Journal of Statistical Theory and Applications
Vol. **18(3)**, September 2019, *pp.* 198–211
DOI: https://doi.org/10.2991/jsta.d.190619.001; ISSN 1538-7887
https://www.atlantis-press.com/journals/jsta

Bivariate and Multivariate Weighted Kumaraswamy Distributions: Theory and Applications

Indranil Ghosh*

Department of Mathematics and Statistics, University of North Carolina, Wilmington, 601 S College Road, Wilmington, North Carolina 28403, USA

ARTICLE INFO

Article History

Received 15 Aug 2017
Accepted 27 Nov 2017

Keywords

Weighted distributions
Bivariate weighted Kumaraswamy distribution
Renyi entropy
Multivariate weighted Kumaraswamy distribution

ABSTRACT

Weighted distributions (univariate and bivariate) have received widespread attention over the last two decades because of their flexibility for analyzing skewed data. In this paper, we derive the bivariate and multivariate weighted Kumaraswamy distributions via the construction method as discussed in B.C. Arnold, I. Ghosh, A. Alzaatreh, Commun. Stat. Theory Methods. 46 (2017), 8897–8912. Several structural properties of the bivariate weighted distributions including marginals, distributions of the minimum and maximum, reliability parameter, and total positivity of order two are discussed. We provide some multivariate extensions of the proposed bivariate weighted Kumaraswamy model. Two real-life data sets are used to show the applicability of the bivariate weighted Kumaraswamy distributions and is compared with other rival bivariate Kumaraswamy models.

1. INTRODUCTION

Recently the construction of continuous bivariate distributions have received a considerable amount of interest in the literature. A vast literature on this topic exists (see, the book by Balakrishnan and Lai [1]). Kumaraswamy [2] argued that the beta distribution does not faithfully fit hydrological random variables such as daily rainfall and daily stream flow and introduced an alternative distribution to the beta distribution, which is known as the Kumaraswamy distribution. According to Nadarajah [3] several papers in the hydrological literature have used this distribution because it is deemed as a better alternative to the beta distribution. The Kumaraswamy distribution is also known as minimax distribution, and generalized beta distribution of the first kind (or beta type I). As a motivation to our current work, we consider a financial risk modeling scenario. In the context of bounded dependent risks, it is desirable to have available flexible models with analytic expressions for the corresponding marginal distributions and densities. In such a context, after rescaling the bounded risks to the interval (0, 1), Kumaraswamy distributions may provide attractive candidate components for such models because of the simplicity of their corresponding density and distribution functions. It thus merits our attention to develop a spectrum of bivariate and multivariate models with Kumaraswamy marginal and/or conditional distributions or at least Kumaraswamy type bivariate and multivariate distributions. For a detailed study, interested readers are referred to Arnold and Ghosh [4], Wagner *et al.* [5]. The usefulness and applications of weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can also be seen in Nanda and Jain [6], Gupta and Keating [7], Oluyede [8], Zelen and Feinleib [9], and the references therein. Recently, Arnold *et al.* [10] introduced a new method of constructing bivariate weighted distributions which they used to model some real-life data sets independently. Al-Mutairi *et al.* [11] developed a new bivariate distribution with weighted exponential marginals and discussed its multivariate generalization. In this paper, we consider the particular weight function considered Arnold *et al.* [10], and in addition, namely the maximum conditioning weight function, and via the modified symmetric bivariate Fa–rlieGumbel–Morgenstern (FGM, henceforth, in short) copula. This is completely different from several other methods of obtaining a bivariate Kumaraswamy distribution as discussed in Arnold *et al.* [4,12,13]. The symmetric bivariate FGM copula based construction approach mentioned in this article, is completely different from Arnold *et al.* [5], where the authors developed several strategies for constructing bivariate Kumaraswamy type distributions via Arnold–Ng copula. We focus our attention on the application of this new weighted bivariate and multivariate Kumaraswamy distribution. We envision a real-life scenario as a genesis of the proposed bivariate weighted distribution in a stress–strength model context.

Let (X, Y) be a two-dimensional absolutely continuous random variable with joint density function (with respect to Lebesgue measure) $f(x, y)$. Also let $(\Omega, \mathcal{F}, \mathcal{P})$ be the common probability space on which X and Y are defined.Using the notation of Arnold and Nagaraja [14], the weighted distribution of (X, Y) denoted by (X^W, Y^W) is given by the following joint density function:

*Email: ghoshi@uncw.edu

$$f^{W}(x,y) = \frac{W(x,y)f(x,y)}{E(W(X,Y))},$$

where $W(x,y)$ is a nonnegative function such that $E(W(X,Y)) < \infty$. The utility of such distributions is well established in literature. Rao [15] employed univariate weighted distributions in various real-life problems such as in analysis of family size, aerial survey and visibility bias, renewal theory, cell cycle analysis, efficacy of early screening for disease, statistical ecology, and reliability modeling. An exhaustive amount of work in this area is available in Patil and Rao [16]. Mahfoud and Patil [17] discussed the properties of bivariate weighted densities based on two different choices of the weight function $W(x,y)$, namely x^{α} and $\max(x,y)$. Arnold and Nagarajah [14] focused on the case of two independent random variables (X, Y) and discussed the dependence structure of the corresponding bivariate weighted distributions. In this article, we focused on the $\max(x,y)$ weight function in more details. In particular, we consider the following different weighted bivariate and multivariate Kumaraswamy models:

- Assume a system has two independent components with strengths W_1 and W_2, and suppose that to run the process each component strength has to overcome an outside stress W_0 which is independent of both (W_1 and W_2). If we define $(X, Y) \stackrel{d}{=} (W_1, W_2 | \min(W_1, W_2) > W_0)$, where the W_i's have absolutely continuous distributions, then the resulting joint distribution of (W_1, W_2) is the type of bivariate weighted distribution to be investigated in this article. Henceforth, we call this as Type I-weighted bivariate Kumaraswamy models. The utility of such models have been discussed in details in Arnold *et al.* [10].
- The motivation for the second type (from now on, Type-II weighted bivariate Kumaraswamy model) can be described as follows: Consider a two component system, and suppose that, component J, $J = 1, 2$ receives outside random shocks, measured as W_i, $i = 1, 2$, and they are independent. Next, consider that the minimum of strength among the two components is measured as W_0, which is independent of W_i, $i = 1, 2$. Furthermore, let us assume that the implicit condition for the associated system to run is that W_0 must exceed the maximum of W_1 and W_2. Then the resulting joint distribution of (W_1, W_2) is the type of bivariate weighted distribution to be investigated next. This is the major motivation to consider this type of models, that we discuss the outside stress structure rather than the inside strength. The genesis of this model is distinct as compared to the previous one, although they might have some similar type properties as we shall see later on.
- The third method considered is based on a symmetric bivariate modified FGM type copula. Since, the quantile function of a Kumaraswamy distribution is available in simple analytical form, we can use this to construct various bivariate and multivariate Kumaraswamy models. A copula $C(x_1, x_2)$ is most simply described as a bivariate distribution with *Uniform* $(0, 1)$ marginal distributions. A bivariate Kumaraswamy distribution can then be obtained from any copula $C(x_1, x_2)$, by using marginal transformations of the form indicated earlier. Thus if (U_1, U_2) has the copula $C(x_1, x_2)$ as its distribution function, then the random vector $(X_1, X_2) = \left(\left(1-(1-U)^{1/b_1}\right)^{1/a_1}, \left(1-(1-U)^{1/b_2}\right)^{1/a_2}\right)$, will have a bivariate Kumaraswamy distribution in which $X_i \sim K(a_i, b_i)$, $i = 1, 2$. This approach thus provides us with a plethora of bivariate Kumaraswamy models. However, it is not clear how to sensibly select the particular copula to be used in the construction in achieving greater flexibility. In a separate article, we will focus our attention in this direction. Henceforth, we will call this method as Type-III weighted bivariate and multivariate copula based Kumaraswamy type distribution.

The remainder of this paper is organized as follows: In Section 2, we briefly describe the method of constructing the bivariate weighted distributions. In Section 3, we introduce a special case of the proposed family, the bivariate weighted Kumaraswamy (BWK) distribution and discuss various properties. In Section 4, we discuss a BWK model via conditioning on maximum as mentioned earlier. In Section 5, some discussion on the multivariate extension of the proposed family is provided. Section 6 deals with the estimation of the BWK distribution parameters. For illustrative purposes, two real-life data seta are fitted to the proposed model in Section 7, and are compared with other bivariate Kumaraswamy and bivariate beta models. Some concluding remarks are provided in Section 8.

2. TYPE-I WEIGHTED BIVARIATE KUMARASWAMY DISTRIBUTION

Let W_1, W_2, and W_0 be independent random variables with density functions $f_{W_i}(w_i)$, $i = 0, 1, 2$. Then according to Arnold *et al.* [10] if we define $(X, Y) \stackrel{d}{=} (W_1, W_2 | \min(W_1, W_2) > W_0)$ then the density function of the corresponding bivariate weighted distribution is given by

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{f_{W_1}(x) f_{W_2}(y) P(\min(W_1, W_2) > W_0 | W_1 = x, W_2 = y)}{P(\min(W_1, W_2) > W_0)} \\ &= \frac{f_{W_1}(x) f_{W_2}(y) P(W_0 < \min(x,y))}{P(\min(W_1, W_2) > W_0)} \\ &= \frac{f_{W_1}(x) f_{W_2}(y) F_{W_0}(\min(x,y))}{P(\min(W_1, W_2) > W_0)}. \end{aligned} \tag{1}$$

Remarks:

i. If W_i's, $i = 0, 1, 2$ are identically distributed with common density function $f_W(w)$, then $P(\min(W_1, W_2) > W_0) = \frac{1}{3}$. Hence, (1) reduces to

$$f_{X,Y}(x, y) = 3f_W(x)f_W(y)F_W(\min(x, y)). \tag{2}$$

ii. In general, we do not have a simple expression for $P(\min(W_1, W_2) > W_0)$, but in some cases it can be obtained in a closed form. For example, it can be computed if W_i's are independent exponential random variables with intensities λ_1, λ_2, and λ_0.

iii. One can obtain a multivariate extension of (1) in the following way: If $W_i \sim f_{W_i}(w_i)$ for $i = 0, 1, \cdots, k$ are independent random variables, then the k-dimensional weighted density analogous to (1) will be of the form

$$f_{X_1,X_2,\ldots X_k}(x_1, x_2, \cdots, x_k) = \frac{\left[\prod_{j=1}^{k} f_{W_j}(x_j)\right] F_{W_0}(\min(x_1, x_2, \ldots x_k))}{P(\min(W_1, W_2, \ldots, W_k) > W_0)}. \tag{3}$$

In the case in which the W_i's are independent and identically distributed (i.i.d.) random variables, (3) reduces to

$$f_{X_1,X_2,\ldots X_k}(x_1, x_2, \ldots x_k) = (k+1)\left[\prod_{j=1}^{k} f_W(x_j)\right] F_W(\min(x_1, x_2, \ldots x_k)). \tag{4}$$

In the next section, we study a special case of the bivariate weighted distribution in (1) where the W_i's are Kumaraswamy with parameters a and b_i, for $i = 0, 1, 2$, respectively.

2.1. Definition and Structural Properties

Consider the scenario in which three W_i's are independent random variables with $W_i \sim$ Kumaraswamy(a, b_i) for $i = 0, 1, 2$. Then from (1), the normalizing constant will be

$$\begin{aligned} C_1 &= P(\min(W_1, W_2) > W_0) \\ &= \frac{b_0}{b_0 + b_1 + b_2}. \end{aligned} \tag{5}$$

Hence, the joint distribution of (X, Y) will be

$$f(x, y) = C_1^{-1} a^2 b_1 b_2 (xy)^{a-1} (1 - x^a)^{b_1 - 1} (1 - y^a)^{b_2 - 1} \left(1 - \left(1 - (\min(x, y))^a\right)^{b_0}\right) \times I(0 < x < 1, 0 < y < 1). \tag{6}$$

In this case the marginals are given by

$$f(x) = C_1^{-1} \frac{ab_0 b_1}{b_0 + b_2} x^{a-1} \left[(1 - x^a)^{b_1 - 1} - (1 - x^a)^{b_0 + b_1 + b_2 - 1}\right] \times I(0 < x < 1). \tag{7}$$

$$f(y) = C_1^{-1} \frac{ab_0 b_2}{b_0 + b_1} y^{a-1} \left[(1 - y^a)^{b_2 - 1} - (1 - y^a)^{b_0 + b_1 + b_2 - 1}\right] \times I(0 < y < 1). \tag{8}$$

Let t_{11}, t_{12}, t_{21}, and t_{22} be real numbers with $0 < t_{11} < t_{12}$ and $0 < t_{21} < t_{22}$. Then, (X, Y) has the total positivity of order two (TP_2) property if for any such set of t_{ij}'s,

$$f_{X,Y}(t_{11}, t_{21}) f_{X,Y}(t_{12}, t_{22}) - f_{X,Y}(t_{12}, t_{21}) f_{X,Y}(t_{11}, t_{22}) \geqslant 0. \tag{9}$$

Theorem 1. The BWK distribution has the TP_2 property.

Proof. Let us consider different cases separately. If $0 < t_{11} < t_{21} < t_{12} < t_{22}$, then for the density function in (6), one can easily show that the condition in (9) is equivalent to

$$\left(1 - t_{21}^a\right)^{b_0} \geqslant \left(1 - t_{12}^a\right)^{b_0}. \tag{10}$$

Now, (10) holds because $t_{21} < t_{12}$ and a and b_0 are positive. The other cases can be shown similarly. Hence the proof.

Theorem 2. The BWK distribution with the density in (6) is log concave.

Proof. Taking negative of the logarithm of (6), we have the following:

Consider $0 < x < y < 1$

$$\begin{aligned}\left(-\log\right) f\left(x,y\right) &= \text{constant} - (a-1)\left[\log x + \log y\right] - (b_1 - 1)\left[\log\left(1 - x^a\right)\right] \\ &\quad - (b_2 - 1)\left[\log\left(1 - y^a\right)\right] - \log\left(1 - \left(\left(1 - x^a\right)^{b_0}\right)\right). \end{aligned} \tag{11}$$

Next, taking partial double derivative w.r.t x and y, of (11), we get,

$$D_y\left[D_x\left(\left(-\log\right) f\left(x,y\right)\right)\right] = 0,$$

where D_x stands for the partial derivative operator for x and similarly for D_y.

Case 2: When $0 < y < x < 1$

Similar result will hold here as well. Hence the proof.

The log-concave property implies the following:

- The Type-I weighted bivariate Kumaraswamy is unimodal.
- The marginals are also log-concave.
- It is closed under weak limits.

Next, note that for any $r \geqslant 1$, the marginal moments of X and Y are given by, respectively

$$E\left(X^r\right) = \frac{r\Gamma\left(\frac{r}{a}\right)\left(\frac{\Gamma(b_1)}{\Gamma\left(\frac{r}{a}+b_1+1\right)} - \frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{r}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2},$$

and

$$E\left(Y^r\right) = \frac{r\Gamma\left(\frac{r}{a}\right)\left(\frac{\Gamma(b_2)}{\Gamma\left(\frac{r}{a}+b_2+1\right)} - \frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{r}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2}.$$

The correlation coefficient ρ for this distribution is given by

$$\rho = \frac{A_1}{B_1 B_2},$$

where

$$\begin{aligned} A_1 &= E\left(XY\right) - \left[E\left(X\right)E\left(Y\right)\right] \\ &= 2C^{-1}\left[\frac{1}{a+1}\left(\frac{\Gamma\left(2+2/a\right)\Gamma\left(b_2\right)}{\Gamma\left(2+2/a+b_2\right)}\right) - \sum_{k=0}^{\infty}\frac{(-1)^k\binom{b_0}{k}}{a\left(b_0+1\right)+1}\left(\frac{\Gamma\left(1+2/a+b_0\right)\Gamma\left(b_2\right)}{\Gamma\left(1+2/a+b_2+b_0\right)}\right)\right] \\ &\quad -C^{-1}\left(\frac{\Gamma\left(\frac{1}{a}\right)\left(\frac{\Gamma(b_1)}{\Gamma\left(\frac{1}{a}+b_1+1\right)} - \frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{1}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2}\right)\times\left(\frac{\Gamma\left(\frac{r}{a}\right)\left(\frac{\Gamma(b_2)}{\Gamma\left(\frac{1}{a}+b_2+1\right)} - \frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{1}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2}\right), \end{aligned}$$

and

$$\begin{aligned} B_1 &= Var(X) \\ &= C^{-1}\left[\frac{2\Gamma\left(\frac{r}{a}\right)\left(\frac{\Gamma(b_1)}{\Gamma\left(\frac{2}{a}+b_1+1\right)}-\frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{2}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2}-\left(\frac{\Gamma\left(\frac{1}{a}\right)\left(\frac{\Gamma(b_1)}{\Gamma\left(\frac{1}{a}+b_1+1\right)}-\frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{1}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2}\right)^2\right]. \end{aligned}$$

Similarly,

$$\begin{aligned} B_2 &= Var(Y) \\ &= C^{-1}\left[\frac{2\Gamma\left(\frac{r}{a}\right)\left(\frac{\Gamma(b_2)}{\Gamma\left(\frac{2}{a}+b_2+1\right)}-\frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{2}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2}-\left(\frac{\Gamma\left(\frac{1}{a}\right)\left(\frac{\Gamma(b_2)}{\Gamma\left(\frac{1}{a}+b_2+1\right)}-\frac{\Gamma(b_0+b_1+b_2)}{\Gamma\left(\frac{1}{a}+b_0+b_1+b_2+1\right)}\right)}{a^2}\right)^2\right]. \end{aligned}$$

Note: From the expression of the correlation coefficient, it is evident that this model will exhibit both positive and negative correlation, depending on the choice of the parameters (a, b_0, b_1, b_2). Thus we seek a bivariate weighted distribution with a positive probability on the unit square $(0, 1)^2$, with marginals are of univariate Kumaraswamy type, and correlation over the full range. The bivariate Kumaraswamy distribution as discussed in Arnold *et al.* [4] does allow correlation to vary over $[-1, 1]$ but it has 5 parameters. In contrary, we propose an alternative (weighted) bivariate Kumaraswamy distribution that has 4 parameters and allows correlations over the full range $[-1, 1]$.

Distribution of the $Z = \min(X, Y)$ and $W = \max(X, Y)$

Suppose, we want to derive the distribution of $Z = \min(X, Y)$ and $W = \max(X, Y)$. Note that, for each $z \in (0 < z < 1)$, we have

$$\begin{aligned} P(Z > z) &= \int_z^1\int_z^y f(x,y)dxdy + \int_z^1\int_z^x f(x,y)dxdy \\ &= C_1^{-1}\left[(1-z^a)^{b_1+b_2} - \frac{b_1+b_2}{b_0+b_1+b_2}(1-z^a)^{b_0+b_1+b_2}\right]. \end{aligned}$$

On differentiating $P(Z \leqslant z) = 1 - P(Z > z)$ w.r.t. z, the density of Z will be

$$f_Z(z) = aC_1^{-1}z^{a-1}(b_1+b_2)\left[(1-z^a)^{b_1+b_2-1} - (1-z^a)^{b_0+b_1+b_2-1}\right] \times I(0 < z < 1).$$

Next, consider the distribution of W. For the distribution of $W = \max(X, Y)$, note that for any $w \in (0, 1)$,

$$\begin{aligned} \overline{F}_W(w) &= P(W > w) \\ &= P(X > w \quad or \quad Y > w) \\ &= \overline{F}_X(w) + \overline{F}_Y(w) - \overline{F}_Z(w). \end{aligned} \tag{12}$$

From (12), the corresponding density will be

$$\begin{aligned} f_W(w) &= f_X(w) + f_Y(w) - f_Z(w) \\ &= aC_1^{-1}w^{a-1}\left[\frac{b_0b_1}{(b_0+b_2)}(1-w^a)^{b_1-1} + \frac{b_0b_2}{(b_0+b_1)}(1-w^a)^{b_2-1}\right. \\ &\quad \left. -(b_1+b_2)(1-w^a)^{b_1+b_2-1} - (1-w^a)^{b_0+b_1+b_2-1}\left(\frac{b_1b_2(2b_0+b_1+b_2)}{(b_0+b_1)(b_0+b_2)}\right)\right] \\ &\quad \times I(0 < w < 1). \end{aligned}$$

Reliability parameter: In this case we have from (6),

$$\begin{aligned} R = P(Y > X) &= \int_0^1\int_x^\infty f(x,y)dydx \\ &= \frac{b_1}{(b_1+b_2)}, \end{aligned} \tag{13}$$

after some algebraic simplification.

3. TYPE II-WEIGHTED BIVARIATE KUMARASWAMY DISTRIBUTION

Let as before, W_1, W_2, and W_0 be independent random variables with density functions $f_{W_i}(w_i)$, $i = 0, 1, 2$. We consider the joint density of (W_1, W_2) given that $W_0 > \max\{W_1, W_2\}$. Now, if we define $(X, Y) \overset{d}{=} (W_1, W_2 | \max(W_1, W_2) < W_0)$, then the density function of the corresponding bivariate weighted distribution is given by

$$\begin{aligned} f_{X,Y}(x,y) &= \frac{f_{W_1}(x) f_{W_2}(y) P\left(\max(W_1, W_2) < W_0 | W_1 = x, W_2 = y\right)}{P(\max(W_1, W_2) < W_0)} \\ &= \frac{f_{W_1}(x) f_{W_2}(y) P\left(W_0 > \max(x,y)\right)}{P(\max(W_1, W_2) < W_0)} \\ &= \frac{f_{W_1}(x) f_{W_2}(y) \overline{F}_{W_0}(\max(w_1, w_2))}{P(\max(W_1, W_2) < W_0)}. \end{aligned} \tag{14}$$

Suppose that we have three independent Kumaraswamy variables. We need to evaluate $P(W_0 > \max\{W_1, W_2\})$. Note that $P(W_0 > \max\{W_1, W_2\}) = P(W_1 < W_2 < W_0) + P(W_2 < W_1 < W_0)$.

Consider

$$P(W_1 < W_2 < W_0)$$

$$\begin{aligned} &= \int_0^1 \left[\int_{w_1}^1 \left[\int_{w_2}^1 a^3 b_0 b_1 b_2 w_1^{a-1} w_2^{a-1} w_0^{a-1} \left(1 - w_1^a\right)^{b_1 - 1} \left(1 - w_2^a\right)^{b_2 - 1} \left(1 - w_0^a\right)^{b_0 - 1} dw_0 \right] dw_2 \right] dw_1 \\ &= \int_0^1 \left[\int_{w_1}^1 a^2 b_1 b_2 w_2^{a-1} \left(1 - w_2^a\right)^{b_0 + b_2 - 1} dw_2 \right] dw_1 \\ &= \frac{a b_1 b_2}{b_0 + b_2} \int_0^1 w_1^{a-1} \left(1 - w_1^a\right)^{b_0 + b_1 + b_2 - 1} dw_1 \\ &= \frac{b_1 b_2}{(b_0 + b_2)(b_0 + b_1 + b_2)}. \end{aligned}$$

Analogously, $P(W_2 < W_1 < W_0) = \frac{b_1 b_2}{(b_0 + b_1)(b_0 + b_1 + b_2)}$, so that $P(W_0 > \max\{W_1, W_2\}) = \frac{b_1 b_2 (2b_0 + b_1 + b_2}{(b_0 + b_2)(b_0 + b_1)(b_0 + b_1 + b_2)}$. For notational simplicity, let us write

$$D = P(W_0 > \max\{W_1, W_2\}) = \frac{b_1 b_2 (2b_0 + b_1 + b_2}{(b_0 + b_2)(b_0 + b_1)(b_0 + b_1 + b_2)}.$$

Therefore,

$$f_{W_0 > \max\{W_1, W_2\}.}(x,y)$$

$$\begin{aligned} &= D^{-1} a^2 b_1 b_2 (xy)^{a-1} (1 - x^a)^{b_1 - 1} \left(1 - y^a\right)^{b_2 - 1} \left(1 - \left(\max(x,y)\right)^a\right)^{b_0} \\ &= \begin{cases} a^2 \left[\dfrac{b_1 b_2 (2b_0 + b_1 + b_2)}{(b_0 + b_2)(b_0 + b_1)(b_0 + b_1 + b_2)} \right]^{-1} w_1^{a-1} w_2^{a-1} w_0^{a-1} \left(1 - w_1^a\right)^{b_1 - 1} \left(1 - w_2^a\right)^{b_0 + b_2 - 1}, & 0 < x < y < 1 \\ a^2 \left[\dfrac{b_1 b_2 (2b_0 + b_1 + b_2)}{(b_0 + b_2)(b_0 + b_1)(b_0 + b_1 + b_2)} \right]^{-1} w_1^{a-1} w_2^{a-1} w_0^{a-1} \left(1 - w_1^a\right)^{b_0 + b_1 - 1} \left(1 - w_2^a\right)^{b_2 - 1}, & 0 < y < x < 1. \end{cases} \end{aligned} \tag{15}$$

In this case the marginals are given by

$$f_X(x) = \left[\frac{(2b_0 + b_1 + b_2}{(b_0 + b_2)(b_0 + b_1)(b_0 + b_1 + b_2)} \right]^{-1} a b_2^{-1} x^{a-1} (1 - x^a)^{b_0 + b_1 - 1} \left[1 - \frac{b_0}{b_0 + b_2} (1 - x^a)^{b_2} \right] I(0 < x < 1). \tag{16}$$

$$f_Y(y) = \left[\frac{(2b_0 + b_1 + b_2}{(b_0 + b_2)(b_0 + b_1)(b_0 + b_1 + b_2)} \right]^{-1} a b_1^{-1} y^{a-1} \left(1 - y^a\right)^{b_0 + b_2 - 1} \left[1 - \frac{b_0}{b_0 + b_1} \left(1 - y^a\right)^{b_1} \right] I(0 < y < 1). \tag{17}$$

Remark 1. From (16) and (17), the marginal densities of Type-II weighted bivariate Kumaraswamy variable are linear combinations of univariate Kumaraswamy densities. In particular, using convenient transparent notation, one may write

$$\begin{aligned} f_X(x) &= \frac{(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)} KW(a, b_0+b_1) - \frac{b_0(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)} KW(a, b_0+b_1+b_2), \\ f_X(x) &= \frac{(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)} KW(a, b_0+b_2) - \frac{b_0(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)} KW(a, b_0+b_1+b_2), \end{aligned}$$

where KW(.,.) stands for the univariate Kumaraswamy distribution.

Distribution of the $Z = \min(X, Y)$ and $W = \max(X, Y)$

In this case, following similar technique as before, the density of Z and W will be, respectively

$$f_Z(z) = a(b_0+b_1+b_2) z^{a-1} (1-z^a)^{(b_0+b_1+b_2)-1} \times I(0<z<1).$$

$$\begin{aligned} f_W(w) &= a\left[\frac{(2b_0+b_1+b_2}{(b_0+b_2)(b_0+b_1)(b_0+b_1+b_2)}\right]^{-1} ab_2^{-1} w^{a-1}\left(b_2^{-1}(1-w^a)^{b_0+b_1-1} + b_1^{-1}(1-w^a)^{b_0+b_2-1}\right) \\ &\quad -aw^{a-1}\frac{b_0(b_0+b_1+b_2)\left(b_0b_2+b_2^2+b_0b_1+b_1^2\right)}{b_1b_2(2b_0+b_1+b_2)}(1-w^a)^{b_0+b_1+b_2-1} \times I(0<w<1). \end{aligned}$$

Note: It is interesting to see here that the distribution of Z is again a Kumaraswamy distribution with parameters a and $b_0+b_1+b_2$.

Now, in this case, for any $r \geqslant 1$, the marginal moments of X and Y are given by, respectively

$$\begin{aligned} E(X^r) &= \frac{(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_1)B\left(b_0+b_1, \frac{r}{a}\right)\right) \\ &\quad -\frac{b_0(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2, \frac{r}{a}\right)\right), \end{aligned}$$

and

$$\begin{aligned} E(Y^r) &= \frac{(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_2)B\left(b_0+b_2, \frac{r}{a}\right)\right) \\ &\quad -\frac{b_0(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2, \frac{r}{a}\right)\right). \end{aligned}$$

The correlation coefficient ρ for this distribution is given by

$$\rho = \frac{M_1}{M_2 M_3},$$

where

$$\begin{aligned} M_1 &= E(XY) - [E(X)E(Y)] \\ &= \sum_{k=0}^{\infty}\binom{1/a+k-1}{k}\frac{1}{b_0+b_1+b_2+k} \\ &\quad -\left[\frac{(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_1)B\left(b_0+b_1, \frac{1}{a}\right)\right) - \frac{b_0(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2, \frac{1}{a}\right)\right)\right] \\ &\quad \times\left[\frac{(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_2)B\left(b_0+b_2, \frac{r}{a}\right)\right) - \frac{b_0(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2, \frac{r}{a}\right)\right)\right], \end{aligned}$$

and

$$\begin{aligned} M_2 &= Var(X) \\ &= \frac{(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_1)B\left(b_0+b_1, \frac{2}{a}\right)\right) - \frac{b_0(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2, \frac{2}{a}\right)\right) \\ &\quad -\left[\frac{(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_2)B\left(b_0+b_2, \frac{1}{a}\right)\right) - \frac{b_0(b_0+b_2)(b_0+b_1+b_2)}{b_2(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2, \frac{1}{a}\right)\right)\right]^2. \end{aligned}$$

Similarly,

$$\begin{aligned} M_3 &= Var(Y) \\ &= \frac{(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_2)B\left(b_0+b_2,\frac{2}{a}\right)\right) - \frac{b_0(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2,\frac{2}{a}\right)\right) \\ &\quad -\left[\frac{(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_2)B\left(b_0+b_2,\frac{1}{a}\right)\right) - \frac{b_0(b_0+b_1)(b_0+b_1+b_2)}{b_1(2b_0+b_1+b_2)}\left((b_0+b_1+b_2)B\left(b_0+b_1+b_2,\frac{1}{a}\right)\right)\right]^2. \end{aligned}$$

Remark 2. We may write the following:

i. Since $\max(W_1, W_2)$ is a TP_2 function, the density corresponding to (15) will also be TP_2. Furthermore, TP_2 is the most rigid dependence property, several other dependency properties will follow immediately. Consequently, we can write the following:
   - X and Y are positive quadrant dependent.
   - $X(Y)$ is a positive regression dependent of $Y(X)$.
   - $X(Y)$ is a left tail decreasing in $Y(X)$.

ii. From the expression of the correlation coefficient, it can be conjectured that, like the Type-I BWK model, Type-II weighted bivariate Kumaraswamy model also allows the correlation coefficient between $[-1, 1]$.

4. COPULA BASED BIVARIATE KUMARASWAMY DISTRIBUTION

In this section we consider two modified versions of the FGM (henceforth, in short) bivariate copula to construct a bivariate Kumaraswamy distribution. We list them as follows:

- We begin by considering a modified version of bivariate FGM copula, given as

$$C(u,v) = uv\left[1+\theta\left((1-u)^{\delta_1}\right)\left((1-v)^{\delta_2}\right)\right], \tag{18}$$

  for $\delta_1, \delta_2 > 0$ and $\theta \in [-1, 1]$. From now on, we call this as modified FGM copula based bivariate Kumaraswamy model (Type I), henceforth in short, Type-III bivariate Kumaraswamy model. Note that (18) is indeed a copula as it satisfies the following:
  - $C(0,0) = 0$; $C(1,1) = 1$.
  - $C(0,1) = 0 = C(1,0)$.
  - For every $u_1 \leqslant u_2$ and $v_1 \leqslant v_2$, $C(u_2,v_2) - C(u_2,v_1) - C(u_1,v_2) + C(u_1,v_1) \geqslant 0$.
    Next, suppose that $X_i \sim K(a_i, b_i)$, for $i = 1, 2$ and they are independent. Then setting $u = F(x_1)$ and $v = F(x_2)$, a bivariate dependent Kumaraswamy model from (18) (hence forth Type-III bivariate Kumaraswamy) can be obtained as (the associated distribution function)

$$H^{Type-III}(x_1,x_2) = \left[1-\left(1-x_1^{a_1}\right)^{b_1}\right]\left[1-\left(1-x_2^{a_2}\right)^{b_2}\right]\left[1+\theta\left(1-x_1^{a_1}\right)^{b_1\delta_1}\left(1-x_2^{a_2}\right)^{b_2\delta_2}\right]. \tag{19}$$

- Another modified version of the FGM copula which can be used to construct a different bivariate Kumaraswamy model is given as follows:

$$C(u,v) = u^{\delta_1}v^{\delta_2}\left[1+\theta\left(\left(1-u^{\delta_1}\right)\right)\left(\left(1-v^{\delta_2}\right)\right)\right], \tag{20}$$

  with $\delta_1, \delta_2 > 0, \quad \theta \in [-1, 1]$. One can easily show that (20) is also a valid copula.Again, we consider two arbitrary independent $X_i \sim K(a_i, b_i)$, for $i = 1, 2$ and they are independent. Then, setting $u = F(x_1)$ and $v = F(x_2)$, a bivariate dependent Kumaraswamy model from (20) (hence forth Type-IV bivariate Kumaraswamy) can be obtained as (the associated distribution function)

$$\begin{aligned} H^{Type-IV}(x_1,x_2) &= \left[1-(1-x^{a_1})^{b_1}\right]^{\delta_1}\left[1-(1-x^{a_2})^{b_2}\right]^{\delta_2} \\ &\quad \left[1+\theta\left(1-\left(1-\left(1-x_1^{a_1}\right)^{b_1}\right)\right)\left(1-\left(1-\left(1-x_2^{a_2}\right)^{b_2}\right)\right)\right]. \end{aligned} \tag{21}$$

4.1. Properties of the Bivariate Copula Based Kumaraswamy Model

- We provide the joint and the conditional copula density function expressions for one of the copula models (Type III) described earlier. For the Kumaraswamy copula (Type III), the corresponding joint copula density

$$\begin{aligned} c(u,v) &= \frac{\partial^2 C(u,v)}{\partial u \partial v} \\ &= \delta_1\delta_2 u^{\delta_1-1}v^{\delta_2-1}\left\{2+\theta\left((1-u)^{\delta_1}\right)\left((1-v)^{\delta_2}\right)+\theta\left((1-u)^{\delta_1-1}\right)\left((1-v)^{\delta_2-1}\right)\right\} \\ &\quad +\delta_2 u^{\delta_1}v^{\delta_2-1}\left\{1-\theta\delta_1\left((1-u)^{\delta_1-1}\right)\left((1-v)^{\delta_2}\right)\right\}+\delta_1 u^{\delta_1-1}v^{\delta_2}\left\{1-\theta\delta_1\left((1-u)^{\delta_1}\right)\left((1-v)^{\delta_2-1}\right)\right\}. \end{aligned}$$

  Also, the conditional copula density of U given $V = v$ is given by

$$c\left(u|v\right) = u^{\delta_1}\left(\delta_2 v^{\delta_2-1}\left\{1+\theta\left((1-u)^{\delta_1}\right)\left((1-v)^{\delta_2}\right)\right\}\right)+\delta_2 u^{\delta_1}v^{\delta_2}\left\{1-\theta\left((1-u)^{\delta_1}\right)\left((1-v)^{\delta_2-1}\right)\right\}.$$

  Similarly, one can get the other conditional copula density function. Following similar logic, one can get the corresponding density and conditional density function(s) for the Type-IV bivariate Kumaraswamy copula model.

- **Dependence structure** Note that the selection of a particular copula function, indeed, depends on a number of factors, among which the dependence parameter is of primary importance. Furthermore, it is obvious from (19) and (21) that the resultant bivariate distributions reduce to the case of independence, when the parameter $\theta = 0$. These singular characteristics make these copulas particularly interesting for empirical analysis, as it is straightforward to compare estimates of the parameters of $F\left(x,y\right)$ and $F_1\left(x\right)$ and $F_2\left(y\right)$ separately. To study the nature of dependence, we consider the following two measures listed below:

  1. Kendall's τ: Let X and Y be continuous random variables with copula C. Then Kendall's τ is given by

$$\tau(X,Y) = 4\iint_{[0,1]^2} C(u,v)\,dC(u,v) - 1 = 4\iint_{[0,1]^2} c(u,v)\,dudv - 1,$$

     where $c(u,v)$ is the corresponding copula density.

  2. Spearman's ρ: Let X and Y be continuous random variables with copula C. Then Spearman's ρ_s is given by

$$\rho_s = 12\iint_{[0,1]^2} uvdC(u,v) - 3.$$

Based on the above, we can write the following:

- For the bivariate Kumaraswamy (Type-III) copula

  1. Spearman's correlation coefficient ρ will be (assuming δ_1 and δ_2 are integers)

$$\rho = 12\left[\frac{1}{\delta_1\delta_2}+\theta B(\delta_1+1,\delta_1+1)\,B(\delta_2+1,\delta_2+1)\right]-3.$$

  2. Kendall's τ will be (assuming δ_1 and δ_2 are integers)

$$\begin{aligned} \tau &= 4\delta_1\delta_2\left\{\frac{2}{\delta_2+1}B(\delta_1+1,\delta_1)+3\theta B(\delta_2+1,\delta_2+1)\,B(\delta_1,\delta_1+1)\right\} \\ &\quad +4\delta_1\delta_2\theta^2\left\{B(\delta_2+1,2\delta_2+1)\,B(\delta_1,2\delta_1+1)+\theta^2 B(\delta_2+1,2\delta_2)\,B(\delta_1,2\delta_1)-1\right\}. \end{aligned}$$

- For the bivariate Kumaraswamy (Type-IV) copula

  1. Spearman's correlation coefficient ρ will be (assuming δ_1 and δ_2 are integers)

$$\rho = 12\left[\frac{1}{(\delta_1+1)(\delta_2+1)}+\theta B\left(2,\frac{1}{\delta_2+1}\right)B\left(1,\frac{1}{\delta_1+1}\right)\right]-3.$$

  2. Kendall's τ will be (assuming δ_1 and δ_2 are integers)

$$\tau = 4\left\{\frac{1}{\delta_1\delta_2}+\frac{\theta}{4}B\left(2,2+\frac{1}{\delta_1}+2\right)-B\left(1,\frac{1}{\delta_1}+1\right)\right\}-1.$$

Next, we discuss the upper tail dependence and lower tail dependence property for these bivariate Kumaraswamy type copula models.

- Tail dependence property: The upper tail dependence coefficient (parameter) λ_U is the limit (if it exists) of the conditional probability that Y is greater than 100α th percentile of G given that X is greater than the 100α th percentile of F as α approaches 1, $\lambda_U = \lim \alpha \uparrow 1 P\left(Y > G^{-1}(\alpha) | X > F^{-1}(\alpha)\right)$. If $\lambda_U > 0$, then X and Y are upper tail dependent and asymptotically independent otherwise. Similarly, the lower tail dependence coefficient is defined as $\lambda_L = \lim_{\alpha\downarrow 0} P\left(Y \leqslant G^{-1}(\alpha) | X \geqslant F^{-1}(\alpha)\right)$. Let, C be the copula of X and Y. Then, equivalently we can write $\lambda_L = \lim_{u\downarrow 0} \frac{C(u,u)}{u}$ and $\lambda_U = \lim_{u\downarrow 0} \frac{\widetilde{C}(u,u)}{u}$, where $\widetilde{C}(u,u)$ is the corresponding survival copula given by $\widetilde{C}(u,u) = 1 - 2u + C(u,u)$.
  For the bivariate Kumaraswamy (Type-III) copula, it is straightforward to see that $\lambda_L = 0$, which implies that X and Y are asymptotically independent. Again, we have $\lambda_U = 0$, thereby implying that X and Y are asymptotically dependent. In a similar way, one can establish these properties for the Type-IV bivariate Kumaraswamy type copula model.
- Left-Tail decreasing property and Right-Tail increasing property
  Nelson [21] showed that $X(Y)$ is left tail decreasing, that is, $LTD\left(Y|X\right)$ and $LTD\left(X|Y\right)$ if and only if for all u, u', v, v' such that $0 < u \leqslant u' \leqslant 1$ and $0 < v \leqslant v' \leqslant 1$, if $\frac{C(u,v)}{uv} \geqslant \frac{C(u',v')}{u'v'}$. Next, we have the following theorem:

Theorem 3. The bivariate (Type-III) Kumaraswamy type copula in (18) has $LTD\left(Y|X\right)$ and $LTD\left(X|Y\right)$ if and only if $\theta \in [0,1]$ and for integer valued δ_1 and δ_2.

Proof. Since, $0 < u \leqslant u' \leqslant 1$ and $0 < v \leqslant v' \leqslant 1$, we may write

$$(1-u)^{\delta_1}(1-v)^{\delta_2} \geqslant (1-u')^{\delta_1}(1-v')^{\delta_2},$$

for any $\delta_1 > 0$, $\delta_2 > 0$ and both are integer valued. Next, for $\theta \in [0,1]$, we can write

$$1 + \theta(1-u)^{\delta_1}(1-v)^{\delta_2} \geqslant 1 + \theta(1-u')^{\delta_1}(1-v')^{\delta_2}.$$

Hence, $\frac{C(u,v)}{uv} = \left(1 + \theta(1-u)^{\delta_1}(1-v)^{\delta_2}\right) \geqslant \left(1 + \theta(1-u')^{\delta_1}(1-v')^{\delta_2}\right)$. This immediately implies the result.

Note that if $\theta \in [-1,0]$, then the above tail dependence property will not hold. Almost identical argument will lead us to the fact that Left-Tail decreasing property and Right-Tail increasing property will also hold for bivariate (Type-IV) Kumaraswamy type copula.

Remark 3.

- When δ_1 and δ_2 are not integers, the expressions for Kendalls τ and Spearmans ρ will involve infinite sums, but, still it will be in a closed form.
- Since, in general, any convex combination of two (or more) is again a copula, one might consider another bivariate Kumaraswamy type copula (say, Type-V) with the following structure: $C^{Type-V}(u,v) = \beta C^{Type-III}(u,v) + (1-\beta) C^{Type-II}(u,v)$, for suitable $\beta \in (0,1)$. For a detailed study on the Arnold–Ng type bivariate copula based Kumaraswamy distribution construction and other associated bivariate copula models, see,Arnold and Ghosh [12].

5. MULTIVARIATE WEIGHTED KUMARASWAMY DISTRIBUTION

Following Arnold *et al.* [13], we consider the model in which in which $T_1, T_2, ..., T_j$ are i.i.d. random variables with distribution and density functions G_0 and g_0; $X_1, X_2, ..., X_k$ are i.i.d. random variables with distribution and density functions F_0 and f_0 and $U_1, U_2, ..., U_\ell$ are i.i.d. random variables with distribution and density functions H_0 and h_0. In this case we have

$$f(x_1, x_2, ..., x_k) \propto \left[\prod_{\beta=1}^{k} f_0\left(x_\beta\right)\right] \left[G_0\left(x_{1:k}\right)\right]^j \left[1 - H_0\left(x_{k:k}\right)\right]^\ell. \tag{22}$$

When the three distributions in (22) are of the Kumaraswamy form, it reduces to

$$f(x_1, x_2, ..., x_k) \propto a^k b^k \prod_{i=1}^{k} x_i^{a-1}\left(1 - x_i^a\right)^{b-1}\left[1 - \left(1 - x_{1:k}^a\right)^{b_0}\right]^j \left(1 - x_{k:k}^a\right)^{\ell b_1}. \tag{23}$$

To identify the required normalizing constant we must evaluate

$$\begin{aligned}&\int_0^1\int_0^1 \cdots \int_0^1 a^k b^k \prod_{i=1}^{k} x_i^{a-1}\left(1 - x_i^a\right)^{b-1}\left[1 - \left(1 - x_{1:k}^a\right)^{b_0}\right]^j \left(1 - x_{k:k}^a\right)^{\ell b_1} dx_1 \cdots dx_k \\ &= \sum_{m=0}^{j} \binom{j}{m} (-1)^m E\left\{\left(1 + X_{1:k}^a\right)^{m b_1}\left(1 + X_{k:k}^a\right)^{-\ell b_0}\right\},\end{aligned} \tag{24}$$

where the X_i's have a K (a, b) distribution. Next, the joint distribution of $X_{1:k}$ and $X_{k:k}$ for a random sample of size k will be

$$f(x_{1:k}, x_{k:k}) = k(k-1)\, a^2 b_0 b_1 x_{1:k}^a x_{k:k}^a \left(1 - x_{1:k}^a\right)^{b_0} \left(1 - x_{k:k}^a\right)^{b_1} I(0 < x_{1:k} < x_{k:k} < 1). \tag{25}$$

From (24) and on using (25), the normalizing constant is

$$C = \sum_{m=0}^{j} \binom{j}{m} (-1)^m \frac{k(k-1)\, b_0}{((b_0 + mb_1 + 1)(b_0(1+\ell)) + b_1(1+m) + b_0 + 2)}.$$

Hence, the k-variate joint density function in (25) can be written as

$$\begin{aligned} f(x_1, x_2, ..., x_k) \;=\; & C^{-1} a^k b^k \prod_{i=1}^{k} x_i^{a-1} (1 - x_i^a)^{b-1} \left[1 - (1 - x_{1:k}^a)^{b_0}\right]^j (1 - x_{k:k}^a)^{\ell b_1} \\ & \times I(0 < (x_1, x_2, ..., x_k) < 1). \end{aligned}$$

Another multivariate extension of the BWK model can be proposed using (3) as follows

$$f(x_1, x_2, ..., x_k) = D^{-1} \sum_{i=0}^{k} \prod_{j=1}^{k} a^k b_j \left[1 - \left(1 - x_{1:k}^a\right)^{b_0}\right] I\left(\underline{0} < \underline{x} < \underline{1}\right),$$

where $D = \dfrac{b_0}{\sum_{j=0}^{k} b_i}$.

6. MAXIMUM LIKELIHOOD ESTIMATION

In this section, we consider the estimation of the model parameters of the BWK distribution.

Suppose we have n observations from the bivariate density in (6). The log-likelihood is given by

$$\begin{aligned} \ell(a, b_0, b_1, b_2) \;=\; & -n \log\left[\frac{b_0}{b_0 + b_1 + b_2}\right] + 2n \log a + 2n\left[\log b_1 + \log b_2\right] \\ & + (a-1) \sum_{i=1}^{n} [\log x_i + \log y_i] \\ & + (b_1 - 1) \sum_{i=1}^{n} \log(1 - x_i^a) + (b_2 - 1) \sum_{i=1}^{n} \log(1 - y_i^a) \\ & + \sum_{i=1}^{n} \log\left(1 - \left(1 - (\min(x_i, y_i))^a\right)^{b_0}\right). \end{aligned} \tag{26}$$

The corresponding likelihood equations are

$$\begin{aligned} \frac{\partial \ell}{\partial b_0} \;=\; & \sum_{i=1}^{n} - \frac{\left(1 - \left(1 - \min(x_i, y_i)\right)^a\right)^{b_0} \log\left(1 - \left(1 - \min(x_i, y_i)\right)^a\right)}{1 - \left(1 - \left(1 - \min(x_i, y_i)\right)^a\right)^{b_0}} \\ & - \frac{(b_0 + b_1 + b_2)\left(\frac{1}{b_0 + b_1 + b_2} - \frac{b_0}{(b_0 + b_1 + b_2)^2}\right) n}{b_0}. \end{aligned} \tag{27}$$

$$\frac{\partial \ell}{\partial b_1} = \sum_{i=1}^{n} \log\left(1 - x_i^a\right) + \frac{2n}{b_1} + \frac{n}{b_0 + b_1 + b_2}. \tag{28}$$

$$\frac{\partial \ell}{\partial b_2} = \sum_{i=1}^{n} \log\left(1 - y_i^a\right) + \frac{2n}{b_2} + \frac{n}{b_0 + b_1 + b_2}. \tag{29}$$

$$\frac{\partial \ell}{\partial a} = \sum_{i=1}^{n} \frac{b_0\left(1-\min(x_i,y_i)\right)^a \log\left(1-\min(x_i,y_i)\right)\left(1-\left(1-\min(x_i,y_i)\right)^a\right)^{b_0-1}}{1-\left(1-\left(1-\min(x_i,y_i)\right)^a\right)^{b_0}} + \frac{2n}{a}$$
$$+ (b_1-1)\sum_{i=1}^{n} -\frac{x_i^a \log(x_i)}{1-x_i^a} + (b_2-1)\sum_{i=1}^{n} -\frac{y_i^a \log(y_i)}{1-y_i^a} + \sum_{i=1}^{n}\log(x_i) + \sum_{i=1}^{n}\log(y_i). \tag{30}$$

Setting (27–30) to 0 and solving these likelihood equations simultaneously, we get the maximum likelihood estimates (MLEs) for b_0, b_1, b_2, and a.

7. APPLICATION

In this section we consider two applications of the proposed BWK distribution based two data sets:

- Data Set I: Earthquakes become major societal risks when they strike on vulnerable populations. We consider the data is obtained from Ozel [18]. Due to the fact that a significant portion of Turkey is subject to frequent earthquakes, destructive mainshocks and their foreshock and aftershock sequences between the longitudes $(39-42°N)$ and latitudes $(26-45°E)$ are investigated. In this particular region, 111 mainshocks with surface magnitude (M_s) of five or more have occurred in the past 106 years. We define the following random variables: X represents the magnitude of foreshocks and Yrepresents the magnitude of the aftershocks. We fit the data to the following bivariate Kumaraswamy models:
- Data Set II: The data on 37 patients were available regarding the hemoglobin content in blood being prone to type II diabetes from a Private Clinic in Tennessee. To see the effect of reducing hemoglobin content in the blood a special type of treatment was administered to those patients. We define the following: X as a random variable which represents the proportion of hemoglobin content in the blood before the treatment, Y as a random variable which represents the proportion of hemoglobin content in the blood after the treatment.

1. Model I: BWK distribution ([6]).
2. Model II: Bivariate Kumaraswamy (absolutely continuous distribution (Wagner *et al.* [5], [5]).
3. Model III: Bivariate Kumaraswamy distribution via conditional specification (Arnold and Ghosh [12]), given by

$$f(x,y) = C\alpha_1\alpha_2 x^{\alpha_1-1}y^{\alpha_2-1}\left(1-x^{\alpha_1}\right)^{\beta_1-1}\left(1-y^{\alpha_2}\right)^{\beta_2-1}\exp\left(\beta_3\log\left(1-x^{\alpha_1}\right)\log\left(1-x^{\alpha_2}\right)\right)$$
$$\times I\left(0<x<1, 0<y<1\right),$$

where C is an appropriate normalizing constant.

4. Model IV: Bivariate Kumaraswamy distribution via conditional survival specification (Arnold and Ghosh [4]), given by

$$f(x,y) = \alpha_1\alpha_2 x_1^{\alpha_1-1}y^{\alpha_2-1}\left(1-x^{\alpha_1}\right)^{\beta_1-1}\left(1-y^{\alpha_2}\right)^{\beta_2-1}\exp\beta_3\log\left(1-x_1^{\alpha_1}\right)\log\left(1-y^{\alpha_2}\right)$$
$$\times\left(\beta_1\beta_2+\beta_2\beta_3\log\left(1-x_1^{\alpha_1}\right)+\beta_3+\beta_3^2\log\left(1-x^{\alpha_1}\right)\log\left(1-y^{\alpha_2}\right)+\beta_1\beta_3\log\left(1-y^{\alpha_2}\right)\right)I\left(0<x_1,x_2<1\right).$$

5. Model V: Nadarajah [19] bivariate F_3-beta distribution, given by

$$f(x,y) = \frac{Cx^{\beta-1}y^{\delta-1}\left(1-x-y\right)^{\gamma-\beta-\delta-1}}{(1-ux)^{\theta_1}\left(1-vy\right)^{\theta_2}},$$

for $0<x<1, 0<y<1, 0<x+y<1, -1<u<1, -1<v<1, (\beta,\delta,\theta_1,\theta_2)>0$ and $\gamma>\beta+\delta$, and C is the normalizing constant.

6. Model VI: Nadarajah [3] bivariate generalized beta distribution given by

$$f(x,y) = \frac{Cx^{\alpha-1}y^{\beta-1}(1-x)^{\gamma-\alpha-1}\left(1-y\right)^{\gamma-\beta-1}}{\left(1-xy\delta\right)^{\gamma}},$$

for $0<x<1, 0<y<1, 0<x+y<1, -1<u<1, -1<v<1, (\beta,\delta,\theta_1,\theta_2)>0$ and $\gamma>\beta+\delta$, and C is the normalizing constant.

7. Model VII: Olkin and Trikalinos [20] bivariate beta distribution given by

$$f(x,y) = \frac{x^{\alpha_1-1}y^{\alpha_2-1}(1-x)^{\alpha_0+\alpha_2-1}\left(1-y\right)^{\alpha_0+\alpha_1-1}}{\left(1-xy\right)^{\alpha_0+\alpha_1+\alpha_2}}.$$

Table 1 | Parameter estimates for Data Set I.

Model	Model I	Model II	Model III	Model IV
Parameter estimates	$\hat{a} = 2.4956\,(0.8010)$ $\widehat{b_1} = 1.4891\,(0.1207)$ $\widehat{b_2} = 0.3165\,(0.0842)$ $\widehat{b_0} = 0.7849\,(0.0229)$	$\widehat{\alpha_1} = 3.5287\,(0.3335)$ $\widehat{\alpha_2} = 1.1845\,(0.9723)$ $\widehat{\alpha_3} = 3.2424\,(0.1065)$ $\hat{\beta} = 1.923\,(0.9537)$	$\widehat{\alpha_1} = 2.4571\,(0.4620)$ $\widehat{\alpha_2} = 2.5183\,(0.5781)$ $\widehat{\beta_1} = 1.472\,(0.1368)$ $\widehat{\beta_2} = 1.1883\,(0.0442)$ $\widehat{\beta_3} = 1.6543\,(0.0279)$	$\widehat{\alpha_1} = 1.1892\,(0.6043)$ $\widehat{\alpha_2} = 2.0367\,(0.1472)$ $\widehat{\beta_1} = 4.352\,(0.4956)$ $\widehat{\beta_2} = 2.1769\,(0.0154)$ $\widehat{\beta_3} = 2.538\,(0.4015)$
Log-likelihood	−162.34	−116.58	−113.36	−113.25
χ^2 goodness p-value	0.9983	0.7146	0.8345	0.9978

Table 2 | Parameter estimates for Data Set II.

Model	Model I	Model II	Model III	Model IV
Parameter estimates	$\hat{a} = 1.2781\,(0.3533)$ $\widehat{b_1} = 1.1139\,(0.0737)$ $\widehat{b_2} = 1.1464\,(0.2102)$ $\widehat{b_0} = 0.8735\,(0.1718)$	$\widehat{\alpha_1} = 2.1258\,(0.4015)$ $\widehat{\alpha_2} = 3.4753\,(0.1216)$ $\widehat{\alpha_3} = 2.3832\,(0.3528)$ $\hat{\beta} = 1.2913\,(0.4583)$	$\widehat{\alpha_1} = 0.547\,(0.6436)$ $\widehat{\alpha_2} = 1.6782\,(0.1635)$ $\widehat{\beta_1} = 0.3587\,(0.1822)$ $\widehat{\beta_2} = 0.7117\,(0.3453)$ $\widehat{\beta_3} = 0.3257\,(0.5726)$	$\widehat{\alpha_1} = 0.723\,((0.5145))$ $\widehat{\alpha_2} = 1.7894\,(0.2051)$ $\widehat{\beta_1} = 0.2291\,(0.0567)$ $\widehat{\beta_2} = 0.5673\,(0.1644)$ $\widehat{\beta_3} = 0.4882\,(0.8293)$
Log-likelihood	−162.34	−116.58	−113.36	−113.25
$\chi^2 p$-value	0.9532	0.8140	0.9123	0.9203

Table 3 | Parameter estimates for Data Set I.

Model	Model V	Model VI	Model VII
Parameter estimates	$\hat{\beta} = 2.427\,(0.6518)$ $\hat{\delta} = 0.980\,(2.490)$ $\hat{\gamma} = 1.0892\,(4.5253)$ $\widehat{\theta_1} = 2.6843\,(0.6439)$ $\widehat{\theta_2} = 0.8735\,(0.1718)$	$\hat{\alpha} = 1.485\,(0.6672)$ $\hat{\beta} = 1.973\,(0.3294)$ $\hat{\gamma} = 7.528\,(0.9897)$ $\hat{\delta} = 2.735\,(0.9892)$	$\widehat{\alpha_0} = 4.016\,(0.6436)$ $\widehat{\alpha_1} = 3.7649\,(2.1873)$ $\widehat{\alpha_2} = 6.172\,(0.5837)$
Log-likelihood	−245.68	−237.18	−186.79
$\chi^2 p$-value	0.5584	0.6283	0.7246

Table 4 | Parameter estimates for Data Set II.

Model	Model V	Model VI	Model VII
Parameter estimates	$\hat{\beta} = 2.893\,(0.753)$ $\hat{\delta} = 1.653\,(1.046)$ $\hat{\gamma} = 3.117\,(3.8931)$ $\widehat{\theta_1} = 1.542\,(0.8568)$ $\widehat{\theta_2} = 0.489\,(0.3294)$	$\hat{\alpha} = 0.891\,(1.248)$ $\hat{\beta} = 1.172\,(1.682)$ $\hat{\gamma} = 6.418\,(1.0583)$ $\hat{\delta} = 1.389\,(2.4015)$	$\widehat{\alpha_0} = 3.184\,(0.776)$ $\widehat{\alpha_1} = 2.457\,(1.2837)$ $\widehat{\alpha_2} = 3.271\,(0.4819)$
Log-likelihood	−262.17	−206.44	−135.43
$\chi^2 p$-value	0.3518	0.3849	0.6722

To check the goodness of fit of all statistical models, several other goodness-of-fit statistics are used and are computed using computational package Mathematica. The MLEs are computed using Nmaximize technique as well as the measures of goodness-of-fit statistics including the log-likelihood function evaluated at the MLEs. Parameter estimates along with several goodness of fit measures are provided in Tables 1 and 3 and in Tables 2 and 4 for the data sets *I* and *II*, respectively.

8. CONCLUSION

In recent times the construction of bivariate and multivariate Kumaraswamy distributions has received a significant amount of attention. While most of the other works focuses primarily on investigating structural properties of the proposed model, in our present work, we try

to provide more emphasize on the application side without undermining the need to discuss structural properties of the developed bivariate and multivariate Kumaraswamy distributions. In this article, we propose a 4 parameter bivariate (weighted) Kumaraswamy distribution, which allows the correlation coefficient to vary over the full range $[-1, 1]$ and it's an improved model in comparison with other bivariate Kumaraswamy type distributions (such as those studied and discussed in Arnold *et al*. [14], [15] with 5 parameters), since we have one parameter less. Thus it merits a separate study. In conclusion, the BWK distributions provides a rather flexible mechanism for fitting a wide spectrum of positive real world data. Additionally, one can easily imagine situations in which observations are made only if the maximum of k variables is less than one particular variable. In such a scenario, within the framework of a multivariate Kumaraswamy joint distribution for the $(k+1)$-dimensional data, efforts will be made. Indeed, one could begin with any one of the many dependent multivariate Kumaraswamy models available in the literature. A separate report on such models will be prepared. However, unless k is small, these models necessarily involve a considerable number of parameters, which can be expected to invite difficulties in estimation in practical settings where sample sizes cannot be expected to be enormous. Furthermore one can envision a semi-parametric model in which $(X, Y) = (h_1(X^*), h_2(Y^*))$, where (X^*, Y^*) has a BWK distribution and h_1 and h_2 are unknown functions to be estimated from the data. Alternatively the functions h_1 and h_2 might be assumed to belong to specific parametric families of functions. The enhanced flexibility of such augmented models may prove to be useful in many applications.

ACKNOWLEDGMENTS

The author is grateful for some invaluable comments made by two anonymous referees which has greatly helped to improve on an earlier version of this manuscript. The author has no conflict of interest. Furthermore, this research received no specific grant from any funding agency in the public, commercial, or not-for-profit sector.

REFERENCES

1. N. Balakrishnan, C.D. Lai, Continuous Bivariate Distributions, second ed., Springer-Verlag, New York, 2009.
2. P. Kumaraswamy, J. Hydrol. 46 (1980), 79–88.
3. S. Nadarajah, METRON - Int. J. Stat. LXV (2007), 153–174.
4. B.C. Arnold, I. Ghosh, Commun. Stat. Theory Methods. 46 (2017), 9335–9354.
5. B.S. Wagner, A.J. Lemonte, Statistics. 47 (2013), 1321–1342.
6. A.K. Nanda, K. Jain, J. Stat. Plan. Inference. 77 (1999), 169–180.
7. C. Gupta, P. Keating, Scand. J. Stat. 13 (1985), 49–56.
8. O. Oluyede, Probab. Eng. Inform. Sci. 13 (1999), 169–185.
9. M. Zelen, M. Feinleib, Biometrika. 56 (1969), 601–614.
10. B.C. Arnold, I. Ghosh, A. Alzaatreh, Commun. Stat. Theory Methods. 46 (2017), 8897–8912.
11. D.K. Al-Mutairi, M.E. Ghitany, D. Kundu, Stat. Pap. 52 (2011), 921–936.
12. B.C. Arnold, I. Ghosh, J. Appl. Stat. Sci. 22 (2016), 37–60.
13. B.C. Arnold, I. Ghosh, RevStat. 15 (2016), 223–250.
14. B.C. Arnold, H.N. Nagarja, Commun. Stat. Theory Methods. 20 (1991), 1853–1860.
15. C.R. Rao, Sankhyā: Indian J. Stat. Ser. A. 27 (1965), 311–324.
16. G.P. Patil, C.R. Rao, in: P.R. Krishnaiah, (Ed.), Applications in Statistics, North-Holland, Amsterdam, 1977, pp. 383–405.
17. M. Mahfoud, G.P. Patil, in: G. Kallianpur *et al.* (eds.), Statistics and Probability Essays in Honor of C.R. Rao, North-Holland, Amsterdam, 1982, pp. 479–492.
18. G. Ozel, in AIP Conference Proceedings 1648, 2015.
19. S. Nadarajah, Commun. Korean Math. Soc. 21 (2006), 363–374.
20. I. Olkin, A.T. Trikalinos, Stat. Probab. Lett. 96 (2015), 54–60.
21. R.B. Nelsen, An Introduction to Copulas, Springer-Verlag, New York, 1999.