Equations on Semidirect Products of Commutative Semigroups

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Abstract:

In this paper; we study equations on semidirect products of commutative semigroups. Let $\mathbf{Com}_{q,r}$ denote the pseudovariety of all finite semigroups that satisfy the equations $xy = yx$ and $x^{r+q} = xr$. The pseudovariety $\mathbf{Com}_{1,1}$ is the pseudovariety of all finite semilattices. We consider the product pseudovariety $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$ generated by all semidirect products of the form $S*T$ with $S \in \mathbf{Com}_{q,r}$ and $T \in \mathbf{Com}_{q',r'}$,. We give an algorithm to decide when an equation holds in $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$. Finite complete sets of equations are described for all the products $\mathbf{Com}_{q,r} * \mathbf{Com}_{1,r'}$ which provide polynomial time algorithms to test membership. Our results imply finite complete sets of equations for $\mathbf{G}_{com} * \mathbf{Com}_{1,1}$ and ($\mathbf{Com}\cap$A) *$\mathbf{Com}_{1,1}$ (among others). Here; $\mathbf{G}_{com}$ denotes the pseudovariety of all finite commutative groups; **Com** the pseudovariety of all finite commutative semigroups and **A** the pseudovariety of all finite aperiodic semigroups.

Article:

1. Introduction

Let $q \geq 1, r \geq 0$ and let **Com**q~ denote the pseudovariety of all finite semigroups that satisfy the equations $xy = yx$ and $x + q = x$. The pseudovariety $\mathbf{Com}_{1,1}$ is the pseudovariety of all finite semilattices. In this paper, we give an equational description of the product $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$, generated by all semidirect products of the form $S * T$ with $S \in \mathbf{Com}_{q,r}$ and $T \in \mathbf{Com}_{q',r'}$. The equational description turns out to be finite for the product **Com**q,r * $\mathbf{Com}_{1,r'}$. As a consequence, we obtain a reasonable algorithm in $O(n^{2r'+2})$ to test whether or not a semigroup of size n given by its multiplication table belongs to $\mathbf{Com}_{q,r} * \mathbf{Com}_{1,r'}$. Our results imply finite equational descriptions for ($\mathbf{Com} \cap \mathbf{G}$) * $\mathbf{Com}_{1,1}$ and ($\mathbf{Com} \cap \mathbf{A}$) * $\mathbf{Com}_{1,1}$ (among others). Here, **Com** denotes the pseudovariety of all finite commutative semigroups, **G** the pseudovariety of all finite groups and **A** the pseudovariety of all finite aperiodic semigroups.

Previous results related to equational descriptions of the products $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$ follow.

- Pin [17] shows that the pseudovariety $\mathbf{Com}_{1,1} * \mathbf{Com}_{1,1}$ is defined by the two equations $xuyvxy = xuyvyx$ and $xux^2 = xux$.
- Irastorza [13] gives equations of the particular products $\mathbf{Com}_{1,1} * \mathbf{Com}_{q,0}$ and shows that, although the two pseudovarieties $\mathbf{Com}_{1,1}$ and $\mathbf{Com}_{2,0}$ are defined by finite sets of equations, their product is not.
- Ash [5] shows that the pseudovariety $\mathbf{Com}_{1,1} * \mathbf{G} = \mathbf{Inv}$ is defined by the equation $x^\omega y^\omega = y^\omega x^\omega$ where Inv is the pseudovariety generated by the inverse semigroups (here, if S is a finite semigroup and $s \in S$, s^ω will denote the idempotent in the subsemigroup generated by s).
- Almeida [3] gives an equational description for the pseudovariety $(\mathbf{Com}_{1,1})^n$ generated by all semidirect products of n finite semilattices. He shows that, for every $n \geq 3$, $(\mathbf{Com}_{1,1})^n$ is not defined by a finite set of equations.

- Blanchet-Sadri and Zhang [8] give (with techniques used in particular by Blanchet-Sadri [6, 7], Brzozowski and Simon [10] and Pin [16, 17]) a complete set of equations for $\mathbf{Com}_{1,1} * \mathbf{Com}_{q,r}$ implying complete sets of equations for $\mathbf{Com}_{1,1} * (\mathbf{Com} \cap \mathbf{G})$, $\mathbf{Com}_{1,1} * (\mathbf{Com} \cap \mathbf{A})$ and $\mathbf{Com}_{1,1} * \mathbf{Com}$.

Almeida [2] proposed a new approach to treat problems that ask for effective algorithms to decide whether a given finite semigroup belongs to the product $\mathbf{V} * \mathbf{W}$ of pseudovarieties $\mathbf{V}$ and $\mathbf{W}$ for which such algorithms are known. Similar problems may be phrased for suitable varieties and the results translated back to pseudovarieties. His method was illustrated in particular with the equality $\mathbf{Com}_{1,1} * \mathbf{D} = \mathbf{LCom}_{1,1}$ [10, 14] and Thérien and Weiss's efficient algorithm for deciding membership in the pseudovariety $\mathbf{Com}*\mathbf{D}$ [18] where $\mathbf{D}$ denotes the pseudovariety of all finite semigroups S such that $se = e$ for $e,s \in S$ with $e^2 = e$. Almeida's method was also used in [9] to show that a semidirect product of $n + 1$ finite semilattices is equivalent to a semidirect product of a finite semilattice by a J-trivial semigroup of height n answering a conjecture of Pin [16] negatively.

In this paper, we use Almeida's method. For $q \geq 1, r \geq 0$, we will denote by $Com_{q,r}$ the variety of all semigroups that satisfy the equations $xy = yx$ and $x^{r+q} = xr$.

1.1. Definitions and notations

Let S and T be two semigroups. We say that S divides T and write $S \prec T$ if S is a morphic image of a subsemigroup of T. A *variety* of semigroups V is a class of semigroups closed under division and direct product. A *pseudovariety* of semigroups $\mathbf{V}$ is a class of finite semigroups closed under division and finite direct product.

For any class C of finite semigroups we denote by $(C)_S$ the least pseudovariety of semigroups containing C. Clearly $S \in (C)_S$ if and only if $S \prec S_1 \times \bullet\bullet\bullet \times S_n$ with $S_1..... S_n \in C$. We call $(C)_S$ the pseudovariety of semigroups *generated* by C.

Let $\mathbf{V}$ be a pseudovariety of semigroups with generators $S_1..... S_n$. Thus

$\mathbf{V} = (S_1.....S_n)S$

Replacing $S_1..... S_n$ by $S_0 = S_1 \times \bullet\bullet\bullet \times S_n$ we obtain $\mathbf{V} = (S_0)_S$. Thus $\mathbf{V}$ is generated by the single semigroup S_0.

Let $S_0 = Z_q$ be the cyclic group of order $q \geq 1$. The pseudovariety $(Z_q)S$ is defined by the pair of equations

$xy = yx, x^q = 1$

The equation $x^q = 1$ abbreviates $x^q y = yx^q = y$. The pseudovariety $\mathbf{G}_{\text{com}}$ of commutative groups is generated by the cyclic groups Z_q and is hence the join of the pseudovarieties $\mathbf{Com}_{q,0}$ with $q \geq 1$. Note that $\mathbf{G}_{com}$ is defined by the equations $xy = yx$ and $x^\omega = 1$. The latter abbreviates $x^\omega y = yx^\omega = y$.

Consider $S_0 = Z_{q,r}$ the cyclic monoid of period $q \geq 1$ and index $r \geq 0$. If a is the generator, then the elements of $Z_{q,r}$ are $1, a, a^2..... a^{r+q-1}$ with $a^r = a^{r+q}$. The semigroup $Z_{q,r}$ is isomorphic with a subsemigroup of Z_q x $Z_{1,r}$. Since further $Z_q \prec Z_{q,r}$ and $Z_{1,r} \prec Z_{q,r}$, it follows that $(Z_{q,r})_S = (Z_q, Z_{1,v})_S$. Thus the pseudovariety $(Z_{q,r})_S$ is the join $(Z_q)_S \vee (Z_{1,r})_S$. The pseudovariety $(Z_{q,r})_S$ is defined by the pair of equations

$xy = yx, x^{r+q} = x^y$

Consider the special case with $q = 1$ and $r = 0$. Then $\mathbf{Com}_{1,0} = \mathbf{I}$ is the pseudovariety consisting of all one-point semigroups. Now if $q = 1$ and $r = 1$, then $\mathbf{Com}_{1,1}$ is the pseudovariety of commutative and idempotent semigroups. The pseudovariety $\mathbf{Com}$ of commutative semigroups is generated by the cyclic semigroups $Z_{q,r}$ with $q \geq 1, r \geq 0$, or $\mathbf{Com}$ is generated by the cyclic groups Z_q with $q > 1$ and by the semigroups $Z_{1,r}$ with $r > 0$. Hence $\mathbf{Com}$ is the join of the pseudovarieties $\mathbf{Com}_{q,0}$ with $q \geq 1$ and $\mathbf{Com}_{1,r}$ with $r > 0$. Note that $(\mathbf{Com} \cap \mathbf{A})$ is the join of the pseudovarieties $\mathbf{Com}_{1,r}$ with $r \geq 0$. Here, **A** denotes the pseudovariety of aperiodic semigroups defined by the equation $x^{\omega+1} = x^{\omega}$ where $x^{\omega+1}$ abbreviates $x^{\omega}x$.

2. Preliminaries

We refer the reader to [4, 11, 15] for terms not explicitely defined here.

2.1. Varieties of semigroups V* W

Let T be a semigroup. The semigroup $T \cup \{1\}$ obtained from T by adjoining a unit element if T does not have one is denoted by T^1.

Let S and T be semigroups. It is convenient to write S additively, without however assuming that S is commutative and T multiplicatively. A left action of T^1 on S is a monoid morphism φ from T^1 into the monoid of (semigroup) endomorphisms of S (composed on the left).

Given a left action φ of T^1 on S we define the semidirect product $S * T$ as follows. The elements of $S * T$ are pairs (s, t) with $s \in S$, $t \in T$. Multiplication is given by the formula

$(s, t)\,(s', t') = (s + ts', tt')$

where ts' represents $\varphi(t)(s')$. The multiplication in $S * T$ is associative. Thus $S * T$ is a semigroup.

We now relate the notion of a variety of semigroups with that of a semidirect product. Given two varieties of semigroups V and W, we denote by $V * W$ the variety generated by all semidirect products $S * T$ with $S \in V$, $T \in W$ and with any left action of T^1 on S. The semidirect product of two pseudovarieties of semigroups is defined analogously. The semidirect product on varieties or pseudovarieties of semigroups is associative. It is easily checked that **I** is the unit element for the semidirect product on pseudovarieties of semigroups.

2.2. Varieties of semigroups defined by equations

Let X be a set. We denote by X^+ (respectively X^*) the free semigroup (respectively monoid) on X. Elements of X^* are viewed as words on the alphabet X. If u,v are words in some X^+, then the equation $u = v$ is said to hold in a semigroup S (or S is said to satisfy the equation $u = v$ and we write $S \models u = v$) in case for every morphism (φ : $X^+ \rightarrow S$, $u\varphi = v\varphi$.

Let $\mathcal{E}$ be a set of equations. The variety of all semigroups that satisfy all the equations in $\mathcal{E}$ is denoted by $[\mathcal{E}]$. If for every semigroup S, we have $S \models \mathcal{E}$ implies $S \models u = v$, then we write $\mathcal{E} \vdash u = v$ and the equation $u = v$ is said to be deducible from $\mathcal{E}$.

By a well-known theorem of Birkhoff, varieties of semigroups are defined by sets of equations. Eilenberg and Sch~utzenberger [12] show that pseudovarieties of semigroups are ultimately defined by equations.

3. An Equational Description of $\mathbf{Com}_{q,v} * \mathbf{Com}_{q',r'}$

In this section we give an equational description of $\mathbf{Com}_{q,v} * \mathbf{Com}_{q',r'}$. In order to do so, we use a semidirect product representation of the free objects in $\mathbf{V} * \mathbf{W}$ in case both the pseudovarieties of semigroups **V** and **W** have finite free objects. This approach to the semidirect product was introduced in [2].

The free object on the set X in the variety of semigroups generated by the pseudovariety of semigroups **V** will be denoted by $F_X(\mathbf{V})$. We will abbreviate $F_{\{x_1,\ldots,x_n\}}(\mathbf{V})$ by $F_n(\mathbf{V})$.

For any $q \geq 1, r \geq 0$, we define on X^* the congruence $\sim_{q,v}$ by $u \sim_{q,v} v$ if for all $x \in X$, either $|u|_x = |v|_x$, or both $|u|_x$, $|v|_x \geq r$ and $|u|_x \equiv |v|_x \bmod q$ (here, $|u|_x$ is the number of times the letter x appears in the word $u \in X^*$). Note the following special cases (here, $u\alpha$ denotes the set of letters in a word $u \in X^*$).

- For all $u,v \in X^*$, $u \sim_{1,0} v$.
- If $u\alpha = v\alpha$, then $u \sim_{1,1} v$.
- If for all $x \in X$, either $|u|_x = |v|_x$ or $|u|_x, |v|_x \geq r$, then $u \sim_{1,v} v$.
- If for all $x \in X$, $|u|_x \equiv |v|_x \bmod q$, then $u \sim_{q,0} v$.

For $n \geq 1$, the free object $F_n(\mathbf{Com}_{q,r})$ can be viewed as a set of representatives of classes modulo $\sim_{q,r}$ of words on $\{x_1..... x_n\}$ (we have $F_n(\mathbf{Com}_{q,r}) \models u = v$ if and only if $u \sim_{q,r} v$) . The set F_n $(\mathbf{Com}_{q,r})$ is then finite and a result of Almeida [2] implies the following representation of free objects for the pseudovariety $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$(the same result of [2] "gives" the decidability of $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$, as a corollary).

Lemma 3.1. *Let $q, q' \geq 1$, $r, r' \geq 0$ and $n \geq 1$. The free object $F_n(\mathbf{Com}_{q,v} * \mathbf{Com}_{q',r'})$ is finite. Moreover, let $Y = (F_n(\mathbf{Com}_{q',r'}))^1 \times \{x_1.....x_n\}$. Then, there is an embedding from F_n $(\mathbf{Com}q,r * \mathbf{Com}_{q',r'})$ into F_Y $(\mathbf{Com}_{q,r}) * F_n$ $(\mathbf{Com}_{q',r'})$ that maps x_i into $((1, x_i), x_i)$ where the left action of $(F_n(\mathbf{Com}_{q',r'}))1$ on $F_Y(\mathbf{Com}_{q,r})$ is given by x_i $(s, x_j) = (x_i s, x_j)$ for $s \in (F_n(\mathbf{Com}_{q',r'}))^1$.*

Let $q, q' \geq 1, r, r' \geq 0$ and $n \geq 1$. We will denote by $\pi_{q,r}$ the canonical projection from $\{x_1.....x_n\}^+$ into $F_n(\mathbf{Com}_{q,r})$ that maps the letter x_i onto the generator x_i of $F_n(\mathbf{Com}_{q,r})$. If $u \in \{x_1.....x_n\}^+$, then $u\pi_{q,r}$ can be viewed as a representative of the class modulo $\sim_{q,v}$ of u. Also we will denote by $\pi_{q,r}^{q',r'}$ the canonical projection from $\{x_1.....x_n\}^+$ into $F_n(\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'})$ that maps the letter x_i onto the generator x_i of $F_n(\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'})$.

Definition 3.2. *Let $q \geq 1, r \geq 0$ and $n \geq 1$. Let $u = x_{i_1}$ ... $x_{i_m} \in \{x_1.....x_n\}^+$. We write $up_{q,r}$ for the word*

$$(1, x_{i_1})\, (x_{i_1}\, \pi_{q,r}, x_{i_2}) ... ((\, x_{i_1} ... \, x_{i_{m-1}})\, \pi_{q,r}, x_{i_m})$$

on the alphabet $(F_n(\mathbf{Com}_{q,r}))^1 \times \{x_1.....xn\}$.

The following lemma provides an algorithm to decide when an equation holds in the pseudovariety $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$.

Lemma 3.3. *Let $q, q' \geq 1$, $r, r' \geq 0$, $n \geq 1$ and let $u, v \in \{x_1.....x_n\}^+$. We have*

$\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'} \models u = v$ *if and only if $up_{q',r'} \sim_{q,r} vp_{q',r'}$ and $u\pi_{q',r'} = v\pi_{q',r'}$.*

Proof. By Lemma 3.1, the free object $F_n(\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'})$ is finite. Moreover, if $Y = (F_n(\mathbf{Com}_{q',r'}))^1 \times \{x_1.....x_n\}$, then there is an embedding from $F_n(\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'})$ into $F_Y(\mathbf{Com}_{q,r}) * F_n(\mathbf{Com}_{q',r'})$ that maps x_i into $((1,x_i),x_i)$ where the left action of $(F_n(\mathbf{Com}_{q',r'}))^1$ on $F_Y(\mathbf{Com}_{q,r})$ is given by $x_i(s,x_j) = (x_i s, x_j)$ for $s \in (F_n(\mathbf{Com}_{q',r'}))^1$. The word $u = x_{i_1} \ldots x_{i_m}$ is mapped into

$$((1, x_{i_1}) + (x_{i_1}, x_{i_2}) + \bullet\bullet\bullet + (x_{i_1} ... \, x_{i_{m-1}}, x_{i_m}), x_{i_1} ... \, x_{i_m})$$

and the word $v = x_{j1} ... \, x_{j_{m'}}$ is mapped into

$$((1, x_{j_1}) + (x_{j_1}, x_{j_2}) + \bullet\bullet\bullet + (x_{j_1} ... \, x_{j_{m'-1}}, x_{j_{m'}}), x_{j_1} ... \, x_{j_{m'}})$$

Suppose that $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'} \models u = v$, or that $u\pi_{q,r}^{q',r'} = v\pi_{q,r}^{q',r'}$. This is equivalent to the two conditions $up_{q'r'} \sim_{q,v} vp_{q'r'}$ and $\mathbf{Com}_{q',r'} \models u = v$. Observe that $\mathbf{Com}_{q',r'} \models u = v$ if and only if $u\pi_{q',r'} = v\pi_{q',r'}$.

The following lemma will be used to give finite equational descriptions of all the products $\mathbf{Com}_{q,r} * \mathbf{Com}_{1,r'}$.

Lemma 3.4. *Let $q \geq 1, r \geq 0$ be such that $(q, r) \neq (1, 0)$. Let $r' > 0, n \geq 1$ and let $u, v \in \{x_1.....x_n\}^+$. Assume that $up_{1,r'} \sim_{q,r} vp_{1,r'}$ Then $u\pi_{1,r'} = v\pi_{1,r'}$. Also, if u_1x is the shortest prefix of u with $u\pi_{1,r'} = (u_1x)\pi_{1,r'}$ then the shortest prefix v' of v with $v\pi_{1,r'} = v'\pi_{1,r'}$ is of the form v_1x. In such a case, there are factorizations $u = u_1xu_2$ with $u_1, u_2 \in \{x_1.....x_n\}^*$, and $v = v_1xv_2$ with $v_1, v_2 \in \{x_1.....x_n\}^*$ satisfying $u_1p_{1,r'} \sim_{q,r} v_1p_{1,r'}$ and u_2 —$_{q,r}$ v_2.*

Proof. To show that $u\pi_{q',r'} = v\pi_{q',r'}$ it suffices to show that for all $x \in \{x_1.....x_n\}$, either $|u|_x = |v|_x$ or $|u|_x, |v|_x \geq r'$. If $|u|_x \neq |v|_x$, then assume that $|u|_x < |v|_x$. If $|u|_x > r'$, there is nothing to prove. Otherwise, $|u|_x = r'' < r'$. Let $v = v_1x...v_{r''}xv_{r''+1}xv_{r''+2}$ be a factorization of v showing the first r'' + 1 x's. Consider the letter $x = ((v_1x...v_{r''}xv_{r''+1})\pi_{1\,r'}, x)$ which occurs exactly one time in $vp_{1,r'}$. This letter does not occur in $up_{1,r''}$. But we have either $|up_{1,r'}|_{\bar{x}} = |vp_{1,r'}|_{\bar{x}}$, or both $|up_{1,r'}|_{\bar{x}}$ and $|vp_{1,r'}|_{\bar{x}} \geq r$ and $|up_{1,r'}|_{\bar{x}} \equiv |vp_{1,r'}|_{\bar{x}}$ mod q. In both cases, we reach contradictions.

Let u_1x be the shortest prefix of u with $u\pi_{1,r'} = (u_1x)\pi_{1,r'}$ Write $u = u_1xu_2$ for some $u_2 \in \{x_1.....x_n\}^*$. Let v_1y be the shortest prefix v' of v with $v\pi_{1,r'} = v'\pi_{1,r'}$, giving a factorization $v = v_1yv_2$ for some $v_2 \in \{x_1.....x_n\}^*$. We have

$up_{1,r'} = u_1p_{1,r'}\,(u_1\pi_{1,r'}, x)u_2p,$
$vp_{1,r'} = v_1p_{1,r'}\,(v_1\pi_{1,r'}, y)v_2p,$

where for every $w = x_{i_1}... x_{i_m}$ on $\{x_1.....x_n\}$, the word

$(u\pi_{1,r'}, x_{i_1})\,(u\pi_{1,r'}, x_{i_2})...(u\pi_{1,r'}, x_{i_m})$

has been abbreviated by wp (recall that $u\pi_{1,r'} = v\pi_{1,r'}$).

First, the sets$(u_1p_{1,r'})\alpha$, $\{(u_1\pi_{1,r'}, x)\}$, $(u_2p)\alpha$, $(v_1p_{1,r'})\alpha$, $\{(v_1\pi_{1,r'}, y)\}$ and $(v_2p)\alpha$, are pairwise disjoint except possibly for the pair $(u_1p_{1,r'})\alpha$, $(v_1p_{1,r'})\alpha$, the pair $\{(u_1\pi_{1,r'}\, x)\}$, $\{(v_1\pi_{1,r'}\, y)\}$ and the pair $(u_2p)\alpha$, $(v_2p)\alpha$, (this fact will be used in the rest of the proof). To see this, the first coordinate of letters in $(u_1p_{1,r'})\alpha$, $\{(u_1\pi_{1,r'}, x)\}$, $(v_1p_{1,r'})\alpha$, and $\{(v_1\pi_{1,r'}, y)\}$ is not $u\pi_{1,r'}$ which is the first coordinate of letters in $(u_2p)\alpha$, and $(v_2p)\alpha$; the letter $(u_1\pi_{1,r'}, x)$ (respectively $(v_1\pi_{1,r'}, y)$) is not in $(u_1p_{1,r'})\alpha$ (respectively $(v_1p_{1,r'})\alpha$) because of the choice of u_1x (respectively v_1y); the letter $(u_1\pi_{1,r'}, x)$ is not in $(v_1p_{1,r'})\alpha$ since every letter $(w_1\pi_{1,r'}, z)$ in $v_1p_{1,r'}$ is such that $|u_1|_y > |w_1|_y$ and $|w_1|_y < r'$ (similarly $(v_1\pi_{1,r'}, y)$ is not in $(u_1p_{1,r'})\alpha$).

Second, if $x \neq y$, then the letter $x = (u_1\pi_{1,r'}, x)$ which is in $(up_{1,r'})\alpha$ is not in $(vp_{1,r'})\alpha$. So since $up_{1,r'} \sim_{q,r} vp_{1,r'}$ and $|up_{1,r'}|_{\bar{x}} \neq |vp_{1,r'}|_{\bar{x}}$, we have both $|up_{1,r'}|_{\bar{x}}, |vp_{1,r'}|_{\bar{x}} \geq r$ and $|up_{1,r'}|_{\bar{x}} \equiv |vp_{1,r'}|_{\bar{x}}$ mod q. If $r > 0$, we get a contradiction since $|vp_{1,r'}|_{\bar{x}} = 0 \ngeq r$. If $r = 0$, then $q > 1$ and we get also a contradiction since $|up_{1,r'}|_{\bar{x}} = 1$ and $|vp_{1,r'}|_{\bar{x}} = 0$ (and so $|up_{1,r'}|_{\bar{x}} \not\equiv |vp_{1,r'}|_{\bar{x}}$ mod q). So $(u_1\pi_{1,r'}, x) = (v_1\pi_{1,r'}, y)$, yielding $x = y$.

Third, we get $u_1p_{1,r'} \sim_{q,r1}p_{1,r'}$ and $u_2 \sim_{q,r} v_2$. To see that $u_1p_{1,r'} \sim_{q,r} v_1p_{1,r'}$, let $z \in \{x_1.....x_n\}$. Assume that $|u_1p_{1,r'}|_{\bar{z}} \neq |v_1p_{1,r'}|_{\bar{z}}$, and hence $|u_1p_{1,r'}|_{\bar{z}} \neq |v_1p_{1,r'}|_{\bar{z}}$ where $\bar{z} = (w_1\pi_{1,\,r'}, z)$. Since $up_{1,\,r'} \sim_{q,r} vp_{1,\,r'}$, both $|up_{1,r'}|_{\bar{z}}, |vp_{1,\,r'}|_{\bar{z}} \geq r$ and $|up_{1,\,r'}|_{\bar{z}} \equiv |vp_{1,\,r'}|_{\bar{z}}$ mod q hold. We then deduce that $|u_1p_{1,r'}|_{\bar{z}}, |v_1p_{1,r'}|_{\bar{z}} \geq r$ and $|u_1p_{1,r'}|_{\bar{z}} \equiv |v_1p_{1,r'}|_{\bar{z}}$ mod q. To see that $u_2 \sim_{q,r} v_2$, let $z \in \{x_1..... x_n\}$. Assume $|u_2|_z \neq |v_2|_z$, and hence $|up_{1,\,r'}|_{\bar{z}} \neq |up_{1,\,r'}|_{\bar{z}}$ where $\bar{z} = (u\pi_{1,\,r'}, z)$. Since $up_{1,\,r'} \sim_{q,r} vp_{1,\,r'}$, both $|up_{1,\,r'}|_{\bar{z}}, |vp_{1,\,r'}|_{\bar{z}} \geq r$ and $|up_{1,\,r'}|_{\bar{z}} \equiv |vp_{1,\,r'}|_{\bar{z}}$ mod q hold. We then deduce that $|u_2|_z$, $|v_2|_z \geq r$ and $|u_2|_z \equiv |v_2|_z$ mod q. ■

The variety $Com_{q,r}$ is locally finite since every finitely generated semigroup in $Com_{q,r}$ is finite. We have $\mathbf{Com}_{q,r} = (Com_{q,r})^F$ where $(Com_{q,r})^F$ consists of all the finite semigroups in $Com_{q,r}$. A result of Almeida [2] implies that

$(Com_{q,r} * Com_{q',r'})^F = (Com_{q,r})^F * (Com_{q',r'})^F$ and that $Com_{q,r} * Com_{q',r'}$ is also locally finite. Thus $Com_{q,r} * Com_{q',r'}$ is the variety of semigroups generated by $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'}$ and so $F_n(\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'})$ is the free object on $\{x_1....x_n\}$ in this variety. Since $\mathbf{Com}_{q,r} * \mathbf{Com}_{q',r'} = (Com_{q,r} * Com_{q',r'})^F$, any set of equations for $Com_{q,r} * Com_{q',r'}$ is also a set of equations for $Com_{q,r} * Com_{q',r'}$.

We now give a finite set of equations for $Com_{q,r} * Com_{q',r'}$.

Theorem 3.5. *Let $q \geq 1$, $r \geq 0$ be such that $(q, r) \neq (1, 0)$ and let $r' > 0$. The variety $Com_{q,r} * Com_{1,r'}$ is defined by the set $\mathcal{E}^{1,r'}_{q,r}$ consisting of the equation*

(1) $$xu_1...xu_{r'}x^{r+q} = xu_1...xu_{r'}x^{v}$$

together with all equations of the form

(2) $$x_1u_1...x_{2r'}u_{2r'}xy = x_1u_1...x_{2r'}u_{2r'}yx$$

where $x_1.....x_{2r'}$ is a list of r' x's and r' y's.

Proof. For the inclusion $Com_{q,r} * Com_{1,r'} \subseteq [\mathcal{E}^{1,r'}_{q,r}]$, we use Lemma 3.3. We have

$$(xu_1...xu_{r'}x^{r+q})p_{1,r'} \sim_{q,r} (xu_1...xu_{r'}x^{r})p_{1,r'}$$

for all equations of the form (1) and

$$(x_1u_1...x_{2r'}u_{2r'}xy)p_{1,r'} \sim_{q,r} (x_1u_1...x_{2r'}u_{2r'}yx)p_{1,r'}$$

for all equations of the form (2) . The equalities

$$(xu_1...xu_{r'}x^{r+q})\pi_{1,r'} = (xu_1...xu_{r'}x^{r})\pi_{1,r'}$$

and

$$(x_1u_1...x_{2r'}u_{2r'}xy)\pi_{1,r'} = (x_1u_1...x_{2r'}u_{2r'}yx)\pi_{1,r'}$$

are implied by Lemma 3.4.

For the reverse inclusion, we want to show that if $u, v \in \{x_1.....x_n\}^+$ are such that $up_{1,r'} \sim_{q,r} vp_{1,r'}$ and $u\pi_{1,r'} = v\pi_{1,r'}$, then $\mathcal{E}^{1,r'}_{q,r} \vdash u = v$. For this purpose, assume that $up_{1,r'} \sim_{q,r} vp_{1,r'}$ (the equality $u\pi_{1,r'} = v\pi_{1,r'}$ follows by Lemma 3.4) . As in Lemma 3.4, let u_1x (respectively v_1x) be the shortest prefix of u (respectively v) satisfying $(u_1x)\pi_{1,r'} = u\pi_{1,r'}$ (respectively $(v_1x)\pi_{1,r'} = v\pi_{1,r'}$), and let u_2 (respectively v_2) be the following segment of u (respectively v). We have $u_1p_{1,r'} \sim_{q,r} v_1p_{1,r'}$ and $u_2 \sim_{q,r} v_2$. The identity $u_2 = v_2$ is deducible from the set $\{x^{r+q} = x^r, xy = yx\}$ since $u_2 \sim_{q,r} v_2$. Each identity in $\mathcal{E}^{1,r'}_{q,r}$ is of the form $wx^{r+q} = wx^r$ where w contains r' x's, or of the form $wxy = wyx$ where w contains r' x's and r' y's. By definition of u_1x and v_1x, every letter in u_2 or v_2 appears at least r' times in u_1x and v_1x. The identity $u_1xu_2 = u_1xv_2$ is hence deducible from $\mathcal{E}^{1,r'}_{q,r}$. Now, since $u_1p_{1,r'} \sim_{q,r} v_1p_{1,r'}$, we repeat the process using Lemma 3.4 again. Since u and v obviously start with the same letter (u and v have the same alphabet and their first letter is the only one to have 1 as first coordinate), the process terminates with a deduction of $u = v$ from $\mathcal{E}^{1,r'}_{q,r}$. ■

Corollary 3.6. *Let* $q \geq 1$, $r \geq 0$ *be such that* $(q, r) \neq (1, 0)$ *and let* $r' > 0$. *The pseudovariety* $\mathbf{Com}_{q,r} * \mathbf{Com}_{1,r'}$ *is defined by* $\mathcal{E}_{q,r}^{1,r'}$.

Our preceding result generalizes a result of Pin.

Corollary 3.7. 17 *The pseudovariety* $\mathbf{Com}_{1,1} * \mathbf{Com}_{1,1}$ *is defined by the two equations* $xuyvxy = xuyvyx$ *and* $xux^2 = xux$.

Corollary 3.8. *Let* $r > 0$. *The pseudovarieties* $\mathbf{G}_{com} * \mathbf{Com}_{1,r}$ *and* $(\mathbf{Com} \cap \mathbf{A}) * \mathbf{Com}_{1,r}$ *are defined respectively by* $\mathcal{E}_{\omega,0}^{1,r}$ *and* $\mathcal{E}_{1,\omega}^{1,r}$, *where* $\mathcal{E}_{\omega,0}^{1,r}$ *consists of*

$$xu_1 \ldots xu_r x^{\omega} = xu_1 \ldots xu_r$$

together with all equations of the form

$$x_1u_1 \ldots x_{2r}u_{2r}xy = x_1u_1 \ldots x_{2r}u_{2r}yx$$

where $x_1 \ldots x_{2r}$ *is a list of r x's and r y's, and* $\mathcal{E}_{1,\omega}^{1,r}$ *consists of*

$$xu_1 \ldots xu_r x^{\omega+1} = xu_1 \ldots xu_r x^{\omega}$$

together with all equations of the form

$$x_1u_1 \ldots x_{2r}u_{2r}xy = x_1u_1 \ldots x_{2r}u_{2r}yx$$

where $x_1 \ldots x_{2r}$ *is a list of r x's and r y's.*

We now derive from Corollary 3.6 a polynomial time algorithm for testing membership in $\mathbf{Com}_{q,r} * \mathbf{Com}_{1,r'}$. We refer the reader to [1] for basic concepts concerning algorithms.

Corollary 3.9. *Let* $q \geq 1$, $r \geq 0$ *be such that* $(q,r) \neq (1,0)$ *and let* $r' > 0$. *Let S be a semigroup of size n given by its multiplication table. Corollary 3.6 provides an* $O(n^{2r'+2})$ *algorithm to decide membership of S in* $\mathbf{Com}_{q,r} * \mathbf{Com}_{1,r'}$.

Proof. By Corollary 3.6, for membership in $\mathbf{Com}_{q,r} * \mathbf{Com}_{1,r'}$, it is enough to test whether S satisfies all equations of the form (1) and (2). Given $x, y, u_1 \ldots u_{2r'} \in S$, we can easily test in $O(1)$ steps whether an equation of the form (1) or (2) is satisfied. Therefore by taking for $x, y, u_1 \ldots u_{2r'}$ all possible values in S, we can test in $O(n^{2r'+2})$ whether S satisfies all equations of the form (1) and (2).

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