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Many students encounter difficulty in their transition to advanced mathematical thinking. Such difficulty may be explained by a lack of understanding of many concepts taught in early school years, especially multiplicative reasoning. Advanced mathematical thinking depends on the development of multiplicative reasoning.

The purpose of this study was to identify indicators of multiplicative reasoning among fourth grade students. Inhelder and Piaget (1958) suggested that children circa age eleven are transitioning from the Concrete Operational Stage to the Formal Operations Stage and that it is not likely for children to demonstrate multiplicative reasoning without the structures of development supporting logical and abstract thinking. By employing a cross-case analysis, this study explores the thinking of fourteen math students from a low socioeconomic school. Through cross-case analysis, the researcher probed for patterns of multiplicative reasoning as students progressed through a test instrument which invoked varying levels of multiplicative reasoning. Section one did not distinguish between multiplicative algorithms and multiplicative reasoning. Section two discriminated with respect to multiplicative scheme extension. Section three discriminated with respect to unequal group identification and manipulation. Section 4 discriminated with respect to proportional reasoning but not with respect to multiplicative reasoning.

The fourth grade subjects fell into three categories: pre-multiplicative, emergent, and multipliers. Those subjects who utilized multiplicative reasoning on less than four questions were considered pre-multiplicative, whereas those subjects who utilized

multiplicative reasoning on six or more questions were considered multipliers. The remaining seven were those subjects who changed their approach from test item to test item, sometimes demonstrating multiplicative reasoning strategies and at other times demonstrating additive reasoning strategies. These subjects were considered emergent in the development of multiplicative reasoning.

This study developed twelve new sub-levels that describe in more detail the multiplicative thinking of these fourth graders. These new sub-levels are Level 1 Non-quantifier, Level 1 Spontaneous Guesser, Level 2 Keyword Finder, Level 2 Counter, Level 2 Adder, Level 2 Quantifier, Level 2 Measurer, Level 3 Repeated Adder, Level 3 Coordinator, Level 4 Multiplier, Level 4 Splitter and Level 5 Predictor.

This paper suggests that when teachers understand a child's method of deriving multiplying schemes and multiplicative reasoning strategies, they are in a better position to provide the appropriate learning environment for the child. Such interaction allows the listening teacher to build on the child's current level of mathematical understanding. Students should be encouraged to discover for themselves the needed theorems, definitions, and mechanics of the number system, and to personally develop any "short cutting" algorithms, rather than simply being handed the algorithms by the instructor with little or no understanding.

INDICATORS OF MULTIPLICATIVE REASONING

AMONG FOURTH GRADE STUDENTS

by
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CHAPTER I

STATEMENT OF RESEARCH PROBLEM

The difficulties encountered by students in their transition to advanced mathematical thinking may be explained by a lack of understanding of many concepts taught in early school years, especially multiplicative reasoning. Advanced mathematical thinking depends on the development of multiplicative reasoning (Confrey, 1994; Dreyfus, 1991; Harel & Sowder, 2005; Lamon, 1994; Tall, 1991). Within the research literature, several approaches of encouraging the development of multiplicative reasoning can be found, including development and cognition, mathematical perspectives, instructional approaches, and the learner and multiplication.

Advanced mathematical thinking embodies certain inherent complexities. By its nature, advanced mathematical thinking relies on a cumulative foundation of prior mathematical experiences. Students cannot comprehend advanced mathematical topics such as differential equations unless they have understood underlying concepts, such as differentiation. Differentiation requires conceptual understanding of the idea of functions and that assumes the student understands variables. Understanding variables is dependent upon the student's understanding of number, which is dependent upon understanding of quantification, which requires comprehension of serial correspondence. In other words, there is a structural order with each previous topic serving as a foundation for the next levels of mathematics, eventually leading toward advanced mathematical understanding.

Definition of Terms

Qualitative Quantification with Serial Correspondence

Qualitative quantification with serial correspondence is the beginning of multiplicative reasoning (Piaget, 1970; Clark and Kamii, 1996; Thompson, 1994; Park and Nunes, 2001). Qualitative quantification with and without serial correspondence can be defined more readily when the statement is broken into its component parts. Quantification means "to enumerate the objects" (Piaget, 1965; Thompson, 1994) or more simply, to count and assign a magnitude to them. Qualitative in this case means that the child has identified the objects within the group as an entity to be counted or enumerated. Serial correspondence means that the child can put the objects in order of magnitude from greatest to least or least to greatest. Once a child can identify objects in a set or group and can seriate the objects, the child has attained qualitative quantification with serial correspondence.

Multiplicative Reasoning

Defining exactly what is meant by multiplicative reasoning is difficult, as evidenced by the varying views provided by some of the experts in the field. Multiplicative reasoning requires reconceptualization of the notion of unit (Hiebert & Behr, 1988). The concept of unit with respect to addition is quite different than the concept of unit with respect to multiplication. Addition and multiplication involve hidden assumptions. The hidden assumption in addition is that the unit is one, which children readily understand; whereas, the hidden assumption in multiplication is that the unit is one as well as more than one simultaneously (Chandler & Kamii, 2009). When children fail to realize the hidden assumptions, learning is made difficult. For example, in the addition problem 4+5=?, the child would assume that the unit is one. Most children in grades one through three would have little difficulty solving this problem by assuming the unit is one. For example, the typical child would begin with five and by using his or her fingers, count: six, seven, eight, nine. However, if the children were asked to write a story problem explaining how 4+5 could equal 7, they would find it extremely difficult unless they were able to reconceptualize their notion of unit. The reconceptualization is demonstrated in the following story problem. A man has four buckets of size A and five buckets of size B. He pours all of the water in the four buckets and five buckets into a bathtub. Next he takes buckets of size C and drains the bathtub into exactly 7 buckets of size C. In this case, four buckets of size A plus five buckets of size B is shown to equal seven buckets of size C. The unit has been reconceptualized in this story problem.

"Multiplication is established when the whole is defined in relation to the objects after the split, and division is defined when the whole is not reinitialized after the split" (Confrey, 1994). Division is splitting or splitting is division when the unit does not change. For example, in a story problem, if the mother bakes a pizza and wishes to share her pizza with her two children and husband, she would cut the pizza in two successive perpendicular cuts. Each of the four created pieces would be called 1/4 of the whole pizza because the unit was not reinitialized after the split. Multiplication is the inverse of division. For example, if the mother wishes to feed the entire family reunion, how many pizzas would she need to make if she were going to feed each person 1/4 of a pizza? How many family members can the mother feed if she bakes 10 pizzas? The child should multiply 4 by 10 to obtain this answer. Splitting assumes that each piece creates equal parts or copies of the original.

Park and Nunes (2001) suggest that children's concept of multiplication originates in their schema of correspondences and not in the concept of addition. Under this definition the concept of multiplication is defined by a constant relationship between two quantities. This constant relationship is known as ratio. Ratio or rate is the core meaning of the concept of multiplicative reasoning. The ratio or rate is the constant unit that is called the multiplicand and acted upon by the multiplier. Children employ the schema of correspondence in order to represent fixed relationships between variables and solve multiplication problems. For example, at the pet store there are four cages with three puppies in each cage. How many bones would be needed to give each puppy a bone? The child would need to correspond the number of bones to the number puppies in each cage to arrive at the correct answer of 12 bones.

Multiplicative reasoning is only one part of a student's understanding of mathematical situations, one aspect of quantitative reasoning. The relationship between the parts of the pizza changes depending on whether the pizza is to be shared within the family or at the family reunion. It is the understanding of the relationships between these quantities that is important. Sharing a pizza often means splitting the pizza into identical units.

Thompson (1994) and Simon & Blume (1994) agree that thinking about the situation as a set of equal quantities and the relationships between those quantities represents multiplicative reasoning. For example, when children are asked to compare the numbers four and eight, those using additive strategies will conclude that eight is four more than four. Students using multiplicative strategies will conclude that four is one half of eight. It is the student's perception of the relationship between the two numbers that demonstrates his or her understanding of multiplicative reasoning.

We can conclude from the discussion above that there are many perspectives concerning multiplicative reasoning. The importance of noting the distinction between multiplicative and additive reasoning was first pointed out by Piaget's pioneering work in the 60s and 70s (Piaget, 1965; Tall, 1991). Understanding multiplication as repeated addition is insufficient for constructing multiplicative understanding (Clark and Kamii, 1996; Tall, 1991; Thompson & Saldanha, 2003), including the concept of exponential functions needed for advanced mathematical thinking and success in college mathematics (Confrey, 1994).

Multiplicative reasoning is important because almost all advanced mathematical thinking is dependent upon the understanding one has of multiplicative reasoning. Without a proper foundation for advanced mathematical thinking, grounded within multiplicative reasoning, the student will not succeed at higher mathematics.

The importance of strengthening students' multiplicative thinking has been demonstrated repeatedly (Harel & Sowder, 2005; Lamon, 1994; Dreyfus, 1991; Tall, 1991; Confrey, 1994). Research has focused on the development of student's multiplicative reasoning from elementary to advanced mathematical thinking at the college level. As the child advances from multiplicative reasoning and progresses to advanced mathematical thinking the concept of unit changes. It is these transitional points, when the units change, that causes complexity and signals the onset of multiplication (Hiebert & Behr, 1988). It is not an easy shift, because it represents a change in what counts as a number. The child who can reconceptualize the notion of unit on demand can likely grasp the advanced mathematical topics with greater ease (Dreyfus, 1991; Thompson & Saldanha, 2003).

Four Factors that Inform the Study

There are four major factors that inform this study: development and cognition, mathematical perspectives, instruction approaches, and the learner and multiplication. Each of these factors informs our understanding of multiplicative reasoning and assists us in identifying and encouraging the development of multiplicative reasoning in young children.

Development and Cognition

From the point of view of development and cognition, Piaget (1970) described the emergence of a concept of speed as quantified motion. Children first notice movement in Piaget's Sensory Motor Stage, birth to two years of age. The child develops action schemas when beginning to understand movement. By the time the child has reached kindergarten or first grade the child is typically able to quantify movement and other entities by magnitude. For example, the child can quantify the motion with a magnitude variable called speed. In gaining the ability to quantify motion, the child develops action schemas or schemas of correspondence, which are mental representations allowing the child to understand the quantification (Piaget, 1965).

The fish feeding problem (see *Figure 1.1*) is a good example of schemas of correspondence. A pilot study was conducted with four fourth grade students in which the students used the computer to feed large yellow fish with the appropriate number of small green fish. The questions presented to the students in the pilot study required various levels of multiplicative reasoning. In the given problem we notice three big yellow fish, to be fed a number of smaller green fish. The problem states that Fish B is 2 times larger than Fish A and that Fish C is 3 times larger than Fish B. Then the problem introduces an "if" statement which requires the student to coordinate all of the information via what we label "schemas of correspondence" to produce solutions to the problem. The student should articulate or demonstrate that since Fish A is one-half the size of Fish B then the correct number of fish to feed Fish A is one-half of six, or three. Similarly, the student should articulate that since Fish C is three times larger than Fish B, then the correct number of fish to feed Fish C is three times six, or eighteen.

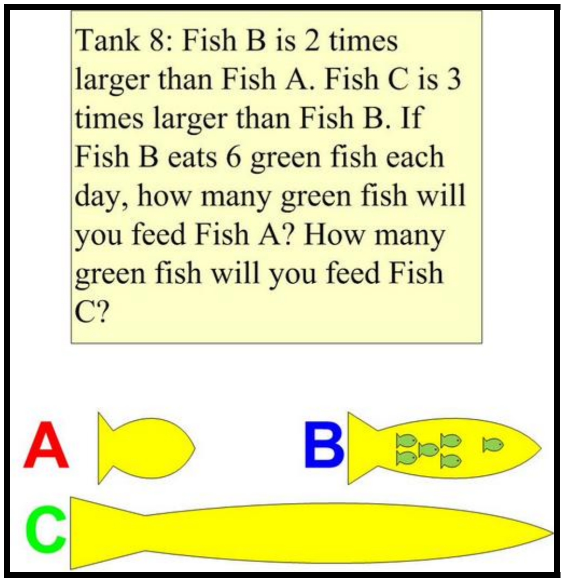

Figure 1.1. Problem Demonstrating Schema of Correspondence.

Students could possibly get the correct answer by utilizing additive strategies in lieu of multiplicative strategies, but additive strategies here will most likely lead to incorrect answers. For example, students who have inadequately constructed schema of correspondence strategies might proceed as follows. Since Fish B eats six fish per day, then Fish A will eat three fish per day because 6-3=3. Now Fish C will eat nine fish per day because 3+3+3=9. As you can see the additive strategies lead to an incorrect answer. Some students may get the solution in this example by employing a build-up strategy. The child employing a build-up strategy might proceed as follows. Fish A will eat three fish per day because 6-3=3. Fish C will eat eighteen fish per day because 6+6+6=18. The solutions are correct, not because the student applied a valid schema of correspondence strategy but because the student employed additive strategies.

Repeated addition can be used to evaluate the result of multiplying, but cannot support conceptualizations of multiplication (Confrey, 1994; Steffe, 1988; Thompson & Saldanha, 2003). Multiplication is not simply repeated addition. The difference between conceptualized multiplication and repeated addition is between envisioning the result of having multiplied, which is a schema of correspondence strategy, and determining that result's value. Imagining the result of having multiplied is to expect and understand multiplication. Determining that result's value can be done through additive strategies.

Mathematical Perspectives

Understanding multiplication, particularly with respect to multi-digit multiplication, which is typically introduced in the fourth grade, we encounter four parts of mathematical knowledge. These four types of mathematical knowledge are identified as intuitive, concrete, computational, and principled knowledge (Lampert, 1986).

Intuitive knowledge.

Intuitive knowledge comes from context (Greer, 1992; Vergnaud, 1988; Lampert, 1986). For example, if a man has a job as a Coca-Cola delivery man, he might soon learn the following by inventing an algorithm. Since there are 24 cans of Coke in a case, and since he can stack 10 cases on one level of a pallet, then if he needs to deliver 528 cans of Coke, he would ask for two levels on the pallet and two cases on top. After a few times, he would no longer need to actually count the 528 cans, but would simply count the two levels on the pallet and then the two cases on top to arrive at the correct number of cans.

Computational knowledge.

Computational knowledge is another category of understanding multiplicative reasoning. With computational knowledge the student can achieve many correct answers involving multi-digit multiplication, but perhaps without conceptual understanding. This is possible because the student has learned to utilize text or visual clues to solve this type of multiplicative reasoning problem. For example, the student may look for the word “times” in a word problem, locate the two numbers mentioned in the problem next, and multiply them together by following the procedure taught in fourth grade.

Another example is beginning with the digits and right hand column one would multiple four times 7 to obtain 28 and write the 8 below the 7 and carry the 2 and add it to the answer for 7 x 5 to obtain 37. The 37 would be written to the left of the 8 yielding an answer of 378.

$$\begin{array}{r} 54 \\ \times\ 7 \\ \hline 378 \end{array}$$

Such computational knowledge relies heavily on visual clues and memory recall of multiplication facts, as well as making many decisions based on the location of a digit. For example, there are two 7s in the problem, but one on the line with the ‘x’ symbol has a place value of ones while the 7s digit between the 3 and the 8 has a place value of tens. The correctness of the answer is often an assessment of how the answer looks (Davis, 1983; Brown & Burton, 1978).

Concrete knowledge.

Multi-digit multiplication can be accomplished through a third type of mathematical knowledge known as concrete knowledge. Concrete knowledge involves knowing how to manipulate objects to find a solution. For example, it can involve grouping data together in sets or sets of sets (Dienes, 1960; Dienes & Golden, 1966). It is important to realize that with respect to multiplication, repeated addition will not suffice. While there exists a conceptual discontinuity between multiplication and addition, there is a procedural connection between these operations. Since multiplication is distributive with respect to addition, repeated addition is often used as a procedure to solve multiplication sums, leading many to believe that multiplication is simply repeated addition (Park & Nunes, 2001; Lampert, 1986). Multiplication involves a new concept, a variable, which counts sets. The multiplier counts sets and the multiplicand counts sets of sets. With this kind of knowledge the student must establish correspondence between sets and set of sets. In other words, the two terms, multiplier and multiplicand, do not represent the same unit. With addition, numbers refer to sets with the unit of one. In multiplication numbers can refer to sets, or sets of sets, or possibly even sets of sets of sets (Dienes & Golden, 1966). It is in the child's understanding of this fundamental change in units that will determine the child's ability to understand the type of problems associated with multiplicative reasoning at this age.

Principled knowledge.

The ability to invent one's own procedures that adhere to the laws of multiplication such that the solutions can be obtained from making sense of the general principles is known as “principled knowledge” (Lampert, 1986). An example of a principled understanding of multi-digit multiplication is as follows. Once again consider the problem:

$$\begin{array}{r} 54 \\ \times\ 7 \\ \hline 378 \end{array}$$

A student may reason: “I know that 7 times 50 is 350. I know that 7 times 4 is 28. Therefore, I know that 7 times 54 is 350 + 28 = 378.” The student has used his own strategy to achieve the solution that could also be arrived at by the standard multi-digit procedural mechanism. Another example: “10 times 54 is 540 and 3 times 54 is 150 +12 = 162. Now we take 162 from 540 to arrive at 378.” The point here is that there are many ways to invent procedures (which demonstrates principled knowledge) to solve this problem other than utilizing the standard multi-digit procedural mechanism. Also, the standard procedure promotes “ones” thinking. Children often say “4 times 7 is 28, carry the 2” instead of “20”. They then say “5 x 7” instead of “50 x 7” and so forth.

Instructional Approaches

Multiplicative reasoning can appear in many forms and with many different indicators. These different forms take on different looks depending on the lens with which we are viewing. Thus far we have looked at multiplicative reasoning through a cognitive lens and through a mathematical lens. This section will allow us to view multiplicative reasoning through an instructional lens.

Not all children can readily learn how to multiply using the algorithmic method for multi-digit multiplication generally taught in fourth grade. Lampert (2001) suggests that multiplication can be taught utilizing a rectangular array as well as other tools to promote conceptual understanding. In fact, there exists an entire list of tools to assist young learners in achieving the goals of the state mandated curriculum.

Finding equivalent groupings and coin problems.

One of the beginnings of multiplicative reasoning is to understand how to group like objects together to form sets of equal quantities. Coin problems provide an excellent tool for accomplishing this objective. For example, asking students to find different combinations of like coins that add up to one dollar will encourage students to group 10 dimes as being equivalent to 4 quarters. Such practice will allow students to form relationships between the coins and the values of the coins. Forming relationships between objects and the associated values of those objects is the beginning of multiplicative reasoning (Chandler & Kamii, 2009; Lampert, 2001; Piaget, 1970; Vergnaud, 1988).

Time-Speed-Distance problems.

Piaget (1970) noted that some of the early indicators of multiplicative reasoning is the movement that children notice, the manner in which they build their action schemas, and how young learners begin to quantify motion as speed. Problems such as "A car is traveling 40 mph. How far will it travel in three and one-half hours?" allow students to form relationships surrounding quantified motion or speed. This development is foundational to multiplicative reasoning because it encourages an understanding of motion as it relates to time.

Finding rectangles with equivalent areas.

It is good practice to find solutions to multiplicative reasoning problems by means other than using the algorithmic method for multi-digit multiplication generally taught in fourth grade. Such strategies encourage students to find many methods that can achieve the correct answer. Students soon learn that to find the area of a rectangle, multiplying the length times the width will save one from having to actually count the squares inside the rectangle to determine the area of the rectangle.

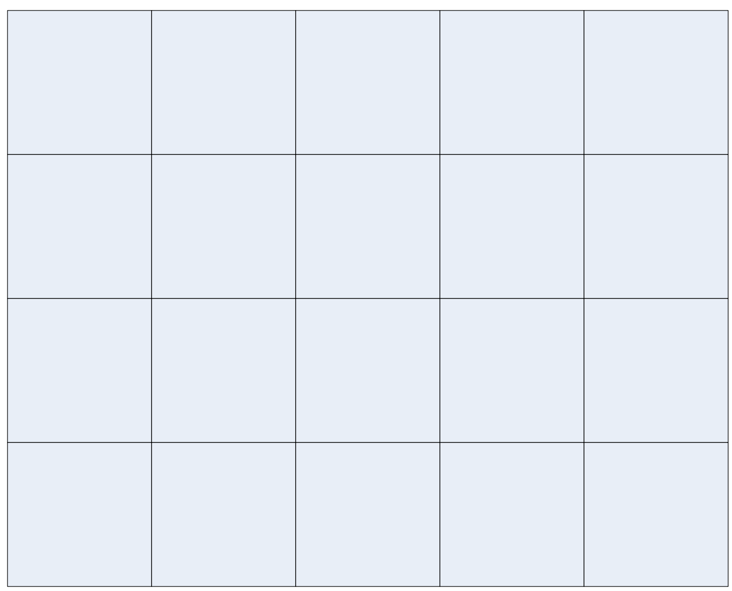

Figure 1.2. Area Model.

Another practice is to allow students to build arrays. A two-dimensional array is a collection of a fixed number of components arranged in rows and columns. Each array element is can be given a unique name or number such as Array(*i*, *j*) where *i* is a variable representing a number describing the row number and *j* is a variable representing a number describing the column number.

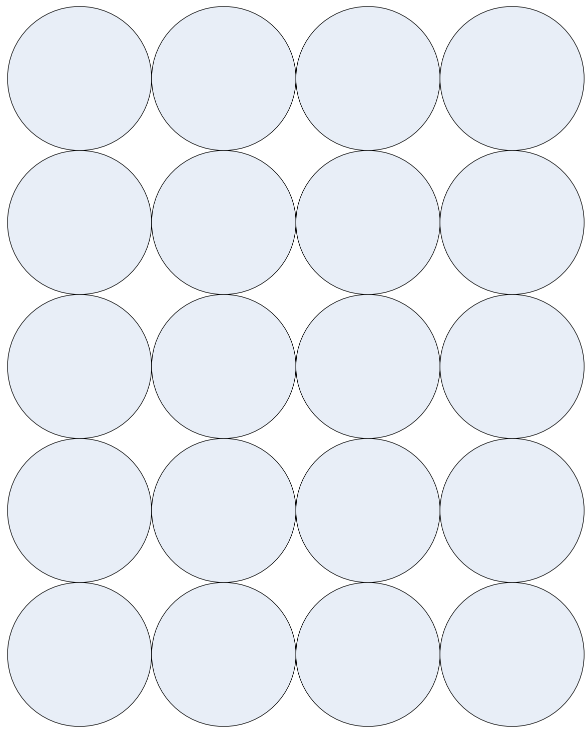

Figure 1.3. Array Model.

Number lines.

Number lines can be especially useful in assisting students in their understanding of multiplicative reasoning. For example, a number line can be readily used to help students understand that "multiplication does not always make bigger," but sometimes actually makes the answer smaller. The idea that multiplication makes bigger and division makes smaller is a misconception that many students acquire in elementary school (Confrey, 1994) and is a misconception teachers should strive to dispel. Using a number line to illustrate the problem of 1/2 times 10 equals 5 can help students understand that 5 is 1/2 of 10 conceptually since it is exactly halfway to 10 on the number line.

Functions and graphing.

The basic function $y = f(x)$ implies that as x varies so does y, but perhaps not in the same magnitude or proportion. By asking the students to keep a "T" table of x and y values, the teacher is encouraging the students to compare and contrast corresponding number values for x and y. Such comparisons are indicators of the beginning of multiplicative reasoning (Piaget, 1970; Clark and Kamii, 1996; Thompson, 1994; Park and Nunes, 2001; Vergnaud, 1988). When table activities are combined with graphing activities, students' understanding of the relationship between the numbers x and y is linked to a visual representation of the relationship, underlying the co-variation (Tall, 1991). Through graphing students understand that a relationship exists between x and y but that it may or may not be linear.

The Learner and Multiplication

Mathematical situations are required for mathematical thinking to emerge; likewise, multiplicative situations are required for multiplicative thinking to emerge. However, it is not in the situations but across them that encourages the development of multiplicative schemes. By constructing learning such that abstraction of mathematical ideas emerge from situations is what makes it possible for students to learn to use mathematics to solve problems in domains that are entirely novel (Lampert, 2001).

A needed event in learning math is play.

The greater the child's collection of actions for attempting to put ideas together, the greater the chance the child has for success (Piaget, 1970). An increase in collection of actions will in turn give more opportunity for the student to make connections to something real. Connecting abstract ideas to real settings is image having and image making (Pirie & Kieren, 1994). In other words, play connects abstract ideas to real world settings. The Kieren-Pirie Model of Mathematical Understanding is a model to enable the listener to "fold back" or "skip ahead" (Davis, 1996). Learners need the freedom to move ahead or fall back and should be allowed time to "play." Piaget has long encouraged play as a mechanism of learning. Davis (1996) notes that there is much to be learned from playing. Davis's meaning of play is variable but generally it means exploring and learning and is not the opposite of seriousness.

The difficulties in learning math.

"Current estimates indicate that approximately 5 to 7% of the school-age population has remarkable difficulty in math achievement, a statistic that presents a challenge for a society that demands at least minimal math competency for success in formal schooling, daily living, and employment" (Geary & Hoard, 2001; Light & DeFries, 1995).

Berch (2005) shows us that the student's differing conceptions of number sense inform the teacher of whether and to what extent a topic may be teachable to the student. We may then gather that it is important for the teacher not only to choose the correct strategy for delivery of math topics to the student, but it is also important that the teacher is aware of the ability of the particular student to understand the topic being taught. It is a combination of student readiness and teacher readiness that makes a successful delivery of a mathematical topic to the student.

Metacognition.

Students in a metacognitive program demonstrated better performance than students who received the algorithm in a the study done by Desoete, Roeyers, and De Clercq, (2003). We may conclude that when both teacher and student are made aware of the processes being taught and the methods being employed (metacognitive program), students are positively affected in their mathematical performance. Awareness of what is occurring has positive effects on student learning.

We may conclude from the work of Kroesbergen, and Van Luit, (2003) that one of the more advanced goals of metacognitive instruction is to *eventually* allow the teacher a less involved role in the instruction of students. At first the teacher must become heavily involved to convey the feelings of success to the students. The teacher's involvement will become cyclical in nature with him/her being the nurturer at the beginning of the student learning curve. The role of the teacher then shifts to facilitator approximately when the two lines intersect as shown in *Figure 1.4*. As the student becomes more self sufficient by understanding the metacognitive process the teacher becomes a guide to the student and asks questions to steer the student in the correct direction.

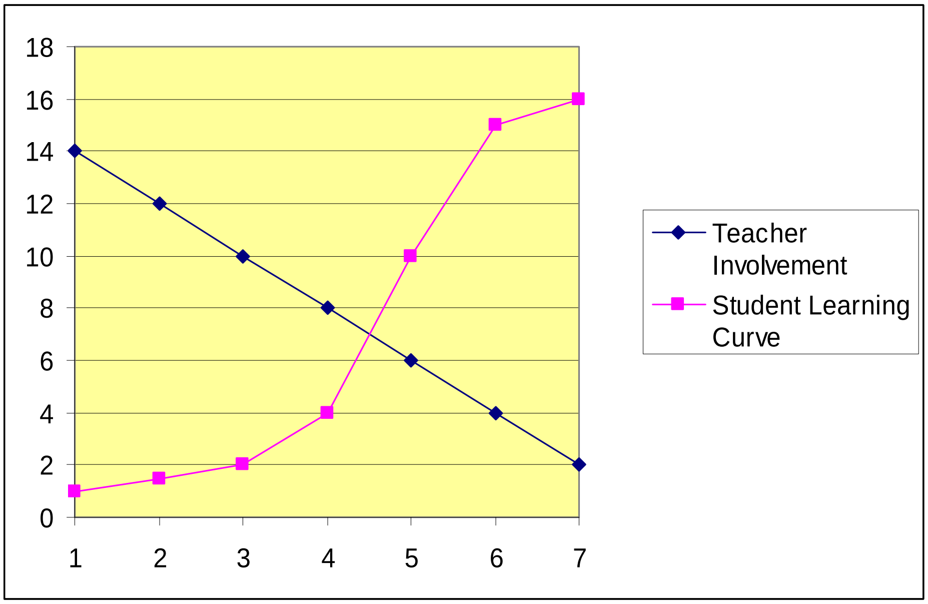

Figure 1.4. Teacher Involvement as Student Learning Increases.

Where are the difficulties within the curriculum?

Within the curriculum there are many pitfalls in which young students can become ensnared. A list of such topics is provided below:

- Quantification
- Seriation
- Regrouping in Addition
- Multiplicative Reasoning
- Proportional Reasoning

Quantification is one of the first topics where students need to have great success. Quantification is the ability to understand that some numbers are larger or smaller than other numbers. For example, the first sign of quantification is typically seen when one sibling says, “She got more pizza than me”. Without quantification, seriation cannot occur. Seriation is the ability to arrange items in order from smallest to largest or vice versa.

After qualitative quantification, additive strategies next appear. Additive strategies are often quite successful for first graders and the teacher often rewards the students who achieve correct answers to problems utilizing additive strategies. This makes some children believe that additive strategies will always work in solving math problems. We find evidence of this when children fall back to additive strategies when problems requiring multiplicative strategies are given. Even though additive strategies are not sufficient, some children insist on using them despite the fact the additive strategies are yielding the wrong solutions.

Counting and adding are certainly very important parts of the learning needed within the mathematics field. But additive strategies are not as useful for higher mathematics as multiplicative strategies are. However, the learning of addition is important and the most difficulties occur when the student is required to regroup. Regrouping in the addition of multi-digit numbers is likely one of the child’s first encounters with the need to reconceptualize the notion of unit.

$$\begin{array}{r} {\scriptstyle 1} \\ 56 \\ +37 \\ \hline 93 \end{array}$$

In the problem above the child must understand that the one that was carried is no longer representing one unit, but is instead representing 10 units, or more abstractly, one unit of ten. The concept of a “one” that represents 10 units is a reconceptualization of the idea of unit. "A change in the nature of the unit is a change in the most basic entity of arithmetic" (Hiebert & Behr, 1988).

Additionally, the language used in regrouping can be a problem. Some teachers say "carry the one" when it should be a ten, or "borrow a one" when it should be a ten. A conceptual breakthrough underlying students' understanding of unit substitutions is their realization that the magnitude of a quantity (its "amount") as determined in relation to a unit does not change even with a substitution of unit (Thompson & Saldanha, 2003).

Children utilize a diversity of superficial cues when attempting to solve multiplicative reasoning word problems or proportional reasoning missing value problems. Sowder (1988) studied students' solutions of simple one-step arithmetic word problems. The study suggests that students associate not only key words or key expressions with arithmetical operations, but that some number combinations guided students toward a certain operation. Van Dooren, De Bock, Evers, and Verschaffel (2009) propose that students' dependence on numbers as superficial cues distracts students with respect to their understanding. Within the study missing-value formatted problems were given to students to determine their clues to their solution strategies. For fourth graders, the missing-value format was an important component in the choice for a proportional strategy. The study suggests that students may benefit in a discussion surrounding the criteria utilized in selecting the appropriate solution strategy. The student should learn to pay closer attention to the real structure that underlies such missing-value format proportion problems.

Summary

According to Piaget, advancing mathematical thinking develops when the formal operations stage appears, typically between the years of eleven and sixteen. The roots of advanced mathematical thinking lie in the makeup of multiplicative reasoning. In order for children to develop advanced mathematical thinking, such as is needed to do well in college math classes, a student must fully understand the derivative and the properties of the derivative. Underlying the properties of the derivative are topics such as rate of change, slope of a tangent line, and the concept of a limit in calculus. Underlying these rate of change ideas are the function, the average rate of change and constant rate of change. Foundational to function and average rate of change and constant rate of change a student would need an understanding of velocity or average velocity. A student's understanding of velocity or average velocity has its foundations in quantification of movement (Piaget, 1970). Piaget further describes the emergence of a concept of speed (quantified motion) as a process where children relate motion to time. According to Piaget, children from birth to 2 years of age start to create mental representations about their universe. These mental representations are called schemas or action schemas. There are many researchers in the multiplicative reasoning field who believe that the beginning of multiplicative reasoning lies in children's schema of correspondence (Park & Nunes, 2001). It is interesting to note this tracing of mathematical development from advanced mathematical thinking to its roots in children's schemas leads not to repeated addition, but instead to children's development of schema of correspondence, or multiplicative reasoning. Multiplicative reasoning depends on the emergence of iterable units, which in turn develop on the basis interiorized, reversible counting (Olive, 2001; Steffe, 1994).

Research Questions

Discussion of the Problem

Advanced mathematical thinking is carried out utilizing mental objects (Harel & Kaput, 1991). The idea that advanced mathematical thinking utilizes conceptual entities formation was proposed as early as the 70s by Piaget. As presented above this need for the ability to understand and utilize conceptual entities begins in a student's schema of correspondence or multiplicative reasoning. Since multiplicative reasoning is foundational to many advanced mathematical thinking topics, multiplicative reasoning is a very important topic with which students and teachers should become familiar. Students, teachers and researchers should begin to identify multiplicative reasoning at the typical onset of using multiplication in school curricula. The questions below should assist teachers and researchers in focusing on multiplicative reasoning.

What are the signs of multiplicative reasoning? Can I tell if students are using multiplicative reasoning from their work, from their words and from their actions? Do they count on their fingers? What symbols do they utilize when they are exhibiting multiplicative reasoning? In depth study has not yet fully identified the need teachers have for understanding, identifying, and utilizing multiplicative reasoning in the classroom. How does in depth study of multiplicative reasoning inform elementary educators?

Research Questions

What are the indicators of multiplicative reasoning among fourth grade students? What strategies do fourth grade students utilize in solving multiplicative reasoning word problems? What multiplicative reasoning strategies do the items on the test instrument invoke in fourth grade students?

CHAPTER II

REVIEW OF LITERATURE

Background

Previous studies have suggested that elementary school teachers within the United States possess a limited knowledge of mathematics they teach (Ball, 1990; Ma, 1999; Post, Harel, Behr & Lesh, 1991; Simon, 1993, Simon & Blume, 1994). Poor understanding of mathematics may in part come from a failure to conceptually understand multiplicative reasoning in elementary school (Ball, 2003; Clark & Kamii, 1996; Harel & Sowder, 2005; NCTM, 2000; Piaget, 1965; Richardson, Berenson, & Staley, 2009; Tall, 1991). The literature supports the existence of four avenues of study surrounding multiplicative reasoning: development and cognition, mathematical perspectives, instructional approaches and the learner and multiplication (Clark & Kammi, 1996; Nunes & Bryant, 1996; Piaget, 1965; Thompson & Saldanha, 2003; Vergnaud, 1983, 1988). For these reasons, I propose to investigate the indicators of multiplicative reasoning among fourth grade students using these four avenues.

Cognitive Development of Multiplicative Reasoning

Overview of the Cognitive Development of Multiplicative Reasoning

The cognitive development of multiplicative reasoning, according to Clark and Kamii (1996), proceeds in the following fashion. First, children learn to quantify. Clark and Kamii (1996) call it qualitative quantification. Qualitative quantification can occur with or without serial correspondence. Qualitative quantification means that children identify items as larger or smaller based on a given unit. Example, “Johnny got more of this pizza than me.” Serial correspondence means that children can put multiple items in order from largest to smallest based on the same unit. Example, “Johnny got more of this pizza than Sally who got more of this pizza than Jane.” Qualitative quantification with serial correspondence generally appears first, often when children are in kindergarten or before (Piaget, 1965).

Repeated Addition versus Schema of Correspondence

There are two different hypotheses suggested to explain the origin of multiplicative reasoning from the point of view of development and cognition. As a result, the use of varying models divides the research directions and caused a thinning rather than a concentration of research. The first hypothesis is that multiplicative reasoning is based upon repeated addition (Fishbein, Deri, Nello, & Marino, 1985; Steffe, 1994). The second hypothesis is that repeated addition is merely a procedure that can assist the student in achieving the correct answer, but understanding multiplicative reasoning lies in correctly perceiving the inherently multiplicative relationships between the objects being enumerated and the numbers representing those objects (Clark & Kammi, 1996; Nunes & Bryant, 1996; Piaget, 1965; Vergnaud, 1983, 1988).

Vergnaud (1988) envisioned multiplication as interconnected concepts and reached conclusions that are in agreement with those of Piaget and Steffe. Vergnaud concluded that multiplication does rely partly on addition but possesses an organizational structure that cannot be broken down entirely into additive structures.

Consider the problem, “How many cans of Coke will you have if you are given 4 six-packs of Coke?” When presented with the problem 4 x 6 = ? the student employing repeated addition would likely demonstrate that 6 + 6 + 6 + 6 = 24 and arrive at the answer via counting. From a developmental perspective, counting to a solution is often a sign of additive strategies, not multiplicative strategies. When employing a multiplicative strategy the student would “correspond” the six-packs of Coke as units of 1 as well as units of 6 and would mentally or concretely note that there are 4 units of 6. The student may understand that the 4 is counting the units of 6. When the student is able to think simultaneously about units of one and units of more than one, the onset of multiplicative reasoning has occurred (Clark & Kamii, 1996).

Steffe (1994) contributed to the advancement of multiplicative reasoning research when he proposed that although the beginning of children’s understanding of multiplication is in the construction of a unit that is repeatedly added, it is what goes on in the mind of the student before the repeated addition of the unit that is crucial for conceptual understanding of multiplicative reasoning. How the student understands the concept of unit and how the student relates the quantities within a problem is labeled a schema. Exactly how children develop their multiplicative reasoning (or schema) should be an area upon which teachers and researchers focus, since children construct their multiplicative structures from their activities and persist in using their own ideas, despite the instruction of teachers (Olive, 2001; Steffe, 1994). It is important to understand the student’s schema of numbers and multiplication as the depth of such understanding can lead to significant new ideas with respect to teaching mathematics to children (Confrey, 1991; Kieren, 1990; Kieren & Pirie, 1991; Piaget, 1973; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983).

Schemes are mental constructs and can consist of up to three parts: (1) recognition, (2) action and (3) results (von Glasersfeld, 1980). Children often engage in physical actions before they are able to mentally perform the same action. For example, a child may begin to count by counting on his fingers, but later he counts by forming an image of counting on his fingers (Wheatly, 1998). It is when the child begins to use the image of counting on his fingers for all counting needs that the child's actions have become interiorized.

Once interiorized, the child can then advance his cognition by reinteriorization. Reinteriorization occurs when the child uses interiorization in an iterative approach to count sets and sets of sets (Dienes & Golden, 1966; Olive, 2001; Steffe, 1994). When such action occurs the child has formed iterable and composite units. Composite units can be thought of as units of one and units of more than one simultaneously (Clark & Kamii, 1996; Olive, 2001; Steffe, 1994). For example, the number seven can be viewed as seven composite units of one or one composite unit of seven. Once the child has conceived of units of one and units of more than one simultaneously, he has begun to understand conceptually the meaning of multiplicative reasoning.

Mathematical Perspectives of Multiplicative Reasoning

Overview of Mathematical Perspectives

From the above discussion it is apparent that multiplicative reasoning is important within the literature from a cognitive prospective. However, multiplicative reasoning is also very important within the literature from a mathematical perspective. Confrey (1994) was one of the first to notice that certain students made a strong connection between the number line as a tool for multiplication and division as well as addition and subtraction. Confrey further illuminated the fact that some students possessed a more conceptual understanding of function in noticing that a function involves covariation as well as correspondence. Correspondence is heavily emphasized within cognitive development literature, but not much is discussed with respect to covariation. Understanding the function as the covariation of two (or more) quantities, together with connections between multiplication and the number line, laid the groundwork for the historical development of logarithms (Confrey, 1994). The historical development of logarithms leads children down a path to advanced mathematical thinking.

The Path to Covariation

Originally, perhaps as far back as the early Greeks, number was based on counting. In those days counting was considered sufficient for indicating the size or the magnitude (Smith & Confrey, 1994). Counting and measuring create a successor action known as addition. This concept of number is distinct from the concept of ratio. The early Greek philosopher Euclid described ratio as a comparison with respect to the size between two magnitudes (Behr, Lesh, Post, & Silver, 1983; Heath, 1956). The chief difference between these two concepts is that number is followed by a successor action of addition, whereas ratio is followed by a successor action of multiplication.

Rizzuti (1991) distinguishes between the traditional definition of function that stresses correspondence between two (or more) variables (Dirichlet-Bourbaki definition which states any correspondence between two sets that assigns to every element in the first set exactly one element in the second set) and the covariation that exists between two or more variables. The main idea of the difference between the two definitions of function is that the traditional definition makes the function more additive or discrete, that is, not continuous. The idea that a function describes the covariation that exists between two or more variables makes the definition of function more multiplicative or continuous. Understanding functions as the correspondence between two (or more) variables requires students to learn a definition that is separated from the functional thinking they do outside of the mathematics class (Rizzuti, 1991; Smith & Confrey, 1994). Conceptualizing functions as covariation models of the relationship between two variables is foundational to student concept attainment surrounding logarithmic functions. For example, given the function $y = 2x$ the students who built two columns of a "T" table learn much from playing with the covarying actions discovered while populating a table such as shown in Table 2.1.

Table 2.1: "T" Table for $y = 2x$

x	y
-3	-6
-2	-4
-1	-2
0	0
1	2
2	4
3	6
4	8
5	10

Smith and Confrey (1994) suggest that such "play" with "T" tables or covariation tables offers the potential of allowing students to construct their personal understanding of covariation and allows them to understand the covariation in the expressions they and others create. Kieren (1994) suggests that Smith's and Confrey's subjects, by utilizing Oresme's concept of ratio, are able to focus on actions and even actions on actions.

For example, we learn as mathematics students that $a^x a^y = a^{x+y}$ and $(a^x)^y = a^{xy}$. We learn other similar rules. We then apply such rules to generate an answer via a set of algorithmic rules. Unfortunately, the repeated application of such rules may lead many down a path of difficulty where we utilize the short-cut of repeated multiplication. This short-cut of repeated multiplication does not prepare us for the conceptual understanding needed to comprehend the next topics in mathematics, such as exponential and logarithmic functions. Confrey (1994) suggests that rather than depend on the actions of repeated multiplication, we should create independent multiplicative structures she calls splitting structures. She argues that because under the splitting model the function is understood with respect to covariation, the understanding of logarithmic functions will appear more naturally.

Smith (1994) argues that this propagation continues up the path to advanced mathematical thinking. He contends that Confrey's work indicates the need for further research involving rate of change for example, the development of conceptual understanding of the mathematical constant *e*.

The Learner and Multiplication

Advanced mathematical thinking is viewed as beginning in elementary school with multiplicative reasoning (Clark & Kamii, 1996; Harel & Sowder, 2005; Tall, 1991; Piaget, 1965). The National Council of Teachers of Mathematics suggests that multiplicative reasoning needs addressing in elementary school (National Council of Teachers of Mathematics [NCTM], 2000). Richardson, Berenson, and Staley (2009) in addition to Ball (2003) acknowledge the requirement for additional research concerning pre-service as well as in-service teachers' understanding of multiplicative reasoning.

Traps within the Curriculum

The importance of guiding young mathematics students onto the path of multiplicative reasoning is easily overlooked if the key distinction between repeated addition and multiplicative reasoning is not addressed in elementary school (Behr, Harel, Post, & Lesh, 1994). Students need to develop connections between selected fundamental mathematical concepts such as multiplication and division or multiplication and ratio. One trap that students and teachers alike fall into exists because multiplicative and additive structures have a common conceptual basis, that of units of quantity. Students and teachers often fail to understand the significance of separating the two ideas conceptually. The teacher should guide the student into understand that 3 x 6 = 18 because 3 is the multiplicand (that counts sets of sets – 3 sets of 6) and the 6 is the multiplier (that counts sets of one) (Dienes & Golden, 1966). Magdalene Lampert (1990) says it best, “When the Problem Is Not the Question and the Solution Is Not the Answer”, it becomes important that the teacher be more concerned with what the student is thinking than whether the student got the “correct” answer.

Hidden Assumptions within the Curriculum

Within the traditional mathematics curriculum it is unfortunate that the first time a child encounters a problem situation in which reunitization is required is when addition of unlike fractions is considered (Behr, Harel, Post, & Lesh, 1994). Consider the problem 3/4 + 7/12 + 2/3 = ?. The child will need to obtain a new common unit, often called a new common denominator. The problem should proceed similar to 9/12 + 7/12 + 8/12 = 24/12 = 2/1. Notice how the common unit changes from 12 to 1 in the problem. Most students do not know that the most common name for the number “two” is simply 2 and are confused by the answer 2/1 and have considerable difficulty in expressing 2/1 as simply 2. The hidden assumption here is that all quantities are represented in terms of units of 1 and this assumption has a negative impact on the elementary and middle school curriculum (Behr, Harel, Post, & Lesh, 1994).

Misconceptions within the Multiplication Curriculum

There are many misconceptions with the mathematics curriculum. For example, as discussed in chapter 1 a common misconception is that multiplication makes bigger and division makes smaller (MMBDMS) (Confrey, 1994; Greer, 1994). The concept that is often taught is the students should check their answers to multiplication problems by making sure their answers are bigger than either of the two numbers they are multiplying. Similarly, the students are often taught that they can check their answer to a division problem by making sure their answer is smaller than either of the numbers they are dividing. This of course only holds true until they encounter a problem in later grades where they must multiply or divide by a number less than 1. Such a problem uncovers the misconception that MMBDMS.

Another major misconception within the curriculum is the idea that if we see the word “times” in a word problem we should always multiply. This is sometimes referred to as a keyword strategy. In applying such a strategy in solving a word problem the student would look for the keyword times and then multiply the two numbers given in the problem to obtain the answer. For example consider the sample problem in Table 2.2: Fish B is 3 times larger than Fish A. If Fish B eats 3 green fish, how many green fish should you feed Fish A each day? The student employing the “Times” keyword strategy sees “times” following the word 3 in the given problem as well as the 3 following the word “eats”.

Table 2.2: The "Times" Keyword Strategy

Fish A	Fish B
?	3

The student then multiplies 3 times 3 to obtain 9, which is of course an incorrect answer. Keyword strategies can be successful, but only if the student is reading for understanding and can decide when and when not to utilize such strategies. The wording of problems and the kinds of quantities used independently of the numbers is what determines the operation that must be used (Bell, 1983; Fishbein, Deri, Nello, & Marino, 1985; Graeber & Tirosh, 1988; Greer, 1994; Luke, 1988; Nesher, 1988; Peled & Nesher, 1988; Resnick, Nesher, Leonard, Magone, Omanson & Peled, 1989). Confrey (1994) carries the idea one step further. She suggests that the assumptions made by these researchers have all focused on the use of numbers as strings of integers to be entered into an "assistance to memory" device such as a calculator or computer. Such a perception contributes to the likelihood of mis-identifying students' actions as inadequate. Instead, she advises researchers to look for simultaneous "operational construction" that must exist between the decimal, ratio and fractional number and the construction of operations that may provide a more useful understanding of the interactions (Confrey, 1994). In other words, we are looking for how the numbers and the operations work together to solve the problem and how the students are conceptualizing those relationships.

Putting It All Together

After reviewing the literature, the following pattern emerges. Each of the four avenues of study surrounding multiplicative reasoning today suggests the need to identify levels of multiplicative reasoning as well as the signs of the existence of multiplicative reasoning in young children (Ball 2003; Behr, Harel, Post, & Lesh, 1994; Clark & Kammi, 1996; Confrey, 1994; Gonzales, Williams, Jocelyn, Roey, Kastberg, & Brenwald, 2008; Greer, 1994; Harel, Behr, Post, & Lesh, 1994; Harel & Sowder, 2005; Kaput & West, 1994; Kieren, 1994; Lamon, 1994; Nunes & Bryant, 1996; Piaget, 1965; Rizzuti, 1991; Schoenfeld, 2004; Smith & Confrey, 1994; Steffe, 1994; Tall, 1991; Thompson, 1994; Thompson & Saldanha, 2003; Vergnaud, 1983, 1988, 1994). What is needed at this point is a framework within which to study levels of multiplicative reasoning and the reasoning of young children.

Theoretical Framework

Young children use an array of superficial clues when attempting to solve multiplicative reasoning word problems. Sowder (1988) studied students' solutions of simple one-step arithmetic word problems. His work suggested that students mindlessly associate key words or key expressions (such as "times") with particular arithmetical operations. Also, his data suggested that certain number combinations pointed students toward a particular operation. In addition, Van Dooren, De Bock, Evers, and Verschaffel (2009) commented that students' reliance on numbers as superficial cues interacts with students' instructional experience.

After reviewing the literature and the definitions of multiplicative reasoning the following themes emerged. Beginning with the work of Clark and Kamii (1996) I identified 5 levels of reasoning in a pilot study conducted in North Carolina over a one week period. Mathematical situational problems were presented to students that required utilization of multiplicative reasoning in arriving at the solution. These situational problems are located in the appendix. The five developmental levels described below emerged from a review of the literature, and were tested in the pilot study.

Consider the following problem: “Fish B is 2 times larger than Fish A. Fish C is 3 times larger than Fish B. If Fish B eats 6 green fish each day, how many green fish will you feed Fish A? How many green fish will you feed Fish C?”

Level 1

Children who are at the lowest level either present no strategy by simply guessing or accept any answer as long as $A < B < C$. “The child who is able to seriate in this task thinks only qualitatively in terms of “more” or “less” and accepts almost any number as long as $A < B < C$” (Clark & Kamii, 1996). Students are exhibiting spontaneous strategy at this level, which means that the subject has selected an answer at random and is not applying any logical thought to arriving at the solution.

Level 2

Students attempt the additive strategy, perhaps because the additive strategy has been successful in the past. Answers such as A=4 because 6-2=4 and C=9 because 6+3=9 are the types of answers received from students utilizing the additive strategy. The students pay attention to the number corresponding to the word "times" in the question, but add or subtract instead of multiplying or dividing. By adding, they demonstrate that they do not understand that an additive strategy is *not* sufficient in solving the problem, perhaps because they have been instructed by their teachers that multiplication is only repeated addition – "…Understanding multiplication as repeated addition keeps it divorced conceptually from measurements, proportionality, and fractions" (Thompson & Saldanha, 2003).

Level 3

Students at level 3 attempt a multiplicative strategy, perhaps because they understand that additive strategies are not sufficient in solving the problem given. However, their attempts at a solution fail because they cannot demonstrate that they understand the inverse of multiplication is division. Answers such as Fish A is12 because 6x2=12 and Fish C is18 because 6x3=18 are the types of answers received from students utilizing an algorithmic strategy, but not necessarily understanding the meaning behind the operations. They pay attention to the number corresponding to the word "times" in the question, but multiply in both situations, when in reality they should divide to calculate the answer for fish A. They do not understand that division is the inverse operation for multiplication or are unable to demonstrate their understanding. Although the answer is correct, the reasoning behind it is incorrect; therefore, this level is considered multiplicative strategy without success.

Level 4

Students at level 4 solve the problem utilizing a multiplicative strategy, but they may not fully understand all the relationships between the numbers. Students who get the correct answer may do so because they saw the word "times" in the given problem and multiplied 6x3=18 to obtain the correct answer for Fish C. The sample question below is designed to filter out those students by also requiring them to perform the inverse operation of multiplication (division) to obtain the answer for Fish A. Since the problem does not explicitly mention the word "division", the student who obtains the correct answer for A and explains that A=3 because half of six equals three satisfactorily demonstrates an understanding of the relationship between the numbers. Students at this level have successfully overcome the MMBDMS (Multiplication Makes Bigger Division Makes Smaller) misconception prevalent in schools today. "In the case of multiplication and division, the lapsing of the MMBDMS rule entails a major conceptual reconstruction…" (Greer, 1994). However, the students scoring at this level do not adequately articulate all the ratios between the fish.

Greer (1992) suggests that some multiplicative reasoning problems are solved by intuitive strategies and internal mental structures (Mulligan & Mitchelmore, 1997) that Greer labels as extensions to positive rational numbers. Extensions occur when the previous definition of the concept does not succeed in solving the problem at hand. Greer's example is the extension that occurs when the student is asked the following question about pizza: "How can fourteen pizzas be divided equally among 3 children?" Under a child's initial beliefs about multiplicative reasoning there is no solution, because each child can be given 4 pizzas with 2 left over. However, by shifting their perspective or, in other words, by employing an extension of multiplicative reasoning that allows the pizza to become something that can be cut into fractions, a solution becomes possible. Such extensions can assist a student in achieving level 5.

Level 5

Unitizing or norming the units within the problem (as discussed above) is the distinguishing factor for level 5. Multiplicative reasoning is evident when a new quantity has emerged as a unitizing factor (Steffe, 1994; Clark & Kamii, 1996; Confrey, 1994; Hiebert and Behr, 1988). Students at level 5 solve the problem utilizing a multiplicative or proportional strategy and they are able to articulate the ratios and proportions needed to correctly solve the problem. Answers such as A=3 because 1/2 of 6 is 3, and C=18 because B is 1/3 of C, indicate the student has introduced a unitizing factor for each ratio needed for a solution within the given problem.

The multiplicative reasoning levels, described by Thorton & Fuller, 1981; Karplus & Lawson, 1974; Clark & Kamii, 1996, are summarized in Table 2.3.

Table 2.3: Multiplicative Reasoning Levels

1. Spontaneous Strategy	Not yet additive/guessing
2. Additive Strategy	Derives answer utilizing addition or subtraction
3. Multiplicative Strategy (w/o success)	Cannot make transition from additive to multiplicative thinking, but understands additive is not sufficient
4. Multiplicative Strategy (w/ success)	Uses multiplicative reasoning successfully with time to reflect in describing the relationship between the numbers
5. Proportional Strategy	Introduces a new quantity as a unitizing factor then successfully completes problem through multiplication or division

(Adapted from Thorton & Fuller, 1981; Karplus & Lawson, 1974; Clark & Kamii, 1996)

Summary

My aim is to better understand the differences between children who can think multiplicatively and those who can only think at additive levels. The five developmental levels can be regularly observed in the progression of a child's learning trajectory as he transitions from Piaget's concrete operational stage to the formal operations stage. Understanding these differences can assist educators in making informed decisions in developing multiplicative curricula. Multiplicative reasoning and the complexities of multiplication as presented by Piaget (1977) may need more in depth description. In other words, it may be useful from an educational perspective to break down the additive strategies and multiplicative strategies into component parts and sub-levels called strategies. Since multiplicative thinking develops out of the levels and sub-levels (strategies) of additive thinking, it may prove helpful to better understand the thought processes of subjects as they transition from additive to multiplicative levels.

CHAPTER III

METHODOLOGY

Pilot Study

In 2008 a pilot study was conducted at the University of North Carolina at Greensboro in hopes of corroborating the work of Clark and Kamii with respect to their studies supporting the five levels describing multiplicative reasoning. Piaget reported that multiplicative reasoning cannot occur until the child has reached the fourth stage of development known as formal operations, which is predominately in the fifth grade, or ages 11-12. The pilot study tested the hypothesis that children in the fourth grade are exhibiting the 5 levels of multiplicative reasoning development, categorized by Clark and Kamii. The testing of this hypothesis was accomplished by the development of mathematical situational word problems, located in the appendix. These word problems were derived and modified from the work of Clark and Kamii (1996), Lamon (2006), and Karplus, Pulos, & Stage (1983). These multiplicative reasoning word problems were presented to four students, ages 9 and 10. The word problems required utilization of multiplicative reasoning to arrive at a solution. The children's thinking was analyzed to determine their fit within the five categories proposed by Clark and Kamii (1996).

Pilot Study Focus

What are the signs of multiplicative reasoning? Do children draw pictures when exhibiting multiplicative reasoning? Do they count on their fingers? What symbols do they utilize when they are exhibiting multiplicative reasoning? The focus of this pilot study was to identify the indicators that fourth grade students utilize when exhibiting multiplicative reasoning.

The goal of this pilot study was to provide the researcher with data concerning what changes needed to be made to the materials, as well as explore the extent to which such materials can be utilized to gain understanding of fourth grade students' multiplicative reasoning.

Subjects

In cooperation with the administration of two local schools, four promising math students in grade four were selected on a volunteer basis. The subjects were nine and ten years old. Subject one was an eleven year old white male. Subjects two and three were eleven year old male and female twins. Subject four was a ten year old white female.

Results

The results of the pilot study showed that students in the fourth grade are in transition with respect to multiplicative reasoning, which means that students who have obtained multiplicative reasoning (Level 3 and beyond) will at times demonstrate it correctly but at other times will fail to correctly demonstrate their understanding of multiplicative reasoning. Therefore, some questions will be worked correctly and then later a similar question will be missed. Additionally, some students have not yet comprehended multiplicative reasoning while a few understand multiplicative reasoning quite well.

Student one indicated he understood Question 4 in depth since he wrote the following mathematical sentence in his written work, 3 x 1 = 3 and 3 / 3 = 1, demonstrating his knowledge that the inverse of multiplication is division. Also, he was not led into multiplying when he should have been adding, as some students would be who employed a procedure of finding the keyword "times" and then multiplying the numbers in the problem. Student one did well with key question number eight by stating and writing the math equation 6 / 2 = 3 and 6 x 3 = 18, indicating he understood when to multiply and when to divide. With respect to Question 9 student one stated and wrote 3/4 is more than 3/8, demonstrating understanding of the relationships between the numbers and objects. On Question 10, student one failed to create a unitizing factor. His equation did not correctly explain the relationship between the numbers and objects. His equation for Question 10 was "2 paper clips = 1.5 buttons." His prediction of the answer was ten paper clips. Student one scored at level four from both his written work and his verbal and on screen actions.

Student two was a twin to student three. Student two explained very clearly that 3 / 3 = 1 concerning Question 4 and that 9 / 3 = 3 for Question 5, clearly demonstrating attainment of at least level three. With respect to Question 8, student three wrote and explained that 6 / 2 = 3 and 3 x 6 = 18, indicating a clear understanding that division is the inverse of multiplication and that he was not following a procedure of finding the keyword "times" and then multiplying the numbers in the problem. Such answers indicate he was at least at level four. With respect to Question 9 the student wrote that "Pizza B has 3 / 4 of the pizza left and Pizza A only has 3 / 8 left" showing a clear understanding that 3 / 4 > 3 / 8. With respect to Question 10, the student was able to estimate the correct answer but was not able to demonstrate a unitizing number in solving the problem. Student two scored at level four from both his written work and his verbal and on screen actions.

Student three was a twin to student three. Student three wrote 3 / 3 = 1 for Question 4 indicating that she was not following a procedure of finding the keyword "times" and then multiplying the numbers in the problem. Such answers indicated she was at least at level three. With respect to Question 8 she wrote 6 / 2 = 3 and 3 x 6 = 18, indicating a clear understanding that division is the inverse of multiplication. With respect to Question 9 the student wrote that 3 / 8 is less than 3 / 4, indicating understanding of simple ratios. On Question 10, she clearly expressed verbally and on paper that 1½ paper clips = 1 button. She then stated and wrote "If 4 buttons equals 6 then 6 buttons equals 9". This statement indicated that student four understood proportions and the relationship between the numbers and the objects given in problem ten. Student three was able to introduce a unitizing factor and then solve the problem utilizing either multiplication or division, placing her clearly at level five or beyond.

With respect to the first four questions, it was clear that student four preferred additive strategies to solve these multiplicative reasoning problems. She was able to obtain the correct answer for the first three questions by employing a build up strategy where repeated addition was employed to achieve the answer. For Question 4 she employed an additive strategy (she actually subtracted 3 – 2 = 1) to obtain the correct answer, but could not explain how she knew the answer was one. Student four continued to employ additive strategies on problems five through eight even when the additive strategies forced her to obtain incorrect answers. She did not give any indication that she had progressed into any form of multiplicative reasoning strategies. This type of behavior is indicative of a student clearly at level two. With respect to Questions 9 and 10, the student was not able to demonstrate understanding of the multiplicative reasoning strategies needed to solve these more difficult problems. The student was able to measure the height of Mr. Tall to obtain the correct answer for Question 10, but could not articulate or demonstrate any multiplicative reasoning strategy to solve this problem.

Implications of Pilot Study

This study revealed that although some children may know *how* to multiply, they often do not know *when* to multiply especially when attempting to solve word problems. One of the student strategies which emerged was that students looked for key words such as “times” and then made the decision to utilize a multiplicative strategy based upon the keyword “times.” Such students are only applying an algorithm and are not demonstrating multiplicative reasoning, despite their use of multiplicative strategies and despite the fact that they obtained the correct answer. One clue that the student had employed an algorithm was made apparent in his explanation. “My teacher tells me to look for the word ‘times’ in the given word problem and if I see one then I should multiply to solve the problem.” Also, some children did not demonstrate that they understood when to divide or that division is the opposite operation of multiplication. For example, when presented with the following problem, “Tank 8: Fish B is 2 times larger than Fish A. Fish C is 3 times larger than Fish B. If Fish B eats 6 green fish each day, how many green fish will you feed Fish A? How many green fish will you feed Fish C?" the student could calculate that the answer for Fish C is 18 by multiplying 3 x 6 to obtain 18. On the other hand, the student could not explain a method of division to obtain an answer for Fish A such as 6 / 2 is 3 fish for Fish A.

Students did not typically understand when they employed the incorrect strategies. For example, a student utilizing additive strategies may not have understood multiplicative reasoning strategies even if presented. Should researchers pursue teaching strategies which encourage the development of multiplicative reasoning strategies? Should teachers become familiar with the indicators that multiplicative reasoning is emerging and know appropriate strategies to employ to further encourage the development of multiplicative reasoning within the minds of the students?

Perhaps teachers should utilize more multiplicative reasoning problems similar to Question 8 to identify those students who do or do not understand the relationships between the numbers and objects. Once identified as needing instruction in multiplicative reasoning, the curriculum planners could develop lesson plans explicitly designed to encourage development in multiplicative reasoning. Teachers could do likewise with questions similar to Question 8 which discriminates between students who comprehend multiplicative reasoning and those who are struggling to understand the concept. Problems similar to Question 4 which discriminates between students who regularly use additive strategies and those who understand the identity element of multiplicative reasoning could be employed to categorize students into levels of readiness for math lessons concerning specific conceptual understanding of multiplicative reasoning.

Methodology

Studies suggest that advanced mathematical thinking is likely dependent on multiplicative reasoning (Confrey, 1994; Dreyfus, 1991; Harel & Sowder, 2005; Lamon, 1994; Tall, 1991), and that poor understanding of mathematics may come from a failure to develop multiplicative reasoning in elementary school (Ball, 2003; Clark & Kamii, 1996; Harel & Sowder, 2005; NCTM, 2000; Piaget, 1965; Tall, 1991). Analyzing the multiplicative strategies utilized by fourth grade students may shed light on why some children succeed at comprehending multiplicative tasks, yet others have difficulty.

Subjects

In cooperation with the administration of a large urban school district in North Carolina, the students were recruited from the fourth grade at a Chapter I school. Approximately, thirty participants were supplied with permission slips in hopes of having ten to fifteen participants. The participants were asked to have their parents or legal guardian sign the "Consent to Act as a Human Participant: Long Form" and the participants signed the corresponding "Children's Assent Form." If the participant became frustrated at any point the interviewer moved to an easier level or terminated the interview. Also, the participants did not receive any incentive for participating. The subjects were age nine or ten.

Instruments

The participants were asked to engage in a dialog while being interviewed. Because the computer can provide an environment that can enhance children's own construction of multiplicative reasoning via interaction with the teacher (Olive, 2000) the computer was employed as an instrument for these interviews with the subjects. As the participant worked on the computer answering questions, he or she was videotaped from behind such that his or her face was not captured on the tape. Ten questions (see appendix) were developed utilizing Microsoft Visio. The questions were coded such that each question distinguished different levels of multiplicative reasoning as well as encouraged the participant to use words indicative of multiplicative reasoning. Below is an outline of the questions, and a discussion of how they measure multiplicative reasoning and encourage the use of multiplicative reasoning words. Words that may be indicators of multiplicative reasoning are: as (twice as big), area, split, half, one-half, one-third, divide, times, cut, more, less, double, larger, smaller, equal, sets, sets of sets, or their synonyms (Clark & Kamii, 1996; Confrey, 1994; Dienes & Golden, 1966; Inhelder & Piaget, 1958; Karplus, Pulos, & Stage, 1983; Steffe, 1994).

Questions one through eight are about fish aquariums or fish feeding tanks. Question one was designed to encourage and warm up the participant, and to ease any fears concerning abilities to answer the questions. The answer for question one could be obtained either by utilizing multiplicative reasoning strategies or by utilizing additive strategies. Lack of success on this question indicated that a subject was not ready to continue this test instrument.

Question two afforded the opportunity to express understanding of the term “half”. Although the question could be solved with either additive or multiplicative reasoning strategies, it was the distinctions between these strategies that became apparent in analyzing the subjects’ thinking. A statement similar to “I knew the answer for fish B is 6 because half of 6 is equal to 3” indicated the use of multiplicative strategies because the unit of growth is the ratio 1/2 (Confrey, 1994). Such statements were especially important because the word “times” did not appear in the problem and thereby eliminated the strategy some children employed where the keyword “times” was located and then regardless of the meaning of the words within the problem, multiplication became the selected strategy.

A solution for question three could employ a simple strategy of multiplying two times four to obtain the answer for fish B. However, since the question did contain the word “times” the student may have utilized a keyword strategy discussed in question two above that would not demonstrate any multiplicative reasoning strategy. The researcher looked for statements similar to “I knew the answer for fish B is eight because I would need four equal sets of two green fish to feed fish B” which would indicate the use of multiplicative strategies (Dienes & Golden, 1966; Steffe, 1994).

Although question four mentions the word “times” the correct solution was obtained by division. However, this question allowed demonstration of advanced multiplicative reasoning ability by multiplying 3 x 1/3 to obtain one. In a closed mathematical system with respect to multiplication and division, the product of reciprocals is one. One is the identity element for multiplication and division. For example, 2 x 1/2 is one or 3 x 1/3 is one. The subject was expected to utilize the identity element for multiplication and division. This was the first opportunity to verbalize the multiplicative reasoning idea that multiplication and division are inverse operations. Success could be demonstrated by rejecting the "times" keyword strategy and instead dividing, even though the problem contained the word "times". Additionally, the researcher considered these questions in light of the subjects' responses. Did the subject get the question correct? What multiplicative reasoning words were used? Did the subject think in the correct direction? Such thinking may have indicated knowledge of greater than and less than.

The idea of fraction or ratio was introduced via the term one-third in Question 5. A strategy of multiplying 3 x 9 yielded 27 but such a strategy would cause too many fish to fit into Fish A, and thus would present a dilemma. The verbal explanation was especially important as it related to the understanding of "Why are 27 fish too many to feed to Fish A?" Success would incorporate an explanation that Fish A should be fed fewer fish than Fish B since it was smaller than Fish B. The researcher looked for strategies employed to determine the correct number of fish to feed Fish A. Successful strategies would explain that nine divided by three or nine times one-third yielded the correct number of fish to feed Fish A. Such an explanation would demonstrate an understanding of the relationships between the numbers and the objects and an ability to coordinate the correct number of objects with each fish (schema of correspondence), an indication of multiplicative reasoning (Park & Nunes, 2001; Piaget, 1965; Steffe, 1994; Vergnaud, 1983).

The design of Question 6 introduced three fish to feed. Up to this point there were only two fish to feed. If multiplicative reasoning was not employed, the likely alternative strategy was an additive strategy producing four fish for Fish B and six fish for Fish C, the correct answer, but perhaps for the wrong reason, depending on the explanation. Such an additive strategy indicated no regard for important key phrases in the problem such as “two times larger” and “three times larger” further indicating little, if any, understanding of proportion. Additionally, both phrases given in the question referenced Fish A as the unitizing factor. In other words, care should have been taken in not confusing the relationships between the numbers and the objects for this question. Were the subjects merely memorizing syntactic rules, or were they able to understand the corresponding relationships between the numbers, fishes, and multiplicative strategies needed to correctly solve and articulate the strategy to solve the problem (Hiebert & Behr, 1988)?

Fish tank seven called for using division to obtain the correct answer, despite the fact that the problem used the keyword “times”. Inhelder and Piaget (1958) and Ramful & Olive, (2008) developed the theory that reversibility of thought plays a very important role in establishing thorough understanding of mathematical concepts. This question probed reversibility of thought surrounding the relationship of Fish C to Fish B while coordinating the relationship of Fish C to Fish A simultaneously. Possessing the ability to think about units of two simultaneously with units of four are signs of the existence of multiplicative reasoning (Clark & Kamii, 1996). For example, a successful solution demonstrated that the value for Fish A was a result of the relationship between Fish A and Fish C whereas the value for Fish B was a result of the relationship between Fish A and Fish B. A successful explanation of the multiplicative strategy to solve this problem was made difficult because the inverse operation of multiplication was needed to demonstrate an understanding of the relationship between multiplication and division.

All of the strategies mentioned in the discussions of the first seven items were included in Question 8. Question 8 was considered a key question on this multiplicative reasoning instrument as success on this question signified an understanding of how to manage all of these strategies simultaneously. To begin with, the question mentioned “times” indicating to those who were utilizing the keyword strategy that such a strategy might be sufficient. Multiplication alone would not solve this problem, it also required division. Additionally, the participant should have explained that the value for Fish A was a result of the relationship between Fish A and Fish B and that division (or multiplication with fractions) was required; whereas the value for Fish C was a result of the relationship between Fish B and Fish C, and multiplication with whole numbers was required. A lens that the researcher used in analyzing the responses to this question was, “Did the subject employ unitizing?” For example, an indication that the subject did employ unitizing could be found in behaviors that identified the “unit entity” of 3 for Fish A first. Then utilizing the unit of 3, the subject should have noted that in order to obtain the correct number to multiply by, one must multiply 2 (the unit size of Fish B) x 3 (the unit size of Fish C compared to Fish B) to obtain 6 (the unit size of Fish C with respect to Fish A). Concurrent successful management of the relationships between the numbers and the objects indicated a higher degree of multiplicative reasoning (Clark & Kamii, 1996; Park & Nunes, 2001).

Questions one through eight described fish tanks. Manipulatives for each question, namely magnetic fish on the black magnetic fish aquarium, were provided, and the researcher noted whether participants prefered to utilize the computer or the manipulatives in their attempts to answer questions one through eight.

Questions 9 and 10 had a different setting. The concept of sharing pizza with siblings or classmates is a familiar scenario for most elementary school children within the Unites States. Question 9 presented two pizzas that were the same in total area but cut or divided differently. Each pizza had exactly three pieces remaining in the pan but the pieces were different sizes. Demonstrating understanding of this multiplicative reasoning problem required stating that pizza pan B had more pizza remaining because 3/8 is less than 3/4. Understanding that the different unit factors were four and eight (the denominators of the fractions) was a sign of multiplicative reasoning (Hiebert & Behr, 1988; Thompson & Saldanha, 2003). A conceptual breakthrough underlying students' understanding of unit substitutions was their realization that the magnitude of a quantity (its "amount") as determined in relation to a unit did not change even with a substitution of unit (Thompson & Saldanha, 2003). For this problem the participants were offered, as manipulatives, a pizza pan and artificial pizza slices representing the same problem as that presented on the computer screen. The researcher noted whether participants preferred to utilize the computer or the manipulatives in their attempts to answer the pizza question.

A classic item in multiplicative reasoning research given to us by Karplus, Pulos, & Stage (1983) was introduced for Question 10. Question 10 required multiplicative reasoning and the development of a unitizing factor, as well as proportional reasoning. The problem (found within the appendix) set the stage for adolescents who were transitioning into Piaget's fourth level of cognitive development, known as the formal operations stage, to articulate their understanding of multiplicative and proportional relationships. Using the computer screen or the manipulatives provided, the subject should have measured the height of Mr. Tall using paper clips to obtain the correct answer of nine. Success was demonstrated by first beginning the measurement at the ground level and then being careful to assure that the paper clips touched end-to-end and that no overlap occurred along a straight line. Hands-on measurement was discouraged until after a prediction of the number of correct paper clips to measure the height of Mr. Tall was solicited. Correct articulation concerning the reasoning to obtain the answer of nine paper clips was as follows. Because six paper clips and four buttons both exactly measured the height of Mr. Short, it appeared that two buttons were equal to three paper clips, which indicated that the student identified a unitizing factor. Once the unitizing relationship was developed, applying the unitizing factor to predict the correct height of Mr. Tall in paper clips could be accomplished. The multiplicative/proportional reasoning should have proceeded as follows. Since two buttons were equal to three paper clips, and since Mr. Tall was six paper clips tall, then three paper clips were needed for each two buttons, yielding a total of nine paper clips to correctly measure Mr. Tall. The participant should then have been encouraged to actually measure Mr. Tall to verify the accuracy of his or her prediction. For Question 10, the preference of the participants with respect to the use of the computer screen over the use of the manipulatives was recorded.

Data

Parents and teachers were not present while I collected the data, as this may have caused undue influence on the students. The researcher employed two cameras, one focused on the computer keyboard, screen and hands of the participant, and the other focused on the participant's written workspace. In addition to the video recordings, the student responses inside Microsoft Visio were stored. The students also provided written work supporting their thinking during the interviews. Transcriptions of the students' verbal responses were also made. By reviewing the video recordings, in conjunction with the Microsoft Visio files, the written work of the student and the transcriptions, the researcher explored indications of the emergence of multiplicative reasoning in the subjects.

Each child's videotapes and work produced in the interview have been stored in a locked file cabinet in the principal investigator's office. Copies of the video tapes and artifacts have been stored in a locked file cabinet in the home office of the student researcher. After three years following the closure of the project, all tapes will be erased, and the consent forms and student work shredded. Subjects were assigned a false name. Evidence of the use of pseudonyms will be erased and/or shredded after three years following the closure of the project.

Analysis

The researcher looked at the data for indications of the emergence of multiplicative reasoning utilizing the following techniques. Student data were analyzed with respect to the framework introduced in Chapters One and Two by observing the manner in which the subject responded to each question and not simply whether the subject "got the correct answer". Subject transcripts, utterances, written work, schemes, and drawings were examined for instances of key words. Additionally, data from the written work were examined for the types of representations used to convey ideas. Table 4.1 presents the strategies employed by the subjects on the test instrument. Table 4.1 assists in answering research question one: what are the indicators of multiplicative reasoning among fourth grade students?

The researcher placed the subjects' key words, transcripts, utterances, written work, schemes, and drawings for each question into one of the twelve strategies based on the following criteria found in the indicators column in Table 4.1. If the subject exhibited non-preservation of the quantification of objects then the subject was placed at Level 1 Non-quantifier. For example, if the subject placed 32 fish into a fish that could clearly not hold 32 fish, this would be an example of non-preservation of the quantification of objects. If the subject arrived at the answer through guessing then the subject was placed at Level 1 Spontaneous Guesser. For example, a demonstration of Level 1 Spontaneous Guesser behavior was exhibited when the student noted that a particular fish should be fed 7 fish because 7 was his favorite number.

When the subject specifically indicated a need to multiply because the problem had the word "times" (or some other keyword such as twice) the subject was placed at Level 2 Keyword Finder, because the subject derived the answer by invoking the multiplication algorithm. When the subject counted the fish and uttered answers where it was easily observed that the answers were related to a one-on-one mapping with the whole number system, the subject was placed at Level 2 Counter.

Subjects who arrived at their answers via addition and plainly indicated so, by writing or saying that they were adding more fish for each fish were placed as a Level 2 Adder. Quite often, because students had success utilizing additive strategies, they tended to utilize additive strategies when they did not have a plan, perhaps because they had been successful in the past. However, utilizing an additive strategy demonstrated more understanding than simply guessing because their answers corresponded to the size of the fish. When subjects made good use of the fact that $A < B < C$ then the students were placed as a Level 2 Quantifier. When students could not find the answer via multiplication they sometimes would measure the relative size of the fish, thus measuring the fish on the computer screen or the manipulatives and derive an answer through measurement. When the subjects derived their answers via measurement then the subjects were placed as a Level 2 Measurer.

Sometimes the subjects understood that additive strategies were not successful and attempted, but did not succeed, at utilizing multiplicative reasoning strategies. Often subjects utilized repeated addition to obtain the correct answer. They would demonstrate this by writing or uttering $3 + 3 + 3 = 9$, for example. When such repeated addition was utilized the subjects were placed as a Level 3 Repeated Adder.

In many cases the subject may have understood that additive strategies were not sufficient and succeeded at utilizing multiplicative reasoning by demonstrating a good ability to coordinate the objects, numbers and operations defined within the word problem. Such subjects often obtained the correct answer but did not or were not able to articulate the method or schema utilized to achieve the correct answer. Such demonstrations provided support for the subject being placed as a Level 3 Coordinator.

If the subject articulated an adequate mathematical sentence, either on paper or verbally, which fully described the mathematical relationship between the fish, the numbers, and the operations then the subjects would be placed as a Level 4 Multiplier.

Additionally, some subjects spoke or wrote the word “cut” or a similar word to indicate the need for division. In such a case, the subjects were placed as a Level 4 Splitter.

When the subjects provided a new quantity as a unitizing factor, articulated an adequate mathematical sentence, and successfully completed the problem using either multiplication or division, then the subjects were placed as a Level 5 Predictor. In the case of Question 10 the subjects exhibiting Level 5 Predictor should have stated something similar to “6 is to 4 as *y* is to 6 and since 6/4 reduces to 3/2 which equals 1.5 (this is the new unitizing factor) then 1.5 x 6 = 9”.

Summary

A comparison of average mathematics content and cognitive domain scores of fourth-grade students by country in 2007 shows ten countries significantly ahead of the score (524) posted by the United States (Gonzales, Williams, Jocelyn, Roey, Kastberg, & Brenwald, 2008). Can proper instruction in multiplicative reasoning positively affect this score? The literature suggests that advanced mathematical thinking cannot be achieved without a proper beginning in multiplicative reasoning. Because math is a complex cognitive process, it is useful for analysis to break down the cognitive requirements into their elements such as “schema of correspondence”. However, even connecting advanced mathematical thinking to its beginning in multiplicative reasoning, the general public as well as some educators may have difficulty in clearly seeing the need for a conceptual understanding of multiplicative reasoning to occur as early as grade four. A better understanding of fourth grade children’s thinking as they emerge from the concrete to the formalized stages of multiplicative reasoning is needed to inform teaching and curriculum development. An examination of students' utterances and representations of different problems is an important step toward that understanding.

CHAPTER IV

RESULTS

Overview

The literature provides a clear baseline for understanding multiplicative reasoning outlined in Table 2.3, yet the data reveal that a more detailed analysis is needed to identify the building blocks underlying the learning of mathematics for fourth grade students. I propose an expansion of the multiplicative reasoning markers listed in Table 2.3 to a more detailed list of markers listed in Table 4.1. Table 4.1 better identifies specific fourth graders whose learning trajectories fall between the cracks of the strategies found in Table 2.3. The expanded results are then employed to analyze each individual subject, and to find categories of multiplicative strategies common to fourth grade students. The expanded results are similarly utilized to perform an item analysis across the ten items on the test instrument, yielding information concerning the ability of each item to evaluate subjects' comfort levels with various markers in learning mathematics at a fourth grade level. This analysis is a cross-case search for patterns, and keeps this investigator from reaching hasty conclusions by ensuring I look at the data from many different perspectives. When a pattern from one data type is supported by the evidence from another, the findings are stronger. If the evidence conflicts then deeper investigation is necessary to identify the origin of the conflict.

Expanded Framework

Level 1 Non-quantifier and Level 1 Spontaneous Guesser

A review of literature supports the five tier structure outlining multiplicative reasoning levels presented in Table 2.3. Although the five levels of Table 2.3 give descriptions to identify subjects who fall within the five categories, the literature is lacking the definition of what it means to be a “Keyword Finder” or “Repeated Adder”. By analyzing the work of those who have come before me, together with the data, I constructed definitions of a more detailed multiplicative reasoning level mastery as outlined in Table 4.1. The subject is Level 1 Spontaneous Guesser when the subject has no valid reasoning for the answer and the work is often labeled “guessing”. This type of guessing is not the same as estimation or an educated guess, but is simply a random guess at the answer with no underlying logic that pertains to the problem presented. Such an answer might take on the form, “I think the answer is 1 because 1 is my favorite number” or “I believe the answer is 7 because 7 is my birthday.” Clark and Kamii (1996) identify this strategy as Level 1 Spontaneous Guesser but include serial correspondence in their definition. I find that subjects are better identified when they are categorized as: Level 1 Spontaneous Guesser, arrives at answer through guessing, and Level 1 Non-quantifier, does not preserve the quantification of objects. In other words, the subject cannot make use of the fact that $A < B < C$. Such an action was demonstrated by Sue (subject 5) when she was giving her solution to question seven. The question contains the keyword “times” and the numbers four and eight. Sue was employing the keyword strategy and decided to multiply four times eight to arrive at the solution for Fish A. From the computer screen it is apparent that $A < B < C$ and that 32 green fish could never fit into Fish A. But even though Sue was questioned by the investigator as to which fish was smaller, Fish C or Fish A, she answered that Fish A is smaller, yet she continued to place 32 fish into Fish A, demonstrating no concern for serial correspondence or qualitative quantification. Subjects 12 and 14 exhibited similar behavior on this same question.

Level 2 Keyword Finder

The expansion needed for Level 2 is more extensive. Level 2 is divided into five different categories: Keyword Finder, Counter, Adder, Quantifier, and Measurer. Children employ a variety of superficial cues when attempting to solve multiplicative reasoning word problems. Sowder (1988) studied students' solutions of simple one-step arithmetic word problems. He found evidence that students mindlessly associated not only key words or key expressions with arithmetical operations, but that particular number combinations pointed students toward a certain operation. Sam (subject 7), as well as Eric (subject 12), and others assisted us in understanding the need for Keyword Finder by directly stating in written work, because problem 3 has a "clue word 'times' and times means to multiply I knew to multiply". Students frequently multiplied when working a problem, and when asked why they chose to multiply the answer was often "the problem said times and times means to multiply."

Level 2 Counter

In addition to Keyword Finder, Level 2 needed an expanded category named Counter. Steffe (1988) researched transition from additive to multiplicative strategies. The basis for Steffe's investigations was the counting scheme, which includes, but is not limited to, counting by ones. When a student counts by ones in a manner that leaves out counting sets of sets or units of more than one, the student is deriving the answer by a one-to-one mapping with the whole number system (Dienes & Golden, 1966; Steffe, 1988). Paul (subject 2) demonstrated such a strategy when on question 7 he stated that the answer was arrived at by understanding that "it goes 2, 3, and 4." Here Paul derived the answer by a one-to-one mapping with the whole number system. Additionally, subjects 4, 5, and 8 employ the counting strategy especially for Question 10 when they lay the paperclips down to measure the height of Mr. Tall but are not careful to place the paperclips end-to-end or start even with the ground.

Level 2 Adder

A third category for Level 2 is Adder. An Adder derives the answer utilizing addition or subtraction, regardless if strategy leads to success. Additive strategies are quite successful for first graders and the teacher often rewards the students who achieve correct answers to problems utilizing additive strategies. This makes some children believe that additive strategies will always work in solving math problems. We find evidence of this when children fall back to additive strategies when problems requiring multiplicative strategies are given. Even though additive strategies are not sufficient, some children insist on using them despite the fact the additive strategies are yielding the wrong solutions. One may engage in repeated addition to evaluate the result of multiplying, but envisioning adding some amount repeatedly cannot support conceptualizations of multiplication without involving additional concepts (Steffe, 1988; Thompson & Saldanha, 2003). The

Level 2 Adder is not to be confused with the Level 3 Repeated Adder. The Level 2 Adder utilizes addition to achieve the answer but does not understand that, although there is a conceptual discontinuity between multiplication and addition, there is a procedural connection between these operations. Nunes and Bryant (1996) further argued that because multiplication is distributive with respect to addition, repeated addition can be used as a procedure to solve multiplication sums without understanding the connection.

Level 2 Quantifier

Clark and Kamii (1996) as well as Lamon (2006) support the idea that children begin to quantify numbers as larger or smaller before they are able to utilize additive strategies. Quantifying numbers means that the subject is able to put objects in order based on A < B < C. This ability is only one step above Spontaneous Guesser. Mark (subject 5) gives us a perfect example of this type of thinking, when in his written response to question one he states, “One more fish would feed fish B because he is larger.” This answer is not correct for problem one, but it does demonstrate that Mark understands that Fish B must eat more than Fish A since Fish B is larger.

Level 2 Measurer

Karplus, Pulos, and Stage (1983) suggest that some subjects making predictions about the height of Mr. Tall in paperclips can utilize measuring tools either mentally or visually such as counting the paperclips. Choosing a strategy of measurement when the investigator suggested using a mathematical strategy to predict the height of Mr. Tall indicates an additive reasoning strategy, although it is possible to use a unit measures approach to solving a missing value proportion problem (Karplus et al., 1983) . An additive reasoning strategy such as the laying of paperclips end-to-end to measure the height of Mr. Tall is indicative of Level 2 Measurer. A Level 2 Measurer is aware of some of the basics of measuring such as understanding that each measurement has a starting and ending point without overlap or gap between unit measures from the ground up, in the case of Mr. Tall. Well over half of the subjects behaved as Level 2 Measurers on question 10 of the instrument. Although measurement yields valid results and is practical, intuitive, and situated, it does not rise to a level three strategy. What measurement alone fails to demonstrate is whether the subject is conceptually utilizing units of one and units of more than one simultaneously.

Level 3 Repeated Adder

Fishbein, Deri, Nello, and Marino (1985) proposed that each mathematics operation has an implied model and that model for multiplication is repeated addition. Steffe (1994) expounded this model by suggesting that the origin of children's understanding of multiplication is the creation of a ‘‘composite unit.’’ A composite unit is a unit containing ones represented as "one thing." It is this composite unit that is repeatedly added. Nunes and Bryant (as cited in Park & Nunes, 2001) explained that despite the conceptual discontinuity between multiplication and addition, there exists an algorithmic connection between these operations, since multiplication is distributive with respect to addition. Therefore, it is possible to achieve multiplicative sums by repeatedly adding the same quantity. This behavior is indicative of a Level 3 Repeated Adder. The behavior of repeatedly adding to solve multiplication sums successfully while demonstrating utilization of schema of correspondence is what separates a Level 2 Adder from a Level 3 Repeated Adder. Piaget (1965) was the first proponent of this view, which was later explored in greater detail by Vergnaud (1983, 1988) and by Nunes and Bryant (1996).

Level 3 Coordinator

Park and Nunes (2001) propose that there is a second hypothesis explaining the concept of multiplication and is defined by an invariant relation between two quantities. This constant relation, known as ratio or rate, is the true meaning of the concept of multiplication. In math it is symbolized as $x = f(y)$. Such a constant relation is not demonstrated when students are utilizing additive strategies. Additive strategies would demonstrate transformations of one quantity ($x + y$ or $x – y$) and not a constant multiplicative relationship between the quantities. Subject 1 on question nine demonstrated excellent coordination of objects to make a statement about the two pizzas, but was not completely successful with the problem, nor did he state a multiplicative sentence. Joe (Subject 1) stated that "it [pizza A] is in eighths and pizza B is in fourths." This statement demonstrated a limited ability to coordinate objects, numbers and operations, and demonstrated he was utilizing a schema of correspondence. In other words, a Level 3 Coordinator demonstrates understanding of multiplicative reasoning, yet fails to complete the problem successfully, possibly due to a calculation error, or some other mitigating factor. Level 3 Coordinator emerged five times during this study.

Level 4 Multiplier

Clark and Kamii (1996) suggest that children who derive correct multiplicative responses utilizing multiplication and state a multiplication sentence such as “7 times 2 is 14” are multiplicative thinkers and in this context are placed in the group Level 4 Multiplier. Subjects in this group are able to demonstrate fluency with respect to coordination of objects, numbers and operations, and to derive solutions. Subjects in this study often made multiplicative sentences 2 x 2 = 4 or 9 / 3 = 3. A Level 4 Multiplier emerged fifty-five times during this study.

Level 4 Splitter

Confrey (1994) proposed that the idea some children utilize when demonstrating multiplicative reasoning is best described by the term “splitting”. Splitting employs the concept of cutting and halving to indicate the need for division or multiplication. Splitting demonstrates a student’s understanding of the relationship between the numbers and operations of the problem presented and is therefore utilizing relational knowledge. For example, when utilizing additive strategies the origin is zero, but when splitting the student understands that the origin is 1, which is the identity element for multiplication. With splitting, the unit of growth is a unit of units or a set of sets (Confrey, 1994). Strong evidence of students employing splitting surfaced five times during this study. Subject 1, Joe, was assessed to be a splitter because of how he employed the term “cut” in explaining “If you cut six in half it will be three.”

Level 5 Predictor

Level 5 Predictor occurred only three times during this study and may indicate that this level may be above the mathematical cognition level of most fourth grade students. This level requires comfort in unitizing and reunitizing quantities. Lamon (2007) suggests that unitizing refers to the mental process of restructuring a given quantity into a more manageable sized piece in order to operate or compare the quantities or to establish a proportional sentence. For example, on Question 9 several subjects mentioned that pizza A was in eighths and that pizza B was in fourths. Only one student was able to determine the denominators and numerators, establish a fraction for each pizza, and then create a proportional sentence stating that 3/8 is less than 3/4. The ability to predict the measure of one piece of pizza given a piece of pizza from the other pan without first measuring is a sign of Level 5 Predictor. Kaput and West (1994) note that conceptual understanding of the situation is required to preserve the part-whole or part-part order needed to write such equations.

Therefore Table 4.1 details what it means to be a Non-quantifier, Spontaneous Guesser, Keyword Finder, Counter, Adder, Quantifier, Measurer, Repeated Adder, Coordinator, Multiplier, Splitter and Predictor. Table 4.1 assists in answering research question one. What are the indicators of multiplicative reasoning among fourth grade students?

Table 4.1: Strategy Table for Level Mastery

Level	Strategy	Indicator	Reference
Level 1	Non-quantifier	Exhibits non-preservation the quantification of objects, i.e. $A < B < C$, meaning that 7 can = 8, can = 9	Clark & Kamii, 1996
Level 1	Spontaneous Guesser	Arrives at answer through guessing	Clark & Kamii, 1996
Level 2	Keyword Finder	Derives answer from keywords such as times and applies the associated algorithm	Sowder, 1988
Level 2	Counter	Enumerates objects with a one-on-one mapping with the whole number system	Dienes & Golden, 1966; Steffe, 1988
Level 2	Adder	Derives answer utilizing addition or subtraction regardless if strategy leads to success	Nunes & Bryant., 1996; Steffe, 1988; Thompson & Saldanha, 2003
Level 2	Quantifier	Makes use of the fact that $A < B < C$	Clark & Kamii, 1996; Lamon, 2006
Level 2	Measurer	Exhibits an understanding of when measurement should be linear or curvilinear and that each measurement has a starting and ending point without overlap or gap between unit measures	Kaput & West, 1994; Karplus, Pulos, Stage 1983
Level 3	Repeated Adder	Demonstrated and understanding that multiplicative answers can be achieved through repeated addition	Fishbein, Deri, Nello, and Marino 1985; Nunes & Bryant, 1996; Piaget, 1965; Steffe, 1994; Vergnaud, 1983, 1988
Level 3	Coordinator	Demonstrates limited ability to coordinate objects, numbers and operations	Park & Nunes, 2001
Level 4	Multiplier	States a multiplication sentence and demonstrates fluency with respect to coordination of objects, numbers and operations	Clark & Kamii, 1996
Level 4	Splitter	Utilizes concept of cutting and halving to indicate the need for division	Confrey, 1994
Level 5	Predictor	Predicts the measure of an object in 1 system given the measure of a proportional or similar object in another system	Kaput & West, 1994; Lamon, 2007

Subject Analysis

Table 4.2 is the frequency of occurrence for each level and strategy by student and begins to answer research question two. What strategies do fourth grade students utilize in solving multiplicative reasoning word problems?

Table 4.2: Student Strategies Frequency Table

	Level 5 Predictor	Level 4 Splitter	Level 4 Multiplier	Level 3 Co-ordinator	Level 3 Repeated Adder	Level 2 Measurer	Level 2 Quantifier	Level 2 Adder	Level 2 Counter	Level 2 Keyword Finder	Level 1 Spon-taneous Guesser	Level 1 Non-quantifier
1		2	6	1				1				
2			1		2	2		2	1		2	
3	1		5		1			1			2	
4			3				1	3	1	1	1	
5			3			1		2	1	2		1
6			3		2	2				3		
7			5		1	2				2		
8		1	3	1	1		1	2	1			
9			4	1		3	1	1				
10		1	5			2		2				
11	1	1	7			1						
12			2		1	2		4				1
13	1		5	1		1				2		
14		3	2	1		1				2		1
Σ	3	8	54	5	8	17	3	18	4	12	5	3

Three Broad Strategies

It became apparent that the strategies utilized by the subjects could be generally categorized into three major categories called pre-multiplicative, emergent, and multiplicative. In the sections which follow, I analyze the transcripts, written work, utterances, schemes, drawings, and manipulations of the subjects to determine which of the twelve strategies found in Table 4.1 the indicators suggest.

Pre-Multiplicative Strategies

Subject 2 – Paul.

In Items 1 and 2 Paul demonstrated characteristics of a Level 3 Repeated Adder, because he utilized repeated addition to achieve multiplicative success:

Line 26: "Because 2 plus 2 equals 4"
Line 36: "Because 3 plus 3 equals 6"

With respect to Item 3, Paul wrote a successful multiplication sentence, exhibiting Level 4 Multiplier characteristics as indicated by the multiplication sentence found in transcript line 46: "Because I did 2 times 4" . On Item 4, although Paul did achieve the correct answer, he was not able to explain why his answer was one with multiplication nor was he able to state a multiplication sentence to represent the problem. Paul had some idea that the smaller fish needed a value less than 3 and used Level 2 Adder strategy to explain his reasoning for this problem. Line 61: "Because Fish A is smaller than Fish B and I just subtracted 3 minus 2"

Item 5 revealed that Paul was not as sophisticated in achieving the multiplicatively correct answer via repeated addition. Paul actually explained his answer using Level 2 Adder strategies as shown in Line 74: "I think you should subtract 9 minus 5"

On Item 6, Paul exhibited the qualities of a Level 2 Counter as he was only able to quantify the number of fish to feed, and made no serious attempt at deriving the solution to this multiplication problem. He understood the bigger the fish, the more it needed to eat, but could not represent the ideas needed to achieve the correct answers. In transcript line number 92 Paul indicated his utilization of a counting strategy when he said "Because it's 2, 3, 4".

On Items 7 and 8, as the difficulty increased, Paul demonstrated less understanding of multiplicative reasoning by suggesting that questions be changed or by giving spontaneous answers, supported by lines 112-114, 140-144, and his written work.

> Lines 112-114: "And change that 8 into a 7… [I: 8 into a 7. Do you like 7 better?] Yeah because it gives you…because it makes the question more challenging"
>
> Lines 140-144: "I think you should change…[I: Change? Yeah, you've filled up that paper, let's work on this paper for awhile, is that okay? (Student nods). I: Do you want to write anything?] Maybe. (Pointing to Fish C) This should be much, much smaller"

For Items 9 and 10, where solutions require unitizing and proportional reasoning to make predictions, Paul was unable to predict the answers multiplicatively. He used less stringent strategies, such as measuring, to achieve solutions. This is substantiated by transcript lines 184, 201, and 223-229.

> Line 184: "Because this…because this there's only two pieces on one piece. And then there's…"
>
> Line 201: "B."

> Lines 223-229: "Because Mr. Short has 6 and Mr. Tall, 6 can stop right here…6, 7, 8, 9. [I: 9? How did you know?] Because, he'll have more than 6, so I'll start at 6, 7, 8, 9. [I: How did you know?] I just compared and contrasted."

Throughout his work on the test instrument Paul illustrated a pattern of selecting a different strategy for almost every problem. He showed no real consistency and utilized no particular strategy on a regular basis.

Subject 4 – Cheryl.

On Questions 1 through 3, Cheryl exhibited qualities of a Level 4 Multiplier in constructing multiplicative sentences, "2 times 4 equals 8" in transcript lines 31 and 47.

> Line 31: "Because I times…2 times 2."
> Line 47: "Because 2 times 4 equals 8."

Cheryl behaved as a Level 2 Adder on Questions 4 and 5, demonstrating a tendency to revert to earlier mathematical strategies, such as addition, by stating that she was using subtraction in transcript lines 78 and 90. Cheryl was successful on Question 4 but not on Question 5.

> Line 78: "I subtracted."
> Line 90: "9 minus 3 equals 6."

With the addition of another fish to feed in Question 6, Cheryl fell back even further to Level 1 Spontaneous Guesser, which may indicate her inability to handle multi-step math problems like those introduced in Questions 6-8 of the test instrument. Even though she did achieve the solution for Fish C, she said that Fish B would have the same number, indicating a spontaneous strategy.

> Line 98: "Because I times 3 times 2 equals 6."
> Line 101: "I think it might have the same amount [fish B has the same amount as C]."
> Line 105: "Because 3 times 2 is also 6."

For Question 7, Cheryl returned to a Level 2 Adder strategy and employed subtraction in ways that were not related to the solution. Transcript lines 118, 120, 122, and her written work support her choice of strategy.

> Line 118: "Because I subtracted."
> Line 120: "4 take away 2"
> Line 122: "Because if you take away 4…take away 2 it's 2."

It appears that with respect to Question 8, Cheryl identified a strategy called "keyword finder," where she identified the keywords in the problem and proceeded to multiply other numbers in this same problem without correctly connecting operations with numbers and objects, in ways that did not lead to the true solutions. These characteristics are revealed by transcript lines 136, 139, and her written work.

> Line 136: "Because 2 times"
> Line 139: "Cause 2 times 4 equals 8."

Cheryl proved to be a Level 2 Quantifier as she correctly selected the pizza with the larger area. Her strategy in doing so shows her ability to quantify objects, and she made use of the fact that A < B < C. In line 150 of the transcript, Cheryl quantified each pizza as having three slices of pizza remaining, but the pizza slices for pizza A were "littler" than those for pizza B, showing qualitative quantification with serial correspondence in line 150: "Because this one has 3 and this one has 3, but it's still little."

Cheryl exhibited qualities of a Level 2 Counter with respect to Mr. Short and Mr. Tall. She supposed that a strategy of counting the paperclips for Mr. Tall was the solution, and no attempt was made to predict the answer for Mr. Tall from the objects supplied in system one. Her counting strategy is demonstrated in lines 157 and 169. While this strategy did yield a correct answer, it is thought to have been achieved through some type of visual measuring.

Line 157: "1, 2, 3, 4…"
Line 169: "Cause I counted."

Cheryl demonstrated a use of pre-multiplicative strategies on 70% of the test questions.

Subject 5 – Sue.

Sue showed pre-multiplicative strategies 70% of the time. For Questions 1-3 she showed a strategy of identifying a keyword "times" and applying a multiplication algorithm, regardless of outcome. On Question 4 it became apparent that Sue was utilizing a keyword strategy because she multiplied three times three to obtain nine when she should have divided three by three to obtain one, the identity element for multiplication. Line 62: "If I multiply 3 times 3 it will come up to 9 fish."

Beginning with Question 5 when fractions were introduced, Sue began to use a strategy of putting the fish tail to tail and look to see how many green fish could fit into one big yellow fish in line 78: "I just put Fish A tail to Fish B tail and look at how many Fish B had and take one." The investigator understood that by "take one" in that sentence, Sue was using an additive strategy.

Beginning with Question 6 Sue began to use spontaneous strategies because Question 6 was the first question requiring multi-step math. She did not have any solutions on the multi-step portion of the test instrument. This is supported by transcript lines 89-90.

> Line 89-90: "Because I just add 2 to each, 2 to A and it put 2, 3…you make 4 then I multiply C and B…two times….and it will be 8."

On Question 7 Sue appeared quite confused. She began utilizing the keyword strategy and invoked an algorithm such as 4 x 8 = 32, because she insisted on inserting 32 fish into Fish A, despite the small size of Fish A. Here, Sue was so focused on using the keyword strategy by putting her answer of 32 into Fish A that she lost track of her ability to put items into corresponding order. In other words, she employed a strategy of Level 1 Non-quantifier, because she paid no attention to $A < B < C$. For Question 8 Sue was able to utilize Level 2 Keyword Finder to find the correct answer for Fish C, but could find no strategy to assist her in achieving an acceptable solution for Fish A.

On Question 9 Sue measured the pizza slices using the manipulatives to obtain the correct answer of Pan B. She demonstrated the characteristics of a Level 2 Measurer. She was not able to state any type of numerator, denominator or other mathematical sentence for problem 9. She did not attempt any type of mathematical prediction concerning Item 9.

Sue decided on Question 10 that the best strategy was to lay the paperclips down and count them on the manipulatives. This may have been a form of measuring, but she was not careful to lay the paperclips end-to-end and was not careful to start at ground level. She did obtain the correct answer of 9, perhaps by chance. Sue demonstrated a Level 2 Counter strategy for Question 10.

Sue exhibited a variety of strategies and no particular strategy emerged as a dominant strategy. Sue's answers on Item 7 showed a lack of ability to quantify objects in a meaningful manner. The fact that she would allow 32 fish to be placed into Fish A (since Fish A is the smallest fish) was a sign that she did not preserve $A < B < C$. This lack of serialization, as well as her numerous attempts at strategies other than multiplicative, places her in the pre-multiplicative category.

Subject 6 – Ava.

Ava is one of three subjects who achieved the correct answer for all the questions on the test instrument. Initially, Ava exemplified characteristics of a Level 3 Repeated Adder for Questions 1 and 2, as evidenced by transcript lines 36, 40, 50, and written work.

> Line 36: "It says, twice as large as Fish A so I added 2 plus 2 and got 4."
> Line 40: "Twice means that you have to add 2 plus whatever number you feed the fish."
> Line 50: "It says half of Fish B so add 3 plus 3 and got 6."

In Question 3, Ava demonstrated her reliance on key word strategies, such as "times", as a clue to her that multiplication was the correct operation, supported by transcript lines 60, 67, and written work.

> Line 60: "Well, it says 4 times. So you multiply 2 times 4…"
> Line 67: "It says 4 times, so you multiply."

Although Ava had difficulty spelling the word "divided", she made excellent use of it in Question 4 by writing the mathematical sentence "3 divided by 3 and get 1". Transcript lines 85, 88, and written work provide a cue that Ava was able to construct multiplicative sentences. Deducing that division was needed in this problem to achieve the identity element showed good understanding of the relationships between the objects and the operations. On this problem Ava was a Level 4 Multiplier since she stated a multiplicative sentence.

> Line 85: "If 1 times 1[3] equals 3, then 3 divided by 3, equals 1."
> Line 88: "If you divide 3 divided by 3, you get 1."

Ava digressed to a Level 2 Keyword Finder as supported by her emphasis on the word "times" in Questions 5 and 6 in transcript lines 97-99, 124-125, 132-133 and written work. Difficulty on these problems may indicate her trouble in dealing with fractions and multi-step math problems. For example, Lines 97-99 illustrate her difficulties: "9 divided by 3 equals 3." [I: How did you know to divide?] "If it says a bigger number first, and the other number is smaller than that, you divide." Here, Ava was clearly utilizing a keyword strategy to decide on her choice of operation but did arrive at a solution for Questions 5 and 6.

> Lines 124-125: "Because it says Fish C is 3 times as large as Fish A, so you do 3 times 2 and get 6. And then it says Fish B is 2 times larger than Fish A so you multiply or add, so you get 4."
>
> Lines 132-133: "It says that Fish C is 3 times so multiply 3 times 6 and get 6. Then it says Fish B is 2 times so multiply 2 times 2 or add 2 plus 2 and get 4."

Ava demonstrated Level 4 Multiplier characteristics and achieved success for Problem 7 as indicated in transcript line number 144 when she stated: "It says 2 times larger than. If it says larger than, you have to subtract or divide…well divide."

Although Question 8 was similar to Question 7, Ava demonstrated a more advanced multiplicative reasoning marker, Level 4 Multiplier, as evidenced by transcript line 174 and written work. Ava gave little explanation on this problem as she expressed that she was tired (line 176), but gave more than sufficient verification supporting her Level 4 Multiplier strategy in her written work by stating two multiplication sentences. Line 174: "You do 2 times 3 and get 6." and in her written work she stated that 6 x 3 = 18.

On Questions 9 and 10, Ava demonstrated characteristics of a Level 2 Measurer as substantiated in transcript line numbers 199-203 and 234-237. She was not able to demonstrate fluency with respect to coordination of objects, numbers, and operations when trying to manipulate abstract quantities. She was more successful in achieving the correct answer through unit measures of pizza slices (laying one pizza slice over another in the manipulatives) and placing the paperclips end-to-end to measure the height of Mr. Tall. Ava may have been tiring, as she did not give any written work after question 8 due to "fatigue in her arm."

> Lines 199-203: "If you put these together on the same pizza, and you're trying to get the whole pizza to equal…there would have to be one more…but add it to this, and you would need to get 5 more.
> [I: How can you tell?] You have to line the pizzas up, then put them near, or on top of each other. Then you see if they're equal or not."
> Lines 234-237: "3 paper clips almost equals 2 buttons. [I: 3 paper clips almost equals 2 buttons….is that what you said? I: How did you know that? Just by…] By lining them up."

Having demonstrated no consistent strategy throughout the test instrument, Ava utilized various strategies to achieve her solutions, all of which were correct. Ava demonstrated pre-multiplicative strategies 70% of the time. Ava may have instinctively known the answers to these math problems but lacked the vocabulary necessary to articulate her reasoning.

Subject 12 – Eric.

Beginning with Question 1 Eric displayed an additive strategy as indicated by the math equation he wrote and in transcript line 53: "You add 2 plus 2 equals 4, so you would give Fish B 4." Eric explained his thinking by using the word "plus" indicating performance at Level 2 Adder.

Eric's behavior on Questions 2 and 3 was consistent as he performed at Level 4 Multiplier for both questions as indicated in the following transcript lines 63 and 80-81.

> Line 63: "You could do um…you could do 3 times 2 which is 6 or you could do 3 plus 3 equals 6."
> Lines 80-81: "I did 2 times 4 which equals 8 because it said Fish B is 4 times larger than Fish A…so that's how I got the answer."

It is obvious when Eric was faced with Question 4 he had little idea how to proceed in finding the identity element for multiplication. From his written work we know that Eric stated that there are no more clues so "you simply add the number again" as in "3 + 3 = 6 because there were no more clues." Since Eric was looking for clue words and since his strategy led to a solution not in alignment with the true solution for the problem, Eric performed at Level 2 Adder for this problem.

In Question 5 it was evident from transcript line number 121 "I think you do…um 9 minus 3 which equals 6" that Eric used an additive strategy to solve this division problem. It may be that he substituted 3 for 1/3 and subtracted it from 9 to obtain the answer of 6. Such behavior would place Eric at Level 2 Adder.

Once the multi-step problems appeared in the series of Questions 6 through 8 Eric's performance level decreased to Level 2 Adder and Level 1 Non-quantifier. This decrease in performance level is consistent with other fourth grade students who are not performing reliably at the multiplicative reasoning level 4. Consider the following transcript lines:

> Lines 140-141: "Because I took one step at a time and I figure out that you can add 3 and 2 and Fish C and A which equals 5 so you put the 5 there, and then…
> Lines 165-166 : "Because um, it says that um, Fish C is 4 times larger than Fish A and If Fish C eats 8 green fish each day and you would probably put 8 times 4 which is 32"
> Line 199: "And you do 6 plus 3, which is 9 and it's 14."
> Line 208: "And then Fish C you do…you put 6 fish…you put 3 more."

Eric was rather confused during his thinking process, especially on Question 7, as he concluded that 32 fish (4 x 8) should go into Fish A. Eric lost his ability to put items into corresponding order. In other words, he demonstrated a strategy of Level 1 Non-quantifier because he paid no attention to A < B < C.

For Questions 9 and 10 Eric demonstrated Level 2 Measurer strategies to obtain correct answers. Eric compared the two pizzas and concluded from a visual comparison (visual measurement) that "Because, it's bigger than those little pieces and so it's got much [more] than that one [pointing to Pan A]." Upon measuring the height of Mr. Tall by utilizing the available manipulatives provided for this problem, he concluded that the answer could be found if he looked "…at the buttons and saw the other side of that place…and I tried to count the paper clips that I was going to have", clearly indicating the behavior of a Level 2 Measurer. Overall, Eric consistently demonstrated the characteristics of being pre-multiplicative. In fact, 80% of the time Eric utilized pre-multiplicative strategies.

Emergent Strategies

Subject 3 – Tim.

In the first three questions, Tim showed consistency in his performance as a Level 4 Multiplier, because he was able to state multiplicative sentences for these items: transcript lines 27, 39, 52, and his written work.

Line 27: "Because they said double…eats twice as many fish as A."
Line 39: "I'm thinking, like, if he's half the size then he must be double three fish."
Line 52: "I'm thinking so he is…B is 4 times bigger than A. So 2 times 4…"

On Question 4, Tim revealed his thinking as that of a Level 2 Adder, in transcript line 72 and his written work, and by utilizing subtraction (an additive strategy) to obtain success in the multiplicative task presented. He was not able to state a multiplicative sentence for this problem as observed in Line 72: "Because Fish B is 3 times bigger so 3 minus 2 equals 1."

For Item 5, Tim seemed to lose his focus and return to Level 1 Spontaneous Guesser as evidenced by line 81 and his written work.

Line 81: "Because Fish B is fed 9 fish each day. Fish A is 1/3 so 8 divided by 4 equals 2."

Tim revealed signs of multiplication in Question 6 by attempting to write multiplicative sentences, but failed to obtain the correct answers, exhibiting the thinking of a Level 3 Repeated Adder: transcript line 95 and his written work.

> Line 95: "Fish A is half the size of Fish B so Fish B eats 4 fish. C is 3 times bigger than A, so 5 fish."

Although Tim does not expressively give a multiplicative sentence in Item 7, he does achieve the correct answer by demonstrating an ability to relate objects, numbers, and operations appropriately. Tim exhibited qualities of a Level 4 Multiplier in transcript line 108 where he states "Fish C is 4 times bigger than A so Fish A eats 2 fish. B eats 4" and his written work.

When presented with Question 8, it appeared that Tim was guessing at the answer, placing him at Level 1 Spontaneous Guesser, exhibited in lines 115-116, 121, and his written work which showed an incorrect answer.

> Lines 115-116: "So Fish B is 2 times larger than Fish A. Fish C is 3 times larger than Fish B. If Fish B eats 6 green fish each day, so I'll say A eats 1 and C eats 5."
>
> Line 121: "Fish B is 2 times bigger than Fish A. Fish C is 3 times bigger than Fish B. So A is 2 fish, C is 5 fish."

In Question 9, Tim showed a good understanding of the need to compare the area of the slices of pizza by laying one over another, and found a common denominator and compared fractions. Tim's behavior demonstrated that he was thinking as a Level 4 Multiplier, supported by line 132: "The pizza is 3/8. B is 3/4. 2/8 are one of the 1/4 so 3/4 are bigger."

In Question 10 Tim demonstrated an excellent ability to predict the measure of an object in one system, given the measure of a proportional object in another system, by concluding that "it was 3 [paperclips] for 2 [buttons]" from transcript line 143. These characteristics show Tim to be performing at the mastery level of a Level 5 Predictor, further evidenced by transcript lines 148-149, and his written work.

> Line 143: "So if Mr. Short is 6 paper clips tall and is 4 buttons, it's going to be 6 right here and it was 3 for 2 so…"
> Lines 148-149: "Mr. Short is 6 paper clips tall. There is 6 paper clips for 4 buttons. 3 for 2. Mr. Tall is 6 buttons so 6 plus 3 is 9 for 6 buttons."

Tim did have some inconsistencies and overall he demonstrated a preference of emergent strategies throughout the test instrument.

Subject 7 – Sam.

Throughout the testing session, Sam's performance between test questions was inconsistent; varying in multiplicative reasoning strategies between levels 2 through 4. Despite his use of varying strategies, Sam obtained the correct answer for every question on the test instrument. Regarding Question 1, Sam performed as a Level 4 Multiplier as evidenced by transcript line numbers 27 -29 and written work as he successfully expressed multiplicative sentences.

> Lines 27-29: "Because when you do 2…it says Fish B is twice as large and when you do 2 twice, you can do multiplication or addition so I'm going to go with multiplication. After that, you do 2 times 2 and that's 4."

For Questions 2 and 3, Sam showed characteristics of a Level 2 Keyword Finder, as indicated in transcript lines 54, 57 and 66, and his written work.

> Line 54: "Because it says 4 times and times means to multiply."
> Line 57: "Fish A has 2 fish and Fish B is 4 times larger and has the word times and times means to multiply."
> Line 66: "Cause it has the word times and you do 2 times 4 it's 8."

Additionally, his written work indicates clue word "times" prompting him to multiply.

Sam showed evidence of a Level 3 Repeated Adder with Question 4 as exhibited by transcript lines 77-78, 81, 85 and his written work.

> Lines 77-78: "Because when you do 3 times…and nothing you can't do anything to multiply to get 3…so when you do 1, you add. You do 1 plus 1 plus 1 is 3."
> Line 81: "4: Fish B is 3 times larger than Fish A, and Fish B has 3 fish but you can't multiply but you can add."
> Line 85: "Because you do 1 plus 1 plus 1 is 3."

Additionally, Sam made the statement "You do 1 plus 1 plus 1 is 3", indicating use of repeated addition to obtain his final answer.

In Question 5, Sam stated in transcript line 98 that "Fish A is 1/3 of Fish B" strongly indicating that he was a Level 4 Multiplier. By concluding that division was needed to achieve 3 from the given amount of 9, and despite the fact that division was not mentioned in the problem, Sam showed a clear ability to coordinate objects, numbers and operations placing Sam at Level 4 Multiplier. Sam displayed a Level 4 Multiplier competency in his written work and transcript line 98: “Fish B is fed 9 fish and A is 1/3 of B so you divide and then you multiply [to check your answer].”

When presented with Question 6, Sam displayed the characteristics of a Level 4 Multiplier, where he again demonstrated his ability to achieve his answer through multiplication, as indicated by transcript lines 116 and 119-120.

> Line 116: “Because when you do…um ,because fish 2 has 2 green fish…so if you do 2 times 3 that’s 6…2 times 2 is 4.”
> Lines 119-120: “Because when you multiply that equals…if you multiply it by 2 that equals…um like 2 times 2 is 4 and then you do 3 times 2 that 6 or you could do 3 plus 3 or for this one you could do 2 plus 2.”

Sam exemplified his competence in Questions 7 and 8 by coordinating the numerical relationships as well as demonstrating that division is the reciprocal operation of multiplication. Sam consistently showed characteristics of a Level 4 Multiplier in his responses as corroborated by transcript lines 131, 161, and 165. Many fourth grade students have difficulty arriving at the conclusion that division is the inverse operation of multiplication.

> Line 131: “…larger number so instead of doing it, you do 8 divided by 2. And for Fish A you do 8 divided by 4.”
> Line 161: “Because 6 divided by 2 is 3.”
> Line 165: “Because when I do 6 times 3 that 18.”

With the presentation of Questions 9 and 10, Sam indicated characteristics of a Level 2 Measurer for both situations, as evidenced by transcript lines 196-197, 257-262 and written work. He laid pizza slices over each other and paperclips end-to-end using the manipulatives provided for these problems.

> Line 196-197: "Because this one is a smaller one compared to this one and if you do this, put another one there, it would equal…"
> Line 257-262: "I: Okay, we've got to get all the way up to the top of Mr. Tall, so how many more do we need?
> Line 258: Um, 5 more…maybe 4 more.
> Line 260: Yes.
> Line 262: 9.

In both questions, he was unable to coordinate operations with respect to quantities, but seemed better able to coordinate operations with respect to areas and ratios of pizza slices, and achieved the answers. Throughout the test instrument Sam was inconsistent with respect to the strategies he chose, varying strategies between multiplicative and pre-multiplicative.

Subject 8 – Jane.

The word "half" implies splitting and multiplicative reasoning (Confrey, 1994). In Questions 1 through 3, Jane demonstrated characteristics of a Level 4 Multiplier and Splitter, as evidenced in transcript lines: 33, 54, 69, 71, and written work. In line 33: Jane stated that "Because 2 times 2 is 4 is twice as much… As big as the fish" and continues with similar reasoning in line 54: "Since it is half size, it will have… A Fish will have 3 and B will have 6 because half of 6 is 3". Additionally, Jane reasoned similarly in Line 69: "Since it is 4 times larger than Fish A, it should get 8" and Line 71: "Because I added 2 four more times." Adding 2 four more times is indicative of a repeated adder (Steffe, 1988).

Beginning with Question 4, Jane reverted to a Level 3 Repeated Adder as supported by the word "subtracted" in line 83: "I subtracted 3 minus… I did… I uhm… Since he's 3 times larger than it, I subtracted 2." Jane was not demonstrating multiplicative reasoning logic and was relying on additive strategies (subtraction for this problem) which were successful in the past.

With Question 5, Jane demonstrated characteristics of a Level 4 Multiplier, as evidenced by transcript line 105 "Because 3 can go in 9, 3 times" and written work. The word "in" implies division and hence multiplication was utilized.

With the addition of Fish C to be fed, beginning in Question 6, Jane dropped to a Level 2 Adder, revealed in transcript line 121 and written work. The complexity of the additional fish to feed resulted in this problem becoming a multi-step problem, perhaps causing her inability to complete the task as line 121: "2 plus 2 is 4 and 4 plus 2 is 8" supports the thinking Jane exhibited.

Continuing with Questions 7 and 8, Jane exhibited characteristics of a Level 2 Quantifier as evidenced by transcript lines 152 and 166-167, and she provided the solution for Problem 7, but an incorrect answer for Problem 8. Jane added the same number each time to arrive at the solution for the next fish. She utilized the concept that A < B < C when assigning amounts to the fish, but was not accurate with the ratios, quantities and relationships needed to achieve success.

> Line 152: "Because since that has 8, then that one would have 4 at least. This one would have 2 at least."
> Line 166-167: "Since… Because each time you go up 3, you can go up 3. Because if you have A it will have 3, then B will have 6, and the large 9. So it can have enough and will not starve."

With the introduction of the area problem in Question 9, Jane demonstrated characteristics of a Level 3 Coordinator by exhibiting a limited ability to coordinate objects, numbers, and operations. Jane was not able to state an accurate mathematical equation to solve this problem such as 3/8 is less than 3/4 or 2/8 equals 1/4. In line 211 Jane stated that "… 1/4 is bigger than 1/8" indicating an ability to coordinate the objects within in the denominator of the fractions, but did not succeed in describing the necessary numerators of the fractions needed for accurate comparison to solve this pizza problem.

From Jane's responses in Question 10, one may infer that Jane was a Level 2 Counter, as there was no indication of any attempt to predict the height of Mr. Tall via calculation. In transcript lines 243-248 and her computer work, Jane's response "3… 5… 6… 9" revealed that her answer was arrived at by counting up to the correct number or by measuring the height of Mr. Tall in paperclips.

Jane was not consistent with her use of strategies and never used the same strategy as much as 50% of the time throughout the test instrument.

Subject 9 – Mark.

Beginning with Item 1 Mark demonstrated his ability to quantify objects as indicated in Line 58: "One more fish would feed Fish B because he is larger." Mark did not offer an additive strategy or a multiplicative strategy as a solution to Problem 1 placing him at a Level 2 Quantifier and yielding an incorrect answer for Problem 1. In Item 2 Mark achieved the solution by making use of the word "double" to multiply, placing him at Level 4 Multiplier for this item. Mark noted in Line 69: "Fish B is double the fish that Fish A is." Similarly, Mark achieved the solution for Item 3 by uttering the phrase "4 times larger" to indicate a need to multiply. Mark noted in Line 88: that "Fish B is 4 times larger than Fish A so Fish B needs 8 fish," placing him at Level 4 Multiplier for this item.

By observing Mark work on Item 4, it was apparent that he was performing some type of mental measurement to obtain the solution of 1 green fish, placing him at a Level 2 Measurer. Mark measured the relative size of the large yellow fish and determined in lines 105-106: "B is 3 times larger than Fish A and also eats 3 fish a day and if you move Fish A back some, then it will look like 3 times the fish, so I would feed Fish A 1 fish."

In Item 5, Mark understood that the number of fish in Fish B (9) must be reduced to a smaller number, but he did not understand how to arrive at the correct answer (3). Clearly he did not know what 1/3 of 9 is, making him a Level 2 Adder, since he believed he needed to give (add) Fish A 4 in line 123: "um…if you give it 3 and um plus a 1/3 makes… you give it 4."

On his first multi-step problem, Item 6, Mark did well utilizing a multiplicative strategy by expressing multiplicative sentences and performed as a Level 4 Multiplier. In lines 134 and 138 he stated "… 2 times 3 is 6" and "… 2 times 2 is 4". In Problem 7, Mark continued to display characteristics of a Level 4 Multiplier when he articulated the two multiplicative sentences needed to explain this problem, as indicated in transcript line 156: "Because 8 divided by 2 equals 4" and "…because 4 times 2 is 8". In Item 8, Mark reverted to an earlier strategy, that of Level 3 Coordinator, where he demonstrated good coordination of objects, numbers, and operations but was not able to achieve a solution for both of the large yellow fish. In transcript line 178 Mark conveyed a sound multiplicative sentence "6 divided by 2 is 3," a solution for Fish A, but failed with his reasoning for Fish C as indicated in Line 182: "Because Fish C is 3 times larger than Fish B and 3 times 3 is 9".

In Question 9 Mark was not able to explain a multiplicative strategy that would lead to a mathematical prediction about the relative sizes of the two pans of pizza. Mark began to compare the sizes of the pizza pieces provided using the manipulatives and declared in lines 244 and 255 of the transcript that Pan B was the answer because of the areas on [Pan] B are larger than the [areas on] Pan A. Such careful measuring led him to a correct answer for this question and placed him at Level 2 Measurer.

Concerning Item 10, Mark stated in line 284: "Well, I remember that Mr. Tall was way up there…so I kind of measured it in my eyes", placing him at a Level 2 Measurer.

In summary for Mark, we see a pattern of various strategies, none of which predominate, placing him in the category of students who utilized a set of emergent strategies.

Subject 10 – Matt.

Beginning with Item 1 and continuing through Item 3, Matt demonstrated the attributes of a Level 4 Multiplier by consistently stating and writing multiplication sentences as evidenced by transcript lines 37, 60-61, 74, and written work. For example, in line 37 Matt demonstrated his understanding of multiplication by explaining that "Because 2 times 2 is 4" that 4 is the number of green fish to feed the large yellow fish. Below we see other examples of multiplication sentences spoken by Matt.

> Line 60: "Because it says Fish A is fed 3 green fish each day and Fish A is half of the size of Fish B, so it would"
> Line 61: "be 6 because 3 is half of 6."
> Line 74: "I'm thinking that 2 times 4 equals 8."

When Matt was introduced to Problem 4, he demonstrated some confusion with how to divide three by three to obtain the identity element for multiplication. The only justification that Matt could find to explain his thinking is found in transcript line 107, where Matt stated "Because I um, subtracted 2 [sic]" (Matt later corrected himself in transcript line 117, meaning he knew he should have subtracted 2 to obtain the correct answer of 1 when utilizing an additive strategy). The fact that Matt used an additive strategy to explain his thinking for Problem 4 (as indicated by the word subtract) places him at Level 2 Adder.

For Problem 5, Matt appeared confused (perhaps by the need to use the fraction one-third) as he concluded that the solution was 19. Matt obviously used an additive strategy that had little correspondence to the solution that was needed. In transcript line 140 he stated "I wrote 9 plus 10 equals 19". Such behavior placed Matt at Level 2 Adder.

Although Matt reverted to less difficult mathematical strategies regarding his solutions to Problems 4 and 5, Matt actually performed better when he encountered the multi-step multiplicative problems presented in Items 6 through 8. On Problems 6 through 8 Matt consistently stated all six multiplicative sentences needed to correctly describe problems 6, 7 and 8 as indicated in the following transcript lines:

> Line 163: "I wrote 2 times 3 equals 6 and 2 times 2 equals 4."
> Line 171: "Fish C is 4 times larger than A and um, for Fish A it would be 2 because 2 times 4 is 8 and for Fish B it would be 4."
> Lines 188-189: "Because it says Fish B is 2 times larger than Fish A and Fish B eats 6 green fish each day and 3 times 2 equals 6."

On Question 9 Matt demonstrated his measuring ability by imagining laying the pizza pieces over the top of one another to form the conclusion that the answer was Pan B "…because that is the bigger piece." Utilizing comparative measuring to arrive at a correct conclusion shows that Matt performed at Level 2 Measurer.

Question 10 again demonstrated that Matt employed Level 2 Measurer strategies to obtain the correct answer of nine. He did make an educated guess of 12 paperclips. However, upon measuring the height of Mr. Tall by utilizing the available manipulatives provided for this problem, he concluded that the answer was "… probably 9…" in line 325 of the transcript. Matt was careful to lay the paperclips end-to-end without overlap and he did begin measuring at the ground level, demonstrating his ability as a Level 2 Measurer.

Matt performed in the emergent category by employing a multiplicative strategy only 60% of the time.

Subject 13 – Linda.

When a student writes or states multiplicative sentences, she is exhibiting behaviors and characteristics of a Level 4 Multiplier. Linda began the test instrument as a Level 4 Multiplier by stating in line 33 of the transcript the answer was "…4… [because] you have to double 2" and in line 35 "2 times 2 is 4." Linda seemed to become confused on Question 2. Perhaps it was Linda's encounter with the word "half" which perplexed her, but for some reason she believed that 3 needed to be divided by 2 when in reality, 3 needed to be multiplied by 2. On line 47 of the transcript she stated: "3 can't be divided by 2." In Question 2 Linda behaved as a Level 3 Coordinator because she was able to coordinate the idea that some type of division needed to occur, as well as properly handle the remainder for a difficult division problem, even though she did not succeed in getting the correct answer for this test item.

On Question 3 Linda performed the test item flawlessly and identified her strategy as searching for clue words as revealed in transcript lines 66 where Linda stated that "…times is a clue word to multiplication so 4 times 2 is 8." Using clue words to define the operation is performing at the stage of a Level 2 Keyword Finder.

On Question 4, Linda performed once again as a Keyword Finder, but this time she was not successful at arriving at the solution. Here Linda made her choice of operation based on the keyword "times" to help her understand that she should apply the multiplication algorithm in line 74 where she stated "Because Fish B is three times as large as Fish A and Fish B is fed 3 fish each day so I thought about multiplying 3 by 2". Her justification placed her at Level 2 Keyword Finder.

On Question 5 Linda rebounded with a good multiplicative sentence, which clearly indicates that Linda understood the relationship between the numbers and objects and places her at Level 4 Multiplier for this problem. Transcript line 88 supports her explanation: "Because if I put 3 fish in the bottom and count all the way until I get 9 then it turns into 1/3 and 1/3 of 9 is 3."

Linda next encountered the first of three multi-step problems found in this test instrument. Question 6 did not give Linda any problems whatsoever, and she clearly and quickly expressed two multiplicative sentences in transcript lines 109 and 114.

> Line 109: "Because Fish A is fed 2 green fish and C is 3 times so I multiplied 3 times 2 equals 6."
> Line 114: "Fish B is 2 times larger than Fish A and Fish A is fed 2 so I multiplied 2 times 2 is 4."

Stating such succinct multiplicative sentences clearly places her as a Level 4 Multiplier.

Linda's behavior for Question 7 was very similar to her behavior in Question 6. She quickly expressed two multiplicative sentences, which clearly defined the problem as evidenced by transcript line numbers 124 and 126. These multiplicative sentences place Linda as a Level 4 Multiplier.

> Line 124: "Because Fish C eats 8 green fish and Fish C is 2 times larger than Fish B so 8 divided by 2 is 4."

Line 126: "Because Fish C is twice as large as B so I thought we should divide and 8 divided by 2 is 4."

Once again Linda's behavior for Question 8 was consistent with her behavior on the previous two multi-step problems. She articulated two multiplicative sentences, which describe the problem in transcript line numbers 140-141 and 146.

Lines 140-141: "Because Fish B is fed 6 green fish each day and Fish C is 3 times larger than Fish B so I did 3 times 6 equals 18."
Line 146: "Because 6 times 3 equals 18."

Linda again performed as a Level 4 Multiplier.

On Question 9 Linda performed as a Level 5 Predictor by identifying the numerators and the denominators of the fractions representing both pans of pizza. Not only did she identify the fractions but she was then able to compare the two fractions correctly in a simple multiplicative reasoning sentence as evidenced by transcript line number 177 where she stated that "3/8 is smaller than 3/4". She then utilized that information to correctly state the solution to the problem as Pan B.

On Question 10 Linda did not offer any multiplicative strategy or proportional strategy that led to a multiplicative sentence that could predict the height of Mr. Tall without the need for measurement. She decided to use the computer screen to measure the height of Mr. Tall using the on-screen paperclips. She was very careful to lay the paperclips end-to-end without overlap starting from the ground up to measure the height of Mr. Tall. Transcript line 221 indicates this as she counts the remaining paper clips "…7, 8, 9" and noted that the correct answer would be 9. These behaviors are clear indications of a student performing at Level 2 Measurer.

Overall, Linda performed most often as a Level 4 Multiplier. Linda also performed once as a Level 5 Predictor on Problem 9. When coupled with her five performances at that of Level 4 Multiplier Linda performed at level 4 or higher 60% of the time making her an emergent performer throughout this test instrument. When faced with Questions 6 through 8 of the test instrument Linda's performance actually rose slightly as she moved from Level 2 Keyword Finder to Level 4 Multiplier for every multi-step problem. Such behavior would indicate that Linda may have been more interested or challenged by the more complex problems.

Subject 14 – Megan.

Megan began the test instrument by choosing her operation based on the word "twice", indicating that Megan performed at Level 2 Keyword Finder, as stated in transcript line number 39 that "Twice means times 2." When Megan moved to Question 2 and was confronted with the fraction 1/2 Megan clearly provided a multiplicative sentence to fully describe Question 2. For example, in transcript line 57 she stated: "Cause half of 6 equals 3 but if you add another 3 it will equal 6." And in transcript line 65, she similarly stated: "Because it says that Fish A is fed 3 green fish each day. Half of … Fish A is half the size of Fish B. And half … so like half of 6 would equal 3."

On Question 3 Megan demonstrated the characteristics of a Level 4 Multiplier, as indicated in transcript line 81 and her written work. She expressed a concise multiplicative statement in transcript line 81when she affirmed that "4 times 2 equals 8."

Megan seemed a little confused on Question 4, perhaps because this question required that the student find the identity element for multiplication by dividing three by itself to obtain one. The length of time that Megan pondered this question may indicate that she was trying to come to a solution but she was not successful. Transcript line 92 states "3 times 3 … cause it says times and Fish B is 3 times larger than Fish A. And Fish A eats 3 green fish." Megan finally settled on the keyword strategy of times as indicated in transcript line 92.

When Megan was presented with the unfriendly fraction 1/3 in Question 5 she again appeared a bit confused. In transcript line 153 Megan reasoned that a solution could be found because "…3 and 3 and 3 equals 9." She continued in transcript line 155 with "yeah, and these go into um…uh… a third of it." She actually was right on track with her thinking at this point but then encountered difficulty in transcript line 158 where she concluded "I think 3 because half of 9 is 3 because it's 3, 6, 9." At the beginning, Megan was able to correctly coordinate the numbers and the relationships between objects and operations, but she did get confused before the problem was concluded. Therefore, she utilized multiplicative strategies and coordination of objects, but without success, placing her clearly at Level 3 Coordinator.

When Megan encountered Problem 6, the first of the multi-step problems for this test instrument, she did well. She quickly stated two multiplicative sentences which described the problems fully, as evidenced in transcript line number 173 "Because if … Fish A has 2 fish and Fish B is two times larger, it would be 2 times 2 equals 4". And transcript line 178 “Um … Fish C is 3 times larger than Fish A and fish has 2 fish and 3 times 2 is 6”. On Item 6 Megan clearly performed at Level 4 Multiplier.

With respect to Question 7 Megan found it a challenge to work backwards from the large fish that was already filled with green fish. Subjects, who are unsure of what to do, will offer an answer of 32 for this question. They are quite often using the keyword strategy. When this happens it is because the subject is looking at the words of the problem and seeing the numbers four and eight and the word “times”. When utilizing a keyword strategy, they will arrive at a solution of 32 fish for either Fish B or Fish A. But Fish B and Fish A should both receive smaller quantities than Fish C since Fish C, is the largest, with eight green fish. Megan did not correctly address this issue and concluded her answers should be 32 and 16, as evidenced by her written work. Megan was so intent on performing the algorithm associated with the keyword “times”, she demonstrated a strategy of Level 1 Non-quantifier because she ignored $A < B < C$.

On Question 8, Megan rebounded soundly by making two multiplicative statements describing the problem. In transcript line 284 Megan stated “… 6 times 3 equals 18.” This correctly described the quantity for Fish C. When Megan calculated the quantity for Fish A, she used the concept of splitting to split the number 6 into two 3s as evidenced by transcript line 309. “… A smaller number like you would split 6”. She explained in more detail in line 313 when she said: “Because um, half of 6 is 3.” She continued her explanation in line 315 and 316 “Yeah, and Fish B eats 6 so you could split Fish B and you could make it … split Fish B and half of 6 equals 3.” Megan clearly performed as a Level 4 Splitter for Problem 8.

In Question 9 Megan utilized the strategy of splitting to split the larger pieces of pizza in Pan B, which were in fourths into smaller, but equivalent, two-eighths. This is evidenced by transcript line 369 and 370.

> Line 369: "Because this has more slices and even though they're bigger you can split it…you can look at this"
> Line 370: "you can split this and this, that equals 2…and that one and one equals 4… equals 6."

She understood that splitting the three pieces in Pan B would yield six pieces of the same size as those pictured in Pan A. She concluded that after splitting the pieces in Pan B, she had six pieces of pizza, which was more than the three equal sized pieces found in Pan A. Clearly Megan was operating as a Level 4 Splitter.

Question 10 proved too difficult for Megan to actually predict the height of Mr. Tall without measuring. She reverted to Level 2 strategies of measuring the height of Mr. Tall by laying the paperclips end-to-end starting from the ground and measuring the height of Mr. Tall on the computer screen. This is evidenced in transcript line 460 as Megan counted the paper clips once they were correctly placed on the computer screen, and derived the answer of nine paperclips. Megan performed as a Level 2 Measurer in this question.

Throughout the test instrument Megan performed at various levels of mathematical reasoning. She was inconsistent in her reasoning at least 50% of the time and reverted to Level 2 Adder strategies when unsure of the correct approach. The other 50% of the time Megan was at Level 4 Multiplier or Level 4 Splitter.

Multiplicative Reasoning Strategies

Subject 1 – Joe.

Beginning with Item 1 and continuing through Item 8, Joe consistently demonstrated the attributes of a Level 4 Multiplier or Level 4 Splitter by stating and writing multiplication sentences as evidenced by transcript lines 35, 55, 75, 88, 99-100, 123, 128, 132, 146, 150, 151, and written work. Line 35 is an example of a multiplication sentence used by the subject. "That if you do like 2 times 2 it will equal 4 and you give Fish B 4 fish each day." Below we see other examples of multiplication sentences spoken by Joe.

Line 55: "If 4 times 2 is 8, then you would feed Fish B 8?"
Line 75: "Because 1 times 3 equals 3"
Line 88: "If you divide 9, divided by 3, it will equal 3."
Lines 99-100: "If you do 2 times 2, it will equal 4 so you give Fish B four. If you do 3 times 2 it will equal 6. So you'll give Fish C six fish."
Line 123: Because 2 times 4 equals 8.
Line 128: If you do 2 times 4, you get 8. So you give A 2 fish.
Line 132: Because 2 times 2 equals 4. Then it would be 2, 4, 8.
Line 146: Because 3 times 6 equals 18.
Line 150-151: Okay, so if you do 2 times 6, if Fish B is two times larger than Fish A, you'll give Fish A 3 fish. You'll give fish A 3 fish, you'll give Fish C 18 fish.

Additionally, Joe demonstrated the characteristics of a Level 4 Splitter, as evidenced in his written work: "if you cut 6 in half, it will be 3". The word "cut" implies splitting and halving, signs of multiplicative reasoning (Confrey, 1994).

When the theme of the questions changed from the relationships of quantities of fish to the relationships comparing sizes of objects, Joe reverted to weaker multiplicative characteristics. For Question 9, Joe reverted to Level 3 Coordinator by demonstrating a limited ability to coordinate objects, numbers, and operations, as evidenced by transcript line number 193: "Pizza B is larger because if you look at Pizza [A] you can see that it is in eighths and pizza B in fourths" and [evidenced by] written work. Joe was able to understand and coordinate the denominators of the two fractions, but did not expressly enumerate the numerator for each fraction necessary for a succinct mathematical sentence for problem 9, such as 3/8 is less than 3/4.

Similarly, on Item 10, Joe reverted to weaker characteristics of a Level 2 Adder, utilizing an unsuccessful additive strategy to achieve an answer to this problem, as evidenced by transcription lines 207-210 and written work: "If Mr. Tall is 8 buttons long and Mr. Short is… I mean, Mr. Tall is 6 buttons tall and Mr. Short is 4 buttons tall, then Mr. Short is 6 buttons… I mean 6 paperclips tall, and Mr. Tall is 8 paperclips tall because if you look over here, there are 4 buttons right here and you add 2 more and if you look over there are 6 paper clips right here and you add 2 more." Joe demonstrated the ability to measure the height of Mr. Tall in paperclips, but did not achieve success, nor was he able to show he could predict the height of Mr. Tall using mathematics. Throughout, Joe demonstrated a very consistent behavior, utilizing a level 4 strategy eight of the ten times on the test instrument.

Subject 11 – Mary.

Mary consistently demonstrated the characteristics of a Level 4 Multiplier, throughout Questions 1 – 8 on the assessment. She expressed multiplicative sentences in transcript lines: 52, 64, 75, 85, 105, 123, 140-141, 155, 161-162, and her written work.

Line 52: I put number 1: 4 fish and 2 times 2 equals 4.
Line 64: I wrote 6 fish cause 3 plus 3 is 6 or 3 times 2 is 6.
Line 75:I put 8 fish cause 4 times 2 is 8 or 6 plus 2 is 8.
Line 85: I mean 3 divided by 3 is 1 in each.
Line 105: I wrote 3 fish because 9 divided by 3 is 3
Line 123: Because 3 times 2 is 6 and 2 times 2 is 4.
Line 140-141: I wrote Fish A gets 2 fish each day and Fish B gets 4 fish each day. And then I wrote 8 divided by 4 is 2 and 8 divided by 2 is 4.
Line 155: Cause 6 divided by 2 is 3 and if C is 3 times larger than B, 6 times 3 is 18.
Line 161-162: Cause if C is 3 times larger than B and B eats 6 fish each day, that means 6 times 3 is 18 or 6 plus 6 plus 6 is 18.

Mary's responses were characteristic of a Level 5 Predictor for Question 9 as substantiated by transcript line 210 and written work: "I said B cause 3/8 is smaller than 3/4". The above sentence indicates the ability of the subject to develop a unitizing factor for each pizza to be measured (the denominator). Similarly, Mary was able to quantify the numerator as 3 for each pizza. Mary was then able to compare the two fractions in a multiplication sentence, stating that 3/8 is a lesser quantity than 3/4 so Pan B has more pizza. Such ability shows that Mary can predict the measure of an object in one mathematical system given the measure of a proportional or similar object in another mathematical system.

On Question 10, Mary reverted to an earlier multiplicative reasoning level: Level 2 Measurer, as demonstrated by transcript lines 276-289 and computer work. Mary showed that each measurement has a starting and ending point without overlap or gap between unit measures in computer work. Lines 276-289: "1, 2, 3, 4, 5. … Yeah. … 1, 2, 3, 4, 5, 6, 7, 8. … Yep. … Well that's a little tall…. 9." The transition from a more visually defined problem, Problem 9, to a more abstract problem, Problem 10, proved too difficult for Mary to achieve the mastery of a Level 5 Predictor for Question 10.

Mary was a very consistent participant as she utilized Level 4 Multiplier strategy 8 out of 10 times. Of the two remaining times she utilized an even higher strategy of Level 5 Predictor once. There were only two other students to perform at Level 5 Predictor. Mary was at or above Level 4 Multiplier strategy 90% of the time.

Summary

When analyzing the subjects it became apparent that they can be sorted into three categories. Those who were consistent in their choice of level 4 strategies or above (more than 70% of the time) I label as multiplicative reasoners. Pre-multiplicative are those students who consistently chose strategies below level 4 in their attempts to solve the problems. Those subjects who used no particular strategy I label as emergent. Emergent subjects are considered non-multipliers. Table 4.3 summarized these findings.

Table 4.3: Multiplicative Reasoning Strategies

Pre-Multipliers (7 times out of 10)	Emergent (5/5 or 4/6)	Multipliers (7 times out of 10)
Subject 2 - Paul	Subject 3 - Tim	Subject 1 - Joe
Subject 4 - Cheryl	Subject 7 - Sam	Subject 11 - Mary
Subject 5 - Sue	Subject 8 - Jane	
Subject 6 - Ava	Subject 9 - Mark	
Subject 12 - Eric	Subject 10 - Matt	
	Subject 13 - Linda	
	Subject 14 - Megan	

Item Analysis

Table 4.3 is the frequency of occurrence for each level and strategy by item. Problems one through eight are multiplicative problems and it is apparent by the large numbers in the column under Level 4 Multiplier that students attempted to use a multiplicative strategy many times for these questions. The pattern of retreating to known safer strategies when faced with a problem where the subject was unsure of the correct strategy was found as a consistent behavior throughout this investigation. Table 4.3 begins answering research question three. What multiplicative reasoning strategies do the items on the test instrument invoke in fourth grade students?

Overview

I sorted the items from the data in Table 4.3 into four categories. Category one (Items 1-3), non-discriminators of reasoning, are those items which failed to separate subjects who chose multiplicative reasoning due to keywords from those who chose multiplicative reasoning because of understanding. Category two (Items 4 and 5), scheme extension discriminators, are those items which separated those subjects who were able to correctly iterate their schemes in a forward and reverse direction from those who could not extend their schemes either forward or in reverse. Category three (Items 6-8), unequal groups discriminators, separated those who could use fractions with odd numbered denominators from those who could not (Sharp & Adams, 2002). Category four (Items 9-10), rational number discriminators, separated those who could work in the rational number domain from those who could not.

Table 4.4: Item Analysis Frequency Table

	Level 5 Predictor	Level 4 Splitter	Level 4 Multiplier	Level 3 Co-ordinator	Level 3 Repeated Adder	Level 2 Measurer	Level 2 Quantifier	Level 2 Adder	Level 2 Counter	Level 2 Keyword Finder	Level 1 Spon-taneous Guesser	Level 1 Non-quantifier
1			9		3		1			1		
2		4	6	1	2					1		
3			11							3		
4			3		2	1		5		3		
5		2	3	1				6		1	1	
6			7		1			3	1	1	1	
7			8				1	1			1	3
8		1	6	1				2		2	2	
9	2	1	1	2		7	1					
10	1					9		1	3			
Σ	3	8	54	5	8	17	3	18	4	12	5	3

Non-Discriminators of Reasoning – Items 1 to 3

Question 1:
Tank 1. Fish B is twice as large as fish A. Fish A eats 2 green fish each day. How many green fish should you feed fish B each day?
Question 2:
Tank 2. Fish A is fed 3 green fish each day. Fish A is half the size of Fish B. How many fish should you feed Fish B each day?
Question 3:
Tank 3. Fish A is fed 2 green fish each day. Fish B is 4 times larger than Fish A. How many green fish do you feed Fish B each day?

Children utilize a variety of clues in their endeavors to solve multiplicative reasoning word problems. Sowder (1988) studied students' solutions to simple one-step arithmetic word problems. Sowder found evidence that children mechanically associate a key word with an arithmetical operation. Because children utilize superficial clues, items 1-3 of the test instrument are non-discriminators; they fail to separate the multipliers from the students who apply multiplication through rote algorithms (Sowder, 1988). Each of the three items had key words such as "twice", "half", and "times". With key words being introduced in the problems it is difficult, if not impossible, to ascertain the motive behind the students' use of multiplicative strategies, meaning that although multiplicative strategies were used, the problems failed to discriminate between those who understood multiplication from those who were just performing algorithms. For example, on Item 1, Sam stated: "I was thinking, because Fish B is twice as large as Fish A and Fish A is feeded 2 fish and twice means add or multiply". In Question 2, Jane said, "Since it is half size it will have 3 and Fish B will have 6 because half of 6 is 3". Eric, on Item 3 stated: "I did 2 times 4 which equals 8 because it said Fish B is 4 times larger than Fish A, so that's how I got the answer". Although the subjects obviously used multiplication, it is unclear why. Was it because they understood the relationships presented or because they were applying a keyword strategy?

Dienes and Golden (1966) studied the makeup of the multiplier. Their study suggested that the multiplier counts sets. The set that becomes the smallest common denominator is the unitizing factor for the item (Behr, Harel, Post & Lesh, 1994). For items 1-3 the students were provided the unitizing factor in Fish A, thus the students were not required to unitize, making the problems simple to do.

Sowder (1988), Fishbein et al. (1985), Vergnaud (1983), Nesher (1988), and Bell et al. (1989) described one-step problems as presenting no significant challenges for fourth grade students. More serious obstacles occur for fourth grade students when multi-step problems are encountered. Items 1-3 on the test instrument were one-step problems, which presented little challenge to these fourth grade students.

Research suggests that while most students are able to solve multiplicative reasoning items involving relatively small whole numbers, they revert to less advanced strategies, such as additive strategies, to solve more complex multiplicative reasoning problems involving larger whole numbers (>8), or those problems where the situation cannot be easily modeled in terms of equal groups (Siemon, Izard, Breed, & Virgona, 2006; Siemon & Virgona, 2001). Items 1-3 allowed for formation of equal groups, making them solvable in a host of methods. For example, many subjects mentioned that you could either multiply 3 x 2 to obtain 6, or you could add 3 + 3 to get 6. For example, Paul understood that Item 1 could be solved because "2 plus 2 equals 4". He saw the idea of two being the size of equal groups. Ava: "It says 'half of Fish B' so add 3 plus 3 and got 6". Ava saw that two equal groups of three could add to six.

Scheme Extension Discriminators – Items 4 and 5

Question 4:
Tank 4. Fish B is 3 times larger than Fish A. If Fish B eats 3 green fish, how many green fish should you feed Fish A each day?
Question 5:
Tank 5. Fish B is fed 9 green fish each day. Fish A is one - third as big as Fish B. How many green fish should you feed Fish A each day?

The subject who is thinking multiplicatively regarding Question 4 will understand that division is required despite the key word "times" given in the problem. The subject should establish a multiplicative sentence, such as 3 / 3 = 1 to obtain the solution. Item 4 discriminates those subjects who can successfully extend their multiplying schemes to accommodate for division by an odd whole number (3) from those who cannot. A subject's multiplying scheme that can be expanded to account for division by an odd whole number is a multiplying scheme well integrated into the mechanics of the whole number system. Sharp and Adams (2002) propose that halving is a more fundamental skill enabling students to divide regions into power-of-two pieces whereas thinking in terms of 1/3 and 1/9 are better comprehended after understanding fractions such as 1/2 and 1/4.

Question 4 demonstrates that only 3 of the 14 subjects could articulate using words, diagrams, or symbolic expressions a multiplicative relationship between the numbers and the operations. For example, Ava, in transcript line #88 stated, “3 divided by 3 equals 1.” Mary had a similar idea when she stated, “I mean 3 divided by 3 is 1 in each.” Joe understood that one way to satisfy the relationships stated in the problem would be to write the math sentence: ___ x 3 = 3, and substituted 1 in the blank. Although Joe did not expressly declare that 3 / 3 = 1 he stated an equivalent mathematical sentence showing his understanding of the relationship between 3 and 1 in this problem and concluded that Fish A should be fed 1 fish, “Because 1 times 3 equals 3.”

Because this problem required the subjects to find the multiplicative identity element, those who utilized additive strategies by subtracting 3 from 3 were confused when they understood that it did not make sense to feed Fish A zero fish. The seven subjects who arrived at the answer using non-multiplicative strategies rendered justifications containing little, if any, connection to the logic of the word problem. For example, Tim said, “Fish B is 3 times biger than A so 3 – 2 = 1”. Jane: “I subtracted 3 minus…since he’s 3 times larger…I subtracted 2”. Matt: “Because 3 – 2 = 1”. Megan expressed that Fish A could not eat zero “cause then the fish would starve”, but 3 – 3 would equal zero. So her dilemma was that Fish A has to eat, so Fish A must eat at least one green fish. The data suggest that problems requiring subjects to use appropriate strategies in finding the identity element for multiplication are difficult for these fourth grade students.

In Question 5, six of fourteen subjects understood that the unit for Fish A was derived by dividing 9 by 3 to obtain 3. Steffe (1988) introduced us to the idea of children's multiplying schemes. Steffe explained that schemes consist of three parts: a trigger, an activity and a result. In the process of developing multiplicative reasoning, children develop schemes, first for counting, then for addition, then for multiplication. But each student's theme develops differently. Students' themes that are correctly integrated into the mechanics of the whole number system are easily expandable to arrive at the solution of more complex problems such as Item 5, where multiplication by a challenging fraction (1/3) is needed. Students whose schemes fall apart when such expansion is required have not adequately integrated their multiplying schemes into the mechanics of the whole number system. The weakness displayed by 8 of the 14 subjects on Item 5 of the test instrument suggests that their multiplying schemes lack the development needed to advance to multiplication by a challenging fraction, such as 1/3. For example, Mary and Linda both stated, "9 / 3 = 3". The 6 subjects who were successful on Item 5 demonstrated their ability to reason multiplicatively by indicating that their schemes were expandable to the next mathematical level when presented as a trigger with a problem covering unfamiliar territory.

Unequal Groups Discriminators – Items 6 to 8

Question 6:
Tank 6. Fish C is 3 times larger than Fish A. Fish B is 2 times larger than Fish A. If Fish A is fed 2 green fish each day, how many green fish should you feed Fish B? How many green fish should you feed Fish C?

Question 7:
Tank 7. Fish C is 4 times larger than Fish A. Fish C is 2 times larger than Fish B. If Fish C eats 8 green fish each day, how many green fish should you feed Fish A? Fish B?

Question 8:
Tank 8. Fish B is 2 times larger than Fish A. Fish C is 3 times larger than Fish B. If Fish B eats 6 green fish each day, how many green fish will you feed Fish A? How many green fish will you feed Fish C?

Items 6, 7, and 8 on the test instrument are multi-step multiplicative reasoning word problems. They are labeled "two-step" because each of the problems requires 2 multiplication operations in order to correctly complete the problem. Question 6 requires multiplication in both steps, Question 7 requires division in both steps, and Question 8 requires a combination of the two.

In Question 6 the subject needs to first multiply 2 x 2 to obtain 4 for Fish B, and multiply 3 x 2 to obtain 6 for Fish C. For this problem the subjects did not need to unitize because the quantity consumed by Fish A was given. Some subjects achieved the solution by stating the even numbers 2, 4, and 6, and others by presenting the correct multiplicative strategies. What I learned from this problem is that failure to achieve success is not specifically due to the multi-step nature of the problem, but when the model or situation cannot be expressed in terms of equal groups (Siemon & Virgona, 2001). The subject reads that Fish B is *two* times larger and Fish C is *three* times larger than Fish A; the 2 and the 3 are the unequal groups. For example, seven of the fourteen subjects managed the unequal groups successfully as indicated in their written work and transcripts when they stated, "Fish C is 3 times so multiply 3 x 2 and get 6 and Fish B is 2 times, so multiply 2 x 2 and get 4".

Similar thinking applies to Question 7 as it, likewise, does not provide for equal groups. The subject reads that Fish C is *four* times larger than Fish A and *two* times larger than Fish B. The same subjects who did not complete Question 6 correctly did not complete Question 7 correctly. In Question 7 the subject should divide 8 by 2 to obtain 4 for Fish B and divide 8 by 4 to obtain 2 for Fish A. The key word "times" led some subjects to multiply to obtain the answer for Fish A. Three subjects considered the solution for Fish A to be 32. Subjects who were not able to unitize, that is, derive the correct value for Fish A, may have done so because they did not understand that division is the inverse operation of multiplication. The subjects who did understand how to unitize and obtain the answer of 2 for Fish A may have been assisted in this problem by the provision of small even numbers (≤ 8). Success here is indicated by proper management of the unequal groups provided in the problem. Four subjects had a clear understanding of the need to organize the unequal groups (groups of 4 and 2) and wrote the multiplicative sentences 4 x 2 = 8 and 2 x 4 = 8. For example, Joe, Ava, Matt and Mary all said, "2 x 4 = 8 and 4 x 2 = 8".

For the most part, the subjects who had problems with Questions 6 and 7 had difficulties with Question 8 for similar reasons. Question 8 is not a problem that is easily modeled in terms of equal groups. The unequal groups in this problem are the *three* (times) and the *two* (times) (Siemon & Virgona, 2001). Question 8, as in Question 7, requires the subject to unitize/divide in order to find the value for Fish A. Several subjects obtained the correct answer for Fish C by using a key word strategy, for example, 3 x 6 = 18. Finding the unitizing factor for Fish A required division, in this case, 6 / 2 = 3. Key words were of little assistance and understanding was required. Subjects who were successful with Question 8 were able to unitize, understand, coordinate multi-step tasks, and utilize multiplicative strategies to solve problems involving relatively small whole numbers. As illustrated in the following transcript line excerpts, six subjects expressed "6 / 2 = 3 and 6 x 3 = 18", indicating the ability to manage unequal groups and to reverse a thought process which is a good demonstration of conceptual understanding (Inhelder and Piaget, 1958).

Items 6-8 became unequal groups discriminators because they separated those subjects who could work with unequal groups from those who could not. The formation of equal groups is a necessary step in processing to obtain a unitizing factor (Siemon & Virgona, 2001). For example, on Question 7 subjects who divided the number 8 into unequal groups of 4 and 2 were successful.

Rational Number Discriminators – Items 9 and 10

Question 9:
Assuming that both pizzas are equal in size, which pan has more pizza (the shaded portion)?
Question 10:
Mr. Short is 6 paper clips, or 4 buttons in height. Mr. Tall is 6 buttons in height. Mr. Tall is 6 buttons in height. How many paper clips would it take to measure the height of Mr. Tall?

Mathematically, Question 9 should be worked by counting all the available pizza slices in Pan A and Pan B, and substituting the number of slices as denominators, that is, denominator of 8 for Pan A and denominator of 4 for Pan B. The subjects should then conclude that 3/8 is less than 3/4, indicating that the subjects correctly formed a unitizing factor for both pans of pizza.

The unitizing factor for Pan A is the denominator 8 and the unitizing factor for Pan B is the denominator 4. The comparison factor is the numerator for each fraction, 3. Subjects not able to unitize mathematically concluded that 3 = 3 without unitizing them over the correct denominator. Most subjects could not explain mathematically why 3 does not equal 3. Those who could explain wrote concise math sentences such as 3/8 is less than 3/4 and utilized splitter, predictor, or multiplier strategies. Those who could not explain multiplicatively reverted to area model comparisons or measurement, signs of pre-multiplicative reasoning. This was typically accomplished by laying a piece of pizza over the top of another, using the manipulatives provided. Because the unitizing factors for this problem were not whole numbers, some subjects were moved to a realm indicative of measurement level of reasoning. Question 9 could be considered a rational number discriminator as it clearly delineated those who could think about rational number quantities from those who had to revert to measuring.

In Question 10, when subjects notice that the buttons and the paper clips are not multiples of each other, some understood the unitizing factor of 1 button = 1.5 paper clips. Once the subject realizes this mathematical extraction, one of two routes can be taken. A missing value proportion can be established, such as 6/x = 4/6, or the subject can state that 1.5 x 6 = 9. Either set up indicates a good understanding of rational numbers for this problem. Question 10 is similar to Question 9 in that it requires the subjects to derive a unitizing factor, which does not exist in the whole number system, but in the rational number system. Most subjects were challenged by this rational unitizing factor and reverted to Level 2 strategies: measuring or counting the paper clips needed to measure the height of Mr. Tall. One subject did as required and stated that 1 button = 1.5 paper clips, therefore 6 x 1.5 = 9. Question 10 is a rational number discriminator, as it requires unitizing and the ability to work in the rational number domain.

Perspective

In this analysis I have looked at multiplicative reasoning and found that some items in the test instrument are discriminators of multiplicative reasoning. Items 1-3 in category cannot distinguish students who are emergent in their multiplicative reasoning from those who are pre-multiplicative.

Items 4 and 5 in category two, scheme extension discriminators, will likely be useful as models from which to build questions assessing subjects who understand multiplicative reasoning well enough to perform reversibility operations from those who cannot. Subjects who demonstrated multiplicative reasoning ability understood that in order to work backwards with multiplication one must divide. Inhelder and Piaget's, (1958) development of the notion of reversibility has sparked the idea of placing reversibility as a central point of focus in the learning of mathematics (Lamon, 2007).

Items 6-8 in category three, unequal groups discriminators, separated those who could use fractions with odd numbered denominators from those who could not. Researchers such as Nunes and Bryant (1996) follow Piaget (1970) in proposing that multiplicative reasoning is primarily different from additive reasoning. Subjects who can only understand multiplication as repeated addition and cannot manage unequal groups, have an under-developed concept of multiplication (Siemon & Virgona, 2001). Items 6-8 will likely be useful as models from which to build questions to assess subjects who can manage equal, as well as unequal groups, and can reason multiplicatively, as well as additively, from those who cannot.

Items 9 and 10 may prove to be useful as exemplary questions in separating those subjects who have obtained the ability to work in the rational number domain from those who cannot. Unfortunately, Items 9 and 10 proved to be non-discriminatory with respect to multiplicative reasoning.

Summary

The responses given by students within this study support the idea that children who are pre-multiplicative in thought are not in a single state of "those who can think only additively". Subject data in this chapter demonstrate the five levels of multiplicative reasoning presented by Clark and Kamii (1996) provide a good framework but leaves many students at Level 2 uncategorized into sublevels called strategies. For example, students at Level 2 under Clark and Kamii (1996) framework could exhibit Level 2 Keyword Finder, Level 2 Counter, Level 2 Adder, Level 2 Quantifier, or Level 2 Measurer when examined under my expanded framework.

Data within this study suggest that fourth graders' levels of multiplicative reasoning are composed of three categories: multiplicative reasoning strategies, pre-multiplicative strategies, and emergent strategies. The data also suggest that fourth graders are in transition with respect to multiplicative reasoning meaning that they at times demonstrate multiplicative reasoning and at other times demonstrate non-multiplicative reasoning strategies on the same or similar problems. Thus, one may conclude that multiplicative reasoning is not either on or off, but instead is both on and off depending on the problem presented and other factors.

The chapter presented item data supporting the capability of the test instrument to bring forth model questions which will be useful in future work as a basis for building mathematics curriculum focusing on eliciting thought process from students which help build strong multiplicative reasoning skills. For example, questions similar to Question 8 may be developed that will encourage students to understand multi-step word problems involving fractions and strengthen both backwards and forward thinking.

CHAPTER V

EDUCATIONAL IMPLICATIONS

Overview

Piaget's (1965) studies suggest that children form a network of mental structures to facilitate their understanding of the surrounding world and their understanding of mathematics. Piaget listed stages of development of the typical child as illustrated in Table 5.1:

Table 5.1: Piaget's Developmental Stages

Stage	Description	Age
1	Sensory Motor Stage	Birth – 2yrs
2	Pre-operational Stage	2yrs – 7yrs
3	Concrete Operational Stage	7yrs – 11yrs
4	Formal Operations Stage	11yrs – 16yrs

Throughout this study I reviewed students' understanding of certain multiplicative reasoning word problems for fourth grade students ages nine and ten. Piaget states that around age eleven children are transitioning into the formal operations stage. One of the purposes of selecting fourth graders for my study is that some of them are at the age just prior, and some are entering this stage of development.

Comprehending the beginnings of students' multiplicative reasoning gains significance when we understand that the roots of advanced mathematical thinking lie in the makeup of multiplicative reasoning (Tall, 1991). Forming relationships between objects and the associated values of those objects is the beginning of multiplicative reasoning (Lampert, 2001; Piaget, 1970; Vergnaud, 1988). Children's concepts of multiplication originate in their schema of correspondences and not in the concept of addition (Piaget, 1965; Vergnaud, 1983, 1988; Nunes & Bryant, 1996). It is important to understand each student's schema of numbers and multiplication, because the depth of such understanding can lead to new ideas with respect to teaching mathematics to children (Confrey, 1991; Kieren, 1990; Kieren & Pirie, 1991; Piaget, 1973; Steffe & Cobb, 1988; Steffe, von Glasersfeld, Richards, & Cobb, 1983). Advanced mathematical thinking is viewed as beginning in elementary school with multiplicative reasoning (Clark & Kamii, 1996; Harel & Sowder, 2005; Tall, 1991; Piaget, 1965). Words that may be indicators of multiplicative reasoning are: as (for example, twice as big), area, split, half, one-half, one-third, divide, times, cut, more, less, double, larger, smaller, equal, sets, sets of sets, or their synonyms (Clark & Kamii, 1996; Confrey, 1994; Dienes & Golden, 1966; Inhelder & Piaget, 1958; Karplus, Pulos, & Stage, 1983; Steffe, 1994). This study supports the findings of those who have come before and answers this question: What are the signs of multiplicative reasoning among fourth grade students?

The Significance of the Development of Multiplicative Reasoning

This paper sheds new light on the development of multiplicative reasoning. Clark and Kamii (1996) aptly developed five levels describing multiplicative reasoning. This research builds on and refines this classification, allowing one to define and diagnose strategies of multiplicative reasoning more comprehensively. This study expands the framework of Clark and Kamii from five levels to twelve levels and sub-levels known as strategies, as illustrated in Table 4.1. Each of the strategies were employed at least three times, certifying the legitimacy of the new strategies.

Levels 1, 2, and 3 and their associated strategies were used frequently. The use of Level 4 Multiplier strategy occurred 54 times during this test instrument, indicating a trend toward multiplication, but the motives behind the use of the strategy may vary, and are sometimes difficult to discern. For example, many subjects used multiplicative reasoning as a result of the word “times” appearing in the problem, but did not explicitly inform the interviewer of the keyword strategy. More research is needed to develop questions for test instruments that more clearly discriminate between those subjects who are using multiplication because of understanding and those subjects who are using multiplication because of a keyword strategy.

Conclusions

Table 5.2 provides the difference in proportion correct of the most and least proficient students (column 4) and proportion correct for each item on the test instrument (column 5). A low value in column 4 indicates the test item does not discriminate with respect to multiplicative reasoning. A high value in column 5 indicates that the item was very easy for these fourth grade students.

Table 5.2: Difference in Most and Least Proficient Students

Item Number	Proportion Correct: Most Proficient Students	Proportion Correct: Least Proficient Students	Difference in Proportion Correct of Most and Least Proficient Students	Proportion Correct
1	1.00	1.00	0.00	0.93
2	1.00	1.00	0.00	0.93
3	1.00	1.00	0.00	1.00
4	1.00	0.40	0.60	0.71
5	0.80	0.00	0.80	0.43
6	1.00	0.20	0.80	0.57
7	1.00	0.00	1.00	0.64
8	1.00	0.20	0.80	0.50
9	1.00	1.00	0.00	1.00
10	0.80	1.00	-0.20	0.93

From the findings of this study, I made the following conclusions. The Multiplicative Reasoning Assessment Instrument contained five questions (Items 4, 5, 6, 7 and 8) that are appropriate discriminators among multiplicative reasoners, pre-multiplicative reasoners and emergent multiplicative reasoners. These questions possess a discrimination index above .35 indicating these items discriminate adequately between the high and low groups. Additionally, Items 5, 6, 7 and 8 have excellent discrimination, possessing indices greater than .51.

The remaining test items (Items 1, 2, 3, 9 and 10) are poor discriminators of multiplicative reasoning. At first glance the data suggest that these items are too easy for the subjects since the difficulty indices are well above .90. In reality, the reason for the high difficulty indices is that subjects used non-multiplicative reasoning strategies that were successful. Thus, these items were not sufficient in leading subjects to utilize multiplicative reasoning strategies alone. For example, Items 1-3 may invoke a multiplication strategy but not a multiplicative reasoning strategy, simply because the keyword "times" appears in the wording. In other words, it is quite possible to get the solutions to Items 1-3 without using multiplicative reasoning and the items do not discriminate between those who are using multiplicative reasoning and those who are using the multiplication algorithm.

Items 9 and 10 likewise have difficulty indices above .90, but only because obtaining the solution was arrived at by non-multiplicative reasoning strategies. In other words, the subjects measured the manipulatives or the computer screen to obtain the solution. But if the definition of solution is modified to mean "expressed a succinct mathematical sentence fully describing the relationships among the numbers and operations present in the word problem", then the difficulty indices are greatly reduced, as indicated in Table 5.3. For Item 9 the difficulty index decreases from 1.00 to .21 indicating that understanding Item 9 is far more difficult than measuring it. Similarly for Item 10, the index decreases from .93 to .07 indicating that understanding Item 10 is far more difficult than obtaining the solution via measurement. Perhaps in future testing the investigator should not allow measurement and counting as a choice of strategies when discriminating for multiplicative reasoning.

Item Number	Proportion Correct: Most Proficient Students	Proportion Correct: Least Proficient Students	Difference in Proportion Correct of Most and Least Proficient Students	Proportion Correct
1	1.00	1.00	0.00	0.93
2	1.00	1.00	0.00	0.93
3	1.00	1.00	0.00	1.00
4	1.00	0.40	0.60	0.71
5	0.80	0.00	0.80	0.43
6	1.00	0.20	0.80	0.57
7	1.00	0.00	1.00	0.64
8	1.00	0.20	0.80	0.50
9	0.20	0.20	0.00	0.21
10	0.00	0.00	0.00	0.07

Table 5.3: Modified Difference in Most and Least Proficient

The findings of this study suggest that mathematics teachers should not only look at whether answers are right or wrong, but also consider the understanding behind the solutions children derive. By expanding the marker table for level mastery (Clark and Kamii, 1996) from its original 5 to its current 12 levels and strategies, this researcher demonstrated that students can obtain the correct answer by several incorrect strategies, as well as correct ones. Therefore, it becomes the responsibility of the listening, caring teacher to guide students utilizing incorrect strategies (yet achieving the correct answer) to a path using a correct strategy.

Future Research

This research suggests that the children in this study gravitated toward one or more of the twelve strategies of mathematical thinking. The levels were created as a result of the utterances, behaviors, and writings of these students. More research is needed to identify additional areas surrounding the development of multiplicative reasoning which can affect a student's progress toward advanced mathematical thinking. What are the critical ages with respect to mathematical development and the development of multiplicative reasoning? What are the clues and signs that a child or a group of children are at a particular developmental stage? How can we construct a reliable test to discriminate multiplicative reasoning from those subjects who are doing multiplication by a memorized algorithm? In what manner should such a test instrument be administered to allow for identification of students' misconceptions? How can identification of multiplicative reasoning levels suggest the necessary curriculum that promotes the development of mathematical reasoning? How can a test instrument be constructed in order to identify the indicators of multiplicative reasoning among fourth grade students? The answers to these questions, together with the findings in this dissertation, have potential to provide important markers in the development of students' multiplicative reasoning and a narrowing of the achievement gap in mathematics.

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APPENDIX

MULTIPLICATIVE REASONING TEST INSTRUMENT

Multiplicative Reasoning

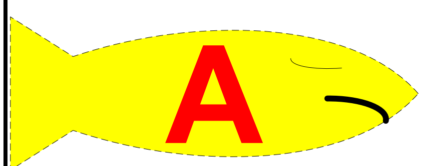
By Jim Carrier

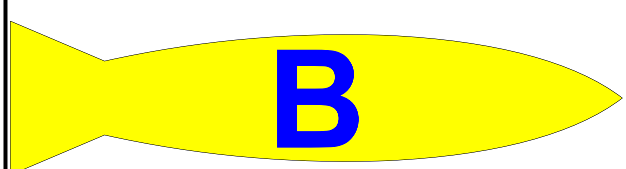
Version
5.12.2008

DIRECTIONS FOR TEST INSTRUMENT

Directions

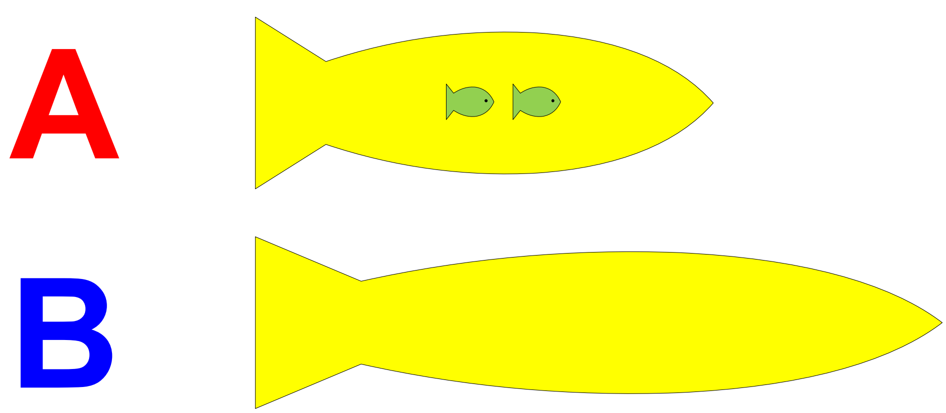
Your job is to feed the yellow fish with the small green fish.
If you feed the yellow fish too much they will get sick.
If you feed the yellow fish too little they will starve.
Figure out how many green fish to feed the yellow fish.



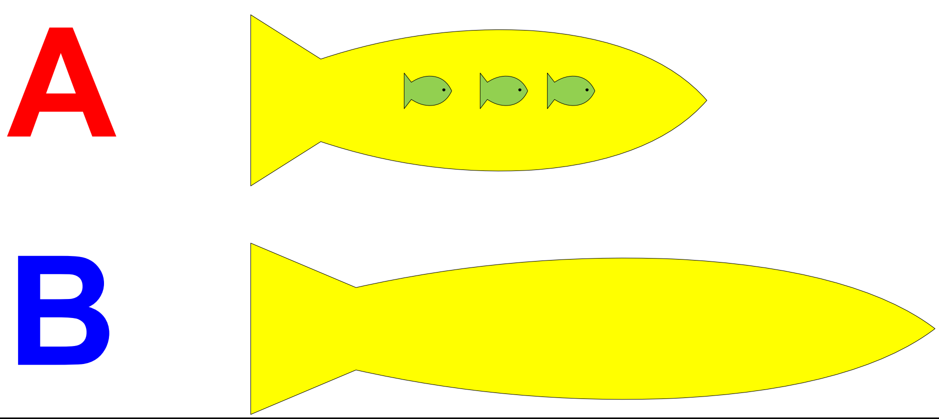


QUESTION 1

Tank 1. Fish B is twice as large as fish A. Fish A eats 2 green fish each day. How many green fish should you feed Fish B each day?

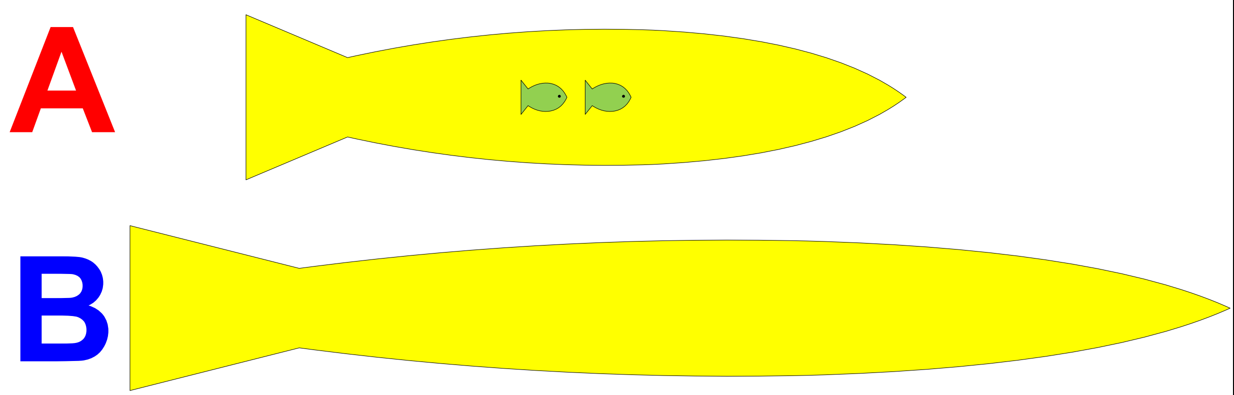


Tank 2. Fish A is fed 3 green fish each day. Fish A is half the size of Fish B. How many fish should you feed Fish B each day?



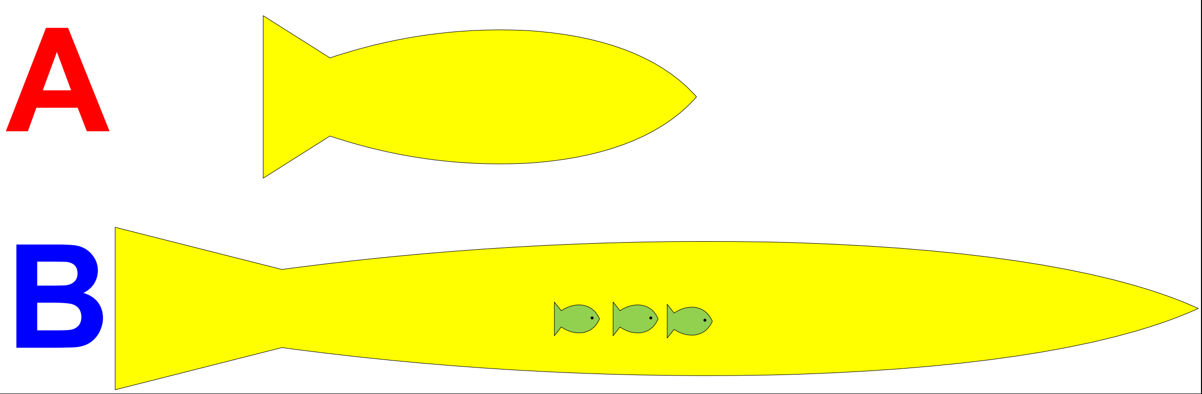
QUESTION 3

Tank 3. Fish A is fed 2 green fish each day. Fish B is 4 times larger than Fish A. How many green fish do you feed Fish B each day?



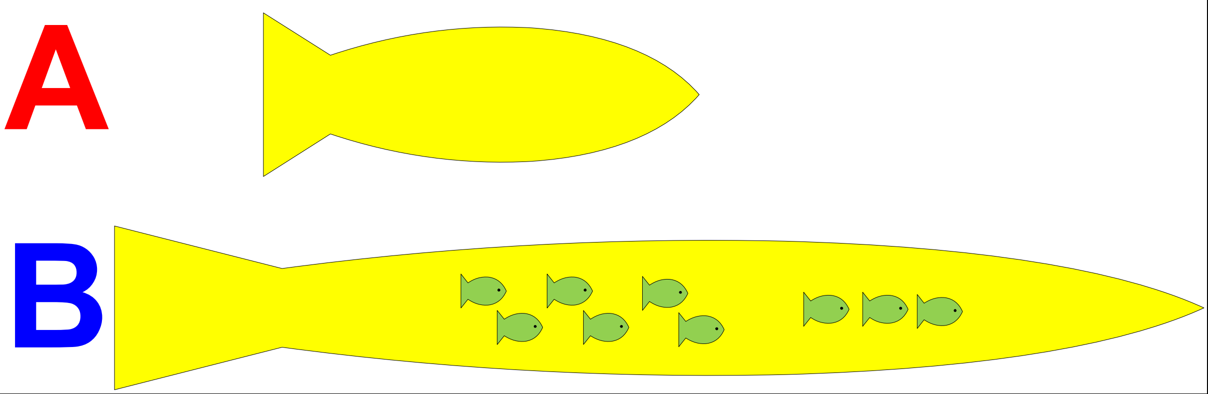
QUESTION 4

Tank 4. Fish B is 3 times larger than Fish A. If Fish B eats 3 green fish, how many green fish should you feed Fish A each day?



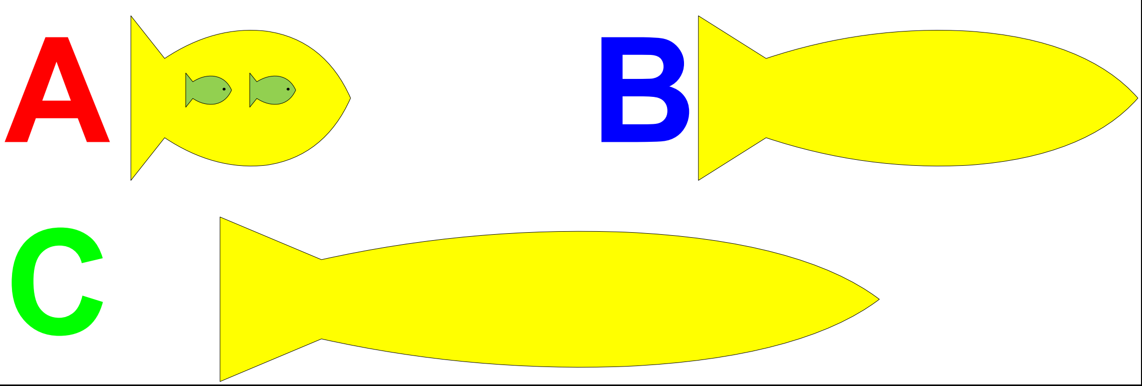
QUESTION 5

Tank 5: Fish B is fed 9 green fish each day. Fish A is one-third as big as Fish B. How many green fish should you feed Fish A each day?



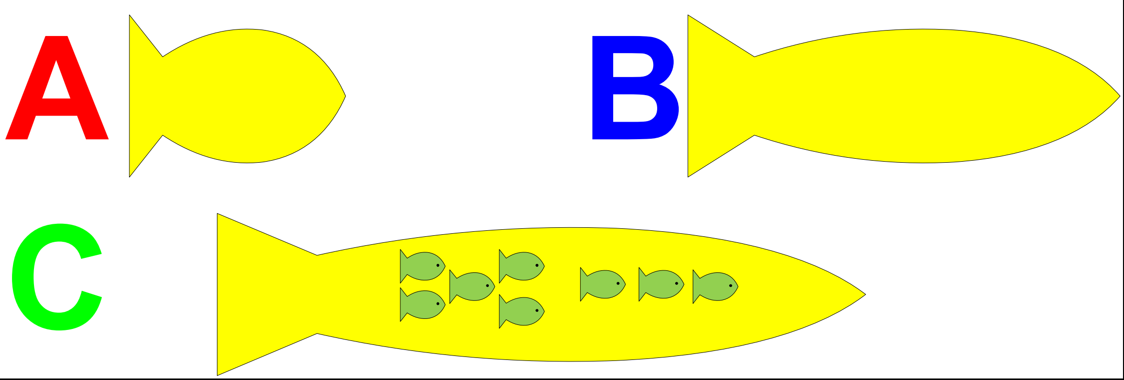
QUESTION 6

Tank 6: Fish C is 3 times larger than Fish A. Fish B is 2 times larger than Fish A. If Fish A is fed 2 green fish each day, how many green fish should you feed Fish B? How many green fish should you feed Fish C?



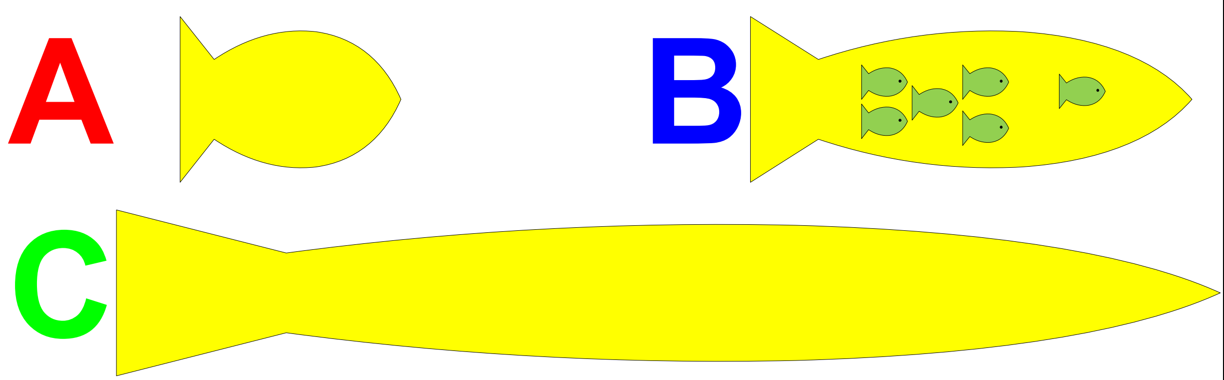
QUESTION 7

Tank 7: Fish C is 4 times larger than Fish A. Fish C is 2 times larger than Fish B. If Fish C eats 8 green fish each day, how many green fish should you feed Fish A? Fish B?

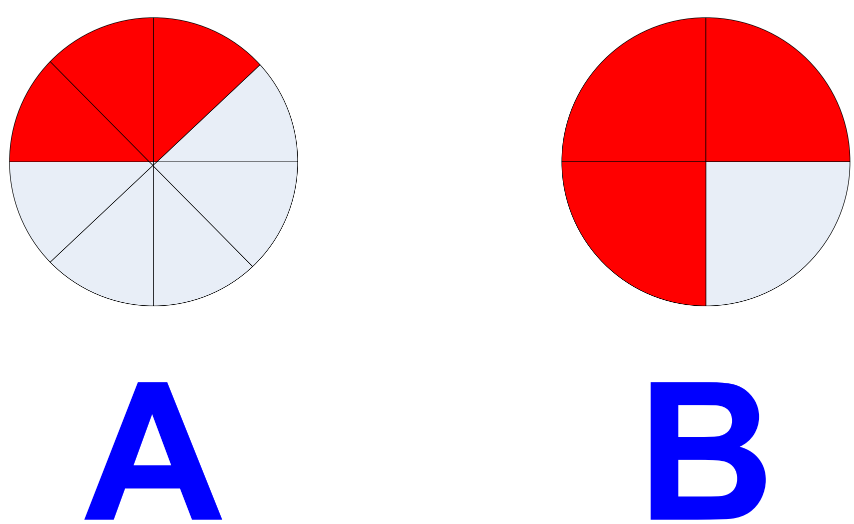


QUESTION 8

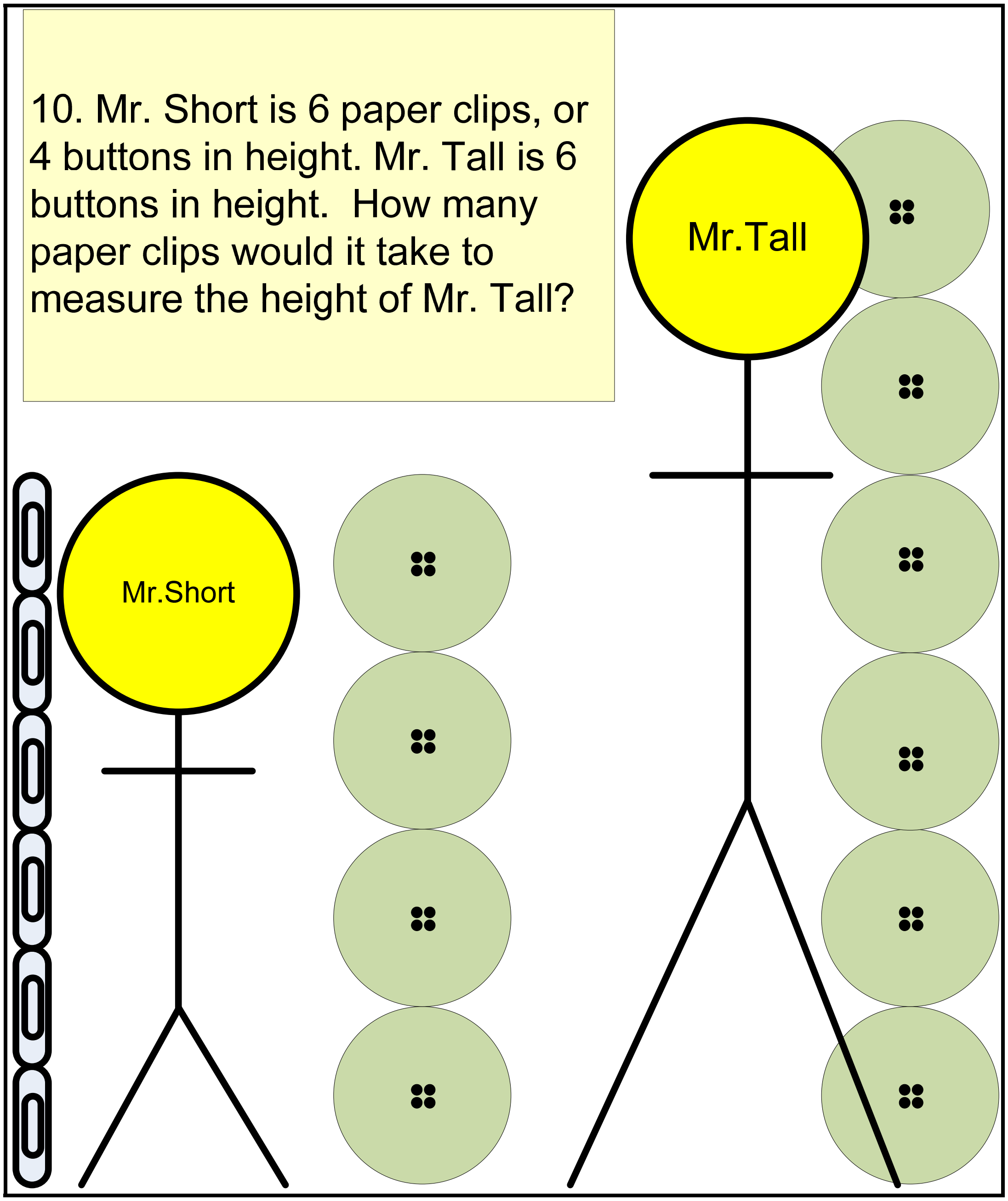
Tank 8: Fish B is 2 times larger than Fish A. Fish C is 3 times larger than Fish B. If Fish B eats 6 green fish each day, how many green fish will you feed Fish A? How many green fish will you feed Fish C?



9. Assuming that both pizzas are equal in size, which pan has more pizza (the shaded portion)?



QUESTION 10


10. Mr. Short is 6 paper clips, or 4 buttons in height. Mr. Tall is 6 buttons in height. How many paper clips would it take to measure the height of Mr. Tall?
Mr.Tall
Mr.Short