Hall numbers of some complete $k-$partite graphs

Julian A. Allagan
School of Science Technology Engineering and Mathematics,
Gainesville State College, Watkinsville, GA- 30677, USA
aallagan@gmail.com

Abstract

The Hall number is a graph parameter closely related to the choice number. Here it is shown that the Hall numbers of the complete multipartite graphs $K(m, 2, \ldots, 2)$, $m \geq 2$, are equal to their choice numbers.

1 Introduction

Throughout this paper, the graph $G = (V, E)$ will be a finite simple graph with vertex set $V = V(G)$ and edge set $E = E(G)$.

A *list assignment* to the graph G is a function L which assigns a finite set (list) $L(v)$ to each vertex $v \in V(G)$.

A *proper $L-$coloring* of G is a function $\psi : V(G) \to \bigcup_{v \in V(G)} L(v)$ satisfying, for every u, $v \in V(G)$,

(i) $(v) \in L(v)$,

(ii) $uv \in E(G) \to \psi(v) \neq \psi(u)$.

The *choice number* or *list$-$chromatic number of* G, denoted by $ch(G)$, is the smallest integer k such that there is always a proper $L-$coloring of G if L satisfies $|L(v)| \geq k$ for every $v \in V(G)$. With χ denoting the chromatic number, it is easy to see, and well known, that $\chi(G) \leq ch(G)$. The extremal equation $\chi(G) = ch(G)$ is a major research interest; see [1], [2], and [3].

1.1 Hall's Theorem

Theorem 1. *(*P. Hall [5]*). Suppose $A_1, A_2, \ldots, A_n$ are (not necessarily distinct) finite sets. There exist distinct elements $a_1, a_2, \ldots, a_n$ such that $a_i \in A_i$, $i = 1, 2, \ldots, n$, if and only if for each $J \subseteq \{1, 2, \ldots, n\}$,
$|\bigcup_{j \in J} A_j| \geq |J|$.*

A list of distinct elements $a_1, \ldots, a_n$ such that $a_i \in A_i$, $i = 1, \ldots, n$, is called a *system of distinct representatives* of the sets $A_1, \ldots, A_n$. A proper $L-$coloring of a complete graph K_n is simply a system of distinct representatives of the finite lists $L(v)$, $v \in V$, and any list $A_1, A_2, \ldots, A_n$ of sets can be regarded as lists assigned to K_n. Therefore, as noted in [6], Hall's theorem can be restated as:

Theorem 2. *(*Hall's theorem restated*). Suppose that L is a list assignment to K_n. There is a proper $L-$coloring of K_n if and only if, for all $U \subseteq V(K_n)$, $|L(U)| = |\bigcup_{u \in U} L(u)| \ \geq \ |U|$.*

Let L be a list assignment to a simple graph G, H a subgraph of G and $\mathcal{P}$ a set of possible colors. If $\psi : V(G) \to \mathcal{P}$ is a proper $L-$coloring of G, then for any subgraph $H \subset G$, ψ restricted to $V(H)$ is a proper $L-$coloring of H.

For any $\sigma \in \mathcal{P}$, let $H(\sigma, L) = < \{v \in V(H) \mid \sigma \in L(v)\} >$ denote the subgraph of H induced by the support set $\{v \in V(H) \mid \sigma \in L(v)\}$. For convenience, we sometimes simply write H_σ.

For each $\sigma \in \mathcal{P}$, $\psi^{-1}(\sigma) = \{v \in V(G) \mid \psi(v) = \sigma\} \subseteq V(G_\sigma)$; $\psi^{-1}(\sigma)$ is an independent set because ψ is a proper $L-$coloring. Further, $\psi^{-1}(\sigma) \cap V(H) \subseteq V(H_\sigma)$. So, $|\psi^{-1}(\sigma) \cap V(H)| \leq \alpha(H_\sigma)$ where α is the vertex independence number. This implies that
$\sum_{\sigma \in \mathcal{P}} \alpha(H_\sigma) \geq \sum_{\sigma \in \mathcal{P}} |\psi^{-1}(\sigma) \cap V(H)| = |V(H)|$ for all $H \subseteq G$.

When G and L satisfy the inequality

$$\sum_{\sigma \in \mathcal{P}} \alpha(H_\sigma) \geq |V(H)| \qquad (3.1)$$

for each subgraph H of G, they are said to satisfy ***Hall's condition***. By the discussion preceding, Hall's condition is a necessary condition for a proper $L-$coloring of G. Because removing edges does not diminish the vertex independence number, for G and L to satisfy Hall's condition it suffices that (3.1) holds for all induced subgraphs H of G.

Hall's condition is sufficient for a proper coloring when $G = K_n$, because if H is an induced subgraph of K_n then for each $\sigma \in \mathcal{P}$,

$$\alpha(H_\sigma) = \begin{cases} 1 & \textit{if } \sigma \in \bigcup_{v \in V(H)} L(v) \\ 0, & \textit{otherwise.} \end{cases}$$

So

$$\sum_{\sigma \in \mathcal{P}} \alpha(H_\sigma) = |\bigcup_{v \in V(H)} L(v)| \ ;$$

therefore Hall's condition, that

$$\sum_{\sigma\in\mathcal{P}} \alpha(H_\sigma) \;\geq\; |V(H)|$$

for every such H, is just a restatement of the condition in Theorem 2. (It is necessary to point out here that if $\sigma \notin L(v)$ for all $v \in V(H)$ then H_σ is the null graph, and $\alpha(H_\sigma) = 0$.) Consequently, Hall's theorem may be restated: For complete graphs, Hall's condition on the graph and a list assignment suffices for a proper coloring.

The temptation to think that there are many graphs for which Hall's condition is sufficient can be easily dismissed. Figure 1 is the smallest graph with a list assignment L_0 for which Hall's condition holds, and yet G has no proper L_0-coloring.

Remark.

It is clear that if H is an induced subgraph of G and $H \neq G$, then $H \subseteq G - v$ for some $v \in V(G)$. So, if $G - v$ has a proper $L-$coloring, then $H \subseteq G - v$ must satisfy (by necessity) (3.1). Thus, in practice, in order to show that G and L satisfy Hall's condition, it suffices to verify that $G - v$ is properly $L-$colorable for each $v \in V(G)$ and that G itself satisfies the inequality (3.1).

Denoted by $h(G)$, the ***Hall number*** of a graph G is the smallest positive integer k such that there is a proper $L-$coloring of G, whenever G and L satisfy Hall's condition and $|L(v)| \geq k$ for each $v \in V(G)$. So, by Theorem 2, $h(K_n) = 1$ for all n. In [6] the following facts are shown:

1. If $|L(v)| \geq \chi(G)$ for every $v \in V(G)$ then G and L satisfy Hall's Condition.

2. $h(G) \leq ch(G)$ for every G.

3. If $ch(G) > \chi(G)$ then $h(G) = ch(G)$.

4. If $h(G) \leq \chi(G)$ then $\chi(G) = ch(G)$.

5. If H is an induced subgraph of G then $h(H) \leq h(G)$.

Facts 3 and 4, are essentially equivalent since χ, $h \leq ch$, make h a parameter of interest of study of the extremal equation $\chi(G) = ch(G)$. These facts and the following theorems underline our findings in the next section.

Theorem A.(Erdös, Rubin and Taylor [2]) Let G denote the complete $k-$partite graph $K(2, 2, \ldots, 2)$. Then $ch(G) = k$.

Theorem B.(Gravier and Maffray [3]) Let G denote the complete $k-$partite graph $K(3,3,2,\ldots,2)$. If $k>2$, then $ch(G)=k$.

When $k=2$, it is shown that $ch(K(3,3))=3$. See [4].

Corollary B. Let G denote the complete $k-$partite graph $K(3,2,\ldots,2)$. Then $ch(G)=k$.

Proof. Since $K(3,2\ldots,2)$ is a complete $k-$partite graph, $k=\chi(K(3,2\ldots,2))\leq ch(K(3,2\ldots,2))$. Further, $K(3,2\ldots,2)$ is a subgraph of the complete $k-$partite graph $K(3,3,2,\ldots,2)$. Therefore $ch(K(3,2\ldots,2))\leq k$ if $k>2$. Thus, $ch(K(3,2\ldots,2))=k$ if $k>2$. When $k=2$, we have $K(3,2)$, of which it is well known that the choice number is 2. See [4], for instance.

□

Theorem C. (Enomoto et al. [1],2002) Let G_k denote the complete $k-$partite graph $K(4,2,\ldots,2)$. Then

$$ch(G_k)=\begin{cases} k & if\ k\ is\ odd \\ k+1 & if\ k\ is\ even. \end{cases}$$

Theorem D. (Enomoto et al. [1]) Let G denote the complete $k-$partite graph $K(5,2,\ldots,2)$. If $k\geq 2$ then $ch(G)=k+1$.

Corollary D. Let G denote the complete $k-$partite graph $K(m,2,\ldots,2)$. If $k\geq 2$ and $m\geq 5$, then $h(G)=ch(G)\geq k+1$.

Proof. Since $ch(G)\geq ch(K(5,2\ldots,2))=k+1>k=\chi(G)$, $h(G)=ch(G)$ by the previous fact 3.

□

2 Hall numbers of some complete multipartite graphs

Throughout this section, L is a list assignment to $V(G)$ such that for each $v\in V(G)$, $L(v)\subset\mathcal{P}$, a set of symbols. If $\sigma\notin L(v)$ for all $v\in V(G)$, then G_σ is the null graph. Further, we denote by ψ, any attempted proper $L-$coloring of G.

2.1 Example

The following example originally appeared in [6]. Consider the complete bipartite graph $K(2,2)$ in Figure 1 with parts $V_i = \{u_i, v_i\}$, $i = 1,2$ and L_0 the list assignment indicated.

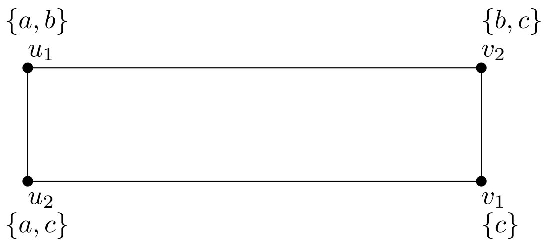

Figure 1: A list assignment to K(2,2).

If v_1 is colored c, as it must be, then u_2 must be colored a and v_2 must be colored b in a proper coloring, so u_1 cannot be properly colored.

However, we will show that G and L_0 satisfy Hall's condition using the argument described in a previous remark. First, for each $v \in V(G)$, it is easy to see that $G - v$ is properly L_0-colorable, meaning every proper induced subgraph $H \subset G$ satisfies, with L_0, the inequality (3.1) in Hall's condition. We now proceed to verify the inequality (3.1) for G itself.

Now, $\alpha(G_c) = 2$ and $\alpha(G_b) = \alpha(G_a) = 1$. So, $4 = \sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) \geq |V(G)| = 4$. Thus, G and L_0 satisfy Hall's condition and yet G has no proper L_0-coloring. Thus, $1 < h(G) \leq 2$ by Fact 2 and Theorem A. Therefore, $h(G) = 2$.

2.2 Some Hall numbers

Theorem 3. *$h(K(2,\ldots,2)) = k$ when $k \geq 2$.*

Proof. Let the partite sets of the complete $k-$partite graph $G = K(2,\ldots,2)$ be $V_1,\ldots,V_k$ with $V_i = \{u_i, v_i\}$, for $i = 1,2,\ldots,k$.

In Example 2.1, we showed that $h(G) = k$ when $k = 2$. So, to complete the proof, we suppose $k \geq 3$.

Let A be a nonempty set of colors with $|A| = k-2$ and a, b, c be distinct colors not in A. We define L a list assignment to G as follows:

1. $L(u_1) = A \cup \{a,b\}$, $L(u_2) = L(u_3) = \ldots = L(u_{k-1}) = A \cup \{a\}$, $L(u_k) = A \cup \{c\}$ and

2. $L(v_1) = A \cup \{b, c\}$, $L(v_2) = L(v_3) = \ldots = L(v_k) = A \cup \{b\}$.

Observe that $|L(v)| \geq k-1$ for every $v \in V(G)$.

Claim 1. *The graph G is not properly $L-$colorable.*

Proof.

In the following cases, we consider all possible distinct ways to properly color the vertices of some part of G, say V_1. We then conclude that the remaining subgraph $H = G - V_1$ is not proper L'-colorable where $L' = L - \{\alpha_1, \alpha_2\}$, $\{\alpha_1, \alpha_2\} \in \bigcup_{v \in V_1} L(v)$. ($\alpha_1, \alpha_2$ are not necessarily distinct colors; they are the colors on V_1.) Let ψ denote the attempted proper coloring.

Case 1: $\psi(u_1) = b$ or $\psi(v_1) = b$.

Let $S =< \{v_2, \ldots, v_k\} >$, an induced subgraph of H. Then $k-2 = |A| = |\bigcup_{v \in V(S)} L'(v)| < |V(S)| = k-1$. Since the subgraph S is a clique, we cannot properly color S from L'.

Case 2: $\psi(u_1) = a$ and $\psi(v_1) = c$.

Similarly as described in case 1, by letting $S =< \{u_2, \ldots, u_k\} >$, it's clear that we cannot properly color S, from L'.

Case 3: $\psi(u_1) = \gamma$ or $\psi(v_1) = \gamma$ for some color $\gamma \in A$.

With S as in case 1, $k-2 = |\bigcup_{v \in V(S)} L'(v)| < |V(S)| = k-1$. Hence we cannot properly color H from L'.

Claim 2. $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) \geq |V(G)|$.

Proof.

It is clear that $\alpha(G_a) = \alpha(G_c) = 1, \alpha(G_b) = 2$; further, $\alpha(G_\sigma) = 2$ for every $\sigma \in A$. Hence $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) = 2(k-2) + 4 = 2k = |V(G)|$.

Claim 3. *Every proper induced subgraph H of G is properly $L-$colorable.*

Proof.

In the following cases we provide a (not necessarily unique) proper coloring for each induced subgraph H of G of the form $G - v$, $v \in V(G)$.

Case 1: $H = G - u_1$.

Let $\psi(v_1) = c$ and color the $2(k-2)$ vertices of the subgraph $G-(V_1 \cup V_2)$ with the colors from A (by coloring vertices of the same part with the same color). Then let $\psi(u_2) = a$ and $\psi(v_2) = b$.

Case 2: $H = G - v_1$.

Let $\psi(u_1) = a$ and color the $2(k-2)$ vertices of the subgraph $G-(V_1 \cup V_k)$ with the colors from A with the same color appearing on u_i and v_i, $i = 2, \ldots, k-1$. Then, let $\psi(u_k) = c$ and $\psi(v_k) = b$.

Case 3: $H = G - u_i$, for some $2 \leq i \leq k$.

Let $\psi(v_i) = b$ and color the remaining $2(k-2)$ vertices of the subgraph $G - (V_i \cup V_1)$ with the colors from A. Then, let $\psi(u_1) = a$ and $\psi(v_1) = c$.

Case 4: $H = G - v_i$, for some $2 \leq i \leq k-1$.

Let $\psi(u_i) = a$ and color the remaining $2(k-2)$ vertices of the subgraph $G - (V_i \cup V_1)$ with the colors from A. Then, let $\psi(u_1) = \psi(v_1) = b$.

Case 5: $H = G - v_k$.

Let $\psi(u_k) = c$ and color the $2(k-2)$ vertices of the subgraph $G-(V_1 \cup V_k)$ with the colors from A. Finally, let $\psi(u_1) = \psi(v_1) = b$.

From the previous claims, we can conclude that $h(G) > k-1$. Thus, by Theorem A and Fact 2, $h(G) = k$. This concludes the proof. □

Corollary 3: $h(K(3, 2 \ldots, 2)) = k = h(K(3, 3, 2 \ldots, 2))$ for $k > 2$.

Proof. From Theorem 3, fact 5 and Theorem B, $k = h(K(2, 2 \ldots, 2)) \leq h(K(3, 2 \ldots, 2)) \leq h(K(3, 3, 2 \ldots, 2)) \leq ch(K(3, 3, 2 \ldots, 2)) = k$. Thus, $h(K(3, 2 \ldots, 2)) = k = h(K(3, 3, 2 \ldots, 2))$. □

We note that when $k = 2$, $h(K(3, 2)) = 2$ since $2 = h(K(2, 2)) \leq h(K(3, 2)) \leq ch(K(3, 2)) = 2$ by Corollary B. Also, since $ch(K(3, 3)) = 3$ by [4], it is clear from Fact 3 that $h(K(3, 3)) = 3$.

Theorem 4. *Let G denote the complete $k-$partite graph $K(4, 2, \ldots, 2)$ with $k \geq 2$. Then*

$$h(G) = \begin{cases} k & if\ k\ is\ odd \\ k+1 & if\ k\ is\ even. \end{cases}$$

Proof. When k is even, from Theorem B we have that $k = \chi(G) < ch(G) = k+1$. Thus, from Fact 3, it is clear that $h(G) = ch(G) = k+1$ for all even $k \geq 2$.

Suppose $k \geq 3$ is odd.

Let the partite sets, or parts, $V_1, V_2, \ldots, V_k$ of the complete $k-$partite graph G be $V_1 = \{x_1, x_2, x_3, x_4\}$ and $V_i = \{u_i, v_i\}$, $i = 2, \ldots, k$, $k \geq 2$.

Let C_1 and C_2 be disjoint $k-2$ sets of colors and 0 an object not in $C_1 \cup C_2$. Let $A = C_1 \cup \{0\}$, $B = C_2 \cup \{0\}$. Let A_1, A_2 and B_1, B_2

be disjoint $(k-1)/2$ sets of colors partitioning A and B respectively, and let $0 \in A_2 \cap B_2$. Let a,b be distinct objects not in $A \cup B$. Define a list assignment L to G as follows:

1. $L(u_2) = A$, $L(v_2) = B$, $L(u_i) = C_1 \cup \{a\}$ and $L(v_i) = C_2 \cup \{b\}$, for every $3 \leq i \leq k$ and

2. $L(x_1) = A_1 \cup B_1$, $L(x_2) = A_1 \cup B_2$, $L(x_3) = A_2 \cup B_1$ and $L(x_4) = A_2 \cup B_2 \cup \{a\}$

Notice that $|L(v)| = k-1$ for every $v \in V(G)$.

Claim 1. *G is not properly $L-$colorable.*

Proof.
Every proper $L-$coloring of $G - V_1 = K(2, \ldots, 2)$ uses $k-1$ elements of $C_1 \cup \{0, a\}$ and $k-1$ elements of $C_2 \cup \{0, b\}$. We proceed by exhausting the possible cases in attempts to properly $L-$color G.

Case 1: suppose $\psi(u_2) \neq 0 \neq \psi(v_2)$. Then all of the colors of $C_1 \cup C_2 \cup \{a, b\}$ will be used to color $G - V_1$. Hence we cannot color x_1 (since $A_1 \cup B_1 \subset C_1 \cup C_2$).

Case 2: suppose $\psi(u_2) = \psi(v_2) = 0$

Case 2.1: $\psi(u_i) \neq a$ and $\psi(v_i) \neq b$ for every $3 \leq i \leq k$.

Then all of the colors of $C_1 \cup C_2$ will be used to color $G - (V_1 \cup V_2)$. Once again we cannot color x_1.

Case 2.2: $\psi(u_i) = a$ and $\psi(v_j) = b$ for some $i, j \neq 2$.

Then there remains exactly one color, say $c_1 \in C_1$ and exactly one color, say $c_2 \in C_2$. If $c_1 \in A_1$ and $c_2 \in B_1$, then we cannot color x_4. Likewise if $c_1 \in A_1$ and $c_2 \in B_2$, then we cannot color x_3. Also if $c_1 \in A_2$ and $c_2 \in B_1$, x_2 cannot be colored and if $c_1 \in A_2$, $c_2 \in B_2$, x_1 cannot be colored.

Case 2.3: $\psi(u_i) \neq a$ for all $i \neq 2$ and $\psi(v_j) = b$ for some $j \geq 3$. Then there remains exactly one color, say $c_2 \in C_2$ and none of C_1. As in the previous case, if $c_2 \in B_1$, then we cannot color x_2. Likewise if $c_2 \in B_2$, then we cannot color either of x_1 and x_3.

Case 2.4: $\psi(u_i) = a$ for some $i \geq 3$ and $\psi(v_j) \neq b$ for all $j \geq 3$. Then there remains exactly one color, say $c_1 \in C_1$ and none of C_2. As before, if $c_1 \in A_1$, then we cannot color either of x_3 and x_4. Likewise if $c_1 \in A_2$, then we cannot color either of x_1 and x_2.

Case 2.5: $\psi(u_i) \neq a$ and $\psi(v_j) \neq b$ for all $3 \leq i, j \leq k$. Clearly the coloring cannot be properly extended to any of x_1, x_2, x_3.

Notice that we can skip the case where $\psi(u_2) = 0$ and $\psi(v_2) \neq 0$ (or vice versa), since if there is a proper $L-$coloring with one of u_2, v_2 colored with 0, then there is a proper $L-$coloring with both colored 0.

From the previous cases we can conclude that G is not properly $L-$colorable.

Claim 2. $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) \geq |V(G)|$.

Proof.

Notice that $\alpha(G_\sigma) = 2$ for every $\sigma \in C_1 \cup C_2$. Also $\alpha(G_0) = 3$ and $\alpha(G_a) = \alpha(G_b) = 1$. Hence $\sum_{\sigma \in \mathcal{P}} \alpha(G_\sigma) = 2(2(k-2)) + 5 = 4k - 3 \geq 2k + 2 = |V(G)|$ for every $k \geq 3$.

Claim 3. *If $k \geq 5$, then every proper induced subgraph H of G is properly $L-$colorable.*

Proof.

We proceed by considering the possible subgraphs of G obtained by deleting a single vertex.

Case 1: $H = G - u_i$, for some i.

Let $\psi(x_2) = \psi(x_3) = \psi(x_4) = 0$. Color $G - V_1$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ (colors a, b included). Hence there remains exactly one unused color of C_1, say c_1, and arrange that $c_1 \in A_1$. Let $\psi(x_1) = c_1$.

Case 2: $H = G - v_i$, for some i. Following the coloring argument in the previous case, there remains exactly one unused color of C_2, say c_2, and arrange that $c_2 \in B_1$. Let $\psi(x_1) = c_2$.

Case 3: $H = G - x_1$. Let $\psi(x_2) = \psi(x_3) = \psi(x_4) = 0$. It is easy to see that we can color the remaining subgraph $G - V_1$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ (a, b included).

Case 4: $H = G - x_2$. Let $\psi(u_2) = \psi(v_2) = 0$, and $\psi(x_4) = a$. Color the vertices of $G - (V_1 \cup V_2)$ with the colors from $C_1 \cup C_2 \cup \{b\}$ (b included). Then there remains exactly one unused color of C_2, say c_2, and arrange that $c_2 \in B_1$. Let $\psi(x_1) = \psi(x_3) = c_2$.

Case 5: $H = G - x_4$. Let $\psi(u_2) = \psi(v_2) = 0$. Color the vertices of $G - (V_1 \cup V_2)$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ (a, b included). Then there remains exactly one unused color of C_1, say c_1, and arrange that $c_1 \in A_1$, and exactly one unused color of C_2, say c_2, and arrange that $c_2 \in B_1$. Let $\psi(x_1) = c_1 = \psi(x_2)$ and $\psi(x_3) = c_2$.

Case 6: $H = G - x_3$. Let $\psi(u_2) = \psi(v_2) = 0$. Color the vertices of $G - (V_1 \cup V_2)$ with the colors from $C_1 \cup C_2 \cup \{a, b\}$ (a, b included). Then there remains exactly one unused color of C_1, say c_1, and arrange that $c_1 \in A_1$, and exactly one unused color of C_2, say c_2, and arrange that $c_2 \in B_2$. Let $\psi(x_1) = c_1 = \psi(x_2)$ and $\psi(x_4) = c_2$.

Notice here that when $k = 3$, $A_2 = B_2 = \{0\}$. Therefore, the attempted coloring of $H = G - x_3$ in case 6 fails, and, in fact H is not properly $L-$ colorable. However, $H = G - x_3$ with the given list assignment L satisfies the inequality (3.1). We can safely end the proof here when $k = 3$.

Still, there follows a list assignment specifically for the case when $k = 3$, which we hope will be of interest.

We define a list assignment L to $G = K(4, 2, 2)$ as follows:

1. $L(u_2) = \{1, 0\}$, $L(v_2) = \{2, 0, c\}$, $L(u_3) = \{1, a\}$, $L(v_3) = \{2, b\}$ and
2. $L(x_1) = \{1, 2\}$, $L(x_2) = \{1, 0\}$, $L(x_3) = \{0, a\}$ and $L(x_4) = \{b, c\}$

It is easy to verify that G and L satisfy the previous claims 1 and 2. We proceed therefore to verify only claim 3 for the subgraphs H of $K(4, 2, 2)$ in the following cases.

Case1: $H = G - u_2$.

Let $\psi(v_2) = 2, \psi(u_3) = a, \psi(v_3) = b$. Also $\psi(x_2) = 0 = \psi(x_3), \psi(x_1) = 1$ and $\psi(x_4) = c$.

Case2: $H = G - v_2$.

Let $\psi(u_2) = 1, \psi(u_3) = a, \psi(v_3) = b$. Also $\psi(x_2) = 0 = \psi(x_3), \psi(x_1) = 2$ and $\psi(x_4) = c$.

Case3: $H = G - u_3$.

Let $\psi(u_2) = \psi(v_2) = 0, \psi(v_3) = b$. Also $\psi(x_1) = 1 = \psi(x_2), \psi(x_3) = a$ and $\psi(x_4) = c$

Case4: $H = G - v_3$.

Let $\psi(u_2) = 1, \psi(v_2) = c, \psi(u_3) = a$. Also $\psi(x_1) = 2, \psi(x_2) = 0 = \psi(x_3)$ and $\psi(x_4) = b$.

Case5: $H = G - x_1$.

Let $\psi(u_2) = 1$, $\psi(v_2) = 2$, $\psi(u_3) = a$ and$\psi(v_3) = b$. Also let $\psi(x_1) = 0 = \psi(x_2)$ and $\psi(x_4) = c$.

Case6: $H = G - x_2$.

Let $\psi(u_2) = 0 = \psi(v_2), \psi(u_3) = 1$ and$\psi(v_3) = b$. Also $\psi(x_1) = 2, \psi(x_3) = a$ and $\psi(x_4) = c$.

Case7: $H = G - x_3$.

Let $\psi(u_2) = 0 = \psi(v_2), \psi(u_3) = a$ and$\psi(v_3) = b$. Also $\psi(x_1) = 1 = \psi(x_2)$ and $\psi(x_4) = c$.

Case8: $H = G - x_4$.

Let $\psi(u_2) = 1, \psi(v_2) = c, \psi(u_3) = a$ and $\psi(v_3) = b$. Also $\psi(x_1) = 2$ and $\psi(x_2) = 0 = \psi(x_3)$.

We conclude that G and L satisfy Hall's Condition. So, $k \leq h(G) \leq ch(G) = k$ by Fact 2 and Theorem B. Therefore, $h(G) = k$ for all $k \geq 3$ odd.

□

Corollary 4: For $m \geq 2$, $k \geq 2$, $h(K(m, 2 \ldots, 2)) = ch(K(m, 2 \ldots, 2))$.

Proof. This follows from Corollaries D and 3, and Theorems C, D, 3 and 4. □

Conjecture: If G is a complete multipartite graph with all parts of size greater than 1, then $h(G) = ch(G)$.

Since $h(K_n) = 1 < n = ch(K_n)$, the conclusion of the conjecture fails if parts of size 1 are allowed. Since $h(G) = ch(G)$ whenever $\chi(G) < ch(G)$, and since $\chi(G) < ch(G)$ for "most" complete multipartite graphs G with part sizes greater than 1, with Theorems 3 and 4 we may be within shouting distance of confirming the conjecture.

Acknowledgement

The author expresses his gratitude to Professor Peter Johnson Jr for communicating this problem and encouraging this work.

References

[1] Hikoe Enomoto, Kyoji Ohba, Katsuhiro Ota, Junko Sakamoto, Choice number of some complete multi-partite graphs *Discrete Math.* 244 (2002), 55-66.

[2] P. Erdös, A.L.Rubin, H. Taylor, Choosability in graphs, *Congressus Numerantium* 26 (1980), 125-157.

[3] S. Gravier and F. Maffray, Graphs whose choice number is equal to their chromatic number, *J. Graph Theory* 27 (1998), 87-97.

[4] D. G. Hoffman and P.D. Johnson Jr, On the choice number of $K_{m,n}$, *Congressus Numerantium* 98 (1993), 105-111.

[5] P. Hall, On representatives of subsets, *J. London Math. Soc.* 10 (1935), 26-30.

[6] A. J. W. Hilton and P. D. Johnson Jr., Extending Hall's theorem, Topics in Combinatorics and Graph Theory: Essays in Honour of Gerhard Ringel, Physica-Verlag, Heidelberg, 1990, 359-371.