FREEDOM FROM WITHIN: SEEKING GENERATING SETS FOR FREE GROUPS

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ABSTRACT

FREEDOM FROM WITHIN: SEEKING GENERATING SETS FOR FREE GROUPS

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Since the 1870's, mathematicians have had only one real tool to help identify when a group contains free subgroups: the Klein Criterion. While powerful and elegant, there are cases when this theorem does not prove helpful. In order to apply it, we must have a candidate set that could generate a free group. This is no longer the case. The *Šunić Criterion* is a new theorem discovered by Zoran Šunić in 2013, and it does not require us to have a candidate for a generating set beforehand! In theory, this could allow mathematicians access to a host of groups which until now have been out of reach.

In this thesis we use this new theorem to reprove that certain groups contain free subgroups and expand upon it showing that it can also be used to find explicit generating sets for locally free groups. Having done this, we find explicit generating sets for locally free subgroups in $PSL_2(\mathbb{Z})$ and $Aut(\mathbb{R}^+)$. We also explore the conditions of this new theorem.

CHAPTER ONE: INTRODUCTION

Consider an object. It might be a triangle in the Euclidean plane, a tree, or perhaps a group of friends. What can we do to this object without fundamentally changing it? While this question seems vague it is precisely the question that gave rise to geometric group theory. For example, consider a square with indistinguishable vertices and indistinguishable sides. How can we map this square to itself? Certainly we can rotate it clockwise by $\pi/2$ radians with respect to the square's center; this would move each vertex. We could also reflect it through one of its diagonals, in essence "fixing" two vertices while swapping the other two. As it turns out, the full collection of these *symmetries*[1] are endowed with all the required properties to form a group. The geometric group theorist would say that these symmetries *act* on the square. In essence, each element of the group of symmetries affects the square in a predetermined way while preserving the structure of the object. More precisely, we define a group action as follows:

Definition 1.1. Let G be a group. The *action* of an element $a \in G$ on a non-empty set X is a homomorphism from $G \to SYM(X)$ (the group of symmetries of X). Alternatively, we can define an action as a map from $G \times X \to X$ such that for all $x \in X$ and all $a, b \in G$ we have that if $a = 1$ then $a \cdot x = x$ and that $(bc) \cdot x = b(c \cdot x)$.

Definition 1.2. We say that G *acts freely on* X if the only element of G that fixes an element of X is the identity.

Not surprisingly, the geometric approach is extremely flexible and grants a certain intuition that might otherwise be opaqued in the cloud of symbols that can

[1] The collection of these symmetries under the operation of composition actually form a well known group: D_4, which has exactly 8 elements and which is generated by the two described symmetries.

accompany a more "traditional" approach to group theory. However, while this approach is useful, it is undoubtedly not new.

Geometric group theory can trace its origins to two nineteenth century mathematical giants: Arthor Cayley and Felix Klein. Cayley began the study of geometric group theory with *Cayley's Theorem*[2]:

Theorem 1.3. *Every group is isomorphic to a subgroup of the symmetric group.*

Proof. Let G be a group. Define the map $\Pi_a : G \to G$ for all $a \in G$ as $\Pi_a(x) = a \cdot x$. Note that

- Π_a is a permutation of G for all $a \in G$ and it forms a group with identity Π_1 where 1 is the group identity of G.

- Let f be the map that takes $a \in G$ to Π_a. Then $f(ab) = \Pi_{ab}$ and for any $x \in G$ $(ab)x = a(bx)$ thus $\Pi_{ab} = \Pi_a \Pi_b = f(a)f(b)$ which implies that $f(ab) = f(a)f(b)$. Also note that if $f(a) = \Pi_1$, then $a = 1$ so f is an isomorphism between G and a subgroup of the Symmetric group.

□

While this theorem in itself does not seem to be linked to geometry, it provides the basis of the next theorem, referred to as *Cayley's Better Theorem* in [4]. However, before we prove this theorem we need to make use of some extra machinery (also courtesy of Cayley).

Definition 1.4. The *Cayley graph*[3] *of G with respect to the generating set S* is the directed graph $\Gamma_{G,S}$ with vertex set $V(\Gamma_{G,S}) = G$ and with a directed edge from a vertex a to a vertex b if there exists an element $s \in S$ such that $as = b$.

[2] We should note that this is not the language that Cayley himself used. Rather, it is the modern interpretation of the theorem.

[3] While some texts from graph theory would call this construction a Cayley digraph we instead use the convention of [4] and [1] and do not distinguish between the directed graph and the underlying undirected graph.

As an example, here is the Cayley graph of D_4 with one generator corresponding to a rotation of $\pi/2$ and another corresponding to a reflection.

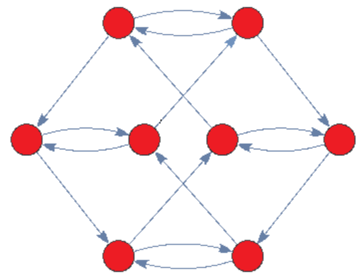

Fig. 1: A Cayley Graph of D4.

To show how the graphs depend on the generating sets, consider the following two Cayley graphs of $\mathbb{Z}/7\mathbb{Z}$. The graph to the left uses $S = \{1\}$ while the graph to the right uses $S = \mathbb{Z}/7\mathbb{Z} - \{0\}$. Notice that we omit arrows when we have bidirectional edges .

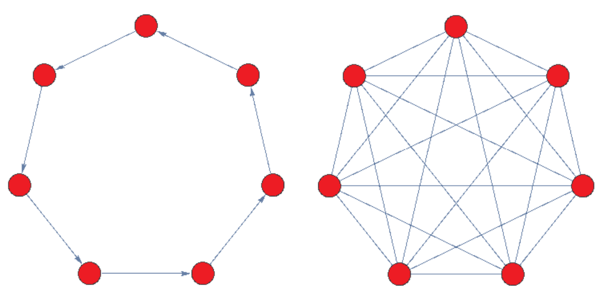

Fig. 2: Two Cayley Graphs of $\mathbb{Z}/7\mathbb{Z} - \{0\}$ With Different Generating Sets.

Definition 1.5. A *symmetry of a (di)graph* Γ is a bijection from Γ to itself that preserves adjacency. So if F is a symmetry and (a, b) is a (directed) edge in $E(\Gamma)$ then $(F(a), F(b))$ is a (directed) edge of $F(\Gamma)$.

Having these definition at our disposal, we can proceed to the proof of Cayley's Better Theorem.

Theorem 1.6 (Cayley's Better Theorem). *Every group is isomorphic to a symmetry group of a directed, connected, locally finite*[4] *graph.*

Proof. Let G be a finitely generated group with finite generating set S and Cayley graph $\Gamma_{G,S}$. We will prove the theorem by first proving that $\Gamma_{G,S}$ is connected, locally finite, and then that G is its group of symmetries.

Fix a vertex $v \in V(\Gamma_{s,g})$ and let $m = |S|$. By the construction of our graph then there is an edge from v to x if and only if there is an element $s \in S$ such that $vs = x$. Therefore there are exactly m edges from v to other elements of G. Likewise, because G is a group, for each s there is a unique element $g \in G$ such that $gs = v$ thus there are exactly m edges terminating in v. From this we see that the degree of v is exactly $2m$ so $\Gamma_{S,G}$ is locally finite.

To prove that the graph is connected we show that there is always a path between the identity vertex and any other vertex. To accomplish this, we again let $v \in V(\Gamma_{G,S}) - \{e\}$. Since by assumption S generates G then there is some minimum combination of elements from S such that $s_1 s_2 ... s_k = v$. Then there is path $e \to s_1 \to ... \to s_k$ that terminates in v. Thus v is connected to the identity.

The only remaining claim from the theorem left to prove is that G is isomorphic to a subgroup of the group of Symmetries of $\Gamma_{S,G}$. To show this, we begin by defining the action of G on $\Gamma_{S,G}$ by left multiplication of vertices. Thus, if $g, h \in G$ we have that $g \cdot v_h = v_{gh}$ where v_h is the vertex of $\Gamma_{S,G}$ with the label h. Note that this action is extended to edges: if (a, as) is an edge, then $g \cdot (a, as) = (ga, gas)$. Thus, G forms a subgroup of the group of symmetries of $\Gamma_{S,G}$.

From here it is easy to see the map that we have constructed from G to $SYM(\Gamma_{G,S})$ is a monomorphism since the kernel of the homomorphism that takes

[4]This means that every vertex has finite degree.

an element of G to its corresponding action contains only the identity. $\square$

The beauty of this theorem is that it shows the great power of the geometric perspective. It is not just that by studying group actions we are able to get a hand on the structure of infinite groups, it is that every group can be looked at geometrically! However, while in hind sight we can appreciate the perspective gained by this statement, Cayley himself only applied his theorems to finite groups [3].

Having examined an example of how a group might act on a specific type of graph we will now introduce a concept which will help us better understand how a group might act on an arbitrary graph.

Definition 1.7. Let a group G act on a connected graph Γ which is embedded in some topological space X. Then a subset $J \subset \Gamma$ is a *fundamental domain* for the action of G if:

1. J is closed in the subspace topology induced from X,
2. the set $\{g \cdot J | g \in G\}$ covers the graph Γ, and
3. no subset of J satisfies properties 1 and 2.

Two observations are that if G acts on a connected graph Γ then there must exist a fundamental domain (even if it is not unique)[5]. Also, if a group G acts on a connected graph then any subgroup H must also act on the same connected graph and thus there would be a fundamental domain of the action of H on the graph that might differ from the fundamental domain of the action of G.

For an example of a fundamental domain, consider the action of a group G with a generating set S on the Cayley graph $\Gamma_{G,S}$. In this case, a fundamental domain would be any single vertex along with a connected half edge for every edge that begins or terminates at that vertex. To show that this does in fact constitute a fundamental

[5] for the unconvinced reader, we will give an algorithm to build a fundamental domain in the next section.

domain, let us label the vertex in question v and the vertex along with its incident half edges by J. Assume that (a, b) is a directed edge in $\Gamma_{G,S}$ Then there exists an element $s \in S$ such that $as = b$. First notice that there exist $c, d \in G$ such that $vc = a$ and $vd = b$, and so both a and b are in the image of J. Now note that $vs = m$ for some $m \in g$ and so the half edge originating at a is in the image of J. Similarly there is some element $n \in G$ such that $ns = v$ and so the half edge terminating in b is also in the image of J. Thus (a, b) is in the image of J under the action of G.

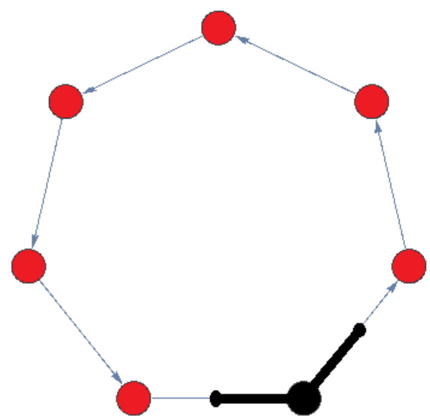

Fig.3 The Cayley Graph of C_7 With Fundamental Domain in Black.

While Cayley was interested in finite groups, Klein was interested in geometry. The nineteenth century saw an explosion of geometric research. Mathematicians were exploring all aspects of properties in new and exciting geometries. What Klein sought was to classify all these new geometries in a meaningful way. In a lecture delivered to the University of Erlangen in 1872 he proposed to view geometry as the study of invariants under the action of groups of transformations [3]. In the same lecture he classified the projective group, the elliptical group, and he gave the first description of what would become known as free groups.

CHAPTER TWO: FREE GROUPS AND THE KLEIN CRITERION

In 1872, Klein described a new type of group acting on the Riemann sphere where no reduced nontrivial combination of elements fixes the sphere. His student Walther von Dyck later pointed out that these groups are the simplest of groups, and that no relations exist among their elements. In a paper in 1882, von Dyck [6] named these groups *free groups* since they are free of any relations. Geometrically, this is equivalent to saying that given a group G which acts on a set X, G is free if no reduced nontrivial combination of the elements of G fixes X. In this chapter we will explore several definitions and theorems for free groups. While a precise definition for free groups can be given from the point of view of universal algebra or pure logic [3] we will instead employ the "constructionist" definition using words.

Definition 2.8. An *alphabet* is a set $S = S^+ \cup S^-$ where S^+ is a set of symbols and $S^{-1} = \{s^{-1} | s \in S^+\}$ (here s^{-1} is called the formal inverse of the symbol s).

Definition 2.9. A *letter* is an element of an alphabet S.

Definition 2.10. Given a alphabet S, we call a finite string of letters from S a *word* and say that it is *reduced* if it does not have any consecutive pairs of letters which are formal inverses of each other. The empty set of letters is referred to as the identity.

Having these definitions, we can now provide a definition for free groups.

Definition 2.11. Given a set S^+ with cardinality n, we say that the group F generated by S is *free* if no nontrivial reduced word represents the identity, and we denote this group by F_n, and call it the *free group* of rank n.

Definition 2.12. Given an alphabet $S = S^+ \cup S^-$ we say that S^+ *freely generates* F if the group generated by S is free.

We should point out that we can freely generate a group with any set of symbols. The operation of the group would then be concatenate and reduce.

2.1 Theorems for Free Groups

Using the *word definition* of a free group we can prove the following theorem.

Theorem 2.13. *Every group is a quotient group of a free group by a normal subgroup.*

Proof. Let $S = G$ and let F be freely generated by S. Define a homomorphism $h : F \to G$ as $h(s) = s$ if $s \in S$ and as $h(s_1 s_2 ... s_n) = h(s_1)h(s_2)...h(s_n)$ for reduced words with more than one letter from S. Clearly h is an epimorphism, and by the first isomorphism theorem we have that $G \cong F/Ker(h)$ where $Ker(h)$ is the kernel of h. □

What this theorem shows is that we can in a way use free groups as "building blocks" to form all groups.

While the word definition of a free group has its advantages, understanding exactly what a free group is might best be accomplished geometrically, as the following theorem illustrates.

Theorem 2.14. *A group F is free if and only if it possesses a generating set S such that the Cayley graph $\Gamma_{F,S}$ is cycle free.*

Proof. Assume that F is freely generated by S. Also, assume that $\Gamma_{F,S}$ contains a cycle. Then for some vertex h, we have that $h = hs_1 s_2 ... s_n$. Thus by left multiplying both sides of the equation by h^{-1} we see that this implies that $e = s_1 s_2 ... s_n$ so there is some reduced word which represents the identity, contradicting the assumption that F is freely generated by S. The other direction of the proof is similar. Assume that there is a cycle free $\Gamma_{F,S}$ and that some reduced word represents the identity. Then let $e = s_1 s_2 ... s_n$ be that word. We see then that this word produces a cycle in the Cayley graph. □

Since, by construction, Cayley graphs are connected, this theorem gives us the following immediate corollary which will be useful later on.

Corollary 2.15. *A group F is free if and only if it has a Cayley graph $\Gamma_{F,S}$ that is a tree.*

The following figure shows a Cayley graph for F_2.

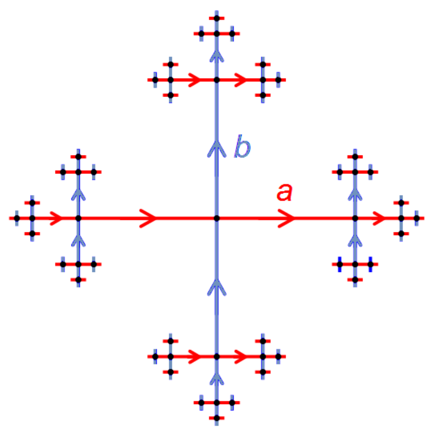

Fig. 4: The Cayley Graph of $\mathbf{F_2 = \langle\ a,b\ \rangle}$.

A variation on the next theorem for deducing when a group is free was first presented by Klein in the context of hyperbolic geometry. We state the theorem without proof for now, as we will prove it in the next section.

Theorem 2.16 (Klein Criterion)**.** *Let G act on X and let $S = S^+ \cup S^{-1}$ be a set of generators and their inverses. For each $s \in S$ let X_s be a subset of X, and let p be a point in $X - \{\cup_{s \in S} X_s\}$ for which the following two conditions hold:*

1. $s(p) \in X_s$ for all $s \in S$; and

2. $s(X_t)$ is a proper subset of X_s, for each $t \in S - \{s^{-1}\}$.

then G is a free group with basis S^+.

Having this theorem at our disposal we can proceed to prove the following lemma.

Lemma 2.17. *A group G is free if and only if it acts freely on a tree.*

Proof. The forward direction is trivial: if a group is free then its Cayley graph is a tree and it acts freely on it. For the reverse direction we first construct a fundamental domain for the action of G on the tree and then apply the Klein criterion to show that it is free.

Let G act freely on a tree T, and let $v \in V(T)$. Now consider the connected subgraph $C \subset T$ where $v \in C$ and if x and y are distinct vertices in C then there is no element $g \in G$ such that $g \cdot x = y$. Since v alone satisfies this condition then we can label the graph v as C_1 and construct a chain of subgraphs $C_1 \subset C_2 \subset C_3 \subset ...$ be a sequence of proper subgraphs each containing the previous. And define[6] $CORE = \cup_{i \in \mathbb{N}} C_i$.

We see that the image of $V(CORE)$ under the action of G is $V(T)$. If the image of $E(CORE)$ under the action of G is $E(T)$ then $CORE$ forms a fundamental domain and we are done. Otherwise we add half edges[7] in the following way: let $(v, w) \in E(T)$ such that $v \in CORE$ and $w \notin CORE$. Then add the closed half edge starting with v. Continue to do this until there are no more edges that meet this description.

Now if there is still an edge (x, y) after doing this such that $(x, y) \notin G \cdot CORE$ then we have elements $g_1, g_2 \in G$ and vertices $v_x, v_y \in V(CORE)$ such that $g_1 \cdot v_x = x$ and $g_2 \cdot v_y = y$ but then g_1 must take the half edge to cover one half of (x, y) and

[6] In doing this, we use the axiom of choice for trees that are not locally finite.

[7] In order to keep the notation consistent, we denote edges as (x, y) for this proof. However, we do not make an assumption as to whether this edge is directed or not.

g_2 must take another half edge to cover the remaining half of (x, y) so we see that $(x, y) \in G \cdot CORE$ which is a contradiction. Thus, $CORE$ is a fundamental domain.

For convenience, we will denote the fundamental domain we constructed by J, and distinguish it from the $CORE$ subgraph without half edges. For each half edge $h_e \in J$ let $g_e \in G$ be the element such that $J \cap g_e \cdot J$ is the midpoint of an edge. We define the set S to be the set of all such elements in G. Then we see that $g_e^{-1} \cdot J \cap J$ is also the midpoint of an edge and so g_e^{-1} is also an element of S. Having constructed S, we will now prove that S is a set of generators of G.

Let $g \in G$ and let $v \in J$. since T is connected, chose a path p joining v to $g \cdot v$. Let $\{J, g_1 J, g_2 J, ...g_n J = gJ\}$ be a finite sequence of images of J under the action of G such that

1. $p \subset \cup g_i J$ and,
2. $g_i J \cap g_{i+1} J \neq \emptyset$ where $g_0 = 1$.

Since $J \cap g_1 J \neq \emptyset$, then $g_1 \in S$. This, in turn, implies that $J \cap g_1^{-1} g_2 J \neq \emptyset$ so $g_1^{-1} g_2 \in S$. Thus g_2 is a product of elements in S. Through a simple induction argument on the length of p, we find that $g = g_n$ is a product of the elements of S.

Having constructed a fundamental domain and found a set of generators for G we are now in a position to apply the Klein Criterion. Fix $v \in V(J)$. For each $g_e \in S$ let T_e be the subtree of T induced by the set of vertices $v' \in V(T) - V(J)$ such that the reduced path from v to v' passes though the half edge h_e. Then,

1. for each $g_e \in S$, let X_s be the associated subtree T_e;
2. take $p = v \in V(J)$ to be any point outside of the union of all T_e.

Condition 1 of the Klein Criterion follows from the construction of T_e. Let $g_e \in S$ and let $T_{e'}$ be one of the subtrees such that $g_{e'} \neq g_e^{-1}$. Let $f = g_e^{-1} \cdot e$ denote the edge connecting $g_e^{-1}(CORE)$ to $CORE$. Given any vertex $w \in T_{e'}$ there is a minimal edge path connecting $g_e^{-1} \cdot v$ to w. Applying g_e we get a minimal edge path from v to $g_e \cdot w$. The original path passed through f, hence the image under g_e will pass through e Thus $g_e \cdot w \in T_e$ by construction. Since w was arbitrary then $g_e(T_{e'}) \subset T_e$. Therefore G is free. □

The beauty of this seemingly particular lemma is that it leads to an extremely simple proof of the Nielsen-Schreier Theorem.

Theorem 2.18 (Nielsen-Schreier Theorem)**.** *Every subgroup of a free group is free.*

Proof. Let H be a subgroup of a free group F . Then F acts freely on its Cayley graphs,one of which is a tree, and therefore H acts freely on the Cayley graph of F. □

One very nice, immediate corollary of this is the following:

Corollary 2.19. *F_n is a subgroup of F_2 for every natural number n.*

In other words every free group contains free subgroups isomorphic to F_n for every natural number n. Having classified all subgroups of free groups and shown how we can "build" other groups from them it might be surprising to know that free groups themselves pop up naturally in several contexts. While this was first noted by Klein himself in the 1870's, it wasn't until Jaque Tits proved his famed *Tits' Alternative* a century later [5] that the exact extent to which this phenomenon occurs was known. What follows is the statement of his theorem. Since the proof is technical and quite long we omit it here.

Theorem 2.20. *Let Γ be a subgroup of $GL_n(\mathbb{K})$ for some integer $n \geq 1$ and some field of characteristic zero, $\mathbb{K}$. Then either Γ contains a free group of rank 2 or a solvable subgroup of finite index.*

Not surprisingly, we can use the word definition of a free group to "combine" different groups in a new way. We call this combination of groups the free product of two groups.

Definition 2.21. Let A and B be groups. Then $A * B$ is the *free product of A and B* where the elements of $A * B$ are the reduced words formed from the alphabet $S = A \cup B$. In this case we call a word reduced if $w = x_1 x_2 ... x_n$ with all $x_i \in S$ and

- no $x_i = 1_A$ *or* 1_B (where 1_A is the identity of A and 1_B is the identity of B); and
- if $x_i \in A$, then $x_{i+1} \in B$. Similarly if $x_i \in B$ then $x_{i+1} \in A$.

The operation of $A * B$ is concatenation and reduction in either A or B.

From this definition, we can see that the free product of two infinite cyclic groups is isomorphic to F_2.

It is simple to check that the free product of two groups is again a group. What is less intuitive is the following theorem that classifies when the free product of two groups is a free group. This theorem will prove vital in understanding applications of the Generalized Klein Criterion.

Theorem 2.22. *Let A and B be groups. Then $A * B$ is free if and only if A and B are free.*

Proof. One direction of this proof follows from the observation that there is a natural monomorphism from both A and B to $A*B$. So A and B are isomorphic to subgroups of $A * B$. Thus if A or B is not free then $A * B$ contains a subgroup that is not free and so by the Nielsen-Schreier theorem, $A * B$ is not free.

On the other hand, assuming that both A and B are free we see then that $A*B$ is equivalent to the word definition of a free group with generators $S = S_A \cup S_B$. □

2.2 The Klein Criterion

This last theorem along with the Klein Criterion gives some insight into two of the more fundamental questions concerning free groups:

- Given a group G and a subset $S \subset G$, is $\langle S \rangle$ free?
- Given a group G, does G contain a free subgroup?

One of the earliest observations combining free groups and geometry is still one of the most powerful tools used to attack these problems. As Johanna Mangahas eloquently said in [2]: "[free] groups play ping pong." The Ping Pong lemma (also known as the Klein Criterion) gives a way to determine if a group is free based on its action on a mathematical object X. When it works, it works beautifully. However, to apply this tool we must have a generating set to begin with (as we will see later, this alone does not guarantee that we can apply the lemma). Thus, while the criterion has been invaluable for solving the first question in multiple contexts, it is inefficient at answering the second. In this section we will prove two alternative versions of this theorem and use it to demonstrate the existence of free subgroups.

We will now proceed with the proof of Theorem 2.16. For completeness, we restate it here.

Theorem 2.16 (The Klein Criterion): *Let G act on X and let $S = S^+ \cup S^{-1}$ be a set of generators and their inverses. For each $s \in S$, let X_s be a subset of X, and let p be a point in $X - \{\cup_{s \in S} X_s\}$ for which the following two conditions hold:*

1. *$s(p) \in X_s$ for all $s \in S$; and*
2. *$s(X_t)$ is a proper subset of X_s, for each $t \in S - \{s^{-1}\}$.*

Then G is a free group with basis S^+.

Proof of the Klein Criterion. This will be an inductive proof on the length of a reduced word $w \in G$ (which we will denote $l(w)$). Assume $l(w) = 1$, then $w = s$ for some $s \in S$. So $w \cdot p = s(p) \in X_s$. Thus $s(p) \neq p$ so no word of length 1 represents the identity. Assume that whenever $l(w) < n$, no reduced word of length w represents the identity. Consider a reduced word w_n of length n. Then $w_n = s_2 w_{n-2} s_1$ for some $s_1, s_2 \in S$ and some reduced word w_{n-2} of length $n-2$. Consider the action w_n on $p : w_n \cdot p = s_2 w_{n-2}(s_1(p))$. Thus, by assumption, $w_n \cdot p \in X_{s_2}$, so we see that w_n is not the identity. Hence, no reduced word represents the identity. □

While this version of the Klein Criterion is quite standard we will now give a generalized version which was used by Tits to prove his alternative [1].

Theorem 2.23. *(Generalized Klein Criterion) Let G be a group acting on a set X, and let Γ_1, Γ_2 be two subgroups of G, such that Γ_1 contains at least three elements and Γ_2 contains at least two elements. Assume that there exists at least two nonempty subsets X_1 and X_2 in X with $X_2 \not\subset X_1$. Furthermore, assume that the following two conditions hold,*

- $a(X_2) \subset X_1$ *for all* $a \in \Gamma_1 - \{1\}$,
- $a(X_1) \subset X_2$ *for all* $a \in \Gamma_2 - \{1\}$.

Then, $\Gamma = \langle \Gamma_1, \Gamma_2 \rangle$ *is isomorphic to* $\Gamma_1 * \Gamma_2$.

Proof. Let w be a nonempty reduced word drawing from the alphabet $S = (\Gamma_1 \cup \Gamma_2) - \{1\}$. We must show that the element of Γ defined by w is not the identity (to distinguish which letters come from which group we will assume that $a_i \in \Gamma_1$ and $b_i \in \Gamma_2$ throughout this proof). This proof can be broken down into four cases based on the first and last letter of the word w.

1. Let $w = a_1 b_1 a_2 b_2 ... b_{k-1} a_k$, then

$$w(X_2) = a_1 b_1 a_2 b_2 ... b_{k-1} a_k(X_2) \subset a_1 b_1 a_2 b_2 ... b_{k-1}(X_1)$$

$$\subset a_1b_1a_2b_2...a_{k-1}(X_2) \subset ... \subset a_1(X_2) \subset X_1.$$

Since by assumption X_2 is not included in X_1, we see that w is not the identity.

2. Let $w = b_1a_1b_2a_2...a_{k-1}b_k$. If w represents the identity, then so does $a_1wa_1^{-1}$. Assuming that this is the case we have reduced the problem to Case 1 and reach a contradiction. Thus w is not the identity.

3. Let $w = b_1a_1b_2a_2...a_k$. If w represents the identity, then so does $a_iwa_i^{-1}$ (where $a_i^{-1} \neq a_k^{-1}$). Assuming that this is the case we have reduced the problem to Case 1 and reach a contradiction. Thus w is not the identity.

4. Let $w = a_1b_1a_2b_2...b_k$. If w represents the identity, then so does $a_iwa_i^{-1}$ (where $a_i \neq a_1$). Assuming that this is the case we have reduced the problem to Case 1 and reach a contradiction. Thus w is not the identity.

In all cases, w does not represent the identity. □

2.3 Some Applications of the Klein Criterion

The following examples illustrate how the Klein Criterion and its generalizations have been used to prove the existence of free subgroups in multiple contexts. The first of these[8] is the special linear group $SL_2(\mathbb{R})$. This is the subgroup of $GL_2(\mathbb{R})$ consisting of matrices with determinant 1.

Proposition 2.24. *The group $SL_2(\mathbb{R})$ contains a free subgroup that is isomorphic to F_2.*

[8]The first example can be proven using the same generating set with slightly different divisions of the plane with many different variations of the Klein criterion. We have essentially expanded the proof found in [1], which uses the Generalized Klein Criterion although the interested reader can look at [4] to see the Klein Criterion used instead.

Proof. To begin this proof we first note that $SL_2(\mathbb{Z}) < SL_2(\mathbb{R})$ acts on the Euclidian plane with the action defined as $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot (x, y) = (ax + by, cx + dy)$. Now, let $S^+ = \{\left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]\}$, and let us denote $a = \left[\begin{smallmatrix} 1 & 0 \\ 2 & 1 \end{smallmatrix}\right]$ and $b = \left[\begin{smallmatrix} 1 & 2 \\ 0 & 1 \end{smallmatrix}\right]$. We will examine the following subsets of the plane: $A = \{(x, y)| \ |y| > |x|\}$, $B = \{(x, y)| \ |x| > |y|\}$.

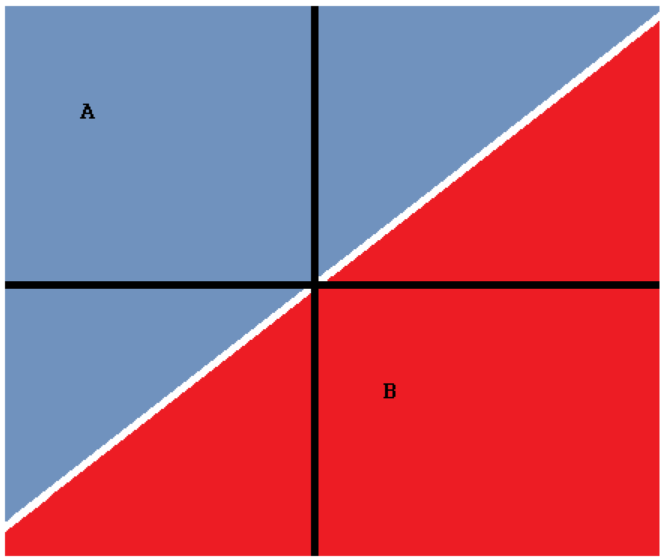

Fig. 5: The Division of the Plane.

It is clear that A does not contain B and that B does not contain A. We will now proceed to apply the Generalized Klein Criterion by showing the following two conditions hold.

1. Let $(x, y) \in B$ and $n \neq 0$. Then $a^n(x, y) = (x, 2nx + y)$. The proof of this condition stems from the assumption that $|x| > |y|$. Using this we see that $|2nx + y| > |2nx| - |y| > 2|x| - |y| > 2|x| - |x| = |x|$. So $a^n(x, y) \in A$.

2. $b^n(A) \subset B$ for $n \neq 0$. The proof of this condition is similar to Condition 1.

Thus by the generalized Klein Criterion and Theorem 2.22 we see that $\langle a, b\rangle$ is a free group of rank two. □

The particular free group that we have just described is the standard free group usually used to prove that $SL_2(\mathbb{R})$ contains free subgroups.

We will now use the Generalized Klein Criterion to prove that $Homeo_+(\mathbb{R})$ (the group of orientation preserving homeomorphisms of the line) contains free subgroups. The White Group (which will be discussed in detail in Chapter 3) is also a subgroup of this group. This proof was taken in great part from [1] .

Proposition 2.25. *The group $Homeo_+(\mathbb{R})$ contains a free subgroup of rank two.*

Proof. Let f be the piecewise linear homeomorphism of the interval $[0,1]$ defined by

$$f(t) = \begin{cases} 4t & \text{if } t \in [0, 0.2] \\ 0.8 + 0.25(t - 0.2) & \text{otherwise.} \end{cases}$$

Observe that for $n \geq 1$ we have that $f^n([0.2, 1]) \subset [0.8, 1]$ and for $n \leq -1$ we have that $f^n([0, 0.8]) \subset [0, 0.2]$.

Let γ_1 be the piecewise linear homeomorphism of the real line defined by

$$\gamma_1(t) = [t] + f(\{t\}) \; for \; all \; t \in \mathbb{R}$$

where $[t]$ denotes the integral part of t and $\{t\} = t - [t] \in [0, 1]$ denote its fractional part. Also, let $T(t) = t - 0.5$ and $\gamma_2 = T\gamma_1 T^{-1}$. We aim to show that $\langle \gamma_1, \gamma_2 \rangle$ is free.

Let $X_1 = \cup_{k\in\mathbb{Z}}[k - 0.2, k + 0.2]$ and $X_2 = T(X_1) = \cup_{k\in\mathbb{Z}}[k + 0.3, k + 0.7]$. Then since for $n \geq 1$ we have that $f^n([0.2, 1]) \subset [0.8, 1]$ and that $f^n([0, 0.8]) \subset [0, 0.2]$, we have that

$$\gamma_1^n(X_2) \subset X_1$$

for all $n \in \mathbb{Z} - \{0\}$. This in turn implies that

$$\gamma_2^n(X_1) = T\gamma_1^n T^{-1}(T(X_2)) \subset T(X_1) = X_2$$

for all $n \in \mathbb{Z} - \{0\}$.

Thus by the Generalized Klein Criterion, Theorem 2.23, $\langle \gamma_1, \gamma_2 \rangle$ is free. $\square$

CHAPTER THREE: THE ŠUNIĆ CRITERION

While powerful, the Klein Criterion is not a catch all solution to finding free groups. The application of this theorem requires that one already have the generators of a free group “in hand.” This is an issue in two respects. First, we might not always possess sufficient knowledge of the group to guess which set of elements might constitute a good candidate for a generating set of a free group. Second, it does not seem likely that we would find unexpected free groups in this way. Fortunately, a recent discovery (which we will refer to as the Šunić criterion) [7] has provided a new way of determining when a group contains a free subgroup. Like the Klien criterion, this new tool examines group actions on a mathematical object X. However, the Šunić Criterion requires additionally that X be a topological space.

The beauty of this new theorem is that we gain the ability to determine if a group contains a free subgroup without needing a candidate for a generating set. This result also provides us with a means of obtaining several explicit “families” of generating sets for free subgroups of different ranks. However, one might ask if this additional restriction on X precludes the theorem’s use in some cases. Luckily, the following theorem (known as Gromov’s Corollary) shows that for every group G, there exists a metric space (and thus a topological space) that G acts on nontrivially.

Theorem 3.26. *Every group is isomorphic to a group of isometries of a metric space.*

Proof. We begin this proof in a similar way to the proof of Cayley’s Better Theorem. First, we construct a metric space from the elements of a group G and then show that G acts on it by isometries. Let us now fix G with a generating set S. Then we will define the function $d_s(g, h)$ as the length of the shortest word representing $g^{-1}h$, which is drawn from the alphabet of S. We will now proceed to prove the theorem by first establishing that d_s is a metric and then showing that the action of G on the space $M = (G, d_s)$ is a group of isometries of M.

Note that since we cannot define words of negative length, then clearly $d_s(h, g) \geq 0$ for all $h, g \in G$. It is also clear that $d_s(h, g) = 0$ if and only if $h = g$. To show that d_s is symmetric, we assume $d_s(h, g) = n$. Then there is a minimum length word w whose length is n which represents $h^{-1}g$. We see that w^{-1} represents $g^{-1}h$ and must also have length n. Therefore $d_s(g, h) \leq n$. If $d_s(g, h) < n$ then we have a word of length less than n who's formal inverse also has length less than n and this formal inverse represents $h^{-1}g$ contradicting that $d_s(h, g) = n$. Thus d_s is symmetric.

To show that d_s satisfies the triangle inequality, let $g, h, k \in G$. Let us define w_{gk} and w_{kh} to be minimum length words such that $gw_{gk} = k$ and $kw_{kh} = h$. Then note that $w_{gk} = g^{-1}k$ and $w_{kh} = k^{-1}h$ thus $d_s(g, k) = l(w_{gk})$ and $d_s(k, h) = l(w_{gk})$. Therefore we see that $g(w_{gk}w_{kh}) = h$ implying $w_{gk}w_{kh} = g^{-1}h$ and so

$$d_s(g, h) \leq l(w_{gk}) + l(w_{gk}) = d_s(g, k) + d_s(k, h)$$

Since the elements of M are the elements of G, we can define the action of $g \in G$ on M to be $g \cdot w = gw$ for all $w \in M$. To prove that this action preserves distance, consider that

$$d_s(h, k) = l(h^{-1}k) = l(h^{-1}g^{-1}gk) = d_s(gh, gk).$$

□

Before proceeding with the statement and proof of the Šunić Criterion we need to first define two terms whose definitions, as used for the purposes of the theorem, vary slightly with what is commonly used in the literature.

Definition 3.27. We say that a group G *acts properly discontinuously* on a topological space X if for every point $x \in X$, there is some open neighborhood of x (which we will denote U_x) such that for all $g \in G$ with $g \neq 1$, we have that $gU_x \cap U_x = \emptyset$.

Definition 3.28. Let $g \in G$ and $Y \subseteq X$ where X is a topological space. Then we say Y is *invariant* under the action of g if $gY = Y$.

3.1 Statement and Proof of the Šunić Criterion

Having these two definitions at our disposal, we can now proceed with the statement of the Šunić Criterion.

Theorem 3.29. *Let G act by left action on a topological space X. The following are equivalent:*

1. *The group G contains a free subgroup of rank 2 such that there exists a nonempty, open F-invariant subspace of $Z \subseteq X$ on which F acts properly discontinuously;*
2. *There exist nonempty, open subsets A and Y of X, with $A \subseteq Y \subseteq X$, and elements a, a_0, a_1, a_2 such that:*
   (a) $Y = A \cup aA$,
   (b) a_0A, a_1A and a_2A are pairwise disjoint,
   (c) Y is invariant under a, a_0, a_1, a_2;
3. *There exist nonempty, open subsets A and Y of X, with $A \subseteq Y \subseteq X$, such that for every $n \geq 3$, there exist $a, a_0, a_1, ..., a_{n-1}$ in G such that*
   (a) $Y = A \cup aA$,
   (b) $a_0A, a_1A, ..., a_{n-1}A$ are pairwise disjoint,
   (c) Y is invariant under $a, a_0, a_1, ..., a_{n-1}$;
4. *There are elements f_1 and f_2 in G and nonempty, open, disjoint subsets*

$$U_0, U_1^+, U_1^-, U_2^+, U_2^-$$

such that:

$$f_1(U_0 \cup U_1^+ \cup U_2^+ \cup U_2^-) \subseteq U_1^+, \quad f_1^{-1}(U_0 \cup U_1^- \cup U_2^+ \cup U_2^-) \subseteq U_1^-,$$
$$f_2(U_0 \cup U_2^+ \cup U_1^+ \cup U_1^-) \subseteq U_2^+, \quad f_2^{-1}(U_0 \cup U_2^- \cup U_1^+ \cup U_1^-) \subseteq U_2^-.$$

Moreover,

(i) if (2) *holds then* $[a, a_1^{-1}a_0]$ *and* $[a, a_2^{-1}a_0]$ *generate a free group of rank 2.*

(ii) if (3) *holds, with* $n = 4$*, then* $a_0aa_1^{-1}$ *and* $a_2aa_3^{-1}$ *generate a free group of rank two;*

(iii) if (3) *holds, with* $n \geq 2k+1$ *for some* $k \geq 2$ *then*

$$F = \langle a_1aa_2^{-1}, a_3aa_4^{-1}, ..., a_{2k}aa_{2k+1}^{-1}\rangle \text{ is a free group of rank } k.$$

Proof. **Condition (1) implies Condition (2).** Let $F = \langle a, b\rangle$ be freely generated by a and b, and A' be the subset of F consisting of elements of F that, when reduced start with a or a^{-1}. Then $F = A' \cup aA'$ and A', bA', b^2A' are disjoint.

Let $z \in Z$. Since the action of F is properly discontinuous, we may choose an open neighborhood U of z such that $(F - \{1\})U \cap U = \emptyset$. The set $Y = FU$ is a nonempty, open, $F-$invariant subset of Z. Let $A = A'U$. Then A is a nonempty, open subset of Y and

$$A \cup aA = A'U \cup aA'U = (A' \cup aA')U = FU = Y.$$

Before we can continue we will prove the following claim:

Claim: For any $h_1, h_2 \in F$ such that $h_1A' \cap h_2A' = \emptyset$ we have that

$$h_1A \cap h_2A = \emptyset.$$

Since $h_1A', h_2A' \subseteq F$, then $h_1A'u \cap h_2A'v \subseteq Fu \cap Fv = \emptyset$ when $u \neq v$, and since the action of F on Z is free, we find the following equalities hold, proving the claim.

$$h_1A \cap h_2A = h_1A'U \cap h_2A'U$$

$$= \cup_{u,v \in U}(h_1A'u \cap h_2A'v) = \cup_{u \in U}(h_1A'u \cap h_2A'u)$$

$$= \cup_{u \in U}(h_1A' \cap h_2A')u = \emptyset.$$

Thus, since A', bA', and b^2A' are pairwise disjoint subsets of F, the open sets A, bA, and b^2A are disjoint subsets of Y and Condition 2 is satisfied for $a_0 = 1, a_1 = b$ and $a_2 = b^2$.

Condition (2) implies Condition (3).

To prove this implication we first need the following lemma.

Lemma 3.30 (Inequality Lemma)**.** *Let G act on X by a left action, A and Y be subsets of X, with $A \subseteq Y \subseteq X$, and, for some $n \geq 3$, let $a, a_0, ..., a_{n-1}$ be elements of G such that*

$$Y = A \cup aA,$$

$$a_0A, a_1A, ..., a_{n-1}A \textit{ are pairwise disjoint},$$

$$Y \textit{ is invariant under } a, a_0, ..., a_{n-1}.$$

Then, for all i, j, l in $\{0, ..., n-1\}$, if $i \neq j$,

$$a_l a^{-1} a_j^{-1}(a_iA) \subseteq a_lA \textit{ and } \; a_l a a_j^{-1}(a_i)A \subseteq a_lA.$$

Proof. Since $Y = A \cup aA$, we see that $Y = a_jA \cup a_jaA$. Since a_jA is disjoint from a_iA it follows that $a_iA \subseteq a_jaA$, or equivalently, $a_la^{-1}a_j^{-1}(a_iA) \subseteq a_lA$.
Since $Y = A \cup a^{-1}A$, applying the same arguments shows that $a_laa_j^{-1}(a_i) \subseteq a_lA$. $\square$

To prove the desired implication, it is sufficient to show that there are n elements $a_0, a_1, ..., a_{n-1}$ such that a_0A, a_1A, , ..., $a_{n-1}A$ are disjoint, and that if we assume that for some $n \geq 3$ we have that for all i such that $0 \leq i \leq n-1$ that $a_iY = Y$ holds, then it holds for $n+1$ as well. So then, let us assume that the statement holds for some $n \geq 3$. Then by the Inequality Lemma we have that

$$a_{n-1}aa_{n-1}^{-1}(a_0A) \subseteq a_{n-1}A \text{ and}$$
$$a_{n-1}aa_{n-1}^{-1}(a_1A) \subseteq a_{n-1}A.$$

Since a_0A and a_1A are disjoint, so are $a_{n-1}a^{-1}a_{n-1}^{-1}(a_0A)$ and $a_{n-1}a^{-1}a_{n-1}^{-1}(a_1A)$, and since they are both in $a_{n-1}A$, they are also disjoint from $a_0A, a_1A, ..., a_{n-2}A$. Thus,

$$a_0A,\ a_1A, ...,\ a_{n-1}A,\ a_{n-1}a^{-1}a_{n-1}^{-1}a_0A,\ a_{n-1}a^{-1}a_{n-1}^{-1}a_1A$$

are $n+1$ disjoint translates of A.

Condition (3) implies Condition (4).

Let $A \subseteq Y \subseteq X$, such that, there exist elements $a, a_0, a_1, a_2, a_4, a_4$ in G and

$Y = A \cup aA$,

$a_0A, a_1A, a_2A, a_3A, a_4A$ *are pairwise disjoint*,

Y *is invariant under* $a, a_0, a_1, a_2, a_4, a_4$.

Then it follows from the Inequality Lemma that

$$a_1aa_2^{-1}(a_0A \cup a_1A \cup a_2A \cup a_3A \cup a_4A) \subseteq a_1A,$$
$$(a_1aa_2^{-1})^{-1}(a_0A \cup a_1A \cup a_2A \cup a_3A \cup a_4A) \subseteq a_2A$$
$$a_3aa_4^{-1}(a_0A \cup a_1A \cup a_2A \cup a_3A \cup a_4A) \subseteq a_3A, \ \textit{and}$$

$$(a_3aa_4^{-1})^{-1}(a_0A \cup a_1A \cup a_2A \cup a_3A \cup a_4A) \subseteq a_4A.$$

Thus, Condition 4 is satisfied with $f_1 = a_1aa_2^{-1}$, $f_2 = a_3aa_4^{-1}$, $U_1^+ = a_1A, U_2^+ = a_3A, U_2^- = a_4A$ and $U_0 = a_0A$.

Condition (4) implies Condition (1)

This is an easy application of the Klein Criterion where p is any point in U_0.

To wrap up the proof of the theorem we need to prove the statements regarding the generating sets under condition (3). If condition (3) is satisfied with $n = 5$, then by the proof (3) $\Rightarrow$ (4), $a_1aa_2^{-1}$ and $a_3aa_4^{-1}$ generate a free group F of rank 2 acting properly discontinuously on Fa_0A. More generally, if condition (3) is satisfied with $n \geq 2k+1$ then $F = \langle a_1aa_2^{-1}, ..., a_{2k-1}aa_{2k}^{-2}\rangle$ is a free group of rank k acting properly discontinuously on Fa_0A.

If condition (2) is satisfied by the proof (2) $\Rightarrow$ (3) condition (3) is satisfied with $n = 4$, and the roles of a_0, a_1, a_2 and a_3 in condition (3) may be played by a_1, a_2, $a_0a^{-1}a_0^{-1}, a^1$, and $a_0a^{-1}a_0^{-1}a^2$. Then using the explicit generators already established when condition (3) is satisfied, we have that $a_0a^{-1}a_0^{-1}a_1aa_1^{-1}$ and $a_0a^{-1}a_0^{-1}a_2aa_2^{-1}$ freely generate a copy of F_2. After conjugation by a_0 we see that $[a, a_1^{-1}a_0]$ and $[a, a_2^{-1}a_0]$ also generate a free group of rank 2. $\square$

3.2 Some Applications of the Šunić Criterion

In [7] Šunić gave several applications of his theorem. What follows are two examples which we have already examined in the previous chapter: $SL_2(\mathbb{R})$ and $Homeo_+(\mathbb{R})$. We will show how Šunić applied his theorem, proving that these groups contain free subgroups. What is interesting is not just that he was able to find the "standard" generating sets usually used to prove that these groups contain free subgroups, but he was also able to find infinite families of generating sets for free subgroups.

Proposition 2.24 (The group $SL_2(\mathbb{R})$ contains free subgroups): Proof 2. We will begin the proof by endowing the plane with a special topology in which we construct our open sets by defining a basis as follows: given two rays beginning at the origin a and b we measure the angle of incidence of each ray with respect to the x-axis. Here, we define our angle as being in the interval $[0, 2\pi)$ .Our basis element is then the set of all points that are elements of a ray with an angle of incidence less than the angle of b and greater than or equal to the angle of a (one can think of this topology as being similar to the half-open topology of the real numbers).

Before we proceed, remember that $SL_2(\mathbb{R})$ acts on the Euclidean plane (which we will denote here by P) with the action being defined as $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot (x, y) = (ax+by, cx+dy)$. Let us define $X = Y = P - \{(0, 0)\}$ in other words, our set Y is the whole of the Euclidean plane except for the origin

Now let $A = \{(x, y) \in Y | x > 0, y \geq 0\} \cup \{(x, y) \in Y | x < 0, y \leq 0\}$. A consists of the first and third quadrants of the plane. Let

$$a = \left[\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right], a_0 = \left[\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right], a_1 = \left[\begin{smallmatrix} 1 & 0 \\ \beta & 1 \end{smallmatrix}\right], a_2 = \left[\begin{smallmatrix} 0 & -1 \\ 1 & -\gamma \end{smallmatrix}\right], a_3 = \left[\begin{smallmatrix} \delta & -1 \\ -1 & 0 \end{smallmatrix}\right],$$

where $\alpha, \beta, \gamma, \delta > 0$, $\alpha\beta \geq 1$, and $\gamma\delta \geq 1$. The next three figures show the action of these elements (for $\alpha = \beta = \gamma = \delta = 1$) on A. Note that for each $a_i \neq a_j$ we have that the image of a_i is disjoint from the image of a_j.

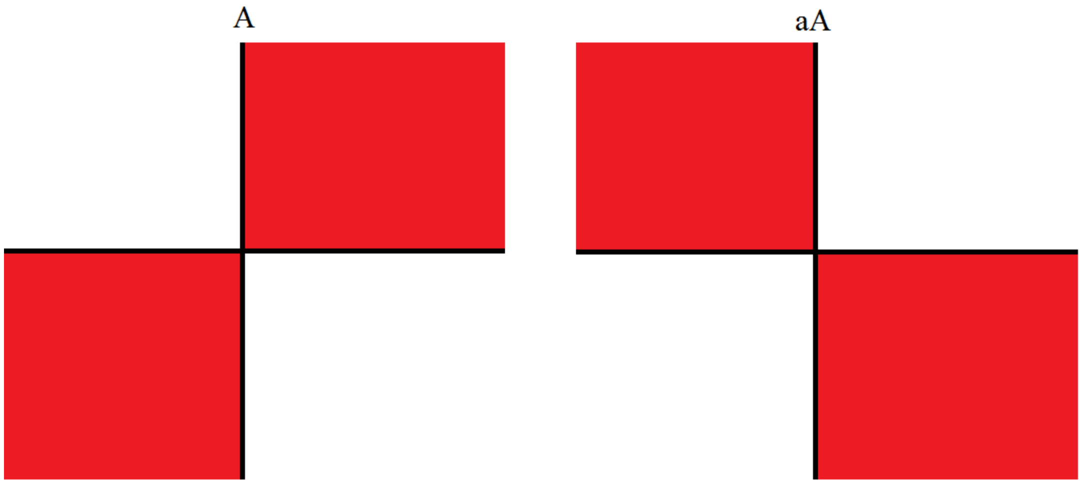

Fig. 6: The Action of a on A.

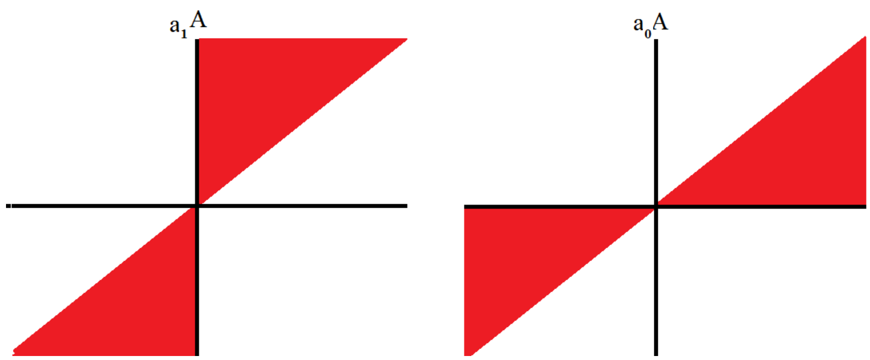

Fig. 7: The Action of a_0 **and** a_1 **on A with** $\alpha = \beta = 1$.

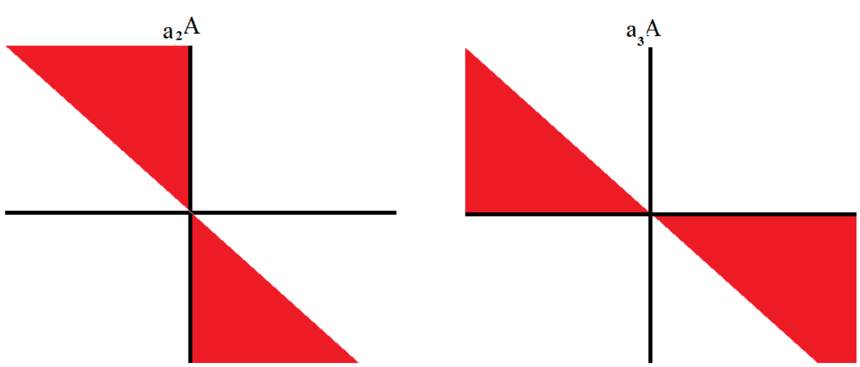

Fig. 8: The Action of a_2 **and** a_3 **on A with** $\delta = \gamma = 1$.

We see that $Y = A \cup aA$ (since a is a rotation by $\pi/2$), $a_0A = -a_0A$, $a_1A = -a_1A$, $a_2A = -a_2A$, and $a_3A = -a_3A$. Since $\alpha\beta \geq 1$ and $\gamma\delta \geq 1$ then $a_iA \cap a_jA = \emptyset$ for $i \neq j$. Thus, by the Šunić Criterion we have that the following pairs of matrices all generate freegroups of rank 2:

- $-a_2aa_0^{-1}$ *and* $-a_3aa_1^{-1}$,
- $a_1aa_0^{-1}$ *and* $a_3aa_2^{-1}$,
- $-a_3aa_0^{-1}$ *and* $-a_2aa_1^{-1}$.

The third pair of matrices simplifies to $-a_3aa_0^{-1} = \left[\begin{smallmatrix} 1 & \alpha+\beta \\ 0 & a \end{smallmatrix}\right]$ and $-a_2aa_1^{-1} = \left[\begin{smallmatrix} 1 & 0 \\ \beta+\gamma & 1 \end{smallmatrix}\right]$. Of course letting $\alpha = \beta = \gamma = \delta = 1$ we obtain the standard generators used in the first proof of Proposition 2.24! □

In the following proof of Proposition 2.25, we revisit the group $Homeo_+(\mathbb{R})$, showing once again that it contains free subgroups. However, the free subgroups that we generate are not those discussed in the Klein Criterion chapter. Šunić did show that his theorem can be used to "rediscover" the generators described in the previous chapter [7] .

Proposition 2.25 (The group $Homeo_+(\mathbb{R})$ contains a free subgroup of rank 2): Proof 2. To reprove this proposition, we will look at the projections from $\mathbb{R}$ to $T = \mathbb{R}/\mathbb{Z}$. For convenience, we will identify each point in T with an element of the real interval $[0, 1)$. We will show that the group of piecewise-linear orientation preserving homeomorphisms of T contains free subgroups and thus, $\mathbb{R}$ contains free subgroups. Let n be even such that $n \geq 4$ and let $A = [0, 1/n)$, then

$$a(t) = \begin{cases} (n-1)t + 1/n, & \text{if } t \in [0, 1/n) \\ \\ \frac{1}{n-1}t - \frac{1}{n(n-1)} & \text{otherwise.} \end{cases}$$

and, for $i = 0, 1, 2, ..., n-1$ we have that $a_i = r^i$,

where $r(t) = t + 1/n$ is the rotation by $1/n$. Thus $a_i A = r^i A = [i/n, (i+1)/n)$ for $i = 0, 1, 2, ..., n-1$, are disjoint. Moreover, $A = [0, 1/n)$ and $aA = [1/n, 1)$ so $T = A \cup aA$ Therefore letting $n = 2k$, we see that by the Šunić Criterion the group generated by

$$\{a_1 a a_2^{-1}, a_3 a a_4^{-1}, ..., a_{2k-1} a a_{2k}^{-1}\}$$

is isomorphic to the free group F_k. Thus, letting $k = 2$ gives us a free subgroup of $Homeo_+(\mathbb{R})$ on two letters. □

We should point out that the algorithm presented in this proof actually gave us an example of how to create a generating set for free groups of any finite rank. In the next chapter we will discuss three examples of groups which Šunić did not explore. Our goal will be to find new generating sets for free subgroups within each group when this is possible.

CHAPTER FOUR: FURTHER INQUIRES INTO THE THE ŠUNIĆ CRITERION

While the application of the Šunić Criterion has advantages over the Klein Criterion, it does not automatically allow us to construct all free subgroups of a given group. The reason for this is that while we have already established that every free group acts on a tree and also that every group acts on a metric space; in reality, to apply this knowledge we first need to know information about the underlying structure of the group and from here, we "build" our space and induce the group action. But what happens when all we know of a group is the group action itself? When a group is defined in terms of its action on a given space it can be difficult (or impossible) to apply the Šunić Criterion.

An example of this are the groups known as *White Groups.* These are the subgroups of bijections over the real numbers generated by two functions: $f(x) = x+1$ and $g(x) = x^p$, such that p is an odd prime [8]. When speaking of a specific White Group, we denote it by WG_p for a fixed odd prime number. This class of groups is of historical importance as it was the first constructive class of subgroups found to be free in the group of order preserving bijections over the reals. The proof that these groups are free was rather unconventional. Instead of applying the Klein Criterion, White used advanced tools from Galois theory. White Groups act on the real numbers with the action defined, naturally, as $a \cdot r = a(r)$ for all $a \in WG_p$ and all $r \in \mathbb{R}$. In this section we will show that these groups do not act properly discontinuously on any open subset of the real line and thus cannot be uncovered by the Šunić Criterion.

Proposition 4.31. *Let $\mathbb{R}$ be endowed with the standard topology and let p be an odd* *prime. Then the group WG_p does not meet the hypothesis of the Šunić Criterion.*

Proof. It is known that the White Groups is a free group of rank two so we must show that it does not act properly discontinuously on any open subset of the real line. Let us thus assume that such a subset does exist; we will call this subset X. Let us further assume that there exists some WG_p invariant subspace Y of X. Then we see that $g(0) = 0^p = 0$ and so $0 \notin Y$. This in turn implies that $\mathbb{Z} \cap Y = \emptyset$ since were this not the case we could simply apply f or f^{-1} repeatedly to whatever value lies in the intersection of $\mathbb{Z}$ and X to show that $0 \in X$. Since $f \in WG_p$, then we see that if $x \in X$ then there is some $y \in Y$ such that $y \in [-1, 1]$. Therefore, there is an open interval $I = (y - d, y + d) \subset [-1, 1]$ such that $I \subset X$, from which we deduce that $I + 2 \in Y$ (here we have applied f twice to I). Thus, for all $y \in I$, $y > 1$. Since $I + 2 \subset X$, then $(I + 2)^{np} \subset X$ for all positive integer values of n (here we have applied g n times). However, this quickly leads us to see that the length of the interval $(I + 2)^{np}$ becomes greater than 1 for some value of n which depends on d and p meaning that there is some $z \in \mathbb{Z}$ such that $z \in X$. □

4.1 Locally Free Groups

It is simple to see that Šunić's theorem can be used to find *locally free* infinitely generated groups. In this section we first define locally free groups and prove that the Šunić Criterion can in fact provide explicit generating sets for these groups. Once this is done, we give two examples of groups with locally free subgroups.

Definition 4.32. A group is *locally free* if every finitely generated subgroup is free.

We can see from this definition that if a locally free group is finitely generated then it must be free and also that all free groups are locally free. What is perhaps less obvious is that there are groups which are not free, but which are locally free. The most commonly encountered example of such a group is the set of rational numbers under addition $(\mathbb{Q}, +)$.

To show that this group is in fact locally free but not free, consider that the infinite cyclic group is free. Also, if a free group is generated by at least two elements then it has a free subgroup isomorphic to F_2, and thus, it is nonabelian. From here, we see that the only free groups which are abelian are isomorphic to the integers under addition, and since $(\mathbb{Q},+)$ is an abelian group which is not isomorphic to the integers then it is not free. To show that it is locally free let $H \leq \mathbb{Q}$ be a finitely generated subgroup with generating set $S^+ = \{\frac{n_1}{d_1}, \frac{n_2}{d_2}, ..., \frac{n_i}{d_i}\}$. Then for each $x \in S^+$ we see that $x \in F_1 = \langle \frac{1}{d_1 d_2 \cdots d_n} \rangle$. Also, note that F_1 is isomorphic to the integers and thus is a free group on 1 letter so by the Nielsen-Schreier Theorem H is free. To show the existence of nonabelian locally free groups we will make use of the following lemma.

Lemma 4.33. *The free product of locally free groups is locally free.*

Proof. Let A and B both be locally free groups and let $H < A*B$ be finitely generated by the words $S^+ = \{w_1, w_2, ..., w_n\}$. Then take from A and B the letters that give rise to the words in S^+. Let us call these sets of letters $W_a \subset A$ and $W_b \subset B$. We see that $H < \langle W_a \rangle * \langle W_b \rangle$. Since S^+ is finite, then both W_a and W_b are finite, so by assumption we have that $\langle W_a \rangle$ and $\langle W_b \rangle$ are free groups and thus, Theorem 1.24 gives us that $\langle W_a \rangle * \langle W_b \rangle$ is free. The Nielsen-Schreier Theorem then confirms that H is also free. □

Applying this lemma we see that $\mathbb{Q}*\mathbb{Q}$ is a nonabelian locally free group which is not free. Having established the existence of locally free groups we now proceed with the following theorem which exploits the Šunić Criterion to find an explicit infinite generating set for locally free groups.

Theorem 4.34. *Let G act properly discontinuously on a topological space X and assume that there exist open sets $A \subseteq Y \subseteq X$ and $a \in G$ such that $Y = aA \cup A$. If there exists an infinite subset $I \subseteq G$ such that for all $a_i \in I$ where $i \in \mathbb{N}$ we have that*

$a_iY = Y$ and that $a_iA \cap a_jA = \emptyset$ for all $i \neq j$, then the group generated by $\mathbf{F}$ *is a locally free group where:*

$$\mathbf{F} = \{a_1 a a_2^{-1}, a_3 a a_4^{-1}, ...\}.$$

Proof. Let us begin by defining $b_i = a_i a a_{i+1}^{-1}$, then we can say that $\mathbf{F} = \{b_1, b_3, b_5, ...\}$. Now assume that $H \leq \langle \mathbf{F} \rangle$ such that H is generated by the following set of words: $S^+ = \{w_1, w_2, ..., w_n\}$. Then let

$$W_{S^+} = \{b_i \in \mathbf{F} | b_i \text{ or } b_i^{-1} \text{ is a letter in some } w_i \in S^+\}.$$

Thus, there exists an element of W_{S^+} of maximum index. Let us call this element b_i. Then $H < \langle b_1, b_3, ..., b_i, b_{i+2} \rangle$. However, by the Šunić Criterion we have that this set of letters generates a free group of rank $(i+3)/2$. Thus, by the Nielsen-Schreier Theorem, H is free. □

4.2 Finding Explicit Generating Sets for Locally Free Subgroups

We will now explore two examples in which we apply Theorem 4.34. The first of these is the group $PSL_2(\mathbb{Z})$ and the second is $Aut(\mathbb{R})^+$. We will define each group and give some history on it and then seek to apply the theorem.

The group $PSL_2(\mathbb{Z})$ was one of the first groups studied by Klein in the nineteenth century. It can be defined in several ways: as a subgroup of the general linear group of dimension 2 over $\mathbb{R}$ or $\mathbb{C}$ we can define it as $SL_2(\mathbb{Z})/\langle -I_2 \rangle$. However, in keeping with the spirit of geometric group theory, we can also choose to define the group in terms of its action on the extended real number line. Using this idea we can view $PSL_2(\mathbb{Z})$ as the group of fractional linear transformations of the extended real line. This group acts on $\widehat{\mathbb{R}} = \mathbb{R} \cup \{\infty\}$ in the following way: let $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \in PSL_2(\mathbb{Z})$ and let $r \in \widehat{\mathbb{R}}$, then $\left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right] \cdot r = \frac{ar+b}{cr+d}$. Now that we have some background on this group, we will now prove the following proposition.

Proposition 4.35. *The group $PSL_2(\mathbb{Z})$ contains an infinitely generated locally free subgroup.*

Proof. Let $S = \{\left[\begin{smallmatrix} 2i & -4i^2-4i+1 \\ 1 & -2i-1 \end{smallmatrix}\right] | i \in \mathbb{N}\}$. We will now show that $\langle S \rangle$ is a locally free group. Let $U = \{a + \frac{1}{b} | a, b \in \mathbb{Z}\}$ then let $Y = \widehat{\mathbb{R}} - U$. We see that Y is open. Let $A = Y \cap (-1, 1)$. Since Y is open and $(-1, 1)$ is open, then A must be open. Let $a = \left[\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right](seeFigure9)$. Then for any $r \in \widehat{\mathbb{R}}$ we have that $a \cdot r = 1/r$ so $Y = A \cup aA$. Now for all $i \in \mathbb{N}$, define $a_i = \left[\begin{smallmatrix} 1 & 2i \\ 0 & 1 \end{smallmatrix}\right]$(Figure 10). We see then that $a_i Y = Y$ for all a_i and that $a_i Y \cap a_j Y = \emptyset$ whenever $i \neq j$. Since each element of S is the product of $a_i a a_{i+1}^{-1}$, then by Theorem 4.34, $\langle S \rangle$ is locally free. □

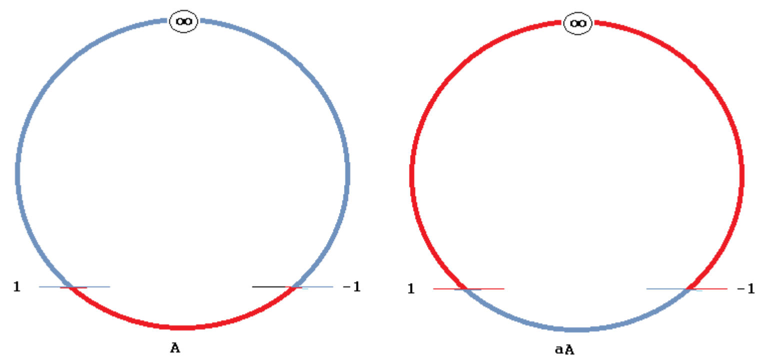

Fig. 9: The Action of a on A$\subset \widehat{\mathbb{R}}$.

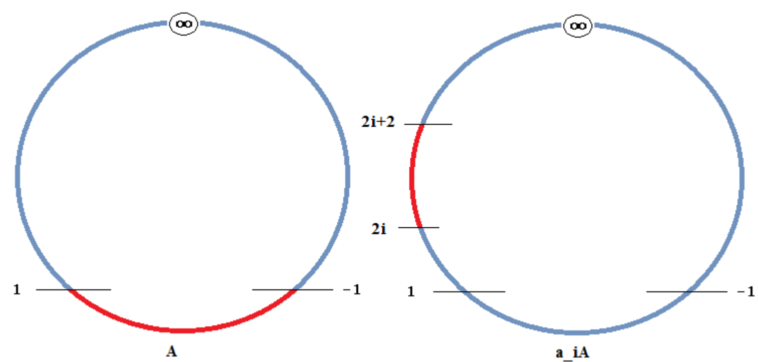

Fig. 10: The Action of a$_i$ on A$\subset \widehat{\mathbb{R}}$.

We denote the group of bijections from $\mathbb{R}^+$ to $\mathbb{R}^+$ under function composition by $Aut(\mathbb{R}^+)$. While this group itself has not been specifically studied, it has been shown that several of its subgroups contain free subgroups. This group acts on the positive real numbers with the action defined naturally as $a \cdot r = a(r)$ for all $a \in Aut(\mathbb{R}^+)$

and all $r \in \mathbb{R}^+$. It is interesting to note that, unlike the previously mentioned groups, $Aut(\mathbb{R}^+)$ is not finitely generated. To show that this group has a locally free subgroup we will actually prove that FIX_1, the subgroup of $Aut(\mathbb{R}^+)$ which fixes the number 1, contains a locally free subgroup.

Proposition 4.36. *The group $Aut(\mathbb{R}^+)$ contains an infinity generated locally free subgroup.*

Proof. Let $Y = \mathbb{R}^+ - \{1\}$. Let $A = (0,1)$, $a(x) = 1/x$ for all $x \in \mathbb{R}^+$. Note that $Y = A \cup aA$ and $a \in FIX_1$. For each $i \in \mathbb{N}$ such that $i \geq 2$ we define a_i as

$$a_i(x) = \begin{cases} x + i, & \text{if } 0 < x < 1 \\ -x + 2, & \text{if } 1 \leq x < 2 \\ i & \text{if } x = 2 \\ (i-1)x + (3-2i), & \text{if } 2 < x < 3 \\ x + (i-2) & \text{if } 3 \leq x \end{cases}$$

Note then that for all a_i and a_j, with $i \neq j$, we see that $a_iY = Y$ and that $a_iA \cap a_jA = \emptyset$. Thus from Theorem 4.34 we conclude that $\langle a_2aa_3^{-1}, a_4aa_5^{-1}, ...\rangle$ is a locally free group. $\square$

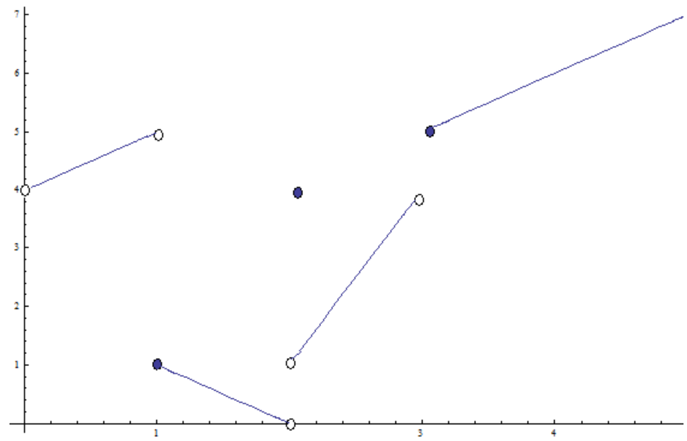

Fig. 11 The Action of $\mathbf{a}_4$ on $\mathbb{R}^+$.

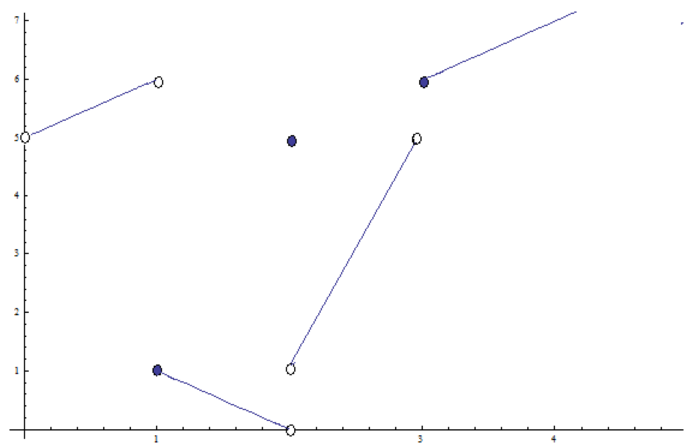

Fig. 12 The Action of $\mathbf{a}_5$ on $\mathbb{R}^+$.

The beauty of FIX_1 is that the action of the functions that we require are easy to visualize (see Figure 11 and Figure 12 for an examples). However, in the spirit of generalization, one might ask if there is something special about the number 1 or if any positive real number would work. The following proposition answers this question.

Proposition 4.37. *Let $n \in \mathbb{R}^+$ then FIX_n contains a locally free subgroup.*

Proof. Let $Y = \mathbb{R}^+ - \{n\}$, $A = (0, n)$ and $a(x) = n/x$. Then for each $i \in \mathbb{N}$ such that $i \geq 2 + n$ define a_i to be the following function:

$$a_i(x) = \begin{cases} n^{-1}x + i, & \text{if } 0 < x < n \\ \\ -nx + n(n+1), & \text{if } n \leq x < n+1 \\ \\ i, & \text{if } x = n+1 \\ \\ (i-n)x + i, & \text{if } n+1 < x < n+2 \\ \\ x + (i-n-1), & \text{if } x+2 \leq x \end{cases}$$

We see then that $Y = A \cup aA$. For all a_i, $a_i Y = Y$, and for all a_i, a_j with $i \neq j$, we have that $a_i A \cap a_j A = \emptyset$.Thus by Theorem 4.34. FIX_n contains a locally free subgroup. $\square$

CONCLUSIONS AND FUTURE RESEARCH QUESTIONS

While we have been able to expand upon the Šunić Criterion, and use it to find locally free groups, we still have questions regarding the structure of the uncovered groups. Namely, are they free? Also, if these specific groups are free, will all such groups that we uncover through the Šunić Criterion be free or could we perhaps uncover locally free but not free groups in one context and F_∞ in another? If it is decided that the groups are free, then since we have a countable group, we should be able to embed it into some F_2 group. Would there be a natural way to find the generators for such a group?

Regarding the specific groups we examined, we also have some questions that would make for interesting research problems going forward. We have shown that FIX_n contains a locally free subgroup for each n. It might be possible to refine our construction to show in fact that $Aut(\mathbb{R}^+)$ contains uncountably infinity many locally free subgroups, the intersection of any two of which is trivial. Since this is just the permutation group of an infinite set, then this might help us further understand the structure of the Symmetric groups.

Since the Šunić Criterion is equivalent to a strengthening of the Klein Criterion, it could be argued that in fact the Šunić Criterion is the latest generalization of the Klein Criterion. However, as we saw in Chapter 4, this new theorem cannot uncover the White Groups. This partially answers the second research question in [1]. However, it still remains to be seen if perhaps a weakening of the Šunić Criterion or an alteration of the topology of $\mathbb{R}$ could prove that the white group is free.

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