NUMERICAL SOLUTIONS OF NONLINEAR ELLIPTIC PROBLEM USING COMBINED-BLOCK ITERATIVE METHODS

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ABSTRACT

This thesis is concerned with iterative and monotone methods for numerical solutions of nonlinear elliptic boundary value problems. The methods we study here are called block iterative methods, which solve the nonlinear elliptic problems in two-dimensional domain in R^2 or higher dimensional domain in R^n. In these methods the nonlinear boundary value problem is discretized by the finite difference method. Two iteration processes, block Jacobi and block Gauss-Seidel monotone iterations, are investigated for computation of solutions of finite difference system using either an upper solution or a lower solution as the initial iteration. The numerical examples are presented for both linear and nonlinear problems, and for both block and pointwise methods. The numerical results are compared and discussed.

DEDICATION

To my family: my parents, my husband and my daughter MeiMei: for their encouragement, love and support.

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I would like to express my sincere appreciation to professor Xin Lu, my advisor, for his guidance and great help on my thesis. I also want to thank professor Wei Feng and professor Sandra McLaurin for serving on my thesis advisory committee and all faculty members in the Mathematics and Statistics Department who gave me valuable advice and help.

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1 INTRODUCTION

1.1 History and Background

Many problems in science and engineering can be represented by mathematical models in the form of partial differential equations (PDE's). The study of PDE's is a fundamental subject area of mathematics which links important strands of pure mathematics to applied and computational mathematics. Indeed PDE's are ubiquitous in many of the applications of mathematics where they provide a natural mathematical description of phenomena in the physical, natural and social sciences.

PDE's and their solutions exhibit rich and complex structures. Unfortunately, closed analytical expressions for their solutions can be found only in very special circumstances, which are mostly of limited theoretical and practical interest. Thus, scientists and mathematicians have been naturally led to seeking techniques for the approximation of solutions. Indeed, the advent of digital computers has stimulated the emerge of Computational Mathematics, much of which is concerned with the construction and the mathematical analysis of numerical algorithms for the approximate solution of PDE's.

Elliptic partial differential equations(EPDE's) arise usually from equilibrium or steady-state problems and their solutions [7], in relation to the calculus of variations, frequently maximize or minimize an integral representing the energy of the system. The well-known physical problems with EPDE's summarization are the St. Venant theory of torsion, the slow motion of incompressible viscous fluid, and the inverse-square law theories of electricity, magnetism and gravitating matter at points where the charge density, pole strength or mass density are non-zero. The most familiar elliptic problems originated in the attempts of nineteenth-century mathematicians like Fourier to develop a science of mathematical physics [8]. Scientists and engineers who solve elliptic problems today usually want to describe some specific physical phenomenon or engineering artifact. The following are some examples of linear and nonlinear elliptic equations [1]:

Poisson's equation:

$$-\nabla^2 u = f(x)$$

Enzyme kinetics models:

$$-\nabla^2 u = -\sigma u/(1-\alpha u)$$

The population genetics problem

$$-\nabla^2 u = \sigma u(u-\theta)(1-u)$$

Models in reactor dynamics and heat conduction

$$-\nabla^2 u = u(a-bu) + q(x)$$

where σ, α, θ, a and b are positive constants.

The science of solving elliptic problems has been revolutionized in the last 35 years [8], and the monotone method has been widely used in the treatment of certain nonlinear elliptic differential equations in recent years [3]. The basic idea of this method is that by using a suitable initial iteration one can construct a monotone sequence from a corresponding linear system, and this sequence converges monotonically to a proximation solution of the nonlinear system either from above or from below, depending on the initial iteration. Based on the monotone iterative method, computational algorithms can be developed for numerical solutions of the problem.

1.2 Purpose and Objective

In the study of numerical solutions of nonlinear boundary-value problems by the finite-difference method, the corresponding discrete problem is usually formulated as a system of nonlinear algebraic equations [4, 7]. A major concern about this system is to obtain reliable and efficient computational algorithms for finding the solution.

In this thesis, we study nonlinear elliptic problem in the form

$$
\begin{aligned}
-\Delta u &= f(x,u) \quad \text{in } \Omega \\
Bu = \alpha_0 \partial u/\partial \nu + \beta_0 u &= h(x) \quad \text{on } \partial\Omega
\end{aligned} \tag{1}
$$

where $\Delta = \nabla^2$, ∇^2 is the Laplace operator and Ω is a bounded domain in $R^p (p = 1, 2, \cdots)$, $\partial\Omega$ is the boundary of Ω and $\partial/\partial\nu$ is the outward normal derivative on $\partial\Omega$. The boundary functions α_0, β_0 are nonnegative on $\partial\Omega$. We assume that Ω is of class $C^{2+\alpha}$, f is Hölder continuous in (x, u), and h is assumed in $C^{\alpha}(\bar{\Omega})$ and $C^{1+\alpha}(\partial\Omega)$[1].

There are many iterative methods which are devoted to the computation of solutions of (1). Some basic methods are the Picard, Jacobi and Gauss-Seidel monotone iterative schemes. However, most of the monotone iterative schemes use point Picard method which is efficient for computation in one space dimension but is not so in two or higher space dimension [4]. In [5] Pao has extended the point monotone iterative schemes to "block" monotone iterative schemes in two or higher dimension. A basic advantage of the block iterative scheme is that the Thomas algorithm can be used for each subsystem in the same fashion as for one-dimensional problem, and the scheme is stable and is suitable to parallel computing.

Our main object is to investigate new block monotone iterative methods introduced by Pao[5] for nonlinear elliptic problem and present a block iterative method for linear elliptic problem. We also conduct numerical simulations using these methods and analyze and discuss the numerical results.

The structure of this thesis is as follows: In Chapter 2, we state and prove some well known results for the general 2nd-order elliptic equation. In Chapter 3, we investigate Jacobi and Gauss-Seidel type of block monotone iterative schemes using upper and lower solutions as the initial iterations for the elliptic problem in two-dimensional domain. Some proofs in [5] are repeated to help us understand the iterative processes. In Chapter 4, numerical simulations are conducted. Six numerical examples with known analytical solutions are given and numerical results are listed in 8 tables in terms of computed solutions, number of iterations, error rate and so on. In Chapter 5, we give the analysis and conclusions based on the numerical results.

2 MONOTONE ITERATIVE METHOD

An iterative method for solving a differential equation is such that an initial approximation is used to calculate the second approximation, which in turn is used to calculate the third and so on [7]. The limit of the sequence constructed converges to a solution of the differential equation. The monotone iterative method, also known as the method of upper and lower solutions, has been widely used in the treatment of nonlinear parabolic and elliptic problem in the recent years. The basic idea of this method is that by using the upper or lower solution as the initial iteration in a suitable iterative process, the resulting sequence of iteration is monotone and will converge to a solution of the problem. The monotone iterative method and the monotone iterative numerical scheme are based on the following well known results [2, 3, 4].

2.1 Preliminaries

The monotone iterative scheme for elliptic boundary-value problem is based on a positivity lemma which is derived from the following maximum principle:

Theorem 1 *Let* $\omega \in C^2(\Omega)$ *satisfy the inequality*

$$-\Delta\omega + c\omega \geq 0 \qquad in\ \Omega \tag{2}$$

where $c \equiv c(x) \geq 0$ *and is bounded in* Ω*. If* w *attains a nonpositive minimum* m_0 *at a point in* Ω *then* $\omega \equiv m_0$*. Furthermore, if* $x_0 \in \partial\Omega$ *is a minimum point of* ω *then* $\partial\omega/\partial\nu < 0$ *at* x_0 *unless* $\omega = m_0$ *in* Ω*. [1]*

Based on the result of above theorem we derive a positive lemma which plays a fundamental role in the nonlinear elliptic boundary-value problem.

Lemma 1 *Let c, β_0 be bounded nonnegative functions which are not both identically zero. If $\omega \in C^2(\Omega)$ satisfies the relation*

$$
\begin{aligned}
-\Delta\omega + c\omega \geq 0 \qquad & in\ \Omega \\
B\omega = \alpha_0 \partial\omega/\partial\nu + \beta_0\omega \geq 0 \qquad & on\ \partial\Omega
\end{aligned} \tag{3}
$$

then $\omega \geq 0$ in $\bar{\Omega}$. Moreover, $\omega > 0$ in Ω unless $\omega \equiv 0$

Proof: By contradiction, suppose the conclusion of the lemma is false, then there exists a point $x_0 \in \bar{\Omega}$ such that $\omega(x_0)$ is negative ($\omega(x_0) < 0$). Precondition here is that α_0 and β_0 are nonnegative.

First by the boundary inequality,

$$B\omega = \alpha_0 \partial\omega/\partial\nu + \beta_0\omega \geq 0$$

when $\alpha_0 = 0 \Rightarrow \beta_0\omega \geq 0 \Rightarrow \omega(x_0) \geq 0 \Rightarrow x_0 \notin \bar{\Omega}$

when $\alpha_0 \neq 0$ and $\beta_0(x_0) > 0$

$$\partial\omega(x_0)/\partial\nu \geq -(\beta_0(x_0)/\alpha_0(x_0))\omega(x_0) > 0 \Rightarrow \omega(x_0) > 0 \Rightarrow x_0 \notin \bar{\Omega}$$

when $\beta_0(x_0) = 0$, $\partial\omega(x_0)/\partial\nu \geq 0$ at $x_0 \Rightarrow x_0 \notin \bar{\Omega}$.

This shows there is no point to satisfy ($\omega(x) < 0$) in the entire domain. The first part thus is proved.

By the maximal principle, the maximal value of ω always appears inside Ω, so if $\omega = 0$ is on the $\partial\Omega$, there is $\omega > 0$ in Ω; or if $\omega = 0$ is in Ω, then $\omega = 0$ holds in the whole domain. The second part is proved.

2.2 Monotone Iterative Method and Existence Of Solution

In a monotone iterative scheme for the problem (1), an upper or lower solution will be used as a suitable initial iteration. The upper and lower solution are defined as following:

Definition 1 *A function $\tilde{u} \in C^{\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ is called an upper solution of (1) if*

$$\begin{aligned} -\Delta \tilde{u} &\geq f(x, \tilde{u}) \qquad && \text{in } \Omega \\ B\tilde{u} &\geq h(x) && \text{on } \partial\Omega \end{aligned} \tag{4}$$

Similarly, $\hat{u} \in C^{\alpha}(\bar{\Omega}) \cap C^2(\Omega)$ is called a lower solution if it satisfies the reversed inequalities in (4). The pair $(\hat{u}, \tilde{u})$ is referred to as ordered if $\tilde{u} \geq \hat{u}$ in $\bar{\Omega}$. For any pair of ordered upper and lower solutions $\tilde{u}, \hat{u}$ we denote by $< \hat{u}, \tilde{u} >$ the sector of all functions $u \in C(\bar{\Omega})$ such that $\hat{u} \leq u \leq \tilde{u}$ in $\bar{\Omega}$.

In order to construct the monotone sequence, we suppose f satisfies the one-sided Lipschitz condition

$$f(x, u_1) - f(x, u_2) \geq -\underline{c}(x)(u_1 - u_2) \text{ for } \hat{u} \leq u_2 \leq u_1 \leq \tilde{u} \tag{5}$$

where $\underline{c}$ is a bounded nonnegative function in Ω. Then by adding the same function $\underline{c}u$ on both sides of the equation of (1) and setting

$$F(x, u) = \underline{c}(x)u + f(x, u) \tag{6}$$

The problem (1) turns into

$$\begin{aligned} -\Delta u + \underline{c}u &= F(x, u) \qquad && \text{in } \Omega \\ Bu &= h(x) && \text{on } \partial\Omega \end{aligned} \tag{7}$$

By condition (5), $F(x,u)$ is monotone nondecreasing in u for $u \in <\hat{u},\tilde{u}>$. We assume $\underline{c} \in C^\alpha(\bar{\Omega})$, so that $F(x,u)$ is Hölder continuous in $\bar{\Omega}\times <\hat{u},\tilde{u}>$. Hence for any given $u^{(0)} \in C^\alpha(\bar{\Omega})$ one can construct a sequence $\{u^{(k)}\}$ from the following iteration process, for $k = 1, 2, \cdots$,

$$
\begin{aligned}
-\Delta u^{(k)} + \underline{c}u^{(k)} = F(x, u^{(k-1)}) &\quad \text{in } \Omega \\
Bu^{(k)} = h(x) &\quad \text{on } \partial\Omega
\end{aligned} \tag{8}
$$

The sequence $\{\bar{u}^{(k)}\}$ is called the upper sequence if the initial iteration $u^{(0)} = \tilde{u}$ and sequence $\{\underline{u}^{(k)}\}$ is called the lower sequence if the initial iteration $u^{(0)} = \hat{u}$. The upper and lower sequences $\{\bar{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ satisfy the following properties(cf. [1]):

Lemma 2 *Let f satisfy condition (5), and let $\underline{c} \geq 0$ and not be identically zero when $\beta_0 \equiv 0$. Then the upper and lower sequences $\{\bar{u}^{(k)}\}$, $\{\underline{u}^{(k)}\}$ are well defined.*

Lemma 3 *Let the hypothesis in Lemma (2) hold. Then the upper and lower sequences possess the monotone property*

$$
\hat{u} \leq \underline{u}^{(k)} \leq \underline{u}^{(k+1)} \leq \bar{u}^{(k+1)} \leq \bar{u}^{(k)} \leq \tilde{u} \ \text{in } \bar{\Omega} \tag{9}
$$

for every k. Moreover, for each k, $\bar{u}^{(k)}$ and $\underline{u}^{(k)}$ are ordered upper and lower solutions.

Theorem 2 *Let $\tilde{u}$, $\hat{u}$ be ordered upper and lower solutions of (1), and let f satisfy (5). Then $\{\bar{u}^{(k)}\}$ converges monotonically from above to a solution $\bar{u}$, and $\{\underline{u}^{(k)}\}$ converges monotonically from below to a solution $\underline{u}$, and $\bar{u}, \underline{u} \in C^{2+\alpha}(\bar{\Omega})$. Moreover, $\hat{u} \leq \underline{u} \leq \bar{u} \leq \tilde{u}$ in $\bar{\Omega}$, and if u^* is any other solution in $<\hat{u},\tilde{u}>$ then $\underline{u} \leq u^* \leq \bar{u}$.*

2.3 Finite Difference and Iterative Scheme

The finite-difference method for approximating a boundary value problem leads to a system of algebraic simultaneous equations. Let $i = (i_1, \cdots, i_p)$ be a multiple index with $i_\nu = 0, 1, \cdots, M_\nu$ and let $x_i = (x_{i_1}, \cdots, x_{i_p})$ be generic mesh point and $h = (h_1, \cdots, h_p)$ the spatial increment of x_i. The set of mesh points in Ω and $\partial\Omega$ are denote by Ω_h and Γ_h , respectively, and the set of all mesh points in $\bar{\Omega} \equiv \Omega \cup \partial\Omega$ are represented by $\bar{\Omega}_h$. For any spatial increment $\Delta x_i = h_\nu e_\nu$ in the x_ν-coordinate direction, where e_ν is the unit vector with its $v - th$ component one and zero elsewhere, we use the central difference approximation

$$L_h[\omega] \equiv -\Sigma_{\nu=1}^{p} \Delta^{(\nu)} \omega(x_i) \equiv -\Sigma_{\nu=1}^{p} h_\nu^{-2} [\omega(x_i + h_\nu e_\nu) - 2\omega(x_i) + \omega(x_i - h_\nu e_\nu)] \quad (10)$$

for the negative Laplace operator $(-\nabla^2)$. Using the standard notation

$$u_i = u(x_i), \qquad f_i = f(x_i, u(x_i)) \quad (11)$$

the boundary operator in (1) is given in the form

$$B_h[\omega_i] = \alpha_i(\omega(x_i) - \omega(\hat{x}))/|x_i - \hat{x}| + \beta_i \omega(x_i) \quad (12)$$

where $\hat{x}$ is a suitable mesh point in Ω_h. Now a finite difference approximation for continuous problem (1) is given by

$$\begin{aligned} -\Sigma_{\nu=1}^{p} \Delta^{(\nu)} u(x_i) &= f(x_i, u(x_i)) \qquad \text{in } \Omega_h \\ \alpha_i(u(x_i) - u(\hat{x}))/|x_i - \hat{x}| + \beta_i u(x_i) &= h(x_i) \qquad \text{on } \Gamma_h \end{aligned} \quad (13)$$

The monotone iterative scheme is used to solve the nonlinear algebraic system (13) with initial iterations, namely upper and lower solutions defined as following:

Definition 2 *A function $\tilde{u}_i$ defined on Ω_h is called an upper solution of (13) if it* *satisfies the inequalities*

$$-\Sigma_{\nu=1}^{p}\Delta^{(\nu)}u(x_i) \geq f(x_i, u(x_i)) \qquad in\ \Omega_h$$
$$\alpha_i(u(x_i) - u(\hat{x}))/|x_i - \hat{x}| + \beta_i u(x_i) \geq h(x_i) \qquad on\ \Gamma_h \qquad (14)$$

Similarly, $\hat{u}_i$ *is called a lower solution of (13) if it satisfies all the reversed inequalities in (14). The pair* $(\hat{u}_i, \tilde{u}_i)$ *is referred to as ordered if* $\tilde{u}_i \geq \hat{u}_i$ *on* Ω_h

Suppose there exists an ordered pair of upper-lower solutions $(\hat{u}_i, \tilde{u}_i)$ and a function $\gamma_i \geq 0$ such that

$$f_i(u_1) - f_i(u_2) \geq -\gamma_i(u_1 - u_2) \quad \text{for } u_1, u_2 \in < \hat{u}, \tilde{u} >, \quad u_2 \leq u_1 \qquad (15)$$

where $< \hat{u}, \tilde{u} >$ denotes the set of functions u with $\hat{u}_i \leq u \leq \tilde{u}_i$ on Ω_h. Then by using the initial iteration $u_i^{(0)} = \tilde{u}_i$ and $u_i^{(0)} = \hat{u}_i$ we construct two sequences $\{\bar{u}_i{}^{(m)}\}$ and $\{\underline{u}_i^{(m)}\}$ respectively from the following iterations:

$$-\Sigma_{\nu=1}^{p}\Delta^{(\nu)}u(x_i)^{(m)} + \gamma_i u(x_i)^{(m)} = \gamma_i u(x_i)^{(m-1)} + f(x_i, u(x_i)^{(m-1)}) \qquad \text{in } \Omega_h$$
$$\alpha_i(u(x_i)^{(m)} - u(\hat{x}))/|x_i - \hat{x}| + \beta_i u(x_i)^{(m)} = h(x_i) \qquad \text{on } \Gamma_h \quad (16)$$

Since for each m the right-hand-side of (16) is known, these sequences can be computed by solving linear algebraic systems. The following theorem gives the monotone convergence property of these sequences

Theorem 3 *Let* $\tilde{u}_i, \hat{u}_i$ *be an ordered pair of upper-lower solutions of (13) such that* $\tilde{u}_i \geq \hat{u}_i$ *and let* f *satisfy (15). Then the maximal and minimal sequences converge monotonically from above and below, respectively, to a maximal solution* $\bar{u}_i$ *and a* *minimal solution $\underline{u_i}$. Moreover,*

$$\hat{u}_i \leq \underline{u_i}^{(m)} \leq \underline{u_i}^{(m+1)} \leq \underline{u_i} \leq \bar{u}_i \leq \bar{u}_i^{(m+1)} \leq \bar{u}_i^{(m)} \leq \tilde{u}_i \tag{17}$$

3 COMBINED BLOCK ITERATIVE METHOD

The monotone iterative scheme (16) is by point Picard method which is efficient for computation in one space dimension but not so for two or higher dimension. In order to improve the efficiency of the monotone iterative scheme in two or higher dimension the monotone iterative scheme (16) has been extended to a Jacob type or a Gauss-Seidel type of "block" monotone iterative scheme(cf.[5]). For the simplicity we limit the space dimension to be two in our investigation. However the methods and results are easily to be extended and applied to higher dimensions. Consider the problem (1) in R^2:

$$\begin{aligned} -\Delta u &= f(x,y,u) \qquad \text{in } \Omega \\ B[u] &= h(x,y) \quad \text{on } \partial\Omega \end{aligned} \tag{18}$$

where Ω is a bounded connected domain in R^2 with boundary $\partial\Omega$, $\Delta = \partial/\partial x^2 + \partial/\partial y^2$ is the Laplace operator, and B is a boundary operator given in the form

$$B[u] = \alpha \partial u/\partial \nu + \beta(x,y)u$$

With $\partial/\partial\nu$ denoting the outward normal derivative on $\partial\Omega$, by choosing the different α and β we may have the different type boundary condition. It is assumed that $f(x,y,u)$, $h(x,y)$ are continuous functions in their respective domains.

3.1 Finite Difference For Two-Dimensional Domain

To derive a finite difference system for the boundary value problem (18), let $h = \Delta x$, $k = \Delta y$ be the increments in the x and y direction, respectively, and let $(x_i, y_j) = (ih, jk)$ be an arbitrary mesh point in $\bar{\Omega}$, where $i = 0, 1, \cdots, M_j$, $j = 0, 1, \cdots, N$, and M_j is the number of intervals in the x-direction for each j. In the case of a rectangular domain, $M_j = M$ is independent of j. Denote by Λ, $\bar{\Lambda}$, $\partial\Lambda$ the sets of mesh points in Ω, $\bar{\Omega}$, $\partial\Omega$, respectively, and set $u_{i,j} = u(x_i, y_j)$, $h_{i,j} = h(x_i, y_j)$ and

$$F_{i,j}(u_{i,j}) = f(x_i, y_j, u(x_i, y_j))$$

Then standard central difference approximations for the operators Δ and B

$$\partial^2 u/\partial x^2 = [(u_{i+1,j}) - 2u_{i,j} + u_{i-1,j}]/h^2$$

$$\partial^2 u/\partial y^2 = [(u_{i,j+1}) - 2u_{i,j} + u_{i,j-1}]/k^2$$

leads to a finite difference system of (18) in the form

$$a_{ij}u_{i,j} - (\alpha_{ij}u_{i-1,j} + \alpha'_{i,j}u_{i+1,j}) - (c_{ij}u_{i,j-1} + c'_{ij}u_{i,j+1}) = hkF_{i,j}(u_{i,j}) + \bar{h}_{i,j} \quad (i,j) \in \bar{\Lambda} \qquad (19)$$

where $a_{ij}, \alpha_{ij}, \alpha'_{ij}, c_{ij}, c'_{ij}$ are positive constants with $a_{ij} = \alpha_{ij} + \alpha'_{ij} + c_{ij} + c'_{ij}$ and $\bar{h}_{i,j}$ is associated with the boundary condition in (18). It is defined that

$$c_{i0} = c'_{iN} = 0 \quad \text{for } i = 0, 1, \cdots, M_j \quad \text{and } \alpha_{0j} = \alpha'_{M_j,j} = 0 \quad \text{for } j = 0, 1, \cdots, N$$

If Ω is a rectangular domain then

$$c_{ij} = c'_{ij} = h/k, \alpha_{ij} = \alpha'_{ij} = k/h \quad \text{and } a_{ij} = 2(k/h + h/k) \quad \text{for all } (i,j)$$

To express the equation (19) in a compact form we define the following vectors and diagonal matrices for each j:

$$\begin{aligned}
U_j &= (u_{0,j}, \cdots, u_{M_j,j})^T \\
H_j &= (\bar{h}_{0,j}, 0, \cdots, 0, \bar{h}_{M_j,j})^T \\
F_j(U_j) &= hk(F_{0,j}(u_{o,j}), \cdots, F_{M_j,j}(u_{M_j,j}))^T \\
C_j &= diag(c_{0,j}, \cdots, c_{M_j,j}) \\
C'_j &= diag(c'_{0,j}, \cdots, c'_{M_j,j}) \qquad (20)
\end{aligned}$$

where $(.)^T$ denotes a column vector. It is clear that $C_0 = C'_N = 0$. Let A_j be the tridiagonal matrix whose diagonal elements are a_{ij} while upper and lower off-diagonal elements are $(-\alpha_{ij})$ and $(-\alpha'_{ij})$, respectively. Then the system (19) can be expressed in the vector form

$$A_j U_j - (C_j U_{j-1} + C'_j U_{j+1}) = F_j(U_j) + H_j, \; (j = 0, 1, \cdots, N) \qquad (21)$$

where

$$A_j = \begin{bmatrix} a_{0j} & -\alpha'_{0j} & & & & 0 \\ -\alpha_{1j} & a_{1j} & -\alpha'_{1j} & & & \\ & .. & & .. & .. & \\ & .. & & .. & .. & \\ & .. & & .. & .. & \\ & & & -\alpha_{M-1,j} & a_{M-1,j} & -\alpha'_{M-1,j} \\ 0 & & & & -\alpha_{M,j} & a_{Mj} \end{bmatrix}$$

$(j = 0, 1, \cdots, N)$

Furthermore, in order to get more compact form, we denote:

$$U = (U_0, \cdots, U_N)^T$$
$$H = (H_0, \cdots, H_N)^T$$
$$F(U) = (F_0(U_0), \cdots, F_N(U_N))^T \tag{22}$$

Let $\Re$ be a N by N block tridiagonal matrix with diagonal submatrices $A_0, \cdots, A_N$ and upper and lower off-diagonal submatrices $-C_1, \cdots, -C_N$ and $-C'_0, \cdots, -C'_{N-1}$, respectively. So we get

$$\Re U = F(U) + H \tag{23}$$

where

$$\Re = \begin{bmatrix} A_0 & -C'_0 & & & & 0 \\ -C_1 & A_1 & -C'_1 & & & \\ & .. & & .. & .. & \\ & .. & & .. & .. & \\ & .. & & .. & .. & \\ & & & -C_{N-1} & A_{N-1} & -C'_{N-1} \\ 0 & & & & -C_N & A_N \end{bmatrix}$$

To obtain a block monotone iterative scheme for the (21), we use the upper and lower solutions as the initial iteration. The upper and lower solutions are defined as following:

Definition 3 *Let $\tilde{U} = (\tilde{U}_0, \cdots, \tilde{U}_N)$. Then $\tilde{U}$ is called an upper solution of (21) if*

$$A_j\tilde{U}_j - (C_j\tilde{U}_{j-1} + C'_j\tilde{U}_{j+1}) \geq F_j(\tilde{U}_j) + H_j,$$
$$(j = 0, 1, \cdots, N) \tag{24}$$

Similarly, the $\hat{U}$ with $(\hat{U}_0, \cdots, \hat{U}_N)$ is called a lower solution if it satisfies (24) in *reserved order.*

3.2 Block-Monotone Iterative Schemes

Let $\gamma_{i,j}$ be any nonnegative function and for each j we define a nonnegative diagonal matrix Γ_j by

$$\Gamma_j = hkdiag(\gamma_{0,j}, \cdots, \gamma_{M_j,j}), \qquad (j = 0, 1, \cdots, N)$$

Then the problem (21) is equivalent to

$$(A_j + \Gamma_j)U_j = C_j U_{j-1} + C'_j U_{j+1} + \Gamma_j U_j + F_j(U_j) + H_j$$
$$(j = 0, 1, \cdots, N)$$

Under the one-sided Lipschitz condition of F_j,we have

$$\Gamma_j U_j + F_j(U_j) \geq \Gamma_j U'_j + F_j(U'_j) \qquad \text{whenever } \tilde{U}_j \geq U_j \geq U'_j \geq \hat{U}_j \tag{25}$$

A) Jacobi type block iteration scheme

Given any initial $U^{(0)}$ we can construct a sequence $\{U^{(m)}\}$ from the Jacobi type of block iteration process

$$(A_j+\Gamma_j)U_j^{(m)} = C_j U_{j-1}^{(m-1)} + C'_j U_{j+1}^{(m-1)} + \Gamma_j U_j^{(m-1)} + F_j(U_j^{(m-1)}) + H_j, (j = 0, 1, \cdots, N) \tag{26}$$

where $U^{(m)} = (U_0^{(m)}, \cdots, U_N^{(m)})$. Denote the sequence by $\{\bar{U}^{(m)}\}$ if $U^{(0)} = \tilde{U}$, and by $\{\underline{U}^{(m)}\}$ if $U^{(0)} = \hat{U}$, and refer to them as maximal and minimal sequence, respectively.

Lemma 4 *The maximal and minimal sequences* $\bar{U}^{(m)}$, $\underline{U}^{(m)}$ *given by (26) with* $\bar{U}^{(0)} = \tilde{U}$ *and* $\underline{U}^{(0)} = \hat{U}$ *possess the monotone property*

$$\hat{U} \leq \underline{U}^{(m)} \leq \underline{U}^{(m+1)} \leq \bar{U}^{(m+1)} \leq \bar{U}^{m} \leq \tilde{U} \qquad (m = 1, 2, \cdots) \tag{27}$$

Moreover for each m, $\bar{U}^{(m)}$ *and* $\underline{U}^{(m)}$ *are ordered upper and lower solutions.*

Proof: 1)Let $W_j^{(0)} = \bar{U}_j^{(0)} - \bar{U}_j^{(1)} = \tilde{U}_j - \bar{U}_j^{(1)}$. By (24),(26)

$$\begin{aligned} (A_j + \Gamma_j) W_j^{(0)} &= (A_j + \Gamma_j)(\tilde{U}_j - \bar{U}_j^{(1)}) \\ &= (A_j + \Gamma_j)\tilde{U}_j - [C_j \bar{U}_{j-1}^{(0)} + C_j' \bar{U}_{j+1}^{(0)} + \Gamma_j \bar{U}_j^{(0)} + F_j(\bar{U}_j^{(0)}) + H_j] \\ &= A_j \tilde{U}_j - [C_j \tilde{U}_{j-1} + C' \tilde{U}_{j+1} + F_j(\tilde{U}_j) + H_j] \geq 0 \end{aligned}$$

The positivity of $(A_j + \Gamma_j)^{-1}$ implies that $W_j \geq 0$ for $j = 0, 1, \cdots, N$, this leads to $\bar{U}^{(0)} \geq \bar{U}^{(1)}$. A similar argument using the property of a lower solution gives $\underline{U}^{(1)} \geq \underline{U}^{(0)}$. Let $W_j^{(1)} = \bar{U}_j^{(1)} - \underline{U}_j^{(1)}$ and $\bar{U}^{(0)} \geq \underline{U}^{(0)}$, by (25)and (26)

$$\begin{aligned} (A_j + \Gamma_j) W_j^{(1)} &= (A_j + \Gamma_j)(\bar{U}_j^{(1)} - \underline{U}_j^{(1)}) \\ &= C_j(\bar{U}_{j-1}^{(0)} - \underline{U}_{j-1}^{(0)}) + C_j'(\bar{U}_{j+1}^{(0)} - \underline{U}_{j+1}^{(0)}) + \Gamma_j(\bar{U}_j^{(0)} - \underline{U}_j^{(0)}) \\ &\quad + F_j(\bar{U}_j^{(0)}) - F_j(\underline{U}_j^{(0)}) \geq 0 \end{aligned}$$

This yields $W_j^{(1)} \geq 0$ for all j. The above conclusion shows that relation (27) holds for $m = 1$ which is $\underline{U}^{(0)} \leq \underline{U}^{(1)} \leq \bar{U}^{(1)} \leq \bar{U}^{(0)}$.

2) Let's suppose that relation (27) holds for $m = k$, so we have $\underline{U}^{(k)} \leq \underline{U}^{(k+1)} \leq \bar{U}^{(k+1)} \leq \bar{U}^{(k)}$ and let $W_j^{(k)} = \bar{U}_j^{(k)} - \bar{U}_j^{(k+1)}$

$$\begin{aligned} (A_j + \Gamma_j) W_j^{(k)} &= (A_j + \Gamma_j)(\bar{U}_j^{(k)} - \bar{U}_j^{(k+1)}) \\ &= (A_j + \Gamma_j)\bar{U}_j^{(k)} - [C_j \bar{U}_{j-1}^{(k)} + C_j' \bar{U}_{j+1}^{(k)} + \Gamma_j \bar{U}_j^{(k)} + F_j(\bar{U}_j^{(k)}) + H_j] \end{aligned}$$

$$
\begin{aligned}
&= C_j\bar{U}_{j-1}^{(k-1)} + C_j'\bar{U}_{j+1}^{(k-1)} + \Gamma_j\bar{U}_j^{(k-1)} + F_j(\bar{U}_j^{(k-1)}) + H_j \\
&\quad -[C_j\bar{U}_{j-1}^{(k)} + C_j'\bar{U}_{j+1}^{(k)} + \Gamma_j\bar{U}_j^{(k)} + F_j(\bar{U}_j^{(k)}) + H_j] \\
&= (C_j\bar{U}_{j-1}^{(k-1)} - C_j\bar{U}_{j-1}^{(k)}) + (C_j'\bar{U}_{j+1}^{(k-1)} - C_j'\bar{U}_{j+1}^{(k)}) \\
&\quad (\Gamma_j\bar{U}_j^{(k-1)} - \Gamma_j\bar{U}_j^{(k)}) + (F_j(\bar{U}_j^{(k-1)}) - F_j(\bar{U}_j^{(k)})) \geq 0
\end{aligned}
$$

The positivity of $(A_j + \Gamma_j)^{-1}$ imply that $W_j^{(k)} \geq 0$ for all $j = 1, 2, \cdots, N$, so we have $\bar{U}^{(k)} \geq \bar{U}^{(k+1)}$. A similar argument using the property of a lower solution gives $\underline{U}_j^{(k+1)} \geq \underline{U}_j^{(k)}$ for $j = 1, 2, \cdots, N$.Furthermore, by (26) and (25) $\bar{U}^{(k)} \geq \underline{U}^{(k)}$, and let $W_j^{k+1} = \bar{U}_j^{(k+1)} - \underline{U}_j^{(k+1)}$

$$
\begin{aligned}
(A_j + \Gamma_j)W_j^{k+1} &= (A_j + \Gamma_j)(\bar{U}_j^{(k+1)} - \underline{U}_j^{(k+1)}) \\
&= C_j(\bar{U}_{j-1}^{(k)} - \underline{U}_{j-1}^{(k)}) + C_j'(\bar{U}_{j+1}^{(k)} - \underline{U}_{j+1}^{(k)}) + \Gamma_j(\bar{U}_j^{(k)} - \underline{U}_j^{(k)}) \\
&+ F_j(\bar{U}_j^{(k)}) - F_j(\underline{U}_j^{(k)}) \geq 0
\end{aligned}
$$

This yields $\bar{U}_j^{(k+1)} \geq \underline{U}_j^{(k+1)}, j = 0, 1, \cdots, N$. The above conclusion shows that relation (27) holds for $m = k + 1$. The monotone property (27) follows from the principle of induction.

3) By (25) and (26), (27), we have

$$
A_j\bar{U}_j^{(m)} \geq C_j\bar{U}_{j-1}^{(m)} + C_j'\bar{U}_{j+1}^{(m)} + F_j(\bar{U}_j^{(m)}) + H_j, \qquad (j = 0, 1, \cdots, N)
$$

it shows $\bar{U}^{(m)}$ is an upper solution.

And

$$
A_j\underline{U}_j^{(m)} \leq C_j\underline{U}_{j-1}^{(m)} + C_j'\underline{U}_{j+1}^{(m)} + F_j(\underline{U}_j^{(m)}) + H_j, \qquad (j = 0, 1, \cdots, N)
$$

which shows $\underline{U}^{(m)}$ is a lower solution.

The monotone property (27) yields the following result.

Theorem 4 *Let $\tilde{U}, \hat{U}$ be a pair of ordered upper and lower solutions of (21). Then the sequence $\{\bar{U}^{(m)}\}$ given by (26) with $\bar{U}^{(0)} = \tilde{U}$ converges monotonically from above to a maximal solution $\bar{U}$ of (21), while the sequence $\{\underline{U}^{(m)}\}$ with $\underline{U}^{(0)} = \hat{U}$ converges monotonically from below to a minimal solution $\underline{U}$. Moreover*

$$\hat{U} \leq \underline{U}^{(m)} \leq \underline{U}^{(m+1)} \leq \underline{U} \leq \bar{U} \leq \bar{U}^{(m+1)} \leq \bar{U}^{m} \leq \tilde{U} \qquad (m = 1, 2, \cdots) \tag{28}$$

and if U^* is any solution in $< \hat{U}, \tilde{U} >$ then $\underline{U} \leq U^* \leq \bar{U}$.

Proof: By the monotone property (27) the limits

$$\lim_{m \to \infty} \bar{U}^{(m)} = \bar{U}, \lim_{m \to \infty} \underline{U}^{(m)} = \underline{U}$$

exist and satisfy relation (27). Letting $m \to \infty$ in (26) shows that $\bar{U}$ and $\underline{U}$ are solutions of (21).

To show the maximal property of $\bar{U}$, we observe that if U^* is a solution of (21) in $< \hat{U}, \tilde{U} >$, then the pair $\tilde{U}, U^*$ are ordered upper and lower solutions. Replacing $\hat{U}$ by U^* in the above conclusion shows that $U^* \leq \bar{U}$. A similar argument using $U^*, \hat{U}$ as the ordered upper and lower solutions gives $U^* \geq \underline{U}$. This proves the theorem.

B)Gauss-Seidel type block iteration scheme

In order to accelerate the rate of convergence of the monotone iterative scheme in (26), we consider an improved iterative scheme, called block Gauss-Seidel iteration, in the form

$$
\begin{aligned}
(A_j+\Gamma_j)U_j^{(m)} &= C_jU_{j-1}^{(m)}+C_j'U_{j+1}^{(m-1)}+\Gamma_jU_j^{(m-1)}+F_j(U_j^{(m-1)})+H_j \\
&\quad (j=1,2,\cdots,N) \qquad (29)
\end{aligned}
$$

Denote the sequences again by $\{\bar{U}^{(m)}\}$ if $U^{(0)}=\tilde{U}$ and by $\{\underline{U}^{(m)}\}$ if $U^{(0)}=\hat{U}$.

Lemma 5 *The maximal and minimal sequences $\{\bar{U}^{(m)}\}$, $\{\underline{U}^{(m)}\}$ given by (29) with $\bar{U}^{(0)}=\tilde{U}$ and $\underline{U}^{(0)}=\hat{U}$ possess the monotone property*

$$
\hat{U}\le \underline{U}^{(m)}\le \underline{U}^{(m+1)}\le \bar{U}^{(m+1)}\le \bar{U}^{m}\le \tilde{U} \qquad (m=1,2,\cdots) \qquad (30)
$$

and for each m, $\bar{U}^{(m)}$ and $\underline{U}^{(m)}$ are ordered upper and lower solutions.

Proof:(1)Let $W_j^{(0)}\equiv \bar{U}_j^{(0)}-\bar{U}_j^{(1)}=\tilde{U}_j-\bar{U}_j^{(1)}$, $j=0,1,\cdots,N$ plug into (29), then

$$
\begin{aligned}
(A_j+\Gamma_j)W_j^{(0)} &= (A_j+\Gamma_j)\tilde{U}_j-[C_j\bar{U}_{j-1}^{(1)}+C_j'\bar{U}_{j+1}^{(0)}+\Gamma_j\bar{U}_j^{(0)}+F_j(\bar{U}_j^{(0)})+H_j] \\
&= A_j\tilde{U}_j-[C_j\bar{U}_{j-1}^{(1)}+C_j'\tilde{U}_{j+1}^{(0)}+F_j(\tilde{U}_j^{(0)})+H_j]
\end{aligned}
$$

From (24),

$$
(A_j+\Gamma_j)W_j^{(0)}\ge C_j\tilde{U}_{j-1}-C_j\bar{U}_{j-1}^{(1)}=C_jW_{j-1}^{(0)}, \qquad j=0,1,\cdots,N
$$

We know $C_0=0$ and the nonnegative property of $(A_j+\Gamma_j)^{-1}$, so we have $W_0^{(0)}\ge 0$. Suppose that $W_{j-1}^{(0)}\ge 0$ for some $j>1$, then $(A_j+\Gamma_j)W_j^{(0)}\ge 0$ since $C_j\ge 0$, it shows that $W_j^{(0)}\ge 0$ from the nonnegative property of $(A_j+\Gamma_j)^{-1}$. By the induction principle, $\bar{U}_j^{(1)}\le \bar{U}_j^{(0)}$ for every j. A similar argument using the property of a lower solution gives $\underline{U}_j^{(1)}\ge \underline{U}_j^{(0)}$ for every j.

Let $W_j^{(1)} \equiv \bar{U}_j^{(1)} - \underline{U}_j^{(1)}$, $j = 1, 2, \cdots, N$, satisfy

$$(A_j + \Gamma_j)W_j^{(1)} = C_j W_{j-1}^{(1)} + C_j' W_{j+1}^{(0)} + \Gamma_j(\bar{U}_j^{(0)} - \underline{U}_j^{(0)}) + F_j(\bar{U}_j^{(0)}) - F_j(\underline{U}_j^{(0)})$$

based on the relation $\bar{U}_j^{(0)} \geq \underline{U}_j^{(0)}$, $C_j' \geq 0$, we have

$$(A_j + \Gamma_j)W_j^{(1)} \geq C_j W_{j-1}^{(1)}$$

Followed by the same induction as for $W_j^{(0)}$, we have $W_j^{(1)} \geq 0$ for every j. By now the conclusion shows that $\underline{U}_j^{(0)} \leq \underline{U}_j^{(1)} \leq \bar{U}_j^{(1)} \leq \bar{U}_j^{(0)}$.
(2)Let's assume that $\underline{U}_j^{(m-1))} \leq \underline{U}_j^{(m)} \leq \bar{U}_j^{(m)} \leq \bar{U}_j^{(m-1)}$ for some $m > 1$ and $W_j^{(m)} \equiv \bar{U}_j^{(m)} - \bar{U}_j^{(m+1)}$, $j = 1, 2, \cdots, N$. Then

$$(A_j + \Gamma_j)W_j^{(m)} = C_j W_{j-1}^{(m)} + C_j' W_{j+1}^{(m-1)} + \Gamma_j(\bar{U}_j^{(m-1)} - \bar{U}_j^{(m-1)}) + F_j(\bar{U}_j^{(m-1)}) - F_j(\bar{U}_j^{(m)})$$

From (25), Γ_j is nonnegative and $C_j\prime \geq 0$

$$(A_j + \Gamma_j)W_j^{(m)} \geq C_j W_{j-1}^{(m)}, \qquad j = 0, 1, \cdots, N$$

By an induction argument, we have $W_j^{(m)} \geq 0$ for all j which means $\bar{U}^{(m+1)} \leq \bar{U}^{(m)}$. A similar argument gives $\underline{U}^{(m+1)} \geq \underline{U}^{(m)}$ and $\bar{U}^{(m+1)} \geq \underline{U}^{(m+1)}$. This proves the monotone property (30).

To show that $\bar{U}^{(m)}$ is an upper solution we apply the iteration process (29)

$$\begin{aligned} A_j \bar{U}_j^{(m)} &= C_j \bar{U}_{j-1}^{(m)} + C_j' \bar{U}_{j+1}^{(m-1)} + \Gamma_j(\bar{U}_j^{(m-1)} - \bar{U}_j^{(m)}) + F_j(\bar{U}_j^{(m-1)}) + H_j \\ &\geq C_j \bar{U}_{j-1}^{(m)} + C_j' \bar{U}_{j+1}^{(m)} + F_j(\bar{U}_j^{(m)}) + H_j \end{aligned}$$

This shows that $\bar{U}^{(m)}$ is an upper solution. The proof for the lower solution $\underline{U}^{(m)}$ is similar.

Theorem 5 *Let $\tilde{U}, \hat{U}$ be a pair of ordered upper and lower solutions of (21). Then the sequence $\{\bar{U}^{(m)}\}$ given by (29) with $\bar{U}^{(0)} = \tilde{U}$ converges monotonically from above to a maximal solution $\bar{U}$ of (21), while the sequence $\{\underline{U}^{(m)}\}$ with $\underline{U}^{(0)} = \hat{U}$ converges monotonically from below to a minimal solution $\underline{U}$. Moreover*

$$\hat{U} \leq \underline{U}^{(m)} \leq \underline{U}^{(m+1)} \leq \underline{U} \leq \bar{U} \leq \bar{U}^{(m+1)} \leq \bar{U}^{m} \leq \tilde{U}(m = 1, 2, \cdots) \tag{31}$$

and if U^* is any solution in $< \hat{U}, \tilde{U} >$ then $\underline{U} \leq U^* \leq \bar{U}$.

Proof: By the monotone property (30), the limits $\bar{U}$ and $\underline{U}$ for the present sequences exist and satisfy the relation (31). Letting $m \to \infty$ shows that $\bar{U}$ and $\underline{U}$ are solutions of (21). The proof for the second part is similar to the proof of Theorem 4.

4 NUMERICAL RESULTS

In this section, we present some numerical examples applying the block monotone iterative schemes given in previous section to demonstrate the efficiency of those methods. We consider six problems in the domain

$$\Omega = \{(x, y) \in \Re^2; 0 < x < 1, 0 < y < 1\}$$

Problem A : Consider the boundary value problem,

$$\begin{aligned} -\Delta u = u(1-u) + q(x,y) \qquad \text{in } \Omega \\ u = 0 \quad \text{on } \partial\Omega \end{aligned} \tag{32}$$

where $f(x, y, u) = u(1 - u) + q(x, y)$ is a nonlinear function of u. Let $u(x, y) = \sin(\pi x)\sin(\pi y)$ be the explicit analytical solution of (32), then we have

$$q(x, y) = 2\pi^2 \sin(\pi x)\sin(\pi y) - \sin(\pi x)\sin(\pi y) + (\sin(\pi x)\sin(\pi y))^2$$

To compute numerical solutions of (32), we consider the corresponding finite difference system where $H = 0$ and

$$F_{i,j}(u_{i,j}) = u_{i,j}(1 - u_{i,j}) + q_{i,j} \text{ with } q_{i,j} = q(x_i, y_j)$$

In order to find the upper solution of this problem, we solve the following linear problem:

$$-\Delta u = u + q(x,y) \quad \text{in} \quad \Omega$$

$$u = 0 \quad \text{on} \quad \partial\Omega$$

The numerical solution of this problem is the upper solution of problem A since $u + q(x, y) \geq u(1 - u) + q(x, y)$. It is also easy to see $\underline{U}^{(0)} = 0$ is the lower solution of problem A. We compute the corresponding sequences $\{\bar{U}^{(m)}\}$, $\{\underline{U}^{(m)}\}$ from (26) and (29) for various M and N. There are two terminate criterions of the iterations depending on the number of solutions of the problem, as for the multiple solution the terminate criterion is

$$\|\bar{U}^{(m+1)} - \bar{U}^{(m)}\| + \|\underline{U}^{(m+1)} - \underline{U}^{(m)}\| \leq \epsilon$$

as for the unique solution, the terminate criterion is

$$\|\bar{U}^{(m+1)} - \underline{U}^{(m+1)}\| \leq \epsilon$$

where $\| \cdot \|$ is the l_2 norm. Notice that every example we choose has the unique solution.

Numerical results of Problem A using Block Jacobi Method and Block Gauss-Seidel Method at $y_j = 0.5$ and various values of x_i for the case $M = N$ and $N = 10, 20$ and 40 are given in Table(1a) and Table(3a), respectively. Included in the tables are the number of iterations for each N, maximal and minimal solutions and the true analytic solution. Tables show that the property $\bar{u}_{i,j} \geq \underline{u}_{i,j}$ holds for every (i, j), and both $\bar{u}_{i,j}$ and $\underline{u}_{i,j}$ compare fairly close to the true solution $u(x_i, y_j)$ at every mesh point (x_i, y_j). We also can see that the number of iterations is approximately proportional to N^2.

Problem B : Consider the boundary value problem,

$$-\Delta u = u(1 - u) + q(x, y) \qquad \text{in } \Omega$$

$$u = xy \quad \text{on } \partial\Omega \tag{33}$$

This problem is as same as Problem A except the boundary condition. Based on the new boundary condition, we choose explicit analytical solution $u(x, y) = \sin(\pi x)\sin(\pi y) + xy$, and we have

$$q(x, y) = (2\pi^2 - 1 - 2xy)\sin(\pi x)\sin(\pi y) + (\sin(\pi x)\sin(\pi y))^2 - xy + (xy)^2$$

The upper solution comes from the solution of the linear problem:

$$-\Delta u = u + q(x, y) \quad \text{in} \quad \Omega$$

$$u = xy \quad \text{on} \quad \partial\Omega$$

and the lower solution is 0. The numerical results are presented in Table(1b) and Table(3b).

Problem C : Consider the boundary value problem,

$$-\Delta u = e^{(-u)}u + q(x, y) \quad \text{in } \Omega$$

$$u = 0 \quad \text{on } \Gamma \tag{34}$$

We choose the explicit analytical solution $u(x, y) = \sin(\pi x)\sin(\pi y)$. Then we have

$$q(x, y) = 2\pi^2 \sin(\pi x)\sin(\pi y) - e^{-\sin(\pi x)\sin(\pi y)}\sin(\pi x)\sin(\pi y)$$

As for this problem, we are going to explore more nonlinear problem. The upper and lower solutions are obtained by the similar ways in Problem A. The numerical results have been shown in Table(1c) and Table(3c).

Problem D : Consider the boundary value problem,

$$-\Delta u = e^{(-u)}u + q(x,y) \qquad \text{in } \Omega$$
$$u = xy \qquad \text{on } \partial\Omega \tag{35}$$

we choose the explicit analytical solution $u(x,y) = \sin(\pi x)\sin(\pi y) + xy$, then we have

$$q(x,y) = 2\pi^2 \sin(\pi x)\sin(\pi y) + e^{-\sin(\pi x)\sin(\pi y) - xy}(\sin(\pi x)\sin(\pi y) + xy)$$

The upper and lower solutions are obtained as the same manner as the problem B. The numerical results are listed in Table(1d) and Table(3d). To demonstrate the monotone property of the iterations, we choose this example to present the numerical results of the maximal and minimal sequences in Table(8), and the numerical results indicate the monotone property of these sequences at every mesh point (x_i, y_j).

To show the efficiency of the block method we also calculate the numerical solutions the following linear problems using block iterative methods:

Problem E : Consider the boundary value problem,

$$-\Delta u = u + q(x,y) \qquad \text{in } \Omega$$
$$u = 0 \qquad \text{on } \partial\Omega \tag{36}$$

The numerical results by block methods are in Table(1e) and Table(3e). From Table(5), the comparison shows this problem has the same level of iteration number as the nonlinear problem. The numerical results by point-wise methods are in Table(2a) and the comparison between point-wise methods and block methods is in Table(6a), it shows that the iteration number of point-wise methods is about two times of the iteration number of block methods. By using different initial value of iteration $u_0 = 1$ and $u_0 = 0.5$, there is a little difference in the iteration number since the distance between the average true value and initial value varies.

Problem F : Consider the boundary value problem,

$$\begin{aligned} -\Delta u = u + q(x, y) \qquad \text{in } \Omega \\ u = xy \quad \text{on } \partial\Omega \end{aligned} \tag{37}$$

The numerical results are in Table(1f) and Table(3f). We also can see the comparison in Table(5), Table(2b) and Table(6b). The numerical results for this problem are similar as the Problem E.

Table 1: Numerical Results Using Block Jacobi Method

(a) Problem A: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.311532	0.592548	0.815543	0.958704	1.008032	
	min sol	0.311473	0.592437	0.815391	0.958526	1.007845	86
20	max sol	0.309643	0.588972	0.810643	0.952963	1.002002	
	min sol	0.309524	0.588746	0.810334	0.952601	1.001622	314
40	max sol	0.309174	0.588082	0.809424	0.951533	1.000501	
	min sol	0.308929	0.587618	0.808789	0.950790	1.000501	1140
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(b) Problem B: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=xy

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.361489	0.692460	0.965412	1.158534	1.258737	
	min sol	0.361433	0.692356	0.965269	1.158368	1.257663	88
20	max sol	0.359633	0.688950	0.960611	1.152920	1.251954	
	min sol	0.359518	0.688735	0.960317	1.152577	1.251595	322
40	max sol	0.359171	0.688076	0.959415	1.151522	1.250488	
	min sol	0.358935	0.687631	0.958808	1.150814	1.249747	1175
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

(c) Problem C: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.311599	0.592688	0.815754	0.958969	1.008317	
	min sol	0.311543	0.592581	0.815608	0.958798	1.008137	89
20	max sol	0.309660	0.589006	0.810695	0.953028	1.002072	
	min sol	0.309536	0.588771	0.810371	0.952647	1.001672	321
40	max sol	0.309178	0.588091	0.809437	0.951549	1.000518	
	min sol	0.308926	0.587612	0.808779	0.950777	0.999707	1166
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(d) Problem D: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=xy

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.361587	0.692663	0.965720	1.158927	1.258270	
	min sol	0.361531	0.692558	0.965575	1.158757	1.258092	92
20	max sol	0.359657	0.689000	0.960687	1.153017	1.252060	
	min sol	0.359533	0.688765	0.960364	1.152638	1.251662	333
40	max sol	0.359177	0.688089	0.959434	1.151546	1.250515	
	min sol	0.358928	0.687616	0.958785	1.150785	1.249715	1216
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

(e) Problem E: Linear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.311762	0.593009	0.816203	0.959504	1.008882	88
	$u_0 = 0.5$	0.311664	0.592820	0.815946	0.959203	1.008565	79
20	$u_0 = 1$	0.309798	0.589270	0.811061	0.953460	1.002527	319
	$u_0 = 0.5$	0.309599	0.588893	0.810541	0.952848	1.001884	280
40	$u_0 = 1$	0.309402	0.588518	0.810025	0.952242	1.001246	1158
	$u_0 = 0.5$	0.309016	0.587784	0.809016	0.951055	0.999998	1274
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(f) Problem F: Linear $F(\bar{x}, u)$ with Boundary Condition= xy

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.361746	0.692976	0.966161	1.159455	1.258831	80
	$u_0 = 0.5$	0.361649	0.692792	0.965908	1.159157	1.258518	87
20	$u_0 = 1$	0.359760	0.689199	0.960963	1.153345	1.252406	291
	$u_0 = 0.5$	0.359561	0.688819	0.960440	1.152729	1.251759	312
40	$u_0 = 1$	0.359326	0.688372	0.959825	1.152006	1.250999	1050
	$u_0 = 0.5$	0.358925	0.687611	0.958777	1.150774	1.249703	1122
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

Table 2: Numerical Results Using Pointwise Jacobi Method

(a) Problem E: Linear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.311831	0.593139	0.816385	0.959720	1.009107	No. of 153
	$u_0 = 0.5$	0.311588	0.592675	0.815748	0.958968	1.008320	132
20	$u_0 = 1$	0.309948	0.589556	0.811454	0.953922	1.003013	560
	$u_0 = 0.5$	0.309427	0.588566	0.810091	0.952320	1.001328	463
40	$u_0 = 1$	0.309710	0.589103	0.810830	0.953188	1.002241	2012
	$u_0 = 0.5$	0.308658	0.587103	0.808078	0.949953	0.998839	1610
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(b) Problem F: Linear $F(\bar{x}, u)$ with Boundary Condition=xy

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.361831	0.693141	0.966385	1.159722	1.259108	129
	$u_0 = 0.5$	0.361587	0.692673	0.965744	1.158965	1.258316	154
20	$u_0 = 1$	0.359946	0.689553	0.961450	1.153917	1.253007	469
	$u_0 = 0.5$	0.359426	0.688564	0.960089	1.152316	1.251324	558
40	$u_0 = 1$	0.359710	0.689103	0.960830	1.153188	1.2522409	1650
	$u_0 = 0.5$	0.358659	0.687105	0.958080	1.149955	1.248842	1999
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

Table 3: Numerical Results Using Block Gauss-Seidel Method

(a) Problem A: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.311532	0.592548	0.815543	0.958704	1.008031	
	min sol	0.311509	0.592505	0.815484	0.958635	1.007960	48
20	max sol	0.309643	0.588972	0.810643	0.952962	1.002002	
	min sol	0.309548	0.588859	0.810489	0.952781	1.001812	171
40	max sol	0.309174	0.588082	0.809423	0.951533	1.000500	
	min sol	0.309052	0.587852	0.809109	0.951165	1.000114	626
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(b) Problem B: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=xy

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.361489	0.692460	0.965411	1.158534	1.257836	
	min sol	0.361465	0.692416	0.965352	1.158465	1.257764	48
20	max sol	0.359633	0.688950	0.960611	1.152920	1.251953	
	min sol	0.359576	0.688843	0.960464	1.152750	1.251775	174
40	max sol	0.359171	0.688076	0.959415	1.151522	1.250488	
	min sol	0.359050	0.687849	0.959105	1.151160	1.250109	638
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

(c) Problem C: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.311599	0.592688	0.815754	0.9589691	1.008317	
	min sol	0.311574	0.592640	0.815689	0.958892	1.008236	49
20	max sol	0.309660	0.589006	0.810695	0.953027	1.002072	
	min sol	0.309601	0.588894	0.81054	0.952846	1.001881	176
40	max sol	0.309178	0.588090	0.809436	0.951549	1.000518	
	min sol	0.309053	0.587854	0.809111	0.951168	1.000117	641
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(d) Problem D: Nonlinear $F(\bar{x}, u)$ with Boundary Condition=xy

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	max sol	0.361587	0.692663	0.965720	1.158927	1.258269	
	min sol	0.361561	0.692615	0.965654	1.158849	1.258188	50
20	max sol	0.359657	0.689	0.960687	1.153017	1.252060	
	min sol	0.359597	0.688887	0.960532	1.152835	1.251869	181
40	max sol	0.359177	0.688089	0.959434	1.151546	1.250515	
	min sol	0.359052	0.687852	0.959109	1.151165	1.250114	663
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

(e)Problem E: Linear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.311733	0.592952	0.816129	0.959417	1.00879	49
	$u_0 = 0.5$	0.311682	0.592855	0.815995	0.959259	1.008625	42
20	$u_0 = 1$	0.309749	0.589177	0.810933	0.953309	1.002368	173
	$u_0 = 0.5$	0.309626	0.588943	0.810611	0.952903	1.001970	147
40	$u_0 = 1$	0.309315	0.588352	0.809797	0.951974	1.000964	628
	$u_0 = 0.5$	0.309054	0.587856	0.809115	0.951171	1.000121	522
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(f) Problem F: Linear $F(\bar{x}, u)$ with Boundary Condition $=xy$

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.361735	0.692954	0.966132	1.15942	1.258794	45
	$u_0 = 0.5$	0.36168	0.692851	0.96599	1.159253	1.258619	47
20	$u_0 = 1$	0.359749	0.689178	0.960935	1.153311	1.252370	155
	$u_0 = 0.5$	0.359623	0.688939	0.960604	1.152923	1.251962	169
40	$u_0 = 1$	0.359314	0.688350	0.959795	1.151971	1.250961	545
	$u_0 = 0.5$	0.359054	0.687856	0.959114	1.151171	1.25012	616
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

Table 4: Numerical Results Using Pointwise Gauss-Seidel Method

(a) Problem E: Linear $F(\bar{x}, u)$ with Boundary Condition=0

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.311757	0.593002	0.816204	0.959513	1.008901	85
	$u_0 = 0.5$	0.311659	0.592806	0.815921	0.959165	1.008517	73
20	$u_0 = 1$	0.309804	0.589288	0.811092	0.953505	1.002585	311
	$u_0 = 0.5$	0.309571	0.588833	0.810452	0.952735	1.001755	261
40	$u_0 = 1$	0.309434	0.588584	0.810124	0.952367	1.001388	1126
	$u_0 = 0.5$	0.308934	0.587622	0.808785	0.950774	0.999693	923
	true sol	0.309017	0.587785	0.809017	0.951057	1	

(b) Problem F: Linear $F(\bar{x}, u)$ with xy Boundary Condition

N	(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)	No. of Iterations
10	$u_0 = 1$	0.361755	0.692998	0.966198	1.159506	1.258893	78
	$u_0 = 0.5$	0.361660	0.692807	0.965923	1.159166	1.258518	83
20	$u_0 = 1$	0.359805	0.688836	0.961094	1.153508	1.252587	275
	$u_0 = 0.5$	0.359572	0.688564	0.960455	1.152739	1.251389	306
40	$u_0 = 1$	0.359434	0.688585	0.960125	1.152368	1.2522409	966
	$u_0 = 0.5$	0.358934	0.687622	0.958785	1.150774	1.249693	1110
	true sol	0.359017	0.687785	0.959017	1.151057	1.25	

Table 5: Comparison On Iteration Number

(a) For Block Jacobi Method

	N=10	N=20	N=40
Problem A	86	314	1140
Problem B	88	322	1175
Problem C	89	321	1166
Problem D	92	333	1216
Problem E	88	319	1158
	79	280	1001
Problem F	80	291	1050
	87	312	1122

(b) For Block Gauss-Seidel Method

	N=10	N=20	N=40
Problem A	48	171	626
Problem B	48	174	638
Problem C	49	176	641
Problem D	50	181	663
Problem E	49	173	624
	42	142	522
Problem F	45	155	545
	47	169	616

Table 6: Comparison Between Pointwise Method and Block Method

(a) For Problem E

	N=10	N=20	N=40
Pointwise Jacobi	153	560	2012
	132	463	1610
Block Jacobi	88	319	1158
	79	280	1274
Pointwise Gauss-Seidel	85	311	1126
	73	261	923
Block Gauss-Seidel	49	173	628
	42	147	522

(b) For Problem F

	N=10	N=20	N=40
Pointwise Jacobi	129	469	1650
	154	558	1999
Block Jacobi	80	291	1050
	87	312	1122
Pointwise Gauss-Seidel	78	275	966
	83	306	545
Block Gauss-Seidel	45	155	545
	47	169	616

Table 7: Comparison On Relative Error Rate

(a) For Block Jacobi Method

		Prob A	Prob B	Prob C	Prob D	Prob E	Prob F
E10/E20	max sol	3.98	4.07	4.01	4.07	3.52	3.72
	min sol	4.84	4.88	4.87	4.94	4.54	4.9
E20/E40	max sol	3.999	4.02	4	4.03	2.02	2.42
	min sol	5.76	6.32	5.69	5.85	4.64	5.97

(b) For Block Gauss-Seidel Method

		Prob A	Prob B	Prob C	Prob D	Prob E	Prob F
E10/E20	max sol	4.01	4.07	4.01	4.07	3.71	3.76
	min sol	4.4	4.43	4.38	4.44	4.37	4.45
E20/E40	max sol	4.00	4.03	4	4.02	2.46	2.48
	min sol	15.89	16.3	16	16.4	16.2	16.3

Table 8: Monotone Sequences of Problem D

(a) Maximal Sequence By Block Jacobi Method

(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)
m=10	0.359317	0.688356	0.959802	1.151979	1.250970
m=50	0.359287	0.688298	0.959721	1.151884	1.25087
m=100	0.359258	0.688243	0.959645	1.151794	1.250775
m=150	0.359236	0.688202	0.959590	1.151729	1.250706
m=200	0.359221	0.688172	0.959548	1.15168	1.250656
m=250	0.359209	0.68815	0.959518	1.151645	1.250618
m=300	0.359201	0.688134	0.959496	1.151619	1.250591
m=350	0.359194	0.688122	0.959479	1.1516	1.250571
m=400	0.35919	0.688113	0.959467	1.151585	1.250556
m=450	0.359186	0.688107	0.959459	1.151575	1.250545
m=500	0.359184	0.688102	0.959452	1.151567	1.250537
m=550	0.359182	0.688098	0.959447	1.151562	1.250531
m=600	0.359181	0.688096	0.959444	1.151558	1.250527
m=650	0.359180	0.688094	0.959441	1.151555	1.250523
m=700	0.359179	0.688093	0.959439	1.151552	1.250521
True Sol	0.359017	0.687785	0.959017	1.151057	1.25

(b) Minimal Sequence By Block Jacobi Method

(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)
m=10	0.017807	0.034172	0.047328	0.055832	0.058777
m=50	0.080307	0.153487	0.212251	0.250749	0.266449
m=100	0.144355	0.276108	0.383575	0.458604	0.499042
m=150	0.195540	0.374884	0.523225	0.630113	0.690837
m=200	0.235818	0.452542	0.632826	0.7635	0.837208
m=250	0.26696	0.512372	0.716595	0.864289	0.946123
m=300	0.290647	0.557697	0.779642	0.939485	1.026542
m=350	0.308448	0.591659	0.826666	0.995242	1.085775
m=400	0.32172	0.61693	0.861552	1.036447	1.129363
m=450	0.331563	0.63565	0.887345	1.06684	1.161428
m=500	0.338841	0.649479	0.906376	1.089231	1.18501
m=550	0.34421	0.659677	0.920399	1.105714	1.202352
m=600	0.348166	0.667188	0.930723	1.117842	1.215104
m=650	0.351078	0.672718	0.93832	1.126763	1.224481
m=700	0.353221	0.676786	0.943909	1.133324	1.231374
TrueSol	0.359017	0.687785	0.959017	1.151057	1.25

(c) Maximal Sequence By Block Gauss-Seidel Method

(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)
m=10	0.359299	0.68832	0.959753	1.151921	1.250909
m=50	0.359252	0.688231	0.959629	1.151775	1.250775
m=100	0.359217	0.688165	0.959539	1.15167	1.250645
m=150	0.359199	0.68813	0.959491	1.151613	1.250585
m=200	0.359189	0.688111	0.959465	1.151582	1.250552
m=250	0.359183	0.688101	0.959451	1.151566	1.250535
m=300	0.35918	0.688095	0.959443	1.151557	1.250526
m=350	0.359179	0.688092	0.959439	1.151552	1.250521
m=400	0.359178	0.688091	0.959437	1.151549	1.250518
m=450	0.359177	0.68809	0.959435	1.151548	1.250516
m=500	0.359177	0.688089	0.959435	1.151547	1.250515
m=550	0.359177	0.688089	0.959434	1.151547	1.250515
m=600	0.359177	0.688089	0.959435	1.151546	1.250515
m=650	0.359177	0.688089	0.959435	1.151546	1.250515
True Sol	0.359017	0.687785	0.959017	1.151057	1.25

(d) Minimal Sequence By Block Gauss-Seidel Method

(x,y)	(0.1,0.5)	(0.2,0.5)	(0.3,0.5)	(0.4,0.5)	(0.5,0.5)
m=10	0.034479	0.066063	0.091458	0.107922	0.113743
m=50	0.14537	0.27916	0.388361	0.465299	0.507664
m=100	0.23728	0.455431	0.637042	0.768829	0.84327
m=150	0.291587	0.55951	0.782192	0.942568	1.029884
m=200	0.32226	0.617962	0.862983	1.038148	1.131172
m=250	0.33914	0.650048	0.907161	1.090157	1.185988
m=300	0.348329	0.667498	0.93115	1.118344	1.215633
m=350	0.35331	0.676954	0.94414	1.133596	1.23166
m=400	0.356005	0.682069	0.951166	1.141843	1.240324
m=450	0.357462	0.684835	0.954965	1.146302	1.245006
m=500	0.35825	0.68633	0.957019	1.148712	1.247538
m=550	0.358676	0.687138	0.958129	1.150014	1.248906
m=600	0.358906	0.687575	0.958728	1.150718	1.249645
m=650	0.359031	0.687811	0.959053	1.151099	1.250045
TrueSol	0.359017	0.687785	0.959017	1.151057	1.25

5 CONCLUSIONS AND DISCUSSION

5.1 Conclusions

In Chapter 4, there are totally six problems studied by using the block Jacobi and block Gauss-Seidel methods. The first four are nonlinear elliptic equations with different boundary conditions, the last two are linear elliptic equations with different boundary conditions. All the problems we choose have the unique analytic solutions. Analyzing the numerical results gives us the following observations and conclusions:

(A) *Monotone and convergence property*

In Table(8), it contains the data of maximal and minimal sequences of Problem D by two block methods. We see the maximal sequence decreases monotonically from the upper solution to the true solution and the minimal sequence increase from the lower solution to the true solution, which shows the monotone property of these sequences holds at every mesh point by two block methods. In Table(1) and Table(3), we see the property $\bar{u}_{i,j} \geq \underline{u}_{i,j}$ holds for every (i, j), and both $\bar{u}_{i,j}$ and $\underline{u}_{i,j}$ are fairly close to the true solution $u(x_i, y_j)$ at every mesh point (x_i, y_j), and those sequences converge very well to the true solution. It should be noted that the slightly larger value of the minimal solution compared with the true solution is mostly likely due to the discretization error of the finite difference system.

(B) *Iteration number*

Iteration numbers for all examples with different methods are listed in Table(5). The data indicates that the number of iterations is approximately proportional to N^2 for each example with different methods applied. Specially we notice that the linear and nonlinear problems require almost the same iteration numbers with the same N and the same method. This clearly indicates the efficiency of the block monotone methods.

(C) *Error ratio*

From Table(7), error ratios are approximate to 4 when the mesh size is doubled. This is consistent with the theoretical results in [5].

(D) *Efficiency*

(1) Less iteration number

From Table(6), the comparison between the point-wise and block method for two linear problems has been presented, it shows the iteration numbers of block method are always less than the point-wise method.

(2) Less computation in each iteration

Compared with the numerical results of the pointwise method, the block method definitely is a reliable and efficient computational algorithm for computing the solution. An advantage of the block method is that the Thomas algorithm can be used to compute numerical solutions in each iteration in the same fashion as for one-dimensional solutions. Theoretically, Thomas algorithm only requires $3N$ of operations (N is the number of mesh points along the x-direction). In a block method, we use Thomas algorithm to calculate the inside block (matrix size is $N \times N$), so for each iteration we have $3N * N$ operations. Also by numerical results, the total iteration number is no more than $O(N^2)$, and therefore the overall operations will be approximately $O(N^4)$. However for the pointwise method, the size of the matrix of the finite difference system is $N^2 \times N^2$, it requires N^4 operations per each iteration, the overall operations will be $O(N^6)$. Therefore the block method is much more efficient than the pointwise method.

(3) Linear and nonlinear problems

From Table(5), we see that both linear and nonlinear problems need almost same numbers of iterations. It implies that it does not require more work to solve a nonlinear problem than a linear problem. This is very significant and more theoretical and numerical studies should be conducted in this regard.

5.2 Open Questions

The efficiency of the block monotone methods are observed from the numerical simulations. The limited numerical results suggest that linear and nonlinear problems have almost same converge rate when we use the block Jacobi or block Gauss-Seidel monotone methods to treat nonlinear problem. More numerical simulations should be conducted with different nonlinear reaction functions to see how many effects of nonlinearity on the convergent rate. The rates of convergence of the block monotone methods need to be investigated theoretically.

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