the partitioning of all the pizzas into sixths whereas Carlos's strategy involved the distribution of one whole before partitioning the remaining pizzas into sixths. Ms. Henry shared that she wanted the students to "compare Carlos's passing out the whole pizzas, compared to [Leo's] who cut every pizza into sixths" and "to be able to find that

6/6 in [Leo's strategy] equaled 1 whole [in Carlos's strategy]." She explained that this lesson was the first time the class had the opportunity to partition into sixths and since the students "know that 3/3 equals 1 whole and they know 4/4 equals one whole," she wanted them to have the opportunity to see that 6/6 equals one whole.

Another example of use of quantities is when the decision making of teachers involved consideration of the opportunity to discuss equivalence of fractions. For example, in the 2-day scenario, Ms. Henry selected both Carly's and John's strategies because of their use of halves or fourths as well as eighths when partitioning the pancakes and their subsequent need to combine pieces of different sizes. For Ms. Henry, this partitioning "brings up this idea of 1/2 plus 1/8, what does that equal?" She further elaborated that she selected John's strategy because he used equivalent fractions to add his quantities of 2/4 and 1/8. She said:

> I chose him not only for the strategy but for his thinking that he did down here—thinking about 2/4 was equal to 4/8. He just started to think about the denominator: "How can I find an equivalent fraction that has a like denominator and then add those together?" Because I didn't have anybody else that went there.

Ms. Henry wanted to highlight that John knew 2/4 was the same as 4/8, a conversion which facilitated adding on the 1/8 from the last pizza. Additionally, no one else in the class had used equivalent fractions to combine fractional amounts for a final answer, and she wanted to highlight this idea in the whole-class discussion.

In the earlier section on holistic strategy groupings, we saw that Ms. Henry selected some strategies based on their prevalence in the class. Here, we see Ms. Henry selected an unusual strategy for the class. This contrast suggests that teachers consider prevalence, but may select common strategies, atypical strategies, advanced strategies, etc.

Ms. Henry's Decision Making Linked to the Author of the Strategy

In addition to considering the main criterion of the mathematics of the strategy, Ms. Henry considered the criterion of the *author of the strategy* which has two sub-criteria, historical patterns of participation and historical patterns of mathematical understandings.

Historical Patterns of Participation

Ms. Henry used the authors' patterns of participation in past discussions to inform her selection decision making so that she could ensure everyone had a regular opportunity to share. Ms. Henry described how, during her selection process, she ensured everyone has a chance to share strategies by asking herself, "Has that person shared in a while? Would it be powerful for that person to share?" By tracking historical patterns of participation, Ms. Henry could recognize who had not recently shared and provide individuals with explicit invitations to participate through her selection of their strategies for whole-class discussions.

Ms. Henry's consideration of students' historical patterns of participation extended beyond their participation in mathematics discussions to also take into account their participation in discussions linked to other subject areas. For example, in the 1-day scenario, Ms. Henry selected Carlos's strategy to share in part because mathematics was a positive experience for him compared to his struggles in other subjects. She shared:

> Carlos has a language processing diagnosis, but just is on fire in math right now, so he might be one I think about using just because that would be a chance. Like literacy, I feel like he crawls into a hole because he doesn't want to be called on…but like math, he's willing to share so it's really a chance for confidence wise and other people can see he's still, he is a valuable member of our classroom.

For Carlos, the selection of his strategy was powerful because he had the opportunity to not only articulate his mathematical thinking but also present himself as a successful student. Additionally, purposeful use of this criterion had the potential to impact how Carlos's felt about his thinking because sharing his strategy could boost his mathematical confidence.

Historical Patterns of Mathematical Understandings

Ms. Henry also selected strategies based on the authors' historical patterns of mathematical understandings. Specifically, she considered students' mathematical understandings from previous lessons and selected strategies that showed advances in understandings for authors or to confirm an advance in understanding. These advanced understandings might be reflected in a new mathematical connection or a new level of strategy sophistication. For example, in the 1-day scenario Ms. Henry selected Sammy's strategy, in part, because she expected him to partition into sixths based on his previous strategies. However, while circulating, Ms. Henry saw that Sammy used thirds and heard him talk comfortably about his use of thirds. She shared her surprise and excitement at this mathematical growth when explaining her selection of his strategy:

> I was really blown away when Sammy talked about 3 plus 3 was 6 and saw a relationship with thirds and sixths because that's not something normally that I get from him at all. Like that was deeper than he's ever gone… so when he talked about that, I thought, I've got, I've got to share his thinking.

Similarly, in the 2-day scenario, Ms. Henry selected Lana's strategy because she partitioned all items into eighths—reflecting the number of sharers—which is a connection she had not previously made. Ms. Henry said:

> Lana is one of my students who struggles the most, so this was huge that she just cut them all into eighths…I mean it's a huge step in the right direction… I don't think she's correctly [drawn] any of our equal share problems for a long, long time.

Ms. Henry considered Lana's mathematical understandings from previous lessons and selected her strategy for the whole-class discussion because it showed an advance in her strategy sophistication and a new understanding for her. In both of these examples, Ms. Henry argued that selecting strategies that represented transitions for authors helped to build their confidence by giving them opportunities to articulate their new ideas.

Additionally, Ms. Henry used this criterion as an opportunity to learn more about students' mathematical understandings that she thought might be advancing. In the 1-day scenario, Ms. Henry wanted to learn more about Carly's thinking to see if she knew partitioning four pancakes in half would give her eight pieces before she actually partitioned the pancakes. Ms. Henry said:

> I'm curious really if Carly understands why that works so I think I might have her share this strategy… Carly is usually [one to use non-anticipatory direct modeling strategies], so to me it's like she's moved up a step or maybe she hasn't…I don't know. I need to talk to her about [it]. I need to dig in deeper and find out if it was just cutting them in half because that's her go-to.

Thus, based on what she knew about Carly's typical strategies, Ms. Henry selected Carly's strategy to hear her talk through the strategy. This sharing opportunity allowed Ms. Henry to learn about Carly's thinking to see if she was transitioning to a more advanced strategy.

Ms. Henry's Decision Making Linked to the Potential of the Strategy to Support Students' Engagement

In addition to considering the main criteria of the mathematics and author of the strategy, teachers might select strategies because of the potential of the strategy to support students' engagement with the mathematics of the strategy. Teachers considered whether the mathematics of the strategy was visible, which provided more opportunities for the rest of the class to engage with selected strategies. For example, in Ms. Henry's 1-day scenario, Tara and Carlos used the same strategy of distributing wholes prior to partitioning the final two pizzas into sixths and distributing the pieces. However, the clarity of their strategies varied, with Carlos's strategy being neater. (See Figure 11 for the two strategies.) Ms. Henry cited this criterion when explaining why she decided *not* to share Tara's strategy:

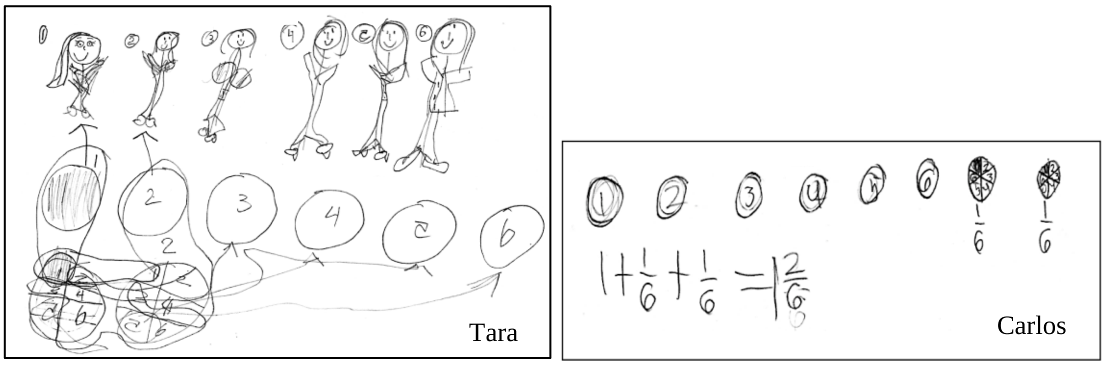
> Tara's is just a mess. I didn't choose it because I thought that's so confusing to look at… I looked at hers then I thought, I know what she's thinking—it's correct. I thought if I put that up there, they're going to go, "Oh Tara you've got to stop circling and drawing arrows."

Ms. Henry opted not to use Tara's strategy because of the limited visibility of the mathematics, and she thought this lack of clarity might distract the students in the whole-class discussion.

Figure 11

Tara's and Carlos's Strategies from Ms. Henry's 1-Day Scenario Discussion

6 friends want to share 8 pizzas. How much pizza will each friend get so that they all get an equal amount?



Conclusion

Ms. Henry's case illustrated the three main criteria of the framework for selecting strategies for whole-class discussions. These criteria included the expected criterion of the mathematics of the strategy as well as potentially less expected criteria, such as the author of the strategy and the potential of the strategy to support students' engagement. This chapter also illustrated the sub-criteria linked to the main criteria, which include the holistic strategy groupings, specific strategy details, historical patterns of participation, and historical patterns of mathematics understandings. The framework identifies criteria to consider when selecting strategies for whole-class discussions, but the criteria are not meant to serve as a checklist. Note that Ms. Henry did not use all criteria in the framework when selecting strategies for her two whole-class discussions. This chapter has focused on selection criteria that apply across mathematical content areas. In the following chapter, I will explore another level of the framework, the sub-categories around specific strategy details, which focus on specific mathematics content—fractions.

CHAPTER V

FINDINGS: SPECIFIC STRATEGY DETAILS FOR FRACTIONS

In the previous chapter, I presented a framework which identified three main criteria (and some sub-criteria) that teachers considered when selecting strategies for whole-class discussions. Teachers can consider these criteria for whole-class discussions on any mathematical content. However, there is another level of the framework that is content specific, and in this chapter, I examine that content-specific level in relation to the content area of fractions. In the previous chapter, I introduced three specific strategy detail categories that are relevant across content areas: final answer, representation, and use of quantities. In this chapter, I expand these detail categories to identify a series of sub-categories specifically related to fraction story problems. Each of the sub-categories will be described and illustrated using data from the teacher cases or PD conversations.

Overview of Categories of Specific Strategy Details

In the previous chapter, I shared how teachers' considerations of specific strategy details fell into three main categories: (a) final answer, (b) representation, and (c) use of quantities. Each of these detail categories includes multiple sub-categories, which are summarized in Figure 12. *Final answer* refers to the teachers' consideration of how the final answer was included in the strategy, including the correctness, the form, or the visibility in the strategy. *Representation* refers to how the student used drawings (i.e., pictures), symbolic notation, or words in solving the story problem. *Use of quantities*, refers to how the quantities were manipulated in the strategy.

The rest of this chapter will illustrate the three specific strategy detail categories using the decision-making underlying the selection of strategies from all three teacher cases and one of the PD conversations. Similar to the previous chapter, this content-specific level of the framework was developed from all of the data, but for clarity, I have chosen to focus on teachers' decision making related to four discussions. One of these discussions was presented in the previous chapter (Ms. Henry's 2-day scenario) and the next section provides an overview of the three new discussions, followed by descriptions and illustrations of the detail categories. See Figure 12 for a summary of which data sources will be used to illustrate each detail category.

Figure 12

Overview of Data Sources Used to Illustrate the Framework Category of Specific Strategy Details

FRAMEWORK		DATA SOURCES			
		Teacher Cases			
Specific Strategy Detail Categories	**Strategy Detail Sub-Categories**	**Ms. Henry's 2-day scenario**	**Ms. Wilbern's 2-day scenario**	**Ms. Dustman's 1-day scenario**	**PD Conversation**
Final Answer	*Correctness of the final answer*	X			
	Form of the final answer		X		
	Visibility of the final answer in the strategy		X		
Representation	*Overall form of the representation*				X
	Use of word labels in the representation				X
	Extent of the representation		X		
	Specific shapes used in the representation	X			
Use of Quantities	*Equivalent quantities*		X		
	Benchmark quantities				X
	Operation used on quantifies			X	
	Fractional quantities created		X		

Background Information for Three Whole-Class Discussions

In the sections that follow, background information will be provided for a whole-class discussion linked to the teacher case of Ms. Wilbern, another whole-class discussion linked to the teacher case of Ms. Dustman, and the PD conversations in which teachers selected strategies for a potential whole-class discussion on the Butter problem. Each description includes the problem and selected strategies.

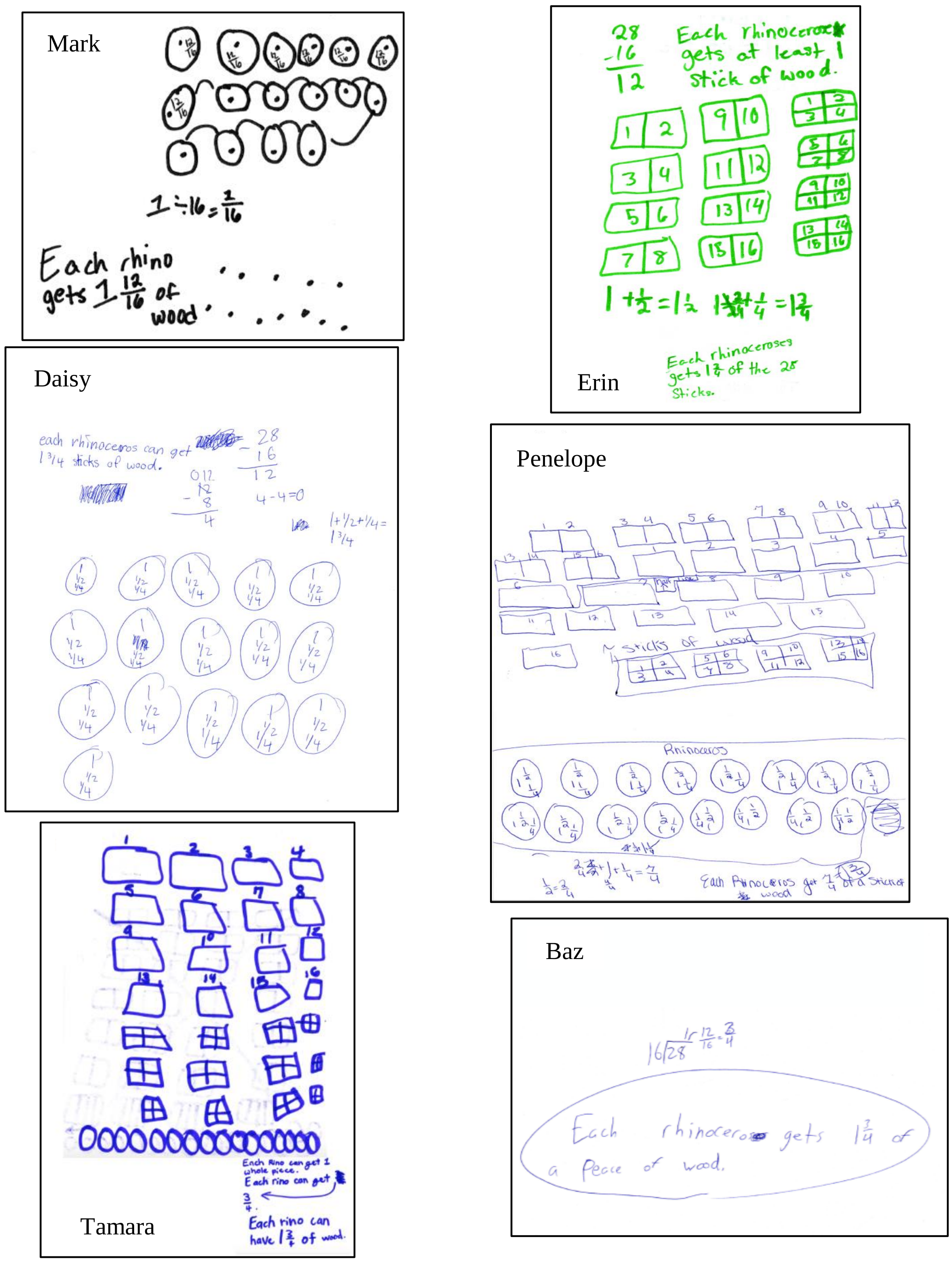
Ms. Wilbern's 2-Day Scenario Discussion

Ms. Wilbern, one of the teacher cases, posed the story problem to her 24 students: *The zookeeper has 28 sticks of wood to feed 16 rhinoceroses. How much wood can each rhinoceros get, if the wood is shared equally with the rhinoceroses?* Ms. Wilbern selected six strategies for her whole-class discussion. Figure 13 displays the six strategies in the order they were shared in the discussion, and Appendix G provides a description of each. All of the strategies selected were valid strategies, had correct answers, and involved some use of symbolic notation. The strategies reflected a range of partitions including halves, fourths, and sixteenths. Specifically, all students but Baz used drawings and distributed 16 whole sticks of wood and fractional pieces from the remaining 12 sticks of wood—Mark distributed sixteenths; Erin, Penelope, and Daisy distributed halves and fourths; and Tamara distributed fourths. Baz used a mental strategy that was set up as the standard division algorithm to solve the problem.

Figure 13

Six Strategies Shared in Ms. Wilbern's 2-Day Scenario Discussion

The zookeeper has 28 sticks of wood to feed 16 rhinoceroses. How much wood can each rhinoceros get, if the wood is shared equally with the rhinoceroses?



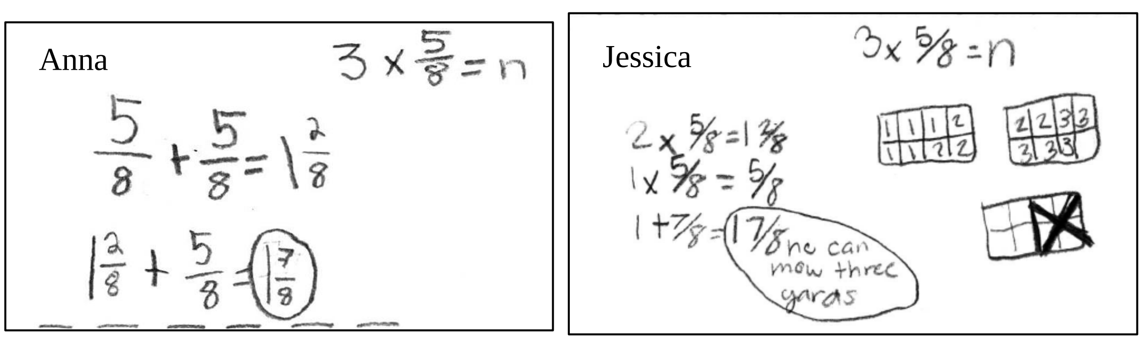
Ms. Dustman's 1-Day Scenario Discussion

Ms. Dustman, one of the teacher cases, posed the story problem to her 16 students: *Sean has a lawn-mowing business. The gas tank on his lawn mower holds 2 1/2 gallons of gas. It takes about 5/8 of a gallon to mow each yard. If his tank is full, does he have enough gas to mow 3 yards?* Ms. Dustman selected a pair of strategies for her whole-class discussion. Figure 13 displays the two strategies in the order they were shared in the discussion, and Appendix H provides a description of each. Both represented the story as a multiplication problem ($3 \times 5/8 = n$), but Anna used repeated addition to solve the problem, and Jessica used multiplication.

Figure 14

Two Strategies Shared in Ms. Dustman's 1-Day Scenario Discussion

Sean has a lawn-mowing business. The gas tank on his lawnmower holds 2 1/2 gallons. It takes about 5/8 gallon to mow each lawn. If his tank is full, does he have enough gas to mow 3 yards?

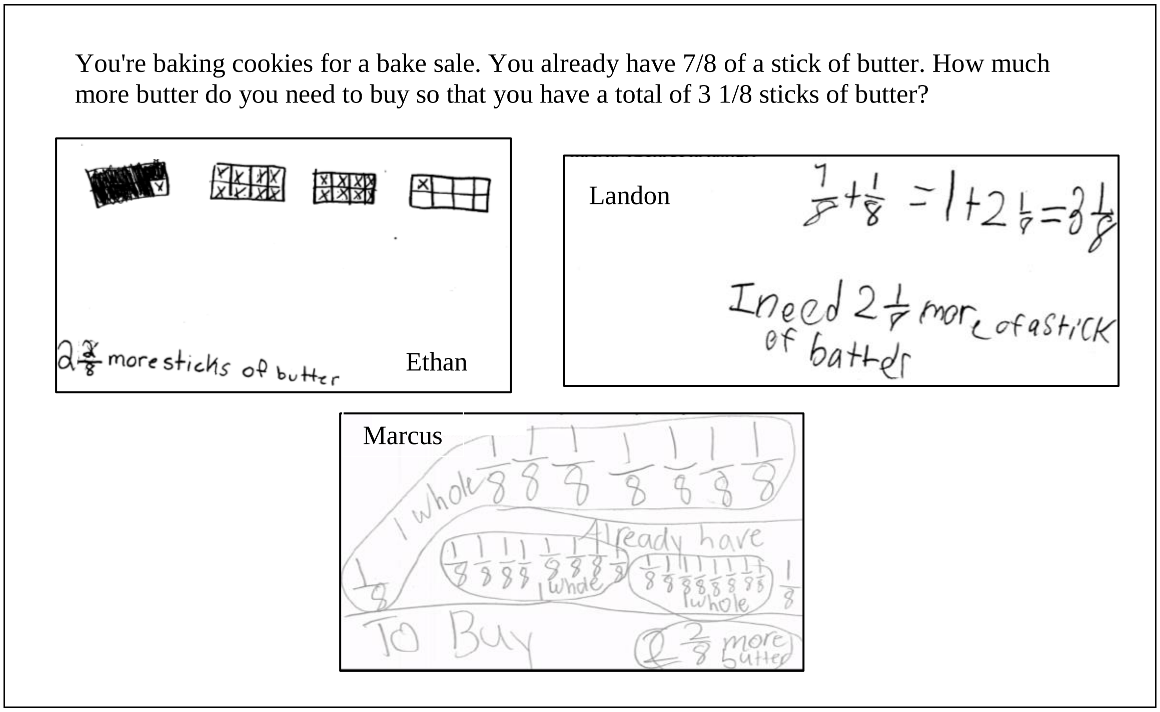


PD Conversation Linked to the Butter Problem

In one of the PD activities, two teachers in their third year of PD selected strategies for the story problem, "*You're baking cookies for a bake sale. You have 7/8 of* *a stick of butter. How much more butter do you need so that you have a total of 3 1/8 sticks of butter?* The teachers selected three strategies for a potential whole-class discussion. Figure 15 displays the three strategies they selected for the discussion, and Appendix I provides a description of each. All of the strategies were valid strategies, with two showing the correct answer (strategies of Ethan and Marcus) and one showing an incorrect answer (strategy of Landon). All used an adding-up strategy in which they first added 1/8 to 7/8 to get to one whole stick of butter, then added additional amounts to get to 3 1/8, and finally derived their answers by combining the amounts they added. How they built to 3 1/8 after the initial 1/8 varied across their strategies as did their representations. Ethan used a drawing whereas Landon and Marcus used symbolic notation.

Figure 15

Three Strategies Selected for a Discussion about the Butter Problem in a PD Conversation



Teachers' Decision Making Linked to the Final Answer

Teachers considered the specific strategy detail of the final answer when selecting strategies to share during whole-class discussions. In particular, teachers considered the correctness of the final answer, form of the final answer, and the visibility of the final answer in the strategy.

Correctness of the Final Answer

Teachers considered the *correctness of the final answer* when selecting strategies to share with the class to broaden assessment of problem solving. Specifically, the final answer was considered in conjunction with the validity of the strategy yielding multiple possibilities: correct answer with a valid strategy, correct answer with an invalid strategy, incorrect answer with a valid strategy, or incorrect answer with an invalid strategy. The following example showcases Ms. Henry's decision making in the 2-day scenario when she selected Hannah's and Lana's strategies with incorrect answers (5 and 8 1/2) for the problem about 8 children sharing 5 pancakes (see Figure 10). These students had valid strategies but did not get the correct answer, and Ms. Henry wanted to show there was value to those strategies:

> I chose Hannah and Lana because they cut every pancake into 8ths and then to get their answer, which is funny because both of them have wrong answers, but they, their [drawing] is correct and so that's why I chose them.

In this quote, Ms. Henry showcased teachers' decisions to share strategies that have incorrect answers with drawings that are accurate for solving the story problems. Because many teachers emphasize helping students reach correct answers, one might assume teachers would typically select strategies that have correct answers with valid strategies or incorrect answers with invalid strategies. The sub-category of correctness of the final answer underscores that there is more complexity to this decision making.

Form of the Final Answer

Teachers selected strategies based on the *form of the final answer* to showcase the different ways the answer can be written and still refer the same quantity. To illustrate this category, we will consider Ms. Wilbern's decision making in her 2-day scenario in which she posed the problem about 28 sticks of wood being used to feed 16 rhinoceroses.

In selecting her six strategies to share during the whole-class discussion, Ms. Wilbern selected strategies to show two different forms of the final answer [1 12/16 and 1 3/4], saying, "I wanted to show them that you can…come up with different answers, but it's still the same." Ms. Wilbern highlighted different forms of the answer, which both referred to the same quantity. Ideas of fraction equivalence are embedded in these different forms but Ms. Wilbern did not pursue how to prove this equivalence, instead choosing only to highlight the idea that a quantity can be labeled in more than one way.

Visibility of the Final Answer in the Strategy

Teachers selected strategies based on the *visibility of the final answer in the strategy* to promote understanding of how an answer can be reached using a drawing-based strategy. When the symbolic notation used to identify the final answer is consistent with the drawing, students are supported in seeing how the final answer was reached. In contrast, at the end of a strategy, students sometimes transform the final answer into an equivalent fraction (e.g., a fraction in lowest terms) and this transformed answer may make it harder to see the connection between the drawing and the final answer.

Ms. Wilbern considered this idea when selecting Tamara's and Erin's strategies in her 2-day scenario in which she posed the problem about 28 sticks of wood being used to feed 16 rhinoceroses. Both students had an answer of 1 3/4 but this answer was easier to see in Tamara's strategy of partitioning into fourths than in Erin's strategy of partitioning into halves and fourths. Ms. Wilbern selected both types of strategies—where the answer was and was not visible in the strategy—but she considered this strategy detail in her selection decision making so that connections could be highlighted in the whole-class discussion.

Teachers' Decision Making Linked to the Representation

Another specific strategy detail category teachers considered when selecting strategies for whole-class discussions was the representation. In particular, teachers considered the overall form of the representation, the use of word labels in the strategy, the extent of the representation, and the specific shapes used in the strategy.

Overall Form of the Representation

Teachers selected strategies based on the *overall form of the representation* to highlight the variety of ways story situations could be represented. In particular, teachers distinguished three types of strategies: strategies based solely on drawing, strategies that relied exclusively on symbolic notation, and strategies that involved a mixture of drawing and symbolic notation. Drawing strategies refer to all of the fractional quantities being represented with a picture and no use of symbolic notation except for the answer. Symbolic-notation strategies refer to strategies in which the amounts are only represented numerically, and no drawings are used. Strategies that involved a mixture of drawing and symbolic notation refer to strategies that have a drawing representing part of the strategy, but other parts of the strategy or the fractional amounts are symbolically notated.

For example, in the PD conversation about starting with 7/8 of a stick of butter when 3 1/8 sticks of butter are needed, the teachers selected Ethan's and Marcus's strategies as a pair to compare a drawing (Ethan's strategy) to symbolic notation (Marcus's strategy). Specifically, the teachers explained that they selected Ethan's strategy because his drawing represented all the 1/8s from the problem and did not include any symbolic notation: “We wanted to have a direct model representation… You could see the 1/8 represented visually in Ethan’s model.” In contrast, they selected Marcus’s strategy because he symbolically notated each of the 1/8s from the problem individually: “We could see the 1/8s here and how he grouped using [symbolic] notation.”

Use of Word Labels in the Representation

Teachers selected strategies based on the *use of word labels in the representation* to explicitly connect the story problem to the strategy. Teachers considered whether or not any labels existed as well as what types of labels. Labels typically identified problem quantities (e.g., children, pizza, etc.) or pieces of the story situation (e.g., amount needed, amount already have, etc.).

For example, in the PD conversation about starting with 7/8 of a stick of butter when 3 1/8 sticks of butter are needed, the teachers considered Marcus’s use of word labels when selecting his strategy. They highlighted how he labeled the amount of butter that he already had by underlining the initial 7 one-eighths and writing the words “already have” underneath. Further, he labeled the rest of the strategy as the amount he would need to add to get 3 1/8 sticks of butter by writing the words “to buy” as well as labeling his answer of 2 2/8 with “more butter.”

Extent of the Representation

Teachers considered the extent to which students represented various problem quantities when they were selecting strategies for whole-class discussions. Each story problem involved multiple pieces (e.g., items to share, items to group, sharers, etc.), and teachers considered how these pieces were or were not represented in strategies. This criterion was important because when students initially start solving problems, they generally represent all of the problem quantities or at least all of the quantities that need to be manipulated (Empson & Levi, 2011).

To illustrate this example, I draw on Ms. Wilbern's 2-day scenario in which she posed the problem about 28 sticks of wood being used to feed 16 rhinoceroses. When selecting strategies, she considered whether all or only part of the sticks of wood and rhinoceroses were represented in the strategy. For example, she considered how Daisy's representation did not include a drawing of the sticks of wood that she partitioned. Daisy drew the 16 rhinoceroses (as circles) and then mentally partitioned and distributed the sticks of wood, recording the amounts each rhinoceros received in the circles. Specifically, she gave each rhinoceros one stick of wood, 1/2 stick of wood, and 1/4 stick of wood. Ms. Wilbern highlighted this mental partitioning when articulating her reasoning for selecting Daisy's strategy: "Daisy didn't need to draw out every single piece [of wood] and give it to every single rhino. So, she just has her rhinos." This focus on the abbreviated nature of Daisy's representation of the sticks of wood was also seen in the whole-class discussion when Ms. Wilbern asked students to identify where Daisy's sticks of wood were located. This excerpt provides an example for how selecting strategies that vary in the extent of the representation can support teachers in providing opportunities for students to become familiar with different types of representations.

Specific Shapes Used in the Representation

Teachers selected strategies based on the specific shapes used in the representation to highlight the importance, or lack of importance, of the shape being used. Specifically, teachers considered whether rectangles, circles, or other shapes were used to represent the quantities in the story situation. Students typically start by matching shapes to the shapes of the objects described in the story problem (e.g. circles for pizza and pancakes, or rectangles for sticks of wood) and eventually move to using shapes that are easiest for them to draw and partition regardless of what those shapes are representing. Research has shown that the specific shape used does not reflect different levels of fraction understandings (Empson & Levi, 2011) but students may initially hesitate to move away from the real-world representations.

To illustrate this idea, I draw on Ms. Henry's decision making around the problem about 8 friends sharing 5 pancakes from her 2-day scenario. She selected Hannah's and Lana's strategies as a pair because both strategies included a drawing of the five pancakes partitioned into eighths, but Lana's strategy included a drawing of rectangles and Hannah's strategy included a drawing of circles (see Figure 10). Ms. Henry explained her decision making:

> I'm tempted really to use Lana's…but I don't know if it's going to throw somebody that she's got rectangles compared to circles… I could put Hannah (who used circles) and Lana's up on the same poster and let the kids talk through that… Is that the same? Does it matter if the pancakes are rectangles?

This quote reflects how Ms. Henry decided to share these strategies as a pair that used different shapes—circles and rectangles—to highlight during the whole-class discussion that a variety of shapes can be used to represent quantities.

Teachers' Decision Making Linked to the Use of Quantities

The third specific strategy detail category teachers considered when selecting strategies for whole-class discussions was the use of the quantities to highlight the ways quantities were manipulated in the strategy. In particular, teachers considered equivalent quantities, benchmark quantities, and operations used on quantities. Additionally, teachers considered the fractional quantities students created, most often for equal sharing story problems.

Equivalent Quantities

Teachers selected strategies based on the inclusion of equivalent quantities to highlight important comparisons of fractions of different sizes or ways to combine fractional quantities of different sizes. For example, in her 2-day scenario, Ms. Wilbern selected Erin's and Penelope's strategies because of the different methods of combining fractional quantities in their strategies for the problem about 16 rhinos sharing 28 sticks of wood. Both used drawings to represent and partition the sticks of wood, and both wrote equations to determine the final amount of wood for each rhinoceros (1 + 1/2 + 1/4). However, Erin added her quantities using a mixed number (1 2/4 + 1/4), whereas Penelope added everything as fractional units (2/4 + 4/4 + 1/4). Ms. Wilbern explained:

> I like how Erin put the number sentence and then showed how she changed it (using equivalent fractions) to change her thinking for her

> pieces. And I also like how Penelope showed her number sentence and how she actually thought about her pieces.

Ms. Wilbern selected these strategies as a pair to highlight different ways fractional quantities can be combined using equivalent quantities. Specifically, both students changed 1/2 to 2/4 to show that they were equivalent quantities and Penelope also changed 1 whole to 4/4 to show that they were equivalent quantities. These adjustments using equivalent quantities facilitated the students' computation to combine the fractional amounts.

Benchmark Quantities

Teachers selected strategies based on the inclusion of benchmarks quantities (also called landmark numbers) to emphasize the power of using familiar amounts (e.g., one whole) when solving problems. Benchmarks played a role in strategies using a drawing, as well as those that relied exclusively on symbolic notation. In the PD conversation about the Butter problem, teachers considered benchmark quantities in their decision making when selecting strategies.

Landon and Ethan used the benchmark quantity of one whole to help them in solving the story problem about starting with 7/8 of a stick of butter when 3 1/8 sticks of butter are needed. Because the story problem started with 7/8 of a stick of butter, adding 1/8 to get to one whole stick of butter was a typical strategy for students to use. The teachers explained this idea in relation to selecting Ethan's strategy by saying, "He's got to be using a landmark number…He has the 7/8 and he needed to add 1/8—he got himself to a whole." For Landon's strategy, they shared similar reasoning: "He got it up to the whole and then just added the 2 1/8." Both Ethan's and Landon's strategies were selected because of their use of the benchmark of one whole in solving the story problem.

Operations Used on Quantities

Teachers selected strategies based on the operations used on quantities so that they had the opportunity to connect students' reasoning to the formal operations of addition, subtraction, multiplication, and division. Teachers also connected operations across strategies to help students make sense of the relationship between operations or to link use of formal operations to drawings.

In Ms. Dustman's decision making in the 1-day scenario, she selected Anna's and Jessica's strategies to highlight the operations of addition and multiplication used in their strategies that both involved use of equations. Anna solved the problem additively, while Jessica solve the problem multiplicatively. Ms. Dustman said:

> Well I saw most of them went straight to the multiplication. There are a couple that used either a direct model with a [drawing] or addition… I want them to see, yeah, I can notate it this way… and try to get them thinking more multiplicatively because when the numbers get bigger they really struggle [using addition]. So I picked the addition one and the multiplication one so that they could see they were the same thing and how they relate to each other so that they can apply that to larger numbers cause when they get to larger numbers, some of them are still trying to repeatedly add and we're trying to get them to think in bigger number chunks.

In this example, Ms. Dustman selected a strategy that used repeated addition and a strategy that used multiplication to allow the class to make sense of the relationship between multiplication and addition in the discussion and encourage the class to move toward multiplication strategies.

Fractional Quantities Created

Teachers selected strategies based on the fractional quantities created in equal sharing problems to give students familiarity with a variety of fractions (e.g., thirds, fourths, eighths, etc.). Teachers often posed story problems that would likely result in the use of different fractional quantities, and then, depending on the goal for the lesson, teachers would select strategies because certain fractional quantities were created. This consideration was most prevalent in equal sharing problems. To illustrate this category, we will look at Ms. Wilbern's selection of strategies for the 2-day scenario problem in which 28 sticks of wood were being used to feed 16 rhinoceroses.

Ms. Wilbern selected her set of strategies to showcase different ways of partitioning—three used halves and fourths, one used sixteenths, and one used only fourths. Specifically, she selected Mark's strategy because he partitioned the sticks of wood into 16ths. She explained, "He shared everything as 16ths." She selected Erin's and Penelope's strategies because of their drawings of partitioning the sticks of wood into halves and fourths, and she selected Tamara's strategy because of her drawing of partitioning the sticks of wood into fourths. The use of sixteenths, halves and fourths, and fourths were fractional quantities Ms. Wilbern wanted to highlight in the whole-class discussion and selecting these strategies allowed her to do so.

Teachers also selected strategies based on the inclusion of an explicit link between the number of sharers and number of partitions to highlight the power in considering this relationship in equal sharing problems. Sometimes the number of partitions corresponded to the number of sharers (e.g., using sixths with 6 sharers) and other times factors of the number of sharers were involved (e.g., using thirds with 6 sharers).

Ms. Wilbern selected Tamara's strategy because of her drawing of partitioning the sticks of wood into fourths. In particular, she was curious if Tamara was intentional about linking the number of partitions to the number of sharers:

> She shared all [the sticks of wood] as fourths which leads me to the impression that she thought about how much she had left over, that she knew there was a relationship between the 12 and the 4—hopefully—and the 16.

Ms. Wilbern selected this strategy to highlight in the whole-class discussion this specific strategy detail of the fractional quantities she created based on numerical relationships.

Conclusion

The specific strategy detail categories (and sub-categories) described in this chapter illuminate specific strategy details teachers might consider when selecting fraction strategies for whole-class discussions. Similar to the criteria discussed in the last chapter, these detail categories are not meant to be a checklist to be executed, but instead they are meant to give a sense of the range of what specific strategy details can inform the selection of strategies for whole-class discussions. In the next chapter, I will discuss the implications of this framework described in the previous two chapters.

CHAPTER VI

CONCLUSION

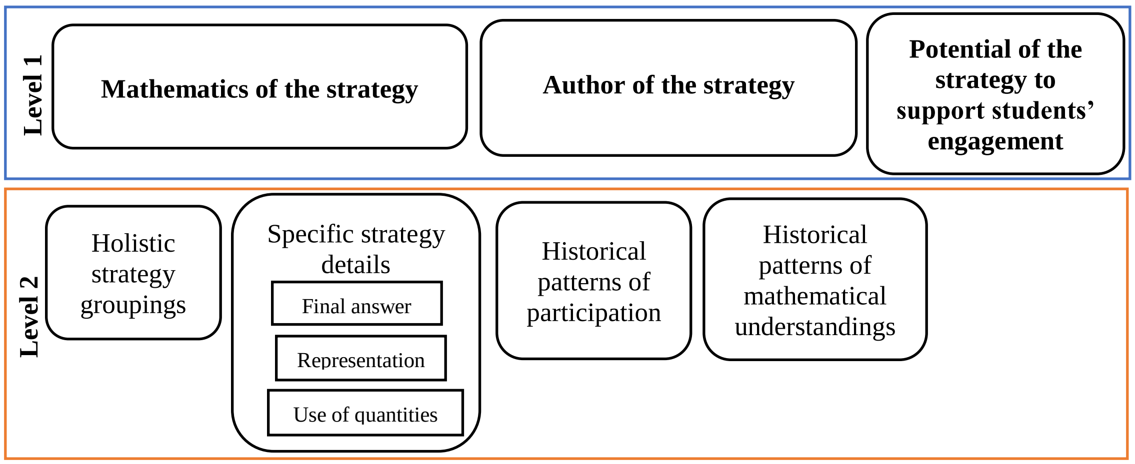
My purpose in creating a framework for selecting strategies for whole-class discussions was to contribute to the field's understanding of this under-appreciated and under-researched teaching practice. My findings reinforce the idea that selecting strategies is a complex practice worthy of more and focused attention, and my framework identifies noteworthy selection criteria and gives them structure.

The structure of the framework was purposefully designed in three levels that progressively include more specificity in the criteria for selecting strategies for whole-class discussions. (See Figure 16 for a version of the framework that makes the levels visible.) *Level 1* includes the three main selection criteria of the mathematics of the strategy, the author of the strategy, and the potential of the strategy to support students' engagement. *Level 2* adds more specificity by identifying sub-categories for two of the main criteria. Both Levels 1 and 2 cut across all content areas whereas *Level 3* provides additional specificity for a particular content area, which in this case is fractions. The intent of the framework is that the Level 3 information for fractions could be replaced with specific strategy details for other content areas as the framework gets extended to other mathematical content. The framework criteria and its structure of progressive specificity provide both theoretical and practical contributions, which are described below.

Figure 16

Framework for Selecting Strategies for Whole-Class Discussions Displayed by Levels

Criteria for Selecting Strategies for Whole-Class Discussions



Level 3

Specific Strategy Details for Fractions	
Final Answer	• Correctness of the final answer • Form of the final answer • Visibility of the final answer in the strategy
Representation	• Overall form of the representation • Use of word labels in the representation • Extent of the representation • Specific shapes used in the representation
Use of Quantities	• Equivalent quantities • Benchmark quantities • Operations used on quantities • Fractional quantities created

Theoretical Implications

Empirical work on selecting strategies for whole-class discussions is limited, and my study provides empirical evidence of the complexity of this practice in three ways.

First, through the framework structure, I elevated the distinction between criteria that cut across content areas and those linked to specific content areas. Second, I collected the disjointed list of criteria presented in the literature, elaborated on the existing criteria, and added criteria to the list. Third, I identified a set of selecting criteria specific to fraction understanding. Each of these contributions is further described below.

Distinction Between Content-General and Content-Specific Criteria

The list of selecting criteria offered in the literature focuses on criteria that are content general in that they should be applicable across content areas. However, responsive teaching is linked to the mathematics of students' ideas and the content-specific strategy details provide an even richer foundation on which teachers can build when being responsive to students' mathematical thinking. My framework for selecting strategies for whole-class discussions includes levels that offer distinctions between content-general and content-specific criteria—Levels 1 and 2 include content-general criteria and Level 3 includes content-specific criteria. Thus, the inclusion of Level 3 draws attention to the importance of attending to strategy details linked to specific content areas when selecting strategies for whole-class discussions.

Criteria Expansion

My framework expanded the list of selection criteria offered in the literature not only by including a Level 3 but also by expanding the criteria included in the content-general criteria of Levels 1 and 2. Specifically, previously identified criteria are reflected in the framework as they typically focused on the mathematics of the strategy (e.g., correct/incorrect answers or mathematical misconceptions) or the author of the strategy (e.g., boosting the confidence of the author or ensuring that all authors have regular opportunities to share). However, I added a third main criteria at Level 1, the potential of the strategy to support students' engagement, and I elaborated on the main categories of mathematics of the strategy and author of the strategy by giving them a 2-level structure.

Criteria for Fractions

Level 3 of the framework articulates selecting criteria focused on specific strategy details for the content area of fractions. Teachers sometimes used these details as the basis for their strategy selection so that they could foreground these details to focus on important fraction ideas in their discussions. The detail categories and sub-categories highlighted in the framework also provide a glimpse into some of the key issues identified in the research on the teaching and learning of fractions, and thus these Level 3 criteria could be used as a support for teacher learning about students' fraction thinking.

Practical Implications

This framework has practical implications that can improve teachers' whole-class discussions and assist teacher educators and professional developers in helping prospective and practicing teachers grow in this area. The overall presence of the framework reinforces the idea that selecting strategies for whole-class discussions should be purposeful. The variety of criteria specified in the framework provides *specific* guidance about what to consider when selecting strategies. Further, as previously mentioned, Level 3 draws attention to mathematical ideas that research has shown to matter in terms of students' development of understanding of fractions. Finally, the leveled structure of the framework provides a way for teachers not only to learn how to be more purposeful when selecting but also to continually grow in their expertise.

The leveling of the framework provides multiple options for use—the framework can be used by considering selection criteria at Level 1 only, at both Levels 1 and 2, or at Levels 1, 2, and 3. For instance, teachers just beginning to learn to be responsive to students' thinking might need to start with Level 1 only. By focusing only on three main categories, they can begin to make sense of the practice of selecting strategies for whole-class discussions and see the power in purposeful selection without becoming overwhelmed. As teachers grow in their comfort with purposeful selection, they might begin connecting Levels 1 and 2 because the combination of these two levels provides more guidance.

Level 3 of the framework provides a way for teachers to *refine* their selecting expertise with attention to specific mathematical content. It would be too overwhelming for teachers to start with this level of the framework so once teachers have developed some expertise in Levels 1 and 2, Level 3 offers a way for teachers to refine their purposefulness by attending to the specific mathematics content. In other words, this level offers a way for teachers to keep learning about ways to be purposeful when selecting strategies for whole-class discussions.

Limitations and Future Research

I view the framework as "initial" because I expect it to be revised as I and others learn more about the ways teachers select strategies for whole-class discussions. In particular, this study has limitations in terms of number of participants and the content focus. First, the number of teacher cases was limited to three teachers, and only the teacher cases were able to consider all of the framework criteria; for instance, the teachers in the PD conversations could not consider criteria related to the author because they did not know the strategy authors in the set of facilitator-selected strategies. Additional teachers' decision making needs to be explored to confirm the selecting criteria as described in the framework. Second, this study focused on selecting strategies for whole-class discussions focused on fractions. Exploring selecting strategies for whole-class discussions focused on different content areas would confirm the leveled nature of the framework. For instance, do Levels 1 and 2 cut across content areas and therefore apply to other content similarly? These additional explorations would also begin to develop the list of specific strategy details needed to inform teachers' strategy selection at Level 3 in other content areas.

Three other issues arose in course of the study but could not be answered with existing data: coordinating of the selecting criteria, the role of planning the lesson goal in selecting strategies, and sequencing strategies for whole-class discussions. These issues point to promising directions for future research and each is explored further below.

Coordinating of Criteria

The goal of this study was to identify the range of criteria teachers had for selecting strategies for whole-class discussion, and the framework explicitly articulates the criteria teachers used for selecting. However, the framework is not meant to be a checklist such that teachers should select strategies according to every criterion for every discussion. Then how do teachers coordinate these criteria into a final decision? For example, a teacher might want to focus on a particular set of specific strategy details, but students who have not shared recently did not use those strategy details. Which criteria should the teacher foreground—the specific strategy details or the historical participation patterns of the authors? Similarly, a teacher might want to elevate the strategy of a student who had made an advance in their mathematical understandings, but the strategy was messy, which might interfere with the rest of the students' engagement with the mathematics of the strategy. Which criteria should the teacher foreground—the historical pattern of the author's mathematical understandings or the potential of the strategy to support students' engagement? Future research needs to investigate how teachers coordinate these criteria into a final decision.

In this study, I informally noticed some patterns in the decision making of the teacher cases and these observations might provide a starting point for exploration. Typically, two of the teacher cases foregrounded the mathematics of the strategy by first identifying specific strategy details they wanted to highlight in the whole-class discussion. If they had more than one student whose strategies had the desired details, teachers would then consider the authors' historical patterns of participation or mathematical understandings. These considerations often led to the selection of strategies belonging to students who either had not shared recently or who seemed to have made an advance in their mathematical understandings. Note that these teachers would also sometimes interweave the criteria of the potential of the strategies to support students' engagement into their considerations of the mathematics of the strategies and the author of the strategies, but this criterion had less prominence. In contrast, the third teacher case only attended to the mathematics of the strategy when selecting strategies for the two whole-class discussions I observed. These examples provide a starting place for future research by suggesting that some teachers may engage in a hierarchical sequence when considering criteria for strategy selection, but there are likely various profiles for teachers' coordination of criteria.

Planning Lesson Goals

Some research has suggested that the selection of strategies is driven by the lesson goal, Smith and Stein's (2018) practice 0. In my study, I found some evidence of this connection, but I also found that teachers' selection of strategies can influence the lesson goal. In other words, when purposefully selecting strategies for whole-class discussion, teachers recognized that the students' strategies necessitated a different or modified lesson goal. This flexibility in lesson goals based on students' strategies is conceptually consistent with the idea of responsive teaching—even the lesson goals are responsive to students' mathematical thinking. Future research needs to investigate the influence of the lesson goal on selecting strategies *and* the influence of selecting strategies on the lesson goal.

The connection between the lesson goal and selecting strategies was not studied systematically in this study because the teacher cases only occasionally explicitly linked selecting strategies to their lesson goals and the issue arose even less frequently in the PD conversations. However, I share one example in which the practice of selecting strategies for whole-class discussions led to the teacher modifying her lesson goal. In Ms. Wilbern's 2-day scenario, she began the lesson with the goal of helping students learn about (a) changing an improper fraction to a mixed number and (b) combining fractions with unlike denominators. However, after seeing the strategies generated by the class, she explained why she abandoned the first goal: "it was a goal that was not met—to see if they would do an improper fraction and change it to a mixed number—but nobody used that strategy." Because Ms. Wilbern abandoned one of her goals, she focused her strategy selection on the second goal of combining fractions with unlike denominators. Here, and in my broader data set, I found that, based on the strategies generated by the class, teachers sometimes abandoned or modified lesson goals rather than force a goal that did not build on the students' strategies or that students would likely not be ready to address. This idea is different than Smith and Stein (2018) and others who have suggested that the mathematical goal is the overall guide for the selection of strategies. Thus, I argue for the need for research to elaborate on the relationship of the goal and strategy selection because the relationship may be complex and multi-directional.

Sequencing Strategies

At the start of this study, sequencing strategies for whole-class discussions was to be explored as well. However, upon investigation, I did not have sufficient data or sufficient variety in my data to include sequencing strategies in the framework. Instead, I argue that future research needs to investigate this practice of sequencing strategies, highlighted in Smith and Stein's (2018) seminal work on the five practices for orchestrating productive mathematics discussions.

I share two issues that arose in my study that could inform future research on this topic. First, what is meant by sequencing needs to be articulated because teachers did not display strategies in just one way (Hewitt, 2018). For example, teachers sometimes had all of the strategies displayed at the start of the discussion, and therefore sequencing referred to the order in which the strategies were discussed. Other times, the strategies were presented and discussed one at a time, and thus sequencing referred to the order the strategies were presented and discussed. Second, when teachers shared their decision making about sequencing, the dominant idea was that the first strategy to be shared should be a strategy that all students would be able to access in the whole-class discussion. These initial strategies that provided access varied in their characteristics, sometimes clearly linking to the story situation, sometimes representing a strategy used by multiple students, and sometimes showing minimal complexity.

For example, in Ms. Dustman's 1-day scenario, she selected two strategies to share in the whole-class discussion—one that used addition and one that used multiplication in the solution. Regarding the sequence she said:

> I thought if we start with…the adding [strategies] then all those kids that were either direct modeling or had some addition errors or something—they had something that they could relate to and they could grasp, so that they could make sense of the story and what was going on, and then jump to the multiplication. So that if [the multiplication strategy] was too much for them, they still got something today. Where sometimes if we started it the other way, they just don't get anything that day. So I'm trying to reach everybody so if we start with the least complex strategy and then grow, then it reaches everybody.

In this example, Ms. Dustman described starting with an addition strategy initially because if she shared a more sophisticated multiplication strategy then some of the class might not have as much opportunity to learn in the discussion. The overarching goal of starting with a strategy that the class would have access to and moving to more sophisticated strategies was a theme across all the teacher cases. However, if teachers always sequence strategies from least to most sophisticated, students will become aware of this sequencing and learn that strategies shared earlier are not as sophisticated as strategies shared later in the discussion. Therefore, I argue for more research on sequencing strategies to capture all of the ways teachers sequence strategies for whole-class discussions and perhaps to envision some new possibilities.

Conclusion

This study focused on the selecting practice of teachers who had demonstrated expertise or were developing expertise in teaching that is responsive to students' mathematical thinking. Thus, this study adds to the field's understanding of responsive teaching by exploring a core teaching practice related to mathematical talk with teachers who value being responsive to students' mathematical thinking and have some expertise in doing so. When considering mathematical talk that takes place in responsive teaching, much attention has been focused on the whole-class discussions that often occur at the conclusion of lessons. However, there are many practices that are needed to set the stage for productive whole-class discussions. Smith and Stein (2018) identified five practices for orchestrating productive mathematics discussions, in which selecting strategies is included. Purposeful selection of strategies is critical for productive discussions because it provides the starting point for the mathematical content of the discussion which lays the foundation for what students can learn.

In conclusion, selecting strategies is a more complex and powerful practice than is often assumed. Without purposeful selection of strategies, certain mathematical ideas are unable to be highlighted in the whole-class discussion, certain authors of strategies can be overlooked, or the strategies shared may not be useful for students' engagement. The framework for selecting strategies for whole-class discussions provides benefits to both researchers and practitioners. For researchers, the framework provides specificity and structure to criteria already identified in the literature, while also incorporating new criteria. For practitioners, the framework can help teachers become more purposeful when selecting strategies, thereby providing students with more opportunities to learn in whole-class discussions. The multiple criteria identified in the framework can inform teachers' purposeful decisions, and the leveled structure of the framework makes the purposeful selection of strategies accessible to teachers at any phase of their development in being responsive to students' mathematical thinking.

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APPENDIX A

QUESTIONS FOR THE GENERAL INTERVIEW

1. We talked about how teachers select student work for whole-class discussions. We discussed two different ways: during the lesson or after the lesson. Can you talk about when you select student work for whole-class discussion in your practice?
    - Do you do both or just one?
    - Is one way more frequent than the other?
2. Describe the process you typically use to select student work for whole-class discussions.
    - How many pieces of student work do you typically share?
    - Tell me about some of the guiding principles you use for selecting student work. *(What are you thinking about when you look at student work?)*
    - What makes selecting student work easy or difficult?
    - How do you decide how to sequence student work for whole-class discussions? *(Do you sequence all of the pieces that will be shared before the discussion starts or do you sequence during the discussion?)*
    - Do you use different number sets? If so, how do you determine which number set to share?
3. How do you know if the student work you selected worked the way you wanted?
    - What do you look for in the discussion to determine if it is working?
4. I'm going to ask a series of logistical questions about what your whole-class discussions typically look like. I know your discussions do not look the same every day, but in general:
    - How often do you have whole-class discussions?
    - When do they occur in the lesson?
    - How long do your discussions last?
    - Where do the discussions occur in the classroom?
5. Why do you have whole-class discussions? (*What is the role of whole-class discussions in your instruction?)*
    - Do your goals for a specific discussion emerge while the discussion is happening or are they planned prior to the discussion? *(Do your goals shift/change/narrow/broaden during the discussion?)*
    - Do you plan goals for the whole class, groups of students, or individual students?
6. What else do I need to know about your whole-class discussions or anything else you want to tell me?
7. Think about a whole-class discussion that you felt was successful. Tell me about that discussion.

- Why did you feel it was successful?
- Tell me about the student work you shared during this discussion.

APPENDIX B

QUESTIONS FOR THE 1-DAY SCENARIO REFLECTION INTERVIEW

1. During the lesson, did you make any changes on the fly?
2. Were there any surprises during the lesson?
3. Tell me about your math instruction leading up to today's story problem.
4. Why did you select this problem? Problem type? Context? Number set?
5. Have the students used a lot, little, or none of these problem types?
6. Summarize the student work selected.
7. Summarize **why** you selected the pieces of student work.
8. How did you think about competing criteria for sharing student work? Were any criteria for selecting student work outweighed by some other criteria?
9. Walk me through the process of how you selected the student work used for discussion.
10. *(**ONLY** if the problem included multiple number sets):* How did you determine which number sets to share?
11. *(Specific teacher resources):* Did you create ___? How did you use ___? Do you save and re-use ___ year-to-year?
12. Tell me about how you decided on the sequence of the strategies used during this discussion.
13. When did you decide how to sequence the selected pieces of student work for whole-class discussions? *(Did you sequence all of the pieces that would be shared before the discussion started or did you sequence during the discussion?)*
14. Did you have any specific goals for the discussion?
    - At what point did your goals emerge? Did any of your goals change during the lesson?
    - When you were planning your goals, were you planning for the whole class, groups of students, or individual students? Tell me about that.
15. Was today's discussion an example of a typical whole-class discussion for you? If not, what was different about the discussion?
16. Did you feel the discussion worked the way you wanted? What is your evidence?

17. If you could go back, would you select the same strategies you used during the discussion? Why or why not?

APPENDIX C

QUESTIONS FOR THE 2-DAY SCENARIO SELECTION INTERVIEW

1. Summarize the pieces of student work you selected.

2. Summarize **why** you selected the pieces of student work.

3. Walk me through the process of how you selected the student work used for discussion. *(So when we started you had a stack of student work, now you have ___ pieces to share, what was the process you went through to get to those ___ pieces?)*

4. Tell me about your math instruction leading up to today's story problem. Why did you select this particular problem type? Context? Number set?

5. ***(ONLY** if the teacher has mentioned the sequencing)* How did you decide the sequence for using the strategies during the discussion?

6. ***(ONLY** if the teacher has mentioned her goals)* Do you have any specific goals for the discussion?

APPENDIX D

QUESTIONS FOR THE 2-DAY SCENARIO REFLECTION INTERVIEW

1. During the lesson, did you make any changes on the fly?

2. Were there any surprises during the lesson?

3. How did you think about competing criteria for sharing student work? Were any criteria for selecting student work outweighed by some other criteria?

4. *(**ONLY** if the problem included multiple number sets)* How did you determine which number set to share?

5. *(**ONLY** if sequencing changed from what was planned the previous day)*
   - How did you decide the sequence for using the strategies during the discussion?
   - When did you decide how to sequence student work for whole-class discussions?

6. Did you have any specific goals for the discussion?
   - Did any of your goals change during the lesson?
   - When you were planning your goals, were you planning for the whole class, groups of students, or individual students? Tell me about that.

7. Was today's discussion an example of a typical whole-class discussion for you? If not, what was different about the discussion?

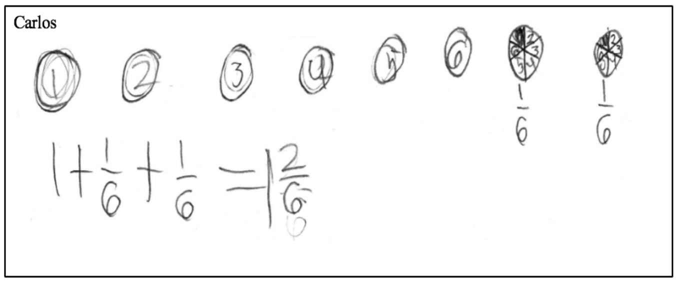
8. Did you feel the discussion worked the way you wanted? What is your evidence?

9. If you could go back, would you select the same strategies you used during the discussion? Why or why not?

APPENDIX E

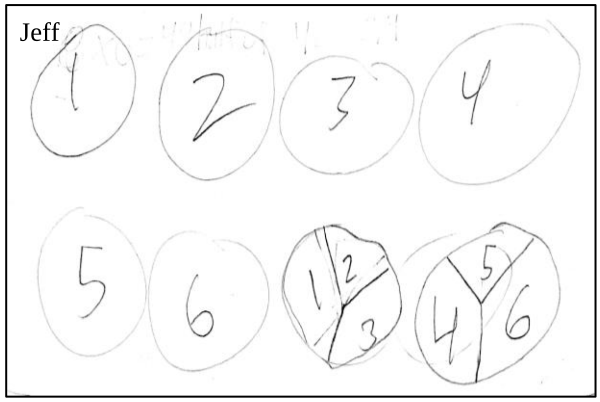
FOUR STRATEGIES SHARED IN MS. HENRY'S 1-DAY SCENARIO DISCUSSION

6 friends want to share 8 pizzas. How much pizza will each friend get so that they all get an equal amount?



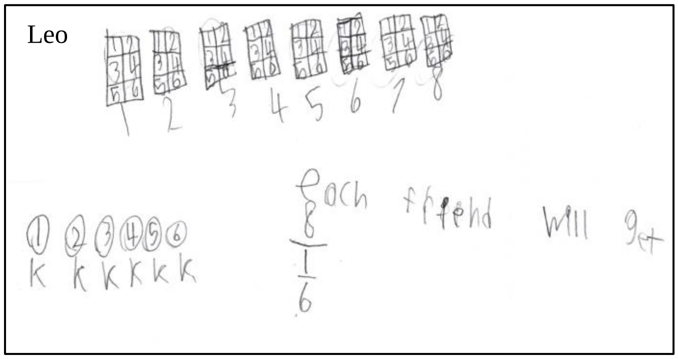
Carlos drew 8 circles to represent the 8 pizzas. He then distributed 1 whole pizza to each of the 6 friends by numbering the pizzas 1–6. Next, he partitioned the 2 remaining pizzas into sixths and distributed 1/6 of each pizza to each friend by numbering the pieces in each partitioned pizza 1–6. Carlos additionally indicated this distribution by writing 1/6 under each of the partitioned pizzas and shaded in the 2 pieces labelled "1." He combined the amounts each friend would receive from the whole and partitioned pizzas by writing the equation: 1 + 1/6 + 1/6 = 1 2/6.

Emergent-anticipatory direct modeling strategy



Jeff drew 8 circles to represent the 8 pizzas. He then distributed 1 whole pizza to each of the 6 friends by numbering the whole pizzas 1–6. Next, he partitioned the two remaining pizzas into thirds and distributed each third to the 6 friends by numbering the pieces 1–6 *across the 2 pizzas*. Jeff did not indicate a final answer.

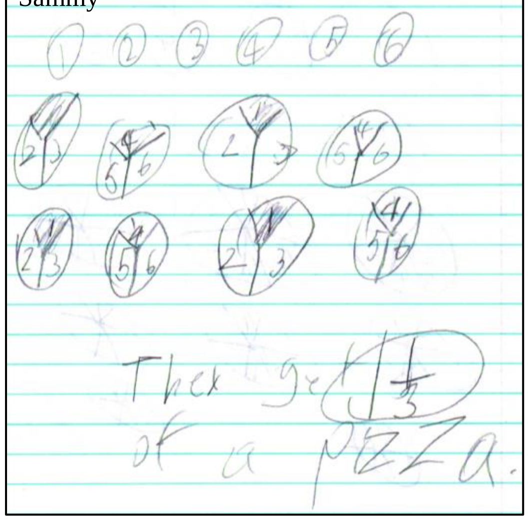
Emergent-anticipatory direct modeling strategy



Leo drew 8 rectangles to represent the 8 pizzas and numbered them 1–8 to indicate the 8 pizzas. He also drew 6 circles to represent the 6 friends, which he indicated by numbering 1–6 and putting a "K" (for kids) beneath each circle. Leo partitioned the 8 pizzas into sixths and then distributed each sixth to the 6 friends by numbering the pieces in each pizza 1–6. Leo indicated each friend would get 8 one-sixths with ambiguous notation "8/1/6."

Emergent-anticipatory direct modeling strategy

Sammy



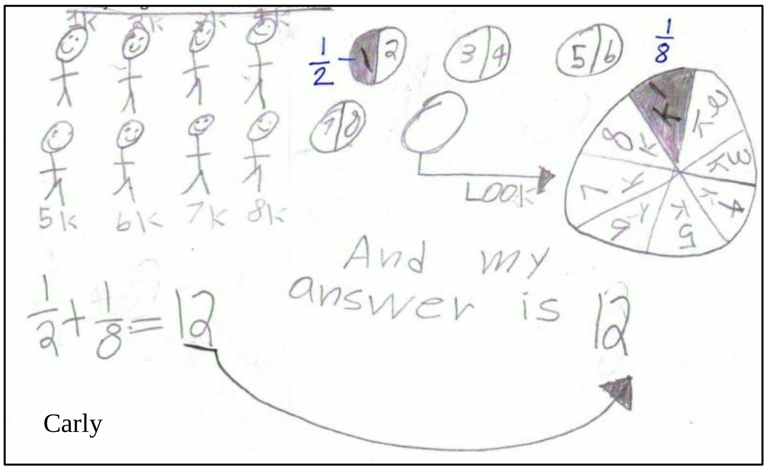
Sammy drew six circles to represent the 6 friends, which he indicated by numbering the circles 1–6, and he then drew 8 circles to represent the 8 pizzas. Sammy partitioned the 8 pizzas into thirds and then distributed each third to the 6 friends by numbering the pieces 1–6 *across 2 pizzas*. Sammy indicated the first friend's share by shading in the 4 pieces (across the 8 pizzas) labeled "1." Finally, Sammy mentally combined the 4 one-thirds and indicated that each friend would get 1 1/3 of pizza.

Emergent-anticipatory direct modeling strategy

APPENDIX F

FOUR STRATEGIES SHARED IN MS. HENRY'S 2-DAY SCENARIO DISCUSSION

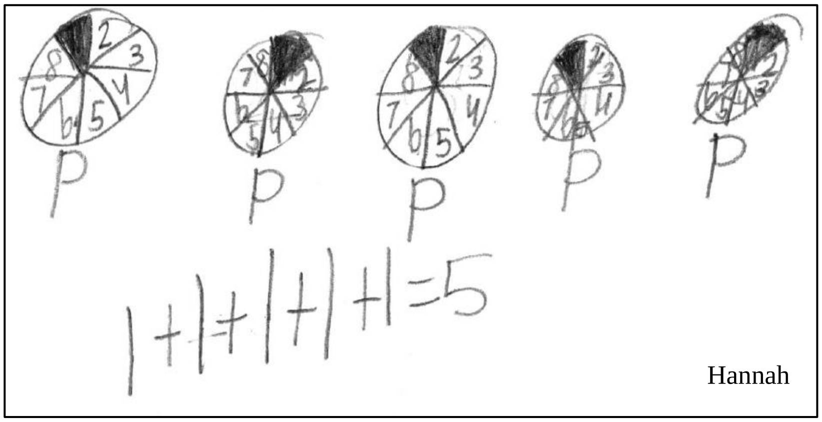
8 friends went to IHOP to celebrate National Pancake Day. They ordered 5 pancakes to share equally. How much pancake will each friend get so that they all get the same amount?



Carly

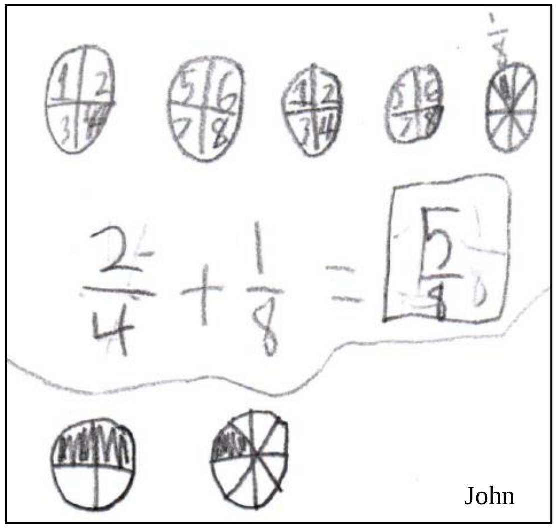
Carly drew 8 stick figures to represent the 8 friends, which she indicated by numbering each stick figure 1K, 2K, 3K, etc., using "K" for kid. Carly then drew 5 circles to represent the 5 pancakes. She partitioned 4 pancakes in half and distributed the halves to the 8 friends by numbering the pieces 1–8 *across the 4 pancakes*. She then redrew the fifth pancake so that it was larger. Carly partitioned this larger pancake into eighths and distributed the eighths to the 8 friends by numbering the pieces 1K–8K. Carly indicated the first friend's amount by shading in two pieces—one half labeled "1" in the first four pancakes and one eighth labeled "1K" in the last pancake. The teacher assisted Carly in identifying the amounts that she had shaded by writing "1/2" and "1/8" next to the partitioned pancakes. Carly then combined the amounts each friend would receive by writing the equation 1/2 + 1/8 = 12 and indicated her answer as 12.

Emergent-anticipatory direct modeling strategy



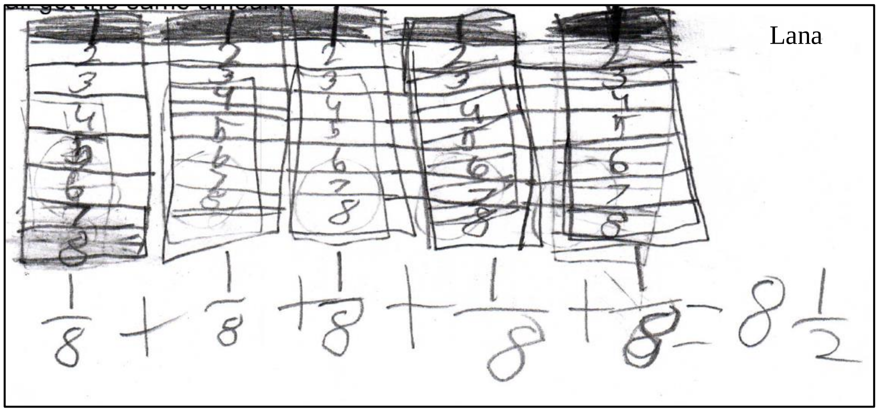
Hannah drew 5 circles to represent the 5 pancakes and wrote a “P” (for pancakes) underneath each circle. She partitioned all the pancakes into eighths and distributed each eighth to the 8 friends by numbering the pieces in each pancake 1–8. Hannah shaded in the piece labeled “1” from each pancake to indicate the first friend’s amount. She then combined the amounts each friend would receive by writing the equation:
1 + 1 + 1 + 1 + 1 = 5.

Emergent-anticipatory direct modeling strategy



John drew 5 circles to represent the 5 pancakes. He partitioned 4 pancakes in fourths and distributed the fourths to the 8 friends by numbering the pieces 1–8 *in every two pancakes*. John partitioned the final pancake into eighths and then indicated distribution of eighths to the 8 friends by shading in one piece and writing 1/8. John then combined the amounts each friend would receive by writing the equation 2/4 + 1/8 = 5/8. He did not show how he combined these amounts, but he did pictorially represent each amount of 2/4 and 1/8 with partitioned circles.

Emergent-anticipatory direct modeling strategy



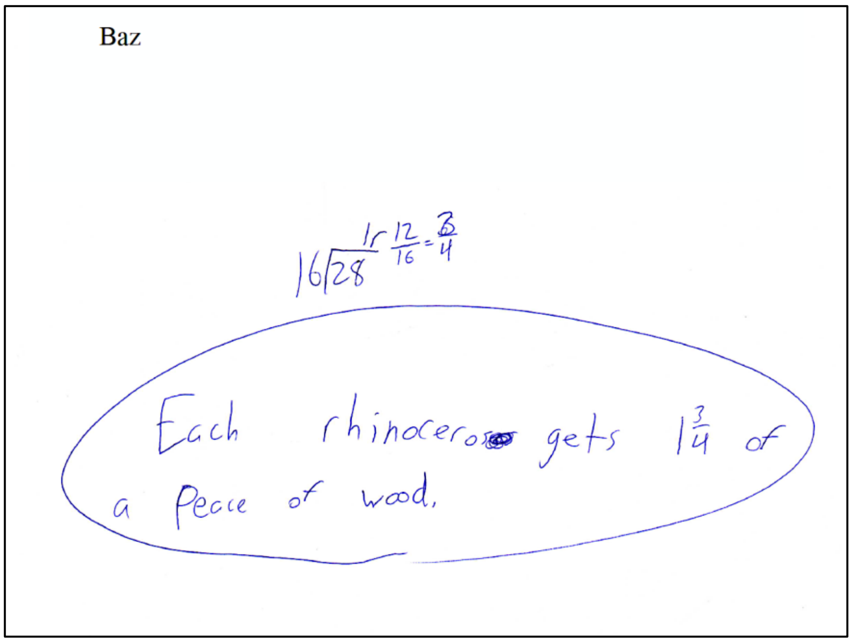
Lana drew 5 rectangles to represent the 5 pancakes. She partitioned all the pancakes into eighths and distributed eighths to the 8 friends by numbering the pieces in each pancake 1–8. Lana shaded in the piece labeled “1” from each pancake to indicate the first friend’s amount. She then combined the amounts each friend would receive by writing the equation: 1/8 + 1/8 + 1/8 + 1/8 + 1/8 = 8 1/2.

Emergent-anticipatory direct modeling strategy

APPENDIX G

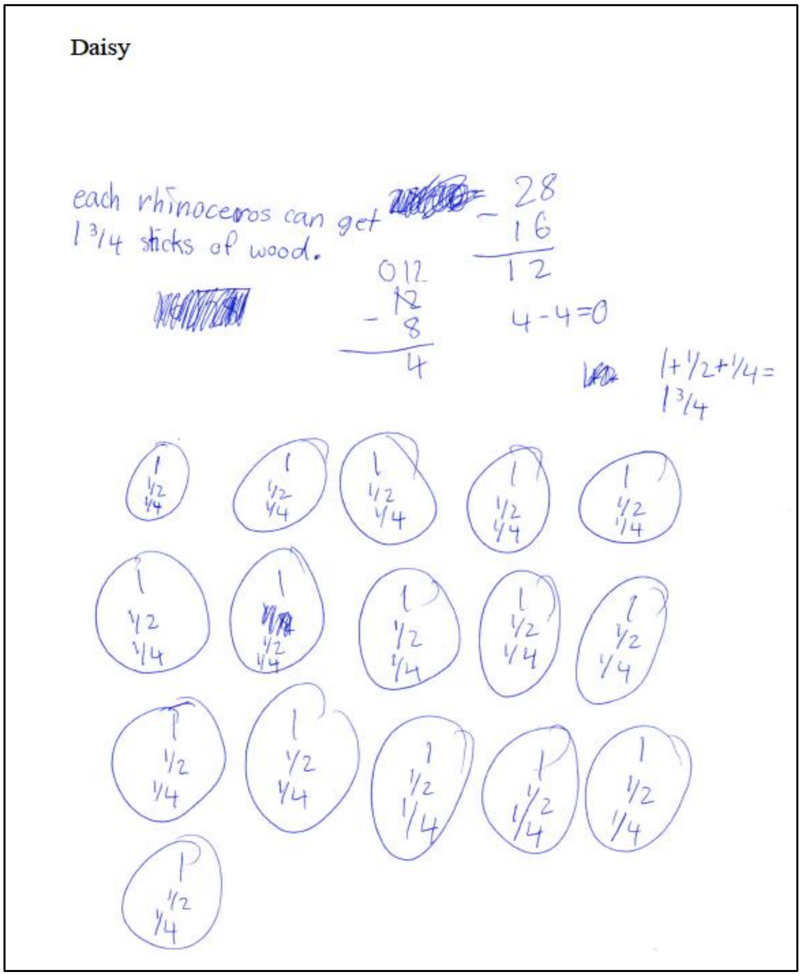
SIX STRATEGIES SHARED IN MS. WILBERN'S 2-DAY SCENARIO DISCUSSION

The zookeeper has 28 sticks of wood to feed 16 rhinoceroses. How much wood can each rhinoceros get, if the wood is shared equally with the rhinoceroses?



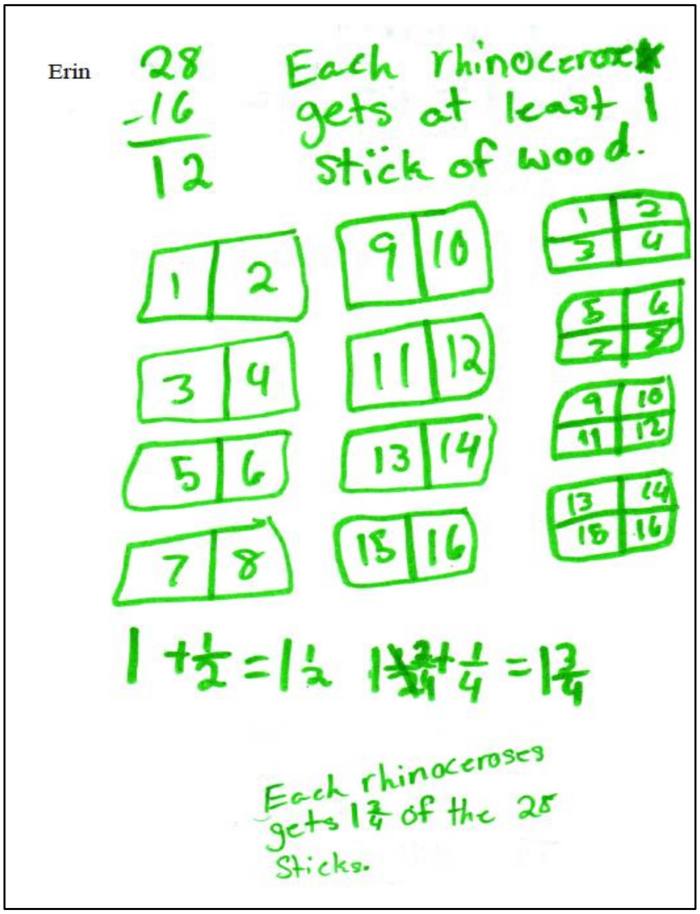
Baz mentally solved the problem and recorded his mental computation in a shortened version of the standard division algorithm. He mentally subtracted 16 sticks of wood from the original 28, leaving 12 sticks of wood. Baz then mentally partitioned the 12 remaining sticks of wood by 16 and got 12/16. Finally, he mentally created the equivalent fraction of 3/4. Baz gave the final answer that each rhinoceros gets 1 3/4 of a piece of wood.

Relational thinking strategy



Daisy drew 16 circles to represent the 16 rhinoceroses. She mentally distributed one whole stick of wood to each of the rhinoceroses, which she indicated by putting a "1" in each circle. She also kept track of the number of sticks she still had to distribute by writing the equation 28 – 16 = 12, which indicated that she had used 16 sticks of wood and had 12 sticks of wood left to distribute. Next, Daisy mentally used 8 sticks of wood to distribute 1/2 stick of wood to each of the rhinoceroses, which she indicated by putting "1/2" in each circle. She again wrote an equation (12 – 8 = 4) to show that out of the 12 remaining sticks of wood, she used 8 so had 4 left. Daisy then mentally used those 4 sticks of wood to distribute 1/4 stick of wood to each of the rhinoceroses, which she indicated by putting "1/4" in each circle. She again wrote an equation (4 – 4 = 0) to show that she had used 4 sticks of wood and therefore had no more sticks of wood remaining to distribute. Daisy combined the amounts that each rhinoceros received by writing the equation 1 + 1/2 + 1/4 = 1 3/4. She did not show how she combined the fractions with unlike denominators.

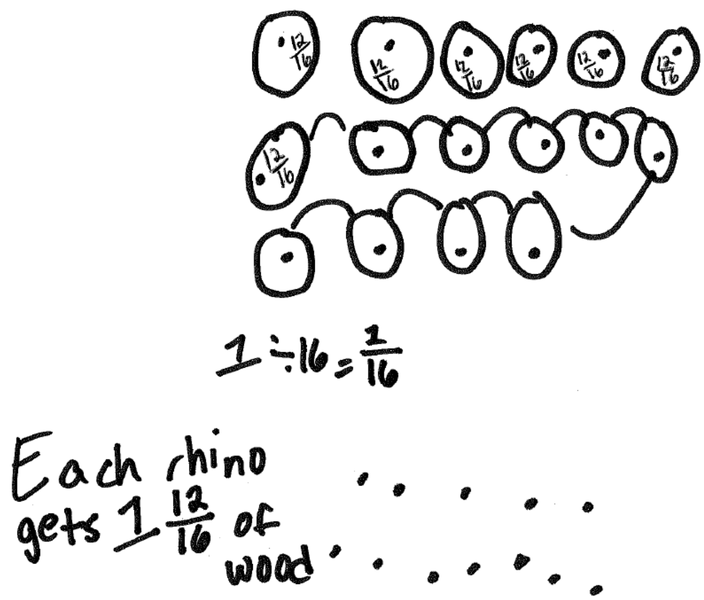
Emergent-anticipatory direct modeling strategy



Erin distributed 1 stick of wood to each of the 16 rhinoceroses, which she recorded by writing the equation 28 – 16 = 12 vertically and writing in words that each rhinoceros would get at least one stick of wood. Next, Erin distributed 1/2 of a stick of wood to each rhinoceros by drawing 12 rectangles to represent the remaining sticks of wood, partitioning 8 of the sticks of wood in half, and distributing the halves to the rhinoceroses by numbering the pieces 1–16 *across the 8 sticks of wood*. She then distributed 1/4 of a stick of wood to each rhinoceros by partitioning the remaining 4 rectangles into fourths and distributing the fourths by numbering them 1–16. Erin determined how much wood each rhinoceros received by combining the three amounts—one whole stick of wood, 1/2 stick of wood, and 1/4 stick of wood. To complete this computation, she first combined the whole and 1/2 stick by writing an equation 1 + 1/2 = 1 1/2. She then added 1 1/2 and 1/4, by finding an equivalent fraction and changing 1 1/2 to 1 2/4, which can be seen in the equation, 1 2/4 + 1/4 = 1 3/4. Erin concluded that each rhinoceros would get 1 3/4 of the 28 sticks of wood, and she wrote this answer in a sentence.

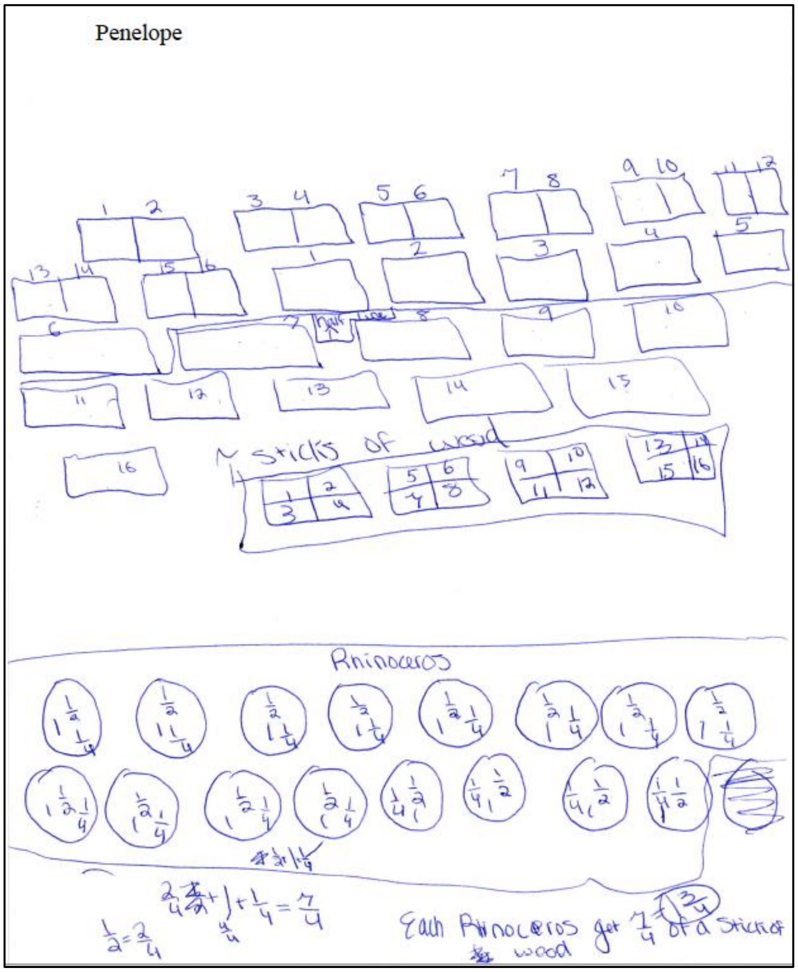
Emergent-anticipatory direct modeling strategy

Mark



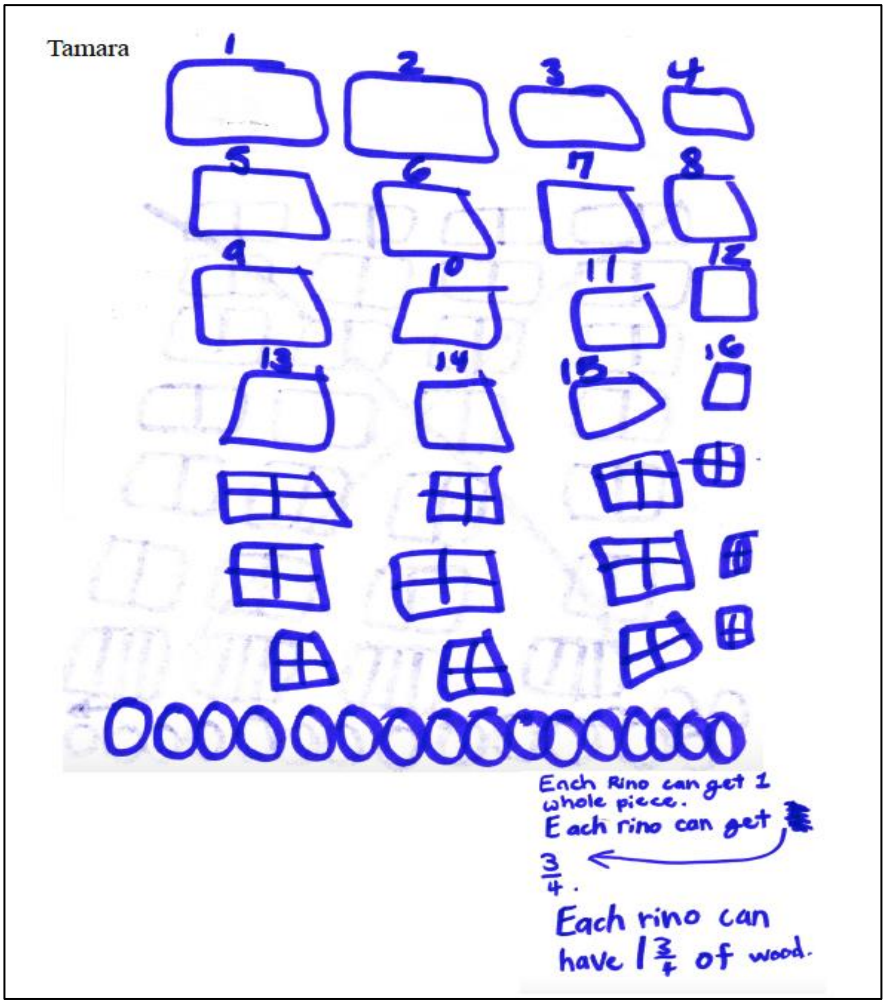
Mark drew 16 circles to represent the 16 rhinoceroses. He then distributed one stick of wood to each rhinoceros, and he represented this distribution by drawing a dot in each circle. After these 16 sticks of wood were distributed, 12 sticks remained, and they were represented as 12 dots at the bottom of the page. Next, Mark determined how to partition a single stick so that the 16 rhinoceroses each could receive a piece of that stick of wood (1/16), which he recorded with the equation 1 ÷ 16 = 1/16. Mark mentally used this partitioning for the 12 remaining sticks and started distributing 12/16—which represents 1/16 from each of 12 sticks of wood—to each rhinoceros. He recorded the 12/16 for the first 7 rhinoceroses and then indicated that the remaining rhinoceroses received the same amount by drawing a curved line connecting the circles. Mark gave his final answer of each rhinoceros receiving 1 12/16 sticks of wood, which could be seen by the dot and 12/16 in each circle.

Relational thinking strategy



Penelope drew 28 rectangles to represent the 28 sticks of wood. She distributed 1/2 stick of wood to each of the rhinoceroses by partitioning 8 sticks of wood in half, and then distributing the halves by numbering the pieces 1–16 *across the 8 sticks of wood*. Penelope then distributed 1 whole stick of wood to each of the rhinoceroses by numbering 16 whole sticks of 1–16. Finally, she distributed 1/4 stick of wood to each of the rhinoceroses by partitioning the 4 remaining sticks of wood into fourths and distributing the fourths by numbering pieces 1–16 *across the 4 sticks of wood*. Throughout the strategy, Penelope kept track of the distributed amounts of 1/2 stick of wood, one whole stick of wood, and 1/4 stick of wood by writing those amounts symbolically in 16 circles that represented the 16 rhinoceroses. Penelope combined the amounts each rhinoceros received by writing the equation $1/2 + 1 + 1/4 = 7/4$. She used the equivalent fractions of $1/2 = 2/4$ and $1 = 4/4$ to add the amounts together. Penelope indicated each rhinoceros would get 7/4 or 1 3/4 of a stick of wood and wrote this answer in a sentence.

Emergent-anticipatory direct modeling strategy



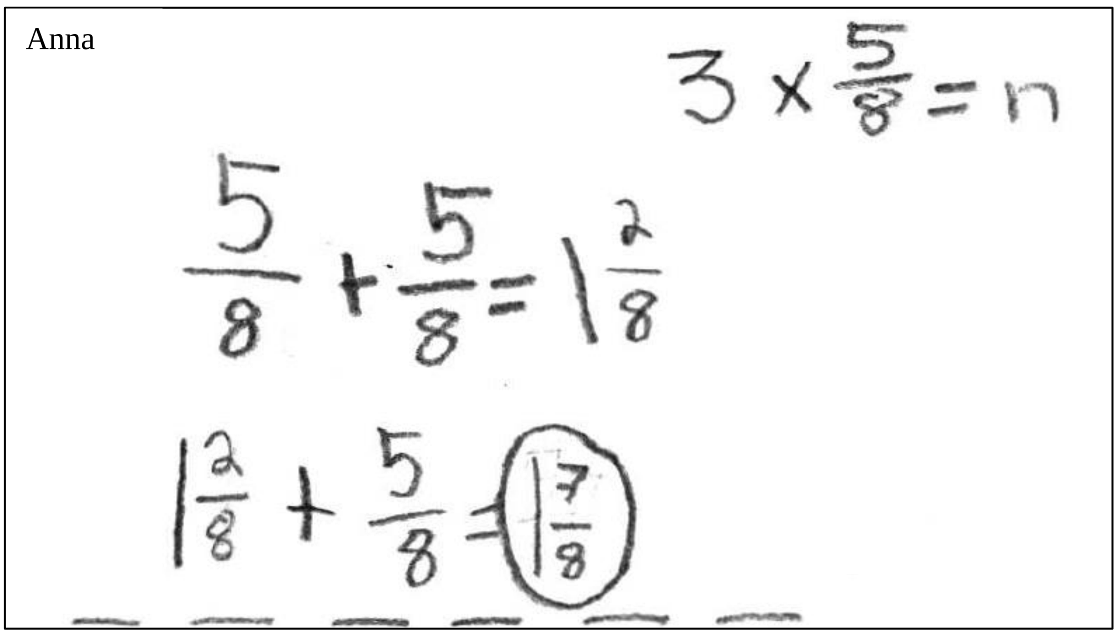
Tamara drew 28 rectangles to represent the 28 sticks of wood and 16 circles to represent the 16 rhinoceroses. She distributed 1 whole stick of wood to each of the rhinoceroses by numbering 16 of the sticks of wood 1–16. She partitioned the remaining 12 sticks of wood into fourths and mentally distributed fourths to the rhinoceroses. Tamara wrote sentences explaining that each rhinoceros received 1 whole stick of wood and 3/4 of a stick of wood as well as her final answer of 1 3/4 (sticks) of wood.

Emergent-anticipatory direct modeling strategy

APPENDIX H

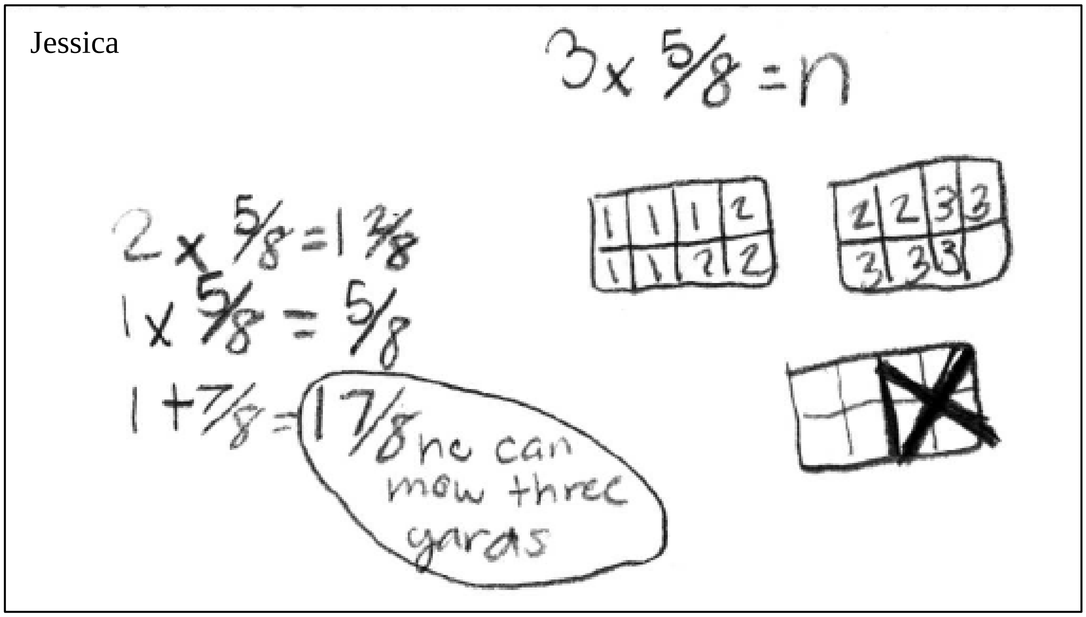
TWO STRATEGIES SHARED IN MS. DUSTMAN'S 1-DAY SCENARIO DISCUSSION

Sean has a lawn-mowing business. The gas tank on his lawnmower holds 2 1/2 gallons. It takes about 5/8 gallon to mow each lawn. If his tank is full, does he have enough gas to mow 3 yards?



Anna represented the problem as a multiplication problem: $3 \times 5/8 = n$. However, she solved the problem using repeated addition. She first added 5/8 + 5/8 which told her that 2 yards could be mowed with 1 2/8 gallons of gas. She then added another 5/8 to that sum, which told her that 3 yards could be mowed with 1 7/8 gallons. She did not show the final step of comparing 1 7/8 gallons to the 2 1/2 gallons currently in the tank to determine that Sean has enough to mow 3 yards.

Counting/adding strategy



Jessica represented the problem as a multiplication problem by writing 3 × 5/8 = *n*. She wrote equations to solve the problem. She first wrote "2 × 5/8," and knew the product was 1 2/8. She then wrote "1 × 5/8" for the third yard, and knew the product was 5/8. Next, Jessica combined all the eighth-sized pieces with numbers and represented this action with the equation 1 + 7/8 = 1 7/8, which was less than 2 1/2 and therefore enough gas to mow three yards. To check her work, Jessica drew 3 rectangles to represent gallons of gas, partitioned them into eighths, and then crossed out 4/8 on the last gallon of gas so that her drawing depicted the 2 1/2 gallons of gas in the tank. She then indicated the 5/8 gallon of gas needed for the first yard by writing a "1" in 5 of the eighth-sized pieces in her drawing. Similarly, she indicated the gas needed for the second yard with a "2" and the third yard with a "3." From her drawing, she concluded that Sean had enough gas to mow 3 yards because the eighth-sized pieces with numbers in them represented the gas needed for the 3 yards, and some of the eighth-sized pieces in her drawing did not have numbers in them, indicating that there would be leftover gas in the tank.

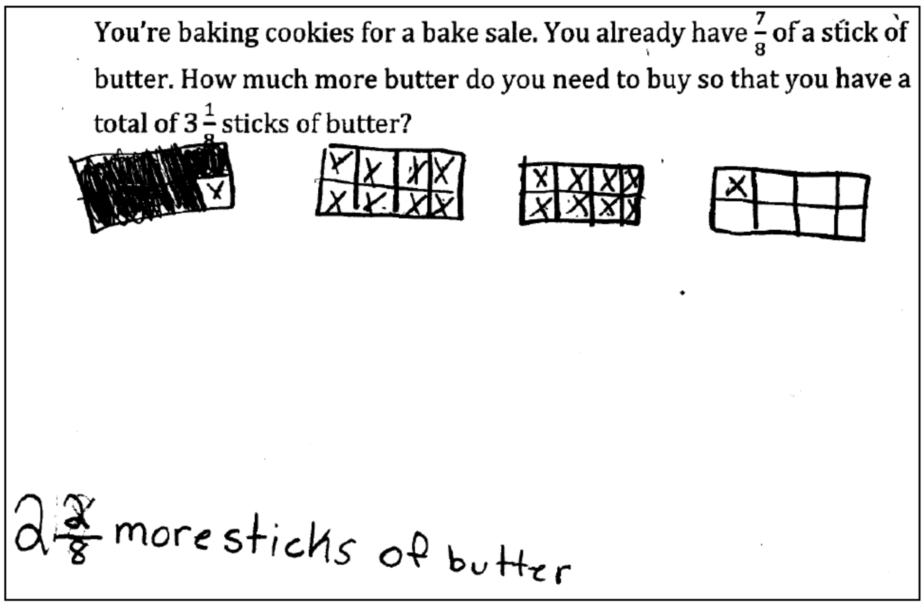
Counting/adding strategy

APPENDIX I

THREE STRATEGIES FOR THE BUTTER PROBLEM USED IN A PD CONVERSATION

You're baking cookies for a bake sale. You already have 7/8 of a stick of butter. How much more butter do you need to buy so that you have a total of 3 1/8 sticks of butter?

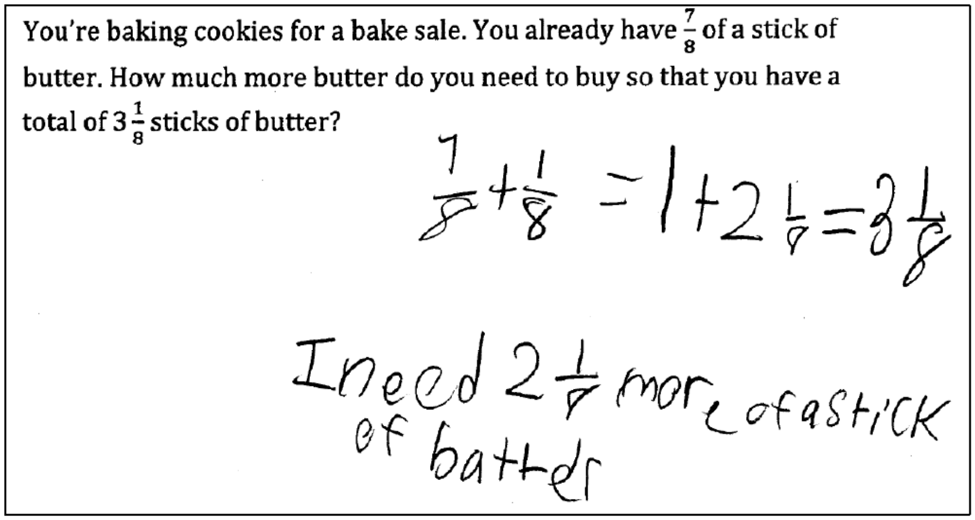
Ethan



Ethan drew four rectangles to represent 4 sticks of butter. He partitioned each stick of butter into eighths and shaded 7/8 of the first stick of butter to indicate the amount of butter he already had. Ethan then built up to 3 1/8 sticks of butter. Specifically, he continued from the 7/8, placing an "X" in each 1/8 until he reached 3 1/8 sticks of butter. Ethan answered that he would need 2 2/8 more sticks of butter, which he determined by combining the number of eighths with Xs in them.

Direct modeling strategy

Landon



You're baking cookies for a bake sale. You already have $\frac{7}{8}$ of a stick of butter. How much more butter do you need to buy so that you have a total of $3\frac{1}{8}$ sticks of butter?

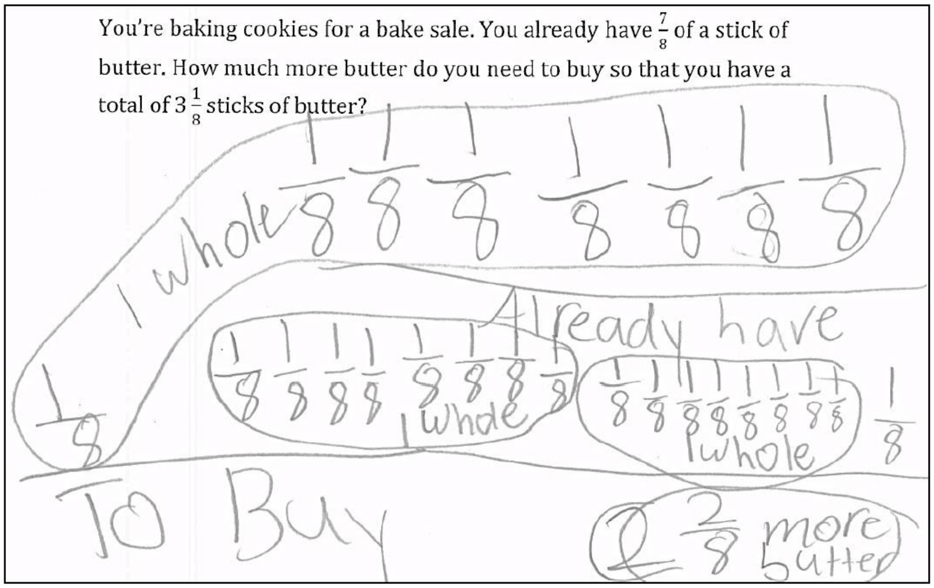
$\frac{7}{8} + \frac{1}{8} = 1 + 2\frac{1}{8} = 3\frac{1}{8}$

I need $2\frac{1}{8}$ more of a stick of batter

Landon started with 7/8 of a stick of butter and added 1/8 of a stick of butter to get to the benchmark of 1 whole stick of butter. He then added 2 1/8 sticks of butter to get to the desired amount of 3 1/8 sticks of butter. Landon answered that he would need 2 1/8 more sticks of butter. He had a valid strategy but an incorrect answer because he only considered the second amount that he added (2 1/8) when he needed to combine both amounts that he added (1/8 + 2 1/8).

Relational thinking strategy

Marcus



Marcus represented 7 individual one-eighths to indicate the 7/8 of a stick of butter that he had at the start. He even underlined this starting amount and labeled the amount with "already have" and the rest of the strategy with "to buy." Marcus then kept adding butter until he reached his desired amount of 3 1/8 sticks of butter. Specifically, Marcus wrote another 1/8 stick of butter and circled that 1/8 with the 7/8 he already had to make 1 whole stick of butter, and he labeled that group of 8 one-eighths as "1 whole." Marcus then represented another whole stick of butter by showing each individual 1/8, again circling the 8 eighths and writing "1 whole." Marcus repeated this step to add another whole stick of butter. At this point 3 whole sticks of butter were represented so Marcus wrote the final 1/8 to represent the desired amount of 3 1/8 sticks of butter. He then mentally combined the amounts he added to the original amount of 7/8 of a stick of butter to give his final answer of 2 2/8 more (sticks of) butter.

Direct modeling strategy