GROUP COVERS AND PARTITIONS: COVERING AND PARTITION NUMBERS

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The set of people to whom I am indebted has roughly the cardinality of $\mathbb{N}$; as such, it is impossible for me to list everyone - hopefully nobody is offended to not see their name here. Omission does not imply lack of significance. But this is a natural place for reflection, and I would like to specifically mention a subset of people to whom I am particularly grateful:[1]

$$\left\{\text{Risto Atanasov, Tuval Foguel, Josh Hiller, Erin McNelis, Muriel Williamson}\right\}$$

Each has been particularly important or influential to me. It's been a wonderful journey that has enriched me immeasurably, both on a personal and intellectual level. One never knows what the future holds, but no matter what happens, this experience has been one I'll always cherish ...

1 This set is not ordered - the names are listed alphabetically.

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ABSTRACT

GROUP COVERS & PARTITIONS

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A group cover is a collection of subgroups whose union is the group. A group partition is a group cover in which the elements have trivial pairwise intersection. Previous work has been done to investigate properties related to the covering number of a group - the minimal number of subgroups necessary to form a cover. Here we define analogously the partition number of a group and examine some of its properties, including its relation to the covering number. In particular, we provide a classification of the covering and partition numbers of the dihedral groups. Also presented is a result concerning a minimal partition of the symmetric group on four elements.

Further, we utilize `GAP` to provide some computational methods for studying partition numbers. Code samples are provided to test conditions relevant to partitions and the partition number; specifically, we are able to use `GAP` to help verify various conjectures about partitions of small finite groups. Included is a naïve and computationally intense method to find the partition number, as well as an attempted refinement designed to deal with the inefficiency of the naïve method.

Part I

MATHEMATICAL PRELIMINARIES

INTRODUCTION

1.1 GROUPS

Much of modern mathematics strives to study things in sufficient generality so as to make it applicable to a wide variety of situations. The idea being that instead of building up individual theories for every specific context that a concept might arise, we can create a general theory that can be immediately applied to the various situations we would like to use it in. Often new applications arise as a result of viewing things in a broader context.

This is at least part of the motivation for group theory. A *group* is a set (collection of objects of some kind) with a rule for combining the members of that set to get another member of the set. In practice, the set will usually contain numbers, and the rule will be an operation such as addition, but in the interest of keeping things sufficiently general, we don't require that in the definition. Now to develop any kind of a theory and to be able to say anything intelligent, it is necessary to place a few more requirements on the set. The motivation for these requirements mostly comes from hindsight and understanding what is not overly restrictive on the applicability of the theory we seek, but at the same time allows for some structure to be imposed on the set. Without rules, there is no structure, and it is not possible to make any inferences.

Before giving a formal list of the properties we will require a group to satisfy, it will be helpful to provide an example of a group and observe some seemingly innocent properties that will be the defining characteristics we require of all groups. The set of integers $\mathbb{Z}$ is a group with the operation of addition (notice that we have to know the set *and* the operation to know the group). Recalling some basic properties of integers, we know that:

A. The set $\mathbb{Z}$ contains 0, and 0 has the property that $0+x=x+0=x$ for any integer x. The number 0 is called the identity of this group.

B. For any integer x, there is another integer $-x$ with the property that $x + -x = -x + x = 0$. In other words, for any x we can find something to combine with x to get the identity of the group.

C. The operation of addition is associative, so for integers x, y, and z, $(x + y) + z = x + (y + z)$. We can combine elements from right to left or left to right and it does not change the outcome.

As seemingly naïve as these properties may be, it turns out that they are sufficiently interesting to generate a remarkable mathematical theory. So we are motivated to make a definition: any set with an operation that contains an identity, has inverses for every member of the set, and whose operation is associative, is a group.

DEFINITION 1.1.1. *A set G, together with a closed binary operation $*$ on G, is a group, if*

A. *There exists an element $e \in G$ with the property that $e * x = x * e = x$ for any $x \in G$,*

B. *For any $x \in G$, there is an element $x^{-1} \in G$ such that $x * x^{-1} = x^{-1} * x = e$, and*

C. *For any three elements $x, y, z \in G$, $(x * y) * z = x * (y * z)$.*

The requirement that G be closed under the operation $*$ just means that by combining elements with $*$, we always get back an element of G. Two quick points on notation: for an arbitrary group G, the identity element is usually written as either 1 or e; and for convenience, we often omit the operation symbol $*$.

Sometimes a non-example is as instructive as an example, so consider again the set of all integers, but instead with the operation of multiplication. This is not a group. It contains an identity, which in this case is the number 1 instead of 0, and multiplication of integers is associative, but the set of integers does not contain the multiplicative inverses of all the integers. For example, the inverse of 2 with respect to multiplication is $\frac{1}{2}$, which is not an integer. As a result, when referring to the integers as a group, the operation is assumed to be addition, unless stated otherwise.

Since groups are sets (collections), it is inevitable that we will need a few basic notions from set theory. Given that we have a group G, it is possible that we could have a subset - a subcollection of the members of G - that forms a group in its own right. For example, since every integer is a real number, the integers are a subset of the set of real numbers $\mathbb{R}$. In this case, we say that they form a *subgroup* of $\mathbb{R}$ under addition, since they are both a subset of $\mathbb{R}$ and a subgroup with respect to the same operation. For various reasons, it is convenient to say that a set G is a subset of itself. A subset is *proper* if it is not the entire set G. Proper subgroups will turn out to be of particular importance to us.

Also of interest will be the idea of a *union*. Union can be thought of as a set which results from combining two other sets together (it unites them). Related to the idea of union is *intersection*. The intersection of two sets is a third set which contains only the things that are in both of the first two sets.

1.2 TYPES OF GROUPS, GROUP PROPERTIES

Certainly, because the definition of group is so general, there are many properties one could look at when attempting to classify them. One of the first ways that comes to mind is size. The size of a group is the *order* of the group, and is simply the number of elements in the underlying set. Naturally, the order is *finite* if the set is finite, and *infinite* if the set is infinite. Infinite groups, while certainly interesting, will not be dealt with here.

It turns out that there is an interesting connection between the order of a group, and the order of any possible subgroups of the group. The result is Lagrange's theorem.

THEOREM 1.2.1. *If H is a subgroup of a finite group G, then $|H|$ divides $|G|$.*

A group G is *cyclic* if there exists some $x \in G$ such that every element of the group can be written as a "power" of x. Here, "power" just means repeated application of the group operation. The classic examples of cyclic groups are the integers modulo n (denoted $\mathbb{Z}_n$) for any value of n. In $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, for example, we can generate all distinct elements by repeatedly adding 1 to itself. So $\mathbb{Z}_5$ is cyclic, generated by the element 1; typical notation for this is $\mathbb{Z}_5 = \langle 1 \rangle$.

"Order" is also used in a slightly different sense, when referring to the *order* of an element of a group. In this sense, it means the smallest positive power of an element that returns the identity.

In many instances, it is possible that groups that appear distinct really have the same (or perhaps very similar) structure. The key tool in determining this structural similarity is *homomorphism*. Groups G and H are homomorphic if there exists a function $\phi : G \to H$ such that $\phi(ab) = \phi(a)\phi(b)$ for any $a, b \in G$. If the function ϕ is also a bijection, then G and H are *isomorphic* and can essentially be regarded as the same group.

Homomorphisms give rise to two natural subgroups, the *kernel* and the *image*. Given $\phi : G \to H$, Ker ϕ is the subgroup of G consisting of all elements that map to e, and Im ϕ is the subgroup of H consisting of everything that ϕ maps to.

An important property that certain subgroups may satisfy is the normality property. A subgroup is *normal* if it is closed under conjugation; that is, a subgroup N of G is normal if for all $g \in G$ and $n \in N$, $gng^{-1} \in N$.

Normal subgroups are typically discussed along with *cosets*. Given a subgroup H of G, we define a coset aH for each $a \in G$, as the set of all products ah where a is fixed, and h ranges over H. In set notation, $aH = \{ah : h \in H\}$. Note that in general, a coset is not a subgroup. However, given a normal subgroup N, we can place a group structure on the set of all cosets of N, in the natural way, by defining $(aN)(bN) = (ab)N$. This group is called the *quotient group*, and is typically written as $\frac{G}{N}$.

Among the important results that arise from looking at quotient groups are the isomorphism theorems; in particular, we will have need for the first isomorphism theorem, which states in part that:

THEOREM 1.2.2. *If* $\phi : G \to H$ *is a group homomorphism, then* $\mathit{Im}\,\phi$ *is isomorphic to* $\frac{G}{\mathit{Ker}\,\phi}$.

It turns out that, up to isomorphism, there are only two distinct groups of order 4. One is $\mathbb{Z}_4$, which will be of little interest to us, but the other, the *Klein Four Group*, will have some significance. The key characteristic of the Klein four group is that each element is its own inverse; in particular, this means that each of the three non-identity elements is of order 2.

Another important class of finite groups is the class of *dihedral groups*. There are a few different ways of thinking about the dihedral groups; from an intuitive perspective, they are traditionally presented as groups of the rotations and symmetries of regular polygons. For example, D_4, the dihedral group of order 8, corresponds to the set of rotations and symmetries of a 4-sided polygon (i.e., a square). The elements of the set are the different ways in which the square can be rotated, and the operation is composition of these rotations. There is a distinct dihedral group for each value of n, each having order $2n$.

We will be dealing with the dihedral groups extensively, and for our purposes, rather than work from the geometric perspective, it will be more convenient to use a precise, symbolic definition. There are different equivalent ways of defining an arbitrary dihedral group; to keep things consistent, the presentation that we will use is

$$\mathrm{D}_n = \left\langle a, b : a^n = b^2 = 1, ab = (ab)^{-1} \right\rangle.$$

One property that we shall record now is that each element of the form ba^i, $0 \le i < n$ has order 2 in D_n. Indeed, if we inductively apply the relation that $(ab) = (ab)^{-1} = ba^{-1}$, we see that

$$(ba^i)^2 = (ba^i)(ba^i) = bb(a^{-1})^i(a^i) = 1.$$

1.3 A QUESTION . . .

Having built up a modest vocabulary, we can now start to direct our focus towards the specific subject of interest. To that end, consider the following. Can we find a group that can be written as a union of two of its subgroups? Trivially of course, any group can be written as a union of itself and any other subgroup, but is there a group that can be written as a union of two of its *proper* subgroups? Surely, of the infinitely many groups that can be constructed, there must be one that can be so written, right? Wrong.

THEOREM 1.3.1. *No group can be written as a union of two of its proper subgroups.*

PROOF. Suppose G were such a group. Let $G = A \cup B$ where A and B are proper subgroups. Since A is proper, there must be an $a \in A$ that is not in B (if not, then A would be all of G). Likewise, since B is proper, there is a $b \in B$ that is not in A. Now look at ab. Since G is a group, by closure, $ab \in G$. In particular, $ab \in A$ or $ab \in B$. But if $ab \in A$, then $a^{-1}ab = b \in A$, a contradiction, and if $ab \in B$, then $abb^{-1} = a \in B$, which is also a contradiction. So there can be no such G. ✠

Theorem 1.3.1 is somewhat of a surprising result, and in many ways, is the starting point for a fascinating topic in group theory, *group covers*. Chapter 2 serves as an introduction to the subject, and lays the groundwork for the results presented in the rest of the thesis.

Part II

COVERS & PARTITIONS

GROUP COVERS AND PARTITIONS

2.1 COVERS

A *group cover* refers to a collection of proper subgroups of a group, whose union is the group. The subgroups contained in the cover are called *summands*.[1] To place the rest of the thesis in proper context, we will survey some of the relevant results and terminology from the theory of group covers.

A group G is *coverable* if there exists some covering of G. Interestingly enough, it turns out that being coverable is equivalent to not being cyclic [6, 13].

THEOREM 2.1.1. *A group G is coverable if and only if G is not cyclic.*

PROOF. Assume G is coverable, and consider any $g \in G$. Then g cannot be a generator for G. Certainly, g is in some proper subgroup. But if g were a generator, then that subgroup would not be proper, it would be all of G.

Conversely, if G is not cyclic, then for any $g \in G$, the subgroup $\langle g \rangle$ is proper. So naturally, $G = \bigcup \langle g \rangle$ for all $g \in G$. ✠

Theorem 2.1.1 allows us to immediately ignore all cyclic groups in our study of coverings. In general, given that a group is coverable, there need not be a unique cover. It is quite possible for a group to have several distinct covers. So given a group G, the question of "what is the minimal number of subgroups required to form a cover?" is generally non-trivial. The minimal number of subgroups required to cover G, called the *covering number* of G, and denoted as $\sigma(G)$, has received attention in the literature.

For example, from Theorem 1.3.1, we know that there is no group such that $\sigma(G) = 2$. This of course begs the question, what about other numbers? What numbers can be covering numbers, and what can be said about a group given that we know its

1 The term "summand" comes from viewing union as a set-theoretic sum.

covering number? Well, in the case of $\sigma(G) = 3$, quite a bit can be said, and we actually have a complete classification of such groups.

THEOREM 2.1.2. *For any group G, $\sigma(G) = 3$ if and only G is homomorphic to K_4, the Klein four group.*

Readers are referred [4, 9] for a proof of Theorem 2.1.2. Other bounds and various minimality conditions have places upon $\sigma(G)$ for various classes of groups. See the work of Cohn [6], Maroti [11], Kappe [9], Garonzi [8], Abdollahi [1, 2], among others.

2.2 PARTITIONS

Of specific interest to us is a particular type of cover known as a *partition*. This is not a true partition in the set-theoretic sense - here the term refers to a collection of subgroups that cover the group and have a trivial pairwise intersection.

DEFINITION 2.2.1. *A collection of subgroups $\{H_1, H_2, \dots, H_n\}$ is a partition of a group G if:*

A. $G = \cup H_i$ *for* $i = 1, \dots, n$

B. $H_i \cap H_j = \{e\}$ *whenever* $i \neq j$.

We say that a group G is *partitionable* if there exists a partition of G. The study of partitions began with the work of Miller in 1906 [12], and culiminated in the complete classification of all partitionable groups through the combined efforts of Baer, Kegel, and Suzuki in 1961 [3, 10, 14]. For our purposes, the key result is the following theorem.

THEOREM 2.2.1. *A group G is partionable if and only if G is isomorphic to one of:*

A. S_4,

B. *a p-group with $H_p(G) \neq G$, where $H_p(G) = \langle x \in G : x^p \neq 1 \rangle$,*

C. *a group of the Hughes-Thompson type,*

D. *a Frobenius group,*

E. $PSL(2, p^n)$*, with* $p^n \geq 4$*,*

F. $PGL(2, p^n)$*, with* $p^n \geq 5$ *and p odd,*

G. $Sz(2^{2n+1})$*,*

for some prime p and some $n \in \mathbb{N}$.

Theorem 2.2.1 is powerful in the sense that it places a decent amount of restriction on the groups that will be of interest.

As a natural extension to the idea of a covering number, it is worth considering the minimal number of subgroups required to form a partition of G, which is henceforth called the *partition number* of G, and denoted $\rho(G)$. As of yet, little is known of the partition number, and it is our specific purpose here to provide some insight in this area. Of course, even making the definition of the partition number presumes that it is interesting enough to merit its own definition, which may initially seem to be overly presumptuous. For it to have any significance, it must be distinct from the covering number at least sometimes, and it is not immediately clear that it would ever have to be.

But this is the question that we seek to answer. One thing to note immediately is that since a partition is a type of cover, we know that $\sigma(G) \leq \rho(G)$, always. So in particular, if our motivation could be condensed into a single question, it would be

When is the covering number strictly less than the partition number?

Again, it may not be initially clear that there is an answer that is at all interesting. It could be that the covering number is always equal to the partition number. But either way, the natural place to start investigating is with small finite groups. We begin with a lemma that will place an immediate restriction on our search.

LEMMA 2.3.1. *Let G be a partitionable group. If* $|G| = pq$ *for some primes* p *and* q*, then* $\sigma(G) = \rho(G)$.

PROOF. The proof will be broken into two cases - when $p = q$, and when $p \neq q$.

If $p = q$, then $|G| = p^2$, so all nontrivial subgroups are of order p. Let H and K be two nontrivial subgroups of G. Suppose there is some $x \in (H \cap K) - \{e\}$. Then $|\langle x\rangle| = p$, and since $x \in H$, all powers of x - there are p distinct such elements - are in H, and therefore, $\langle x\rangle = H$. Likewise, since $x \in K$, $\langle x\rangle = K$, and thus, $H = K$. So nontrivial subgroups of G either intersect at the identity only, or are equal. Hence, any covering by distinct nontrivial subgroups is necessarily a partition, so $\sigma(G) = \rho(G)$.

If $p \neq q$, then all nontrivial subgroups are of either order p or order q. Again, let H and K be two nontrivial subgroups of G. Suppose $|H| = |K| = p$. Then any nontrivial subgroups of H or P must have order p. So for any $x \in (H \cap K) - \{e\}$, $|\langle x\rangle| = p$, so using the same argument as in the first case, $\langle x\rangle = H = K$. So all nontrivial subgroups of equal order must intersect at e only, or be equal.

If $p \neq q$ and $|H| \neq |K|$, then without loss of generality, let $|H| = p$ and $|K| = q$. Now for some $x \in (H \cap K) - \{e\}$, it follows that $|\langle x\rangle|$ divides $|H| = p$ and $|\langle x\rangle|$ divides $|K| = q$. Since p and q are prime, they are relatively prime, meaning that $|\langle x\rangle| = 1$, and so $\langle x\rangle = \{e\}$. So all nontrivial subgroups of distinct order are either equal or have trivial intersection.

Thus, whenever $p \neq q$, all distinct nontrivial subgroups have trivial intersection, which means that every cover is a partition, and $\sigma(G) = \rho(G)$. ✠

Theorem 2.1.1 excludes groups of prime order, and Lemma 2.3.1 excludes (among others) orders 4 and 6. So, the smallest order we have not immediately excluded is 8. So the question arises, is there a group of order 8 for which $\sigma(G) < \rho(G)$? It turns out that the answer is yes. It will be instructive to explictly demonstrate this example, which we can then generalize to a whole family of examples.

LEMMA 2.3.2. *The dihedral group of order 8, D_4, is the smallest group G for which $\sigma(G) < \rho(G)$.*

PROOF. Consider $D_4 = \{e, a, a^2, a^3, b, ba, ba^2, ba^3\}$. There are eight nontrivial subgroups, which are:

A. $\langle a \rangle = \{e, a, a^2, a^3\}$

B. $\langle a^2 \rangle = \{e, a^2\}$

C. $\langle b \rangle = \{e, b\}$

D. $\langle ba \rangle = \{e, ba\}$

E. $\langle ba^2 \rangle = \{e, ba^2\}$

F. $\langle ba^3 \rangle = \{e, ba^3\}$

G. $\langle a^2, b \rangle = \{e, a^2, b, ba^2\}$

H. $\langle a^2, ba \rangle = \{e, a^2, ba, ba^3\}$

Let $\mathscr{C} = \{\langle a \rangle, \langle a^2, b \rangle, \langle a^2, ba \rangle\}$. Inspection reveals that $\mathscr{C}$ is a cover of D_4, and therefore, $\sigma(D_4) = 3$ since no group can be covered by two subgroups (Theorem 1.3.1). However, $\mathscr{C}$ is a not a partition. Notice that any partition (or covering) must include $\langle a \rangle$ since this is the only subgroup containing the element a. By including $\langle a \rangle$ in a partition, we cannot include $\langle a^2 \rangle$, $\langle a^2, ba \rangle$, $\langle a^2, b \rangle$, or $\langle a^2, ba \rangle$. The remaining five subgroups do give a partition $\mathscr{P} = \{\langle a \rangle, \langle b \rangle, \langle ba \rangle, \langle ba^2 \rangle, \langle ba^3 \rangle\}$, and so $\rho(D_4) = 5$. ✠

The result above does not seem particularly deep or impressive, but it turns out to be very representative, and the insight it gives us really does generalize nicely. What we learn from this example is a natural springboard to understanding the situation for arbitrary dihedral groups. And so in Chapter 3, we build upon it, and study the covering and partition numbers of dihedral groups in general.

THE DIHEDRAL GROUPS

3.1 BACKGROUND

Here, the aim is to present a mostly complete classification of the partition numbers of dihedral groups. The subgroup structure of the dihedral groups is consistent enough that we are able to generalize observations from a few specific cases to effectively classify them all.

Many of the results presented in this chapter rely on an understanding of the subgroup structure of an arbitrary dihedral group. So to begin, we present a theorem due to Cavior [5] that helps give us that understanding.

THEOREM 3.1.1. *For a dihedral group D_n of order $2n$, D_n has $\sum_{d \mid n}(d+1)$ subgroups, and each subgroup is either of the form:*

A. $\langle a^d \rangle$, *where* $d \mid n$, *or*

B. $\langle a^d, ba^i \rangle$, *where* $d \mid n$ *and* $0 \leq i < d$.

3.2 KEY RESULTS

Our first lemma draws upon the wisdom gained from Lemma 2.3.2 and shows that the case of D_4 is really part of a much larger phenomenon.

LEMMA 3.2.1. *If $n > 2$ is even, then $\sigma(D_n) = 3$.*

PROOF. Consider $D_n = \langle a, b : a^n = b^2 = e, ab = (ab)^{-1} \rangle$.

First, we claim that $\sigma(D_n) = 3$. Observe that $\langle a^2 \rangle$ is normal in D_n, and that $[D_n : \langle a^2 \rangle] = 4$. So there exists a homomorphism $\phi : D_n \to \frac{D_n}{\langle a^2 \rangle}$.

By construction, we know that $\left|\frac{\mathrm{D}_n}{\langle a^2\rangle}\right| = 4$, and so $\frac{\mathrm{D}_n}{\langle a^2\rangle}$ is isomorphic to either $\mathbb{Z}_4$ or K_4. Were it isomorphic to $\mathbb{Z}_4$, then it would have two elements of order 4 and one element of order 2. However, it has two elements (cosets) of order 2 and therefore, must be isomorphic to K_4. ✠

It turns out that the partition number of D_n can be expressed as a closed formula that holds without restriction on n. As before, the argument follows closely to the specific case of D_4 that we saw in Lemma 2.3.2.

LEMMA 3.2.2. *For any n, $\rho(D_n) = n+1$.*

PROOF. Observe that any covering of D_n must include the subgroup $\langle a\rangle$, since this is the only subgroup that contains a. Then by including $\langle a\rangle$, to form a partition, we cannot include any other subgroups of the form $\left\langle a^d, ba^i\right\rangle$ for $d > 0$, since these each contain powers of a that lie in $\langle a\rangle$. The remaining subgroups will then have the form $\langle ba^i\rangle$, where $0 \le i < n$. Each of these subgroups has order 2, and they interesect trivially. So the only partition we can form is $\mathscr{P} = \{\langle a\rangle\} \cup \left\{\langle ba^i\rangle : 0 \le i < n\right\}$, and clearly, $|\mathscr{P}| = n+1$. ✠

Lemma 3.2.1 and Lemma 3.2.2 give us immediately a large class of groups for which the covering number is strictly less than the partition number.

COROLLARY 3.2.1. *If $n > 2$ is even, $\sigma(D_n) < \rho(D_n)$.*

And so our motivating question has at least a countably infinite collection of affirmative answers. One may wonder whether the condition that n be even in Corollary 3.2.1 is truly necessary. What if n is odd? Hindsight will reveal that looking at odd n doesn't reveal much about the relation between the covering and partition number of D_n. What we will want to look at is whether or not n is prime. It is from this perspective that our next lemma is motivated.

LEMMA 3.2.3. *If N is a normal subgroup of G, then $\sigma(G) \leq \sigma\left(\frac{G}{N}\right)$.*

PROOF. Let $\mathscr{C}' = \left\{\frac{H_1}{N}, \frac{H_2}{N}, \ldots, \frac{H_n}{N}\right\}$ be a cover for $\frac{G}{N}$. Then the claim is that $\mathscr{C} = \{H_1, H_2, \ldots, H_n\}$ is a cover for G. Every $g \in G$ must be in some coset aN, since the cosets of N partition G; and each coset aN must be contained in some $\frac{H_i}{N}$, since $\mathscr{C}'$ is a cover of the set of all cosets. Hence, every $g \in G$ must be in some H_i, making $\mathscr{C}$ a cover for G. ✠

The importance of Lemma 3.2.3 becomes clear from the following corollary. Note that if p is a prime that divides n, $\frac{\mathrm{D}_n}{\langle a^p \rangle}$ is isomorphic to D_p, and so we have:

COROLLARY 3.2.2. *If p is a prime such that $p \mid n$, then $\sigma(D_n) \leq \sigma(D_p)$.*

One might wonder under what conditions equality can hold in the relation expressed in Corollary 3.2.2. It turns out that a necessary condition for equality is that p be the smallest prime divisor of n.

LEMMA 3.2.4. *If p is the smallest prime such that $p \mid n$, then $\sigma(D_n) = \sigma(D_p) = p+1$.*

PROOF. Let $\mathscr{C}$ be a covering of D_n, and suppose that $|\mathscr{C}| < p+1$, where p is the smallest prime divisor of n. Observe that $\mathscr{C}$ can have at most $p-1$ subgroups of the form $\langle a^k, ba^i \rangle$ since any covering must contain $\langle a \rangle$. Also, any element that is not a power of a must lie in one these $\langle a^k, ba^i \rangle$.

For each of these subgroups, $\langle a^k, ba^i \rangle = \mathrm{D}_t$, where $t = |a^k|$. Also, for any such subgroup in $\mathscr{C}$, we cannot have $t = n$ since $\mathscr{C}$ contains proper subgroups. Now write $n = p_1 p_2 \cdots p_k$, where all p_i are prime and $p_i \leq p_{i+1}$ (so that $p_1 = p$). Then since $p \mid n$, we must have $1 \leq t \leq \frac{n}{p}$.

Note that D_t has t elements of order 2 that are not powers of a. Hence each $D_t \in \mathscr{C}$ has at most $\frac{n}{p}$ elements of order 2 that aren't powers of a. The whole group D_n has n such elements, which we can only find in $|\mathscr{C}|-1$ subgroups (remember that $\langle a\rangle \in \mathscr{C}$, and contains no such elements). Let $|\mathscr{C}|_2 = \left|\{x \in D_n : x^2 = 1, x \neq a^i\right.$ for Then

$$n = |\mathscr{C}|_2 \leq \left(\frac{n}{p}\right)(|\mathscr{C}|-1) < \left(\frac{n}{p}\right)(p) = n,$$

and so we have the contradiction that $n < n$.

To show that $\sigma(D_n) \leq p+1$, first note that if n is even, then $p = 2$, and by Lemma 3.2.1, the result is immediate. Now suppose n is odd. Then D_n has p subgroups of the form $\langle a^p, ba^i\rangle, 0 \leq i < p$. The claim is that $\mathscr{C} = \{\langle a^p, ba^i\rangle : 0 \leq i < p\} \cup \{\langle a\rangle\}$ is a cover for D_n.

Certainly, any power of a is in $\langle a\rangle$, and $b \in \langle a^p, ba^0\rangle$. Consider some ba^k for $1 \leq k < n$. Note that $k \equiv i \pmod{p}$ for some $0 \leq i < p$, and for that i, we have $ba^k \in \langle a^p, ba^i\rangle$. So $\mathscr{C}$ is a cover of D_n, and $|\mathscr{C}| = p+1$. ✠

The significance of Lemma 3.2.4 is that it completely classifies the relation between the covering and partition number of D_n, on the basis of whether or not n is prime. We now have another necessary condition for which $\sigma(D_n) < \rho(D_n)$ which is stronger than the one we obtained in Corollary 3.2.1.

COROLLARY 3.2.3. *If n is composite, then $\sigma(D_n) < \rho(D_n)$.*

PROOF. If n is composite, then n has a prime divisor strictly less than n. Let p be the smallest such divisor. Then by Lemma 3.2.4, $\sigma(D_n) = p+1 < n+1 = \rho(D_n)$. ✠

Furthermore, Lemma 3.2.4 tells us that a necessary condition for $\sigma(D_n) = \rho(D_n)$ is that n is prime.

COROLLARY 3.2.4. *If p is prime, then $\sigma(D_p) = p + 1 = \rho(D_p)$.*

PROOF. That $\rho(\mathrm{D}_p) = p+1$ was established in Lemma 3.2.2. The other equality is immediate from Lemma 3.2.4 because, if p is prime, then p is the smallest prime divisor of p. ✠

In Chapter 4, we step back from our focus on the dihedral groups, and take a more computational approach to studying covering and partition numbers in general.

EFFICIENT COVERINGS AND PARTITIONS

4.1 MOTIVATION AND DEFINITION

Having seen some interesting results surrounding the dihedral groups, there are other interesting questions we might ask. But before asking those questions, we make a couple of definitions.

Once we have found the covering number of a group, we know that a covering of order $\sigma(G)$ exists. This certainly does not imply that such a covering exists uniquely, but it is reasonable to expect that any covering of order $\sigma(G)$ may have similar properties. The structure of the group, and the restriction that a covering be as efficient as possible is bound to restrict what that covering could look like. And so, we define an *efficient covering* (ϵ-covering) of a group as follows.

DEFINITION 4.1.1. *A covering $\mathscr{C}$ for a group G is an ϵ-covering if $|\mathscr{C}| = \sigma(G)$.*

Similarly motivated is the idea of an *efficient partition.*

DEFINITION 4.1.2. *A partition $\mathscr{P}$ for a group G is an ϵ-partition if $|\mathscr{P}| = \rho(G)$.*

With these definitions in hand, one observation we may extract from our study of the dihedral groups in Chapter 3 is that the subgroups contained in the ϵ-partitions we constructed were always abelian.

LEMMA 4.1.1. *Any ϵ-partition of a dihedral group consists only of abelian summands.*

PROOF. Indeed, as we saw in Lemma 3.2.2, the only way to construct a partition of D_n was with a cyclic group $\langle a \rangle$ and subgroups of order $\langle ba^i \rangle$, all of which are abelian. ✠

Table 1: An ϵ-partition for S_4

$\alpha_3 = \{(), (14)\}$	$\delta_4 = \{(), (234), (243)\}$
$\alpha_5 = \{(), (24)\}$	$\gamma_1 = \{(), (1234), (1432), (13)(24)\}$
$\alpha_6 = \{(), (34)\}$	$\gamma_2 = \{(), (1243), (1342), (14)(23)\}$
$\delta_2 = \{(), (124), (142)\}$	$\gamma_3 = \{(), (1324), (1423), (12)(34)\}$
$\delta_3 = \{(), (134), (143)\}$	$\tau_1 = \{(), (12), (13), (23), (123), (132)\}$

4.2 COMPUTATION OF THE PARTITION NUMBER

One thing that would be to our benefit would be to have a way to easily compute the covering or partition number of a group. In general, this can be a difficult problem. It is possible to use `GAP` to compute the covering or partition number of a group directly. Without some real insight, though, this approach will be computationally inefficient because we must look at all possible combinations of subgroups. But given adequate computing power, the problem can solved via sledgehammer, as we demonstrate naïvely in with the `pnumber` function, the code for which is contained in Section A.1.3.

Alternately, we can try to be more efficient in using `GAP` to help answer questions related to the partition number and ϵ-partitions of groups. As a motivating example, let us consider the symmetric group on 4 elements.

4.3 THE SYMMETRIC GROUP S_4

Using `GAP`, it can be verified that $\rho(S_4) = 10$.[1] We can define an ϵ-partition of S_4 as

$$\mathscr{P} = \{\tau_1, \alpha_3, \alpha_5, \alpha_6, \delta_2, \delta_3, \delta_4, \gamma_1, \gamma_2, \gamma_3\}$$

with permutations defined as in Table 1.

1 See Appendix B and Corollary B.1.1 for details.

Note that τ_1 is of course isomorphic to S_3, and is therefore nonabelian. So in the case of S_4, we have something different than we encountered with the dihedral groups: a group with a nonabelian summand in an ϵ-partition. The natural question to ask is if S_4 is the smallest example of such a group. A partial answer is the following.

THEOREM 4.3.1. *Either the symmetric group S_4 or the generalized dihedral group of order* 18 *is the smallest non-p group with a nonabelian summand in an ϵ-partition.*

Theorem 4.3.1 is still a work in progress, as the complexity of the statement suggests. There is hope that work in GAP will soon confirm which of the two candidates is the smallest with a nonabelian summand, and also hope that the non-p group condition can be dropped after addressing groups of order 16. At any rate, what follows is the work that has been done to progress Theorem 4.3.1 to its current state.

We know that S_4 has a nonabelian summand in an ϵ-partition, and we are interested in whether any smaller group has this property. Clearly, to get an answer, we need to look at all groups of order 23 or less. Many orders can be eliminated quite easily, with two simple observations:

A. All groups of order 5 or less are abelian, and

B. Groups of prime order are abelian.

In particular, combining (A) with Lagrange's theorem (Theorem 1.2.1) implies that no group of order 10 or less can have a nonabelian summand (since all proper subgroups must be abelian). Groups of order 14, 15, 21, and 22 can be eliminated since their proper subgroups must be of prime order, and hence, abelian.

This leaves us to consider groups of order $12, 16, 18$, and 20. For a group of order 12, the only possible nonabelian summand would be isomorphic to S_3, which is the only nonabelian group of order 6. The three nonabelian groups of order 12 are Dic_{12} (the dicyclic group of order 12), A_4, and D_6. We can easily check for a subgroup isomorphic to S_3 in GAP [7]. One way to do this is to define the groups - the GAP IDs for referencing from the Small Group library are (12,1), (12,3), and (12,4), respectively - and run the following code:

```
# % Sonata package is needed for the IsIsomorphicGroup function
LoadPackage("sonata");
s3 := SymmetricGroup(3);
subs := Subgroups(g);
n := Length(subs);
s3check := [1 .. n];

# % IsIsomorphicGroup returns true or false depending on
# % whether or not the groups are isomorphic
for i in [1 .. n] do
    s3check[i] := IsIsomorphicGroup(subs[i],s3);
od;

# % so s3check is a list of booleans corresponding to
# % whether each subgroup of g is isomorphic to s3
s3check;
```

From this computation, we can observe that only D_6 has a subgroup isomorphic to S_3. And of course, from Lemma 4.1.1, we know D_6 contains only abelian summands.

4.3.1 *Groups of order 20*

Moving on to consider groups of order 20, again we have 3 nonabelian groups. These are Dic_{20}, F_{20} (the Frobenius group of order 20), and D_{10}, with GAP IDs of (20,1), (20,3), and (20,4). Here, we can observe that a nonabelian summand would have to be isomorphic to D_5, the only nonabelian group of order 10. Running a similar check as the one above reveals that Dic_{20} does not contain a copy of D_5, although F_{20}, and D_{10} each do. D_{10} does not concern us, and neither will F_{20}, as we demonstrate below.

LEMMA 4.3.1. *The Frobenius group of order 20 does not contain a nonabelian summand in an ϵ-partition.*

PROOF. Since F_{20} is Frobenius, it admits a partition $\mathscr{P}$ consisting of its Frobenius kernel K, and the conjugates H^x of its Frobenius complement H. In this case, K is isomorphic to $\mathbb{Z}_5$, and each H^x is isomorphic to $\mathbb{Z}_4$, so $\mathscr{P}$ is not a problem. Now suppose we have some other partition $\mathscr{P}^*$ with a nonabelian summand $N \in \mathscr{P}^*$. Then N is isomorphic to D_5, and so there is an $\alpha \in N$ such that $|\alpha| = 2$. Note that in $\mathscr{P}$, we must have had $\alpha \in H^x$ for some x (α could not have been in K). For that x, there is some $\tilde{f} \in H^x$ such that $|\tilde{f}| = 4$, since H^x is cyclic of order 4. In particular, $\alpha = (\tilde{f})^2$. Note that $\tilde{f} \notin N$, so $\tilde{f}$ is in some $T \in \mathscr{P} - \{N\}$. But this gives a contradiction, since we have $\alpha \in T \cap N$. ✠

4.3.2 *Groups of order 18*

The three nonabelian groups of order 18 are D_9, $S_3 \times \mathbb{Z}_3$, and $E_9 \rtimes \mathbb{Z}_2$,[2] with GAP IDs $(18,1)$, $(18,3)$, and $(18,4)$. As in the case of order 12, the only possible nonabelian summand would be isomorphic to S_3. Ignoring D_9, running a check in GAP reveals that $S_3 \times \mathbb{Z}_3$ does contain a subgroup isomorphic to S_3, and $E_9 \rtimes \mathbb{Z}_2$ contains 12 such subgroups. First we will exclude $S_3 \times \mathbb{Z}_3$ from consideration.

LEMMA 4.3.2. *$S_3 \times \mathbb{Z}_3$ does not contain a nonabelian summand in a ϵ-partition.*

PROOF. First, recall from above that any nonabelian summand would necessarily be isomorphic to S_3. Denote $G = S_3 \times \mathbb{Z}_3 = \{(a,b) : a \in S_3, b \in \mathbb{Z}_3\}$. Note that there is $\alpha \in S_3$ such that $|\alpha| = 2$. Hence, there is $(\alpha, 1) \in G$, which has order 6. Then we may consider $(\alpha,1)^3 = (\alpha, 0)$ as an element of S_3. If $\mathscr{P}$ is a partition, and $S_3 \in \mathscr{P}$, then $(\alpha,1)^3 \in S_3$, but $(\alpha,1) \notin S_3$. So we must have $(\alpha,1)$ in some other

2 The generalized dihedral group, formed from the semidirect product of the elementary abelian group of order nine with a cyclic group of order two acting via the inverse map.

$H \in \mathscr{P}$. Since $\mathscr{P}$ is a partition, we must have $H \cap \mathrm{S}_3 = \{(\alpha, 0)\}$. But since $(\alpha, 1) \in H$, we must have $(\alpha, 1)^3 \in H$, which is a contradiction. ✠

That leaves us to consider $\mathrm{E}_9 \rtimes \mathbb{Z}_2$. At this point, we have not yet determined whether this group provides a smaller example than S_4. We do know it has a partition, though. Recall from Theorem 2.2.1 that one of the relevant conditions for testing whether a group is partitionable was whether or not $G = H_p(G) = \langle x \in G : x^p \neq 1 \rangle$. Specifically, if there exists a prime p that divides $|G|$, such that $G \neq H_p(G)$, then G is partitionable. Once again, we can employ `GAP` to check this condition. The `hpg` function listed in Section A.2 checks exactly this condition.

Running `hpg(g)` on $G = \mathrm{E}_9 \rtimes \mathbb{Z}_2$ reveals that $G \neq H_2(G)$, so G has a partition. So further work remains to either show that G contains a nonabelian summand, or that it doesn't. After that, the next objective will be to examine groups of order 16 to see if the non-p group condition can be dropped from Theorem 4.3.1.

Part III

APPENDIX

NAÏVE ROUTINES

In the text, we made reference to several methods which could be run in `GAP` to help our study of the partition number. Full code examples are listed below, along with an overview of the idea behind each method.

A.1 PARTITION NUMBER

The `pnumber` function takes in a group, and returns its partition number (or 0 if the group is not partitionable). The approach is simply to take all combinations of subgroups, filter out the combinations that contain trivial subgroups or are not a partition, and pick the minimal combination.

A.1.1 *is_partition*

This `is_partition` function is designed to check if a particular collection of subgroups `subs` is a partition. The function assumes that the group we are checking against exists in memory, stored as g (which it certainly will in the event that this function is called by `pnumber`).

```
LoadPackage("sonata");

 is_partition := function(subs)
# % is_partition takes a collection of subgroups subs, and checks
# % if they are a partition for a group g.  it returns true if
# % they are, and false if they are not.

        local u,flag,i;

        flag := true;

        # % first find all pairs of subgroups
        u := Combinations(subs,2);

        # % if the union is not all of g, then we can't have a partition
```

```
        if Size(Union(subs)) <> Size(g) then
                flag := false;
        fi;

        # % or if any pairwise intersection is greater than 1, no
           partition
        if flag = true then
                for i in [1 .. Length(u)] do
                        if Size(Intersection(u[i])) <> 1 then
                                flag := false;
                        fi;
                od;
        fi;

        return flag;
end;
```

A.1.2 *is_nontrivial_subs*

The purpose of `is_nontrivial_subs` is to check for the existence of nontrivial subgroups in a given collection. Strictly speaking, the `is_partition` function doesn't make this check, so it does need to happen.

```
is_nontrivial_subs := function(subs)
# % is_nontrivial_subs checks if any of the subgroups in subs is trivial,
# % meaning either the identity or all of g

        local flag,i;

        flag := true;

        for i in [1 .. Length(subs)] do
                if Size(subs[i]) = 1 then
                        flag := false;
                elif Size(subs[i]) = Size(g) then
                        flag := false;
                fi;
        od;

        return flag;
end;
```

A.1.3 *pnumber*

Now we can use the above routines to find the partition number. The method is completely brute-force, and completely inefficient. However, at least it is accurate, and could be easily tweaked to output all ϵ-partitions.

Listing 1: The `pnumber` function

```
pnumber := function(g)
# % pnumber computes the partition number of g

        local subs,u,pnumber,i;

        # % first we gather all subgroups of g
        subs := Subgroups(g);

        # % then all possible combinations
        u := Combinations(subs);
        # % filter out any combinations with trivial subgroups
        u := Filtered(u,is_nontrivial_subs);
        # % filter out any that are not a partition
        u := Filtered(u,is_partition);

        # % then check the sizes of our partitions
        for i in [1 .. Length(u)] do
                u[i] := Length(u[i]);
        od;

        # % if we have any partitions, pick the smallest
        # % otherwise return 0 as the pnumber
        if Length(u) <> 0 then
                pnumber := Minimum(u);
        else
                pnumber := 0;
        fi;

        return pnumber;
end;
```

A.2 HUGHES THOMPSON GROUPS

The hpg function implemented below tests whether $G = H_p(G)$ for some prime divisor of $|G|$. The code is fairly straightforward; for each prime divisor p, we generate a collection of all $x \in G$ such that $x^p \neq 1$. Then we form the subgroup generated by each collection, and build an array that indicates with true/false whether that subgroup is equal to G. So in searching for a partitionable group, we are looking for a “false" entry in the output list `isg`.

Listing 2: hpg function

```
hpg := function(G)

# % First we find the prime divisors of G
n := Size(G);
divs := DivisorsInt(n);
count := 0;

# % here we check if each divisor is prime
# % if it is, we store it in primedivs
for i in [1 .. Length(divs)] do
    if IsPrime(divs[i]) = true then
        count := count + 1;
    fi;
od;

# % the loop below seems redundant to the one above, but GAP seems to want
# % primedivs to exist before I can assign values to it
primedivs := [1 .. count];
count := 0;

for i in [1 .. Length(divs)] do
    if IsPrime(divs[i]) = true then
        count := count + 1;
        primedivs[count] := divs[i];
    fi;
od;

r := Length(primedivs);

# % Now we want to find how many elements have the property
# % that x^p <> 1, for each p:

# % e is a list of all elements of G
e := Elements(G);
```

```
axp := [1 .. r];

# % main loop to get all x such that x^p <> 1
for i in [1 .. r] do
    count := 0;
    for j in [1 .. n] do
    # % b compares element j^p = 1
    b := e[j]^primedivs[i] <> e[1];
        if b = true then
            count := count + 1;
        fi;
    od;

    xp := [1 .. count];
    count := 0;

    for j in [1 .. n] do
    b := e[j]^primedivs[i] <> e[1];
        if b = true then
            count := count + 1;
            xp[count] := e[j];
        fi;
    od;
    axp[i] := xp;
od;

# % now axp is an array that stores all the x^p <> 1 for each p | n
# % we should check if the group generated by those things is G

# % isg will be a boolean list that for each p, says true if the
# % group generated by all x^p <> 1 is G
isg := [1 .. r];

for i in [1 .. r] do
    Gt := Subgroup(G,axp[i]);
    t := Size(Gt);
    if t = n then
        isg[i] := true;
    else
        isg[i] := false;
    fi;
od;

# % So at the end we can look at isg.  If isg contains a "false",
# % G has a partition.

return isg;

end;
```

COMPUTATIONAL INEFFICIENCY

The `pnumber` function and its associated subroutines were given as one method for computing the partition number of a function in `GAP`. The downside to this method, as was alluded to earlier, was that the method was very computationally inefficient, computing all possible combinations of subgroups and filtering through them to find ρ.

In fact, this method is so inefficient, that without a tremendous amount of memory (and a way of telling `GAP` how to use it), `pnumber` will not return an answer for even moderately small finite groups (e.g., order 18 or more). `GAP` will simply run out of memory and not be able to extend the workspace any further to accomodate the demands of the routine. What we present here is a strategy for dealing with this issue, even on moderately powerful machines.

The solution is to subdivide the work into small enough pieces, storing the results outside of temporary memory (i.e., on the hard disk instead of in RAM), and processing the individual pieces to get an answer. This is still a work in progress, and there is further improvement that can be made. We will exhibit explicitly how to do this in the case of $\rho(S_4)$.

B.1 BACK TO S_4

In Chapter 4, we demonstrated a partition of S_4 that contained 10 subgroups, which bounds $\rho(S_4)$ above by 10. Noting that S_4 is not homomorphic to K_4, our only concern at this point becomes whether or not S_4 has a partition $\mathscr{P}$ with $|\mathscr{P}| \in \{4,5,6,7,8,9\}$.

Our first goal is to isolate combinations consisting of i subgroups for each $i \in \{4,5,6,7,8,9\}$. A quick check in `GAP` shows that S_4 has 28 nontrivial subgroups, so in the worst case where $i = 9$, we will have to consider $\binom{28}{9} = 6,906,900$ combinations.

This is admittedly a lot, but still within GAP's capabilities on a relatively modern machine.

In GAP we can run the following code to generate all the subgroups of S_4, and store all combinations of 9 nontrivial subgroups in a text file.

```
LoadPackage("sonata");

g := SymmetricGroup(4);
subs := Subgroups(g);
n := Length(subs);
tsubs := [1 .. n-2];

for i in [2 .. n-1] do
        tsubs[i-1] := subs[i];
od;

AppendTo("E:\\thesis\\sfour\\nine\\subs.txt", "u :=", Combinations(tsubs
    ,9), ";");
```

This will take a while; the resultant text file weighs in at 1.82 gigabytes. For smaller binomial coefficients, we can read the above file directly into GAP and begin to process it. However, in the interest of making the method stronger, that may not be the ideal approach. Instead, what we can do is split this file into several smaller files that will be easier to handle.

There are at least two ways we could approach the splitting problem. The more robust method, and the one that will ultimately be required if we want to handle larger and larger binomial coefficients is to write a new function to compute combinations. The built-in `Combinations` functions uses recursion and does not lend itself well to providing partial results. If we could write a method that used for loops, we could compute a portion of the combinations, save them, and then come back and compute more on subsequent passes to ease the memory burden.

For now, though, the approach is just to use another utility to manually split the file. There are various third-party wares available to handle such a problem; without wishing to show any preference to one in particular, the author notes that `GSplit` (`http://www.gdgsoft.com/gsplit/`) is a freeware application that is able to handle the task.

After splitting the file, we can then format the resultant files in such a way as to make them directly readable by GAP . In particular, we can structure each file so that when read by GAP , it stores the combinations as an array. This is how the original (large) file was structured, but because we lack control over the output of the Combinations function, we do need to tweak the split files to make sure each smaller file is structured similarly. Again, there are many methods available for manipulating text files; on a Windows machine, for example, the user can run

```
for /f %f in ('dir /b e:\thesis\sfour\nine\ndisk') do ( ren %f temp.txt
echo.u := [>%f
echo.asfdsasdf>tmp.txt
type temp.txt tmp.txt>>%f
del temp.txt
del tmp.txt
)
```

in a command window, from the same directory as the split files are stored to achieve the desired effect. The code above places `u := [` at the beginning of each file in the ndisk directory (where the split files are stored), and inserts an arbitrary string at the end of each file. The purpose of the arbitrary string is to give us a placeholder which we can then use to search and replace to get the appropriate ending syntax of a GAP command at the end of each file. The reason for doing this is because GSplit, or whatever method is used to split the files, will need to split the file after the ending character of a particular combination, which is a comma. So each split file ends in a comma, but we need it to end in `];` so that the contents of the file are exactly the GAP code for assignment of an array of combinations.

Having formatted the split files appropriately, all we need to do is pass their contents to the is_partition function to see if we have a partition. Depending on the number of split files, this is not necessarily a trivial task. For example, to achieve a manageable average split file size of around 15 megabytes requires splitting the subs.txt file generated above into 222 files. A nice way to address this issue is (of course) using GAP . We can copy all of the split file names into a text editor, then use a search and replace function to generate something like this:

```
Read("E:\\thesis\\ndisk\\ndisk1.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk2.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk3.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk4.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk5.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk6.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk7.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk8.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk9.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
Read("E:\\thesis\\ndisk\\ndisk10.txt"); for i in [1 .. Length(u)] do b := is_partition(u[i]); if b = true then AppendTo("E:\\thesis\\bndisk.txt", b, "\n"); fi; od;
```

In the above code, the split files are the `ndisk{num}.txt` files. What each line of code is doing is looping through the array stored in `ndisk{num}.txt` (recall that each element of the array is a combination of subgroups), and running checking if we have a partition. If for some combination we do have a partition, then a "true" is written to the `bndisk.txt` file. At the end, if `bndisk.txt` is a blank text file, then we found no partitions.

If we run the above code for all 222 split files (this can be done more easily by saving the above code into a text file, and reading *that* file into `GAP`), we confirm that no combination of 9 subgroups of S_4 is a partition. Then we need to rinse and repeat the above technique for $i \in \{4,5,6,7,8\}$. The method is admittedly tedious and not at all elegant, but in so doing, we can confirm that:

COROLLARY B.1.1. *The partition number of S_4 is* 10.

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