The Generalized Transmuted Poisson-G Family of Distributions: Theory, Characterizations and Applications

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Abstract

In this work, we introduce a new class of continuous distributions called the generalized Poisson family which extends the quadratic rank transmutation map. We provide some special models for the new family. Some of its mathematical properties including Rényi and q-entropies, order statistics and characterizations are derived. The estimations of the model parameters are performed by maximum likelihood method. The Monte Carlo simulation is used for assessing the performance of the maximum likelihood estimators. The flexibility of the proposed family is illustrated by means of two applications to real data sets.

Keywords: Entropy; Generating function, Maximum likelihood estimation, Order statistics, Characterizations.

1. Introduction

In many practical situations, classical distributions do not provide adequate fit to real data. For example, if the data are asymmetric, the normal distribution will not be a good choice. So, several generators based on one or more parameters have been proposed to generate new distributions. Some well-known generators are Marshal-Olkin generated family by Marshal and Olkin (1997), the beta-G by Eugene et al. (2002), Jones (2004), Kumaraswamy-G by Cordeiro and de Castro (2011), McDonald-G by Alexander et al. (2012), gamma-G (type 1) by Zografos and Balakrishanan (2009), gamma-G (type 2) by Ristić and Balakrishanan (2012), gamma-G (type 3) by Torabi and Montazari (2012), log-gamma-G by Amini et al. (2012), logistic-G by Torabi and Montazari (2012), exponentiated generalized-G by Cordeiro et al. (2013), Transformed-Transformer (T-X) by Alzaatreh et al. (2013), exponentiated (T-X) by Alzaghal et al. (2013), Weibull-G by Bourguignon et al. (2014), Exponentiated half logistic generated family by Cordeiro et al. (2014), Kumaraswamy odd log-logistic by Alizadeh et al. (2015), Lomax Generator by Cordeiro et al. (2014), Kumaraswamy Marshal-Olkin family by Alizadeh et al. (2014), generalized transmuted-G family by Nofal et al. (2015) and transmuted exponentiated generalized-G family by Yousof et al. (2016), another generalized transmuted-G by Merovci et al. (2015), transmuted geometric-G by Afify et al. (2016a), Kumaraswamy transmuted-G by Afify et al. (2016b), beta transmuted-H by Afify et al. (2016c), Zografos-Balakrishnan odd log-logistic by Cordeiro et al. (2016b), type I half-logistic-G by Cordeiro et al. (2016a), Burr X-G by Yousof et al. (2016) and odd-Burr generalized-G families by Alizadeh et al. (2016), a new generalized two-sided class of distributions by Korkmaz and Genç (2017), Marshall-Olkin generalized-G family of distributions by Yousof et al. (2018a), new extended G famil by Hamedani et al. (2018), Burr-Hatke-G by Yousof et al. (2018b), generalized odd Weibull generated family by Korkmaz et al. (2018a), the Marshall-Olkin generalized G Poisson family by Korkmaz et al. (2018c), extended odd Frechet family by Yousof et al. (2018c), new Weibull class of distributions by exponential Lindley Odd Log-Logistic-G Family by Korkmaz et al. (2018b), the extended Weibull-G family by Yousof et al. (2018b), Korkmaz (2019), Type II general exponential class of distributions by Hamedani et al. (2019) and Weibull Marshall-Olkin family Korkmaz et al. (2019), among others.

Consider the probability density function (pdf) $p(t)$ of a random variable $T \in [a,b]$ for $-\infty < a < b < \infty$ and consider a function of the cumulative distribution function (cdf) of a random variable X, $W[G(x)]$, where $W[G(x)]$ satisfies the following conditions:

$$\begin{cases} (i) \quad W[G(x)] \in [a,b], \\ (ii) \quad W[G(x)] \text{ is differentiable and monotonically non-decreasing, and} \\ (iii) \quad W[G(x)] \to a \text{ as } x \to -\infty \text{ and } W[G(x)] \to b \text{ as } x \to \infty. \end{cases} \tag{1}$$

Recently, Alzaatreh et al. (2013) defined the *T-X family* of distributions by

$$F(x) = \int_a^{W[G(x)]} p(t)\, dt, \tag{2}$$

where $W[G(x)]$ satisfies the conditions. The pdf corresponding to is given by

$$f(x) = \{dW[G(x)]/dx\}\; p\{W[G(x)]\}. \tag{3}$$

Further details about the *T-X family* were explored by Alzaatreh et al. (2013). Based on complementary power series distribution, first we define the transmuted complementary Poisson (TCP) distribution with cdf and pdf given, respectively, by

$$F_{TCP}^{(\theta,\beta,\lambda)}(x) = \left[e^{\theta\left(1-e^{-\beta x}\right)} - 1\right]\left[1 + \lambda \frac{e^{\theta} - e^{\theta\left(1-e^{-\beta x}\right)}}{e^{\theta}-1}\right] / \left(e^{\theta}-1\right)$$

and

$$f_{TCP}^{(\theta,\beta,\lambda)}(x)=\theta\beta\, e^{-\beta x}e^{\theta\left(1-e^{-\beta x}\right)}\left[1-\lambda+2\lambda\frac{e^{\theta}-e^{\theta\left(1-e^{-\beta x}\right)}}{e^{\theta}-1}\right]/\left(e^{\theta}-1\right),$$

where $\theta,\beta>0$ and $|\lambda|\leq 1$. For $W[G(x)]=-\log\left[1-G(x;\phi)\right]$ and $p(t)$ the pdf of transmuted complementary exponential Poisson with scale equal one, we define the cdf of *the new* generalized transmuted Poisson family (GTP-G for short) of distributions by

$$F_{GTP-G}^{(\lambda,\theta,\phi)}(x)=\int_{0}^{-\log[1-G(x;\phi)]}\frac{\theta e^{-t}e^{\theta\left[1-e^{-t}\right]}}{e^{\theta}-1}\left[1-\lambda+2\lambda\frac{e^{\theta}-e^{\theta\left[1-e^{-t}\right]}}{e^{\theta}-1}\right]dt$$

where $G(x;\phi)$ is the baseline cdf depending on a parameter vector ϕ and $\theta>0$ and $|\lambda|\leq 1$ are two additional shape parameters. GTP-G is a wider class of continuous distributions. It includes *the* transmuted family of distributions when $\theta\to 0$.

The paper is organized as follows. In Section 2, we define the GTP-G family, describe the shape of the pdf and hazard rate function (hrf) analytically, derive a useful mixture representation for its pdf and present two special models and plots of their pdf's and hrf's. In Section 3, we derive some of its general mathematical properties including Rényi and q-entropies, order statistics and some useful characterizations. Maximum likelihood estimations of the model parameters are addressed in Section 4. In section 5, simulation results to assess the performance of the proposed maximum likelihood estimation procedure are discussed and two applications to real data to illustrate the importance and flexibility of the new family are provided. Finally, some concluding remarks are presented in Section 6.

2. The new family and its motivation

2.1 Genesis

The cdf of the GTP-G family is now defined by

$$F_{GTP-G}^{(\lambda,\theta,\phi)}(x)=F\left(x;\lambda,\theta,\phi\right)=\left[e^{\theta G(x;\phi)}-1\right]\left[1+\lambda-\lambda\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\right]/\left(e^{\theta}-1\right). \tag{4}$$

The pdf corresponding to (cdfgt) is given by

$$f_{GTP-G}^{(\lambda,\theta,\phi)}(x)=f\left(x,\lambda,\theta,\phi\right)=\theta e^{\theta G(x;\phi)}g(x;\phi)\left[1+\lambda-2\lambda\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\right]/\left(e^{\theta}-1\right). \tag{5}$$

The reliability function (rf) and hrf of X are, respectively, given by

$$R_{GTP-G}^{(\lambda,\theta,\phi)}(x)=R(x;\lambda,\theta,\phi)=1-\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\left[1+\lambda-\lambda\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\right],$$

and

$$\tau_{GTP-G}^{(\lambda,\theta,\phi)}(x)=\tau(x;\lambda,\theta,\phi)=\frac{\theta e^{-\theta\bar{G}(x;\phi)}g(x;\phi)\left[1+\lambda-2\lambda\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\right]}{1-\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\left[1+\lambda-\lambda\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\right]}.$$

We are motivated to introduce the GTP-G family because it exhibits the increasing, the decreasing and the upside-down hazard rates as illustrated in Figures 1 and 2. It is shown in Subsection 2.2 that the GTP-G family can be viewed as a mixture representation of the exponentiated G (Exp-G) densities. The new family can also be viewed as a suitable model for fitting the symmetric, the right-skewed, and bimodal data (see Subsection 5.2). The peneralized transmuted Poisson Lindley is much better than the Kumaraswamy Lindley, beta Lindley, Lindley-Poisson, power Lindley, Transmuted Lindley and Lindley models in modeling the relief times data as well as the generalized transmuted Poisson Weibull is much better than the beta transmuted Weibull, transmuted exponentiated generalized Weibull, Kumaraswamy Weibull, McDonald Weibull, transmuted modified Weibull, beta Weibull and Weibull models in modeling the nicotine data.

2.2 Special models

In this section, we provide some examples of the GTP-G family. The pdf (pdfgt) will be most tractable when $G(x) = G(x;\phi)$ and $g(x) = g(x;\phi)$ have simple analytic expressions. These special models generalize some well-known distributions reported in the literature. Here, we provide two special models of this family corresponding to the baseline Weibull (W) and Lindley (Li) distributions to show the flexibility of the new family.

The GTPW distribution

Consider the cdf and pdf (for $x > 0$) $G(x) = 1 - e^{-(\alpha x)^{\beta}}$ and $g(x) = \beta\alpha^{\beta} x^{\beta-1} e^{-(\alpha x)^{\beta}}$, respectively, of the Weibull distribution with positive parameters α and β. Then, the pdf of the GTPW model is given by

$$f_{GTP-G}^{(\lambda,\theta,\alpha,\beta)}(x) = \frac{\theta\beta\alpha^{\beta}}{e^{\theta}-1} x^{\beta-1} e^{-(\alpha x)^{\beta}} e^{\theta\left[1-e^{-(\alpha x)^{\beta}}\right]} \left\{1+\lambda-2\lambda\frac{e^{\theta\left[1-e^{-(\alpha x)^{\beta}}\right]}-1}{e^{\theta}-1}\right\}.$$

The GTPW density and hrf plots for selected parameter values are displayed in Figure 1.

The GTPLi distribution

The Lindley distribution with parameter $\alpha > 0$ has pdf and cdf (for $x > 0$) given by $g(x) = \frac{\alpha^2}{1+\alpha}(1+x)e^{-\alpha x}$ and $G(x) = 1 - \frac{1+\alpha+\alpha x}{1+\alpha} e^{-\alpha x}$, respectively. Then, the pdf of the GTPLi model is given by

$$f_{GTP-G}^{(\lambda,\theta,\alpha)}(x) = \frac{\theta\alpha^2(1+x)e^{-\alpha x}}{\left(e^{\theta}-1\right)\left(1+\alpha\right)} e^{\theta\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}e^{-\alpha x}\right]} \left\{1+\lambda-2\lambda\frac{e^{\theta\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}e^{-\alpha x}\right]}-1}{e^{\theta}-1}\right\}.$$

The plots of the GTPLi pdf and hrf are displayed in Figure 2 for some parameter values.

2.3 Mixture representation

We provide a useful representation for (pdf) using the concept of exponentiated distributions.

The density (pdf) can be expressed as

$$f_{GTP-G}^{(\lambda,\theta,\phi)}(x) = (1+\lambda)\frac{\theta g(x)e^{\theta G(x)}}{e^{\theta}-1} - 2\lambda\frac{\theta g(x)e^{2\theta G(x)}}{\left(e^{\theta}-1\right)^{2}} + 2\lambda\frac{\theta g(x)e^{\theta G(x)}}{\left(e^{\theta}-1\right)^{2}}.$$

The last equation can be rewritten as

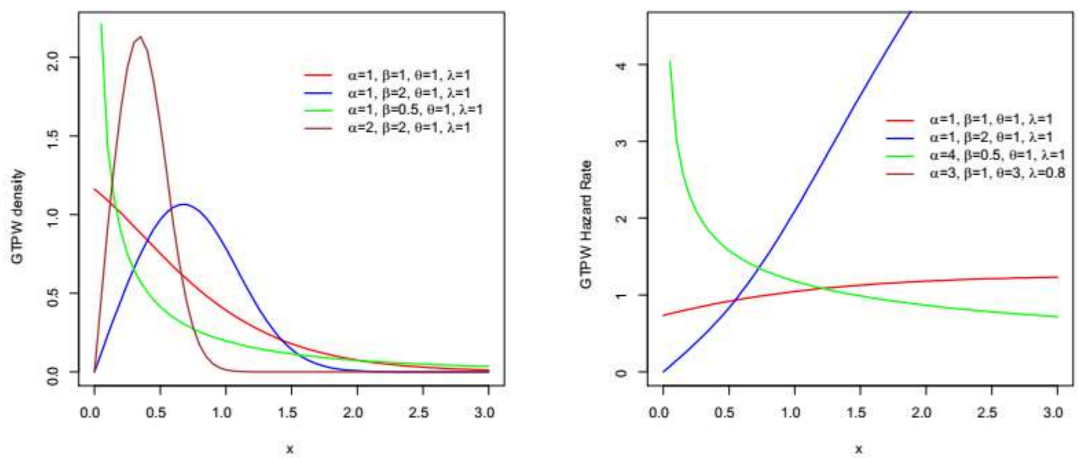

Figure 1: Plots of the GTPW distribution (left panel) density function and (right panel) hrf

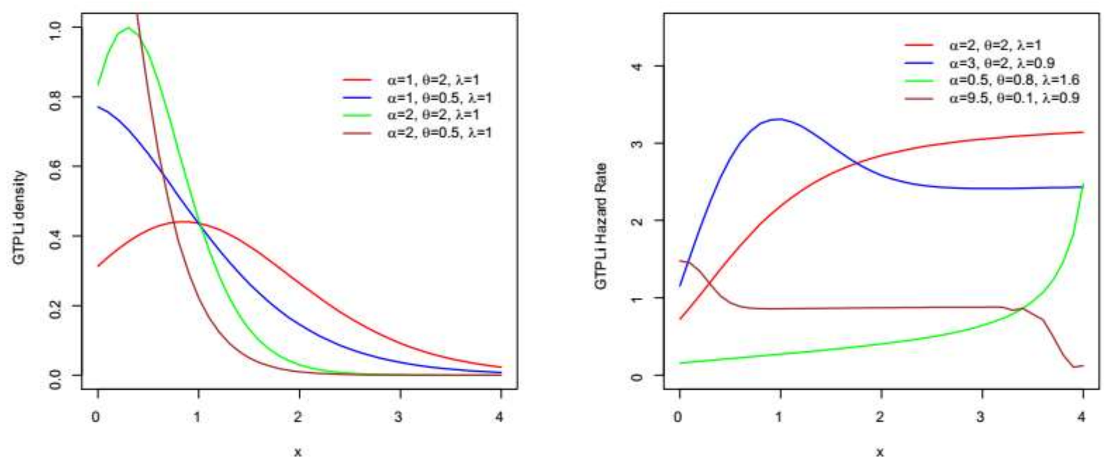

Figure 2: Plots of the GTPLi distribution (left panel) density function and (right panel) hrf

$$f_{GTP-G}^{(\lambda,\theta,\phi)}(x) = \sum_{k=0}^{\infty}\frac{(1+\lambda)\theta^{k+1}g(x)G(x)^{k}}{(k+1)!\left(e^{\theta}-1\right)} - \sum_{k=0}^{\infty}\frac{\lambda(2\theta)^{k+1}g(x)G(x)^{k}}{(k+1)!\left(e^{\theta}-1\right)^{2}} + \sum_{k=0}^{\infty}\frac{2\lambda\theta^{k+1}g(x)G(x)^{k}}{(k+1)!\left(e^{\theta}-1\right)^{2}}.$$

Then, the GTP-G density can be expressed as

$$f_{GTP-G}^{(\lambda,\theta,\phi)}(x) = \sum_{k=0}^{\infty} t_k h_{k+1}(x), \tag{6}$$

where $h_{\delta}(x) = \delta\, g(x)G(x)^{\delta-1}$ and

$$t_k = \frac{(1+\lambda)\theta^{k+1}}{(k+1)!\left(e^{\theta}-1\right)} - \frac{\lambda(2\theta)^{k+1}}{(k+1)!\left(e^{\theta}-1\right)^{2}} + \frac{2\lambda\theta^{k+1}}{(k+1)!\left(e^{\theta}-1\right)^{2}}.$$

Equation (mixture) reveals that the GTP-G density function is a mixture of Exp-G densities. Thus, some mathematical properties of the new family can be derived from those of the Exp-G class.

The cdf of the GTP-G family can also be expressed as a mixture of Exp-G cdfs. By integrating (mixture), we obtain the same mixture representation

$$F_{GTP-G}^{(\lambda,\theta,\phi)}(x)=\sum_{k=0}^{\infty}t_k\, H_{k+1}\left(x\right),$$

where $H_{k+1}\left(x\right)$ is the cdf of the Exp-G family with power parameter $(k+1)$.

3. Properties

3.1 Entropies

The Rényi entropy of a random variable X represents a measure of variation of the uncertainty. The Rényi entropy is defined by

$$I_{\delta}\left(X\right)=\frac{1}{1-\delta}\log\int_{-\infty}^{\infty}f\left(x\right)^{\delta}dx,\ \delta>0 \text{ and } \delta\neq 1.$$

Using the pdf (6), we can write

$$\left[f_{GTP-G}^{(\lambda,\theta,\phi)}(x)\right]^{\delta}=\sum_{k=0}^{\infty}v_k g(x)^{\delta}G(x)^{k},$$

where

$$v_k=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\left(-1\right)^{j+i}\Gamma\left(\delta+1\right)\Gamma\left(i+1\right)\theta^{\delta+k}\left(1+\lambda\right)^{\delta-i}\left(2\lambda\right)^{i}\left(\delta+i-j\right)^{k}}{k!\,j!i!\Gamma\left(\delta-i+1\right)\Gamma\left(i-j+1\right)\left(e^{\theta}-1\right)^{\delta+i}}.$$

Then, the Rényi entropy of a random variable X following the GTP-G family is given by

$$I_{\delta}\left(X\right)=\frac{1}{1-\delta}\log\left\{\sum_{k=0}^{\infty}v_k\int_{-\infty}^{\infty}g(x)^{\delta}G(x)^{k}dx\right\}.$$

The q-entropy, say $H_q\left(X\right)$, is defined by

$$H_q\left(X\right)=\frac{1}{q-1}\log\left\{1-\int_{-\infty}^{\infty}f\left(x\right)^{q}dx\right\},\ q>0 \text{ and } q\neq 1,$$

and then

$$H_q\left(X\right)=\frac{1}{q-1}\log\left\{1-\left[\sum_{k=0}^{\infty}v_k^{*}\int_{-\infty}^{\infty}g(x)^{q}G(x)^{k}dx\right]\right\},$$

for $q>0$ and $q\neq 1,$ where

$$v_k^{*}=\sum_{i=0}^{\infty}\sum_{j=0}^{i}\frac{\left(-1\right)^{j+i}\Gamma\left(q+1\right)\Gamma\left(i+1\right)\theta^{q+k}\left(1+\lambda\right)^{q-i}\left(2\lambda\right)^{i}\left(q+i-j\right)^{k}}{k!\,j!i!\Gamma\left(q-i+1\right)\Gamma\left(i-j+1\right)\left(e^{\theta}-1\right)^{q+i}}.$$

3.2 Order statistics

Order statistics make their appearance in many areas of statistical theory and practice. Let $X_1, \ldots, X_n$ be a random sample from the GTP-G family. The pdf of the i th order statistic, say $X_{i:n}$, is given by

$$f_{i\,:\,n}(x) = \frac{f(x)}{\mathrm{B}(i, n-i+1)} \sum_{j=0}^{n-i} (-1)^j \binom{n-i}{j} F^{\,j+i-1}(x).$$

Using (5), we can write

$$\left[F_{GTP-G}^{(\lambda,\theta,\phi)}(x)\right]^{j+i-1} = \frac{\left(e^{\theta G(x;\phi)}-1\right)^{j+i-1}(1+\lambda)^{j+i-1}}{\left(e^{\theta}-1\right)^{j+i-1}}\left[1-\frac{\lambda}{1+\lambda}\frac{e^{\theta G(x;\phi)}-1}{e^{\theta}-1}\right]^{j+i-1}.$$

Using the power series expansion, the last equation can be expressed as

$$\left[F_{GTP-G}^{(\lambda,\theta,\phi)}(x)\right]^{j+i-1} = \sum_{w=0}^{j+i-1} d_w \left[e^{\theta G(x;\phi)}-1\right]^{j+i-1+w},$$

where

$$d_w = \frac{(-1)^w\,\Gamma(j+i)(1+\lambda)^{j+i-1-w}\,\lambda^w}{w!\,\Gamma(j+i-w)},$$

then

$$f_{GTP-G}^{(\lambda,\theta,\phi)}(x)\left[F_{GTP-G}^{(\lambda,\theta,\phi)}(x)\right]^{j+i-1} = \sum_{k=0}^{\infty}(b_k - s_k)\,h_{k+1}(x),$$

where

$$b_k = \sum_{m=0}^{\infty}\sum_{w=0}^{j+i-1} \frac{(-1)^{w+m}\,\Gamma(j+i)\Gamma(j+i+w)\theta^{k+1}(1+\lambda)^{j+i-w}\,\lambda^w\,(j+i+w-m)^k}{w!\,m!\,(k+1)\left(e^{\theta}-1\right)^{j+i+w}\Gamma(j+i-w)\Gamma(j+i+w-m)},$$

and

$$s_k = \sum_{m=0}^{\infty}\sum_{w=0}^{j+i-1} \frac{(-1)^w\,\Gamma(j+i)\Gamma(j+i+w+1)\,2(1+\lambda)^{j+i-w-1}\,\lambda^{w+1}\,(j+i+w+1-m)^k}{w!\,\Gamma(j+i-w)\Gamma(j+i+w+1-m)(k+1)\theta^{-(k+1)}\left(e^{\theta}-1\right)^{j+i+w+1}}.$$

Then, the pdf of $X_{i\,:\,n}$ follows as

$$f_{i\,:\,n}(x) = \sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^j}{\mathrm{B}(i,n-i+1)}(b_k - s_k)\binom{n-i}{j}h_{k+1}(x),$$

Thus, the density function of the GTP-G order statistics is a mixture of Exp-G densities. Based on last equation, we can obtain some structural properties of $X_{i\,:\,n}$ from those of the Exp-G model. The r th moment of $X_{i\,:\,n}$ is given by

$$E\left(X_{i\,:\,n}^r\right) = \sum_{k=0}^{\infty}\sum_{j=0}^{n-i}\frac{(-1)^j}{\mathrm{B}(i,n-i+1)}(b_k - s_k)\binom{n-i}{j}E\left(Y_{k+1}^n\right),$$

where Y_δ has the Exp-G density with power parameter δ

3.3 Characterizations

In this subsection we present characterizations of the GTP-G distribution in terms of a simple relationship between two truncated moments. We like to mention here the works of Glänzel (1987,1990), Glänzel and Hamedani (2001) and Hamedani (2010) in this direction. Our characterization results presented here will employ an interesting result due to Glänzel (1987) (Theorem 3.1 below). We believe that our characterizations of GTP-G distribution may be the only ones possible due to the nature of the distribution function of GTP-G.

Theorem 3.1. Let (Ω, P) be a given probability space and let $H=[a,b]$ be an interval for some $a<b$ ($a=-\infty$, $b=\infty$ might as well be allowed). Let $X:\Omega\to H$ be a continuous random variable with the distribution function F and let q and h be two real functions defined on H such that

$$\mathbf{E}[q(X) \mid X\geq x]=\mathbf{E}[h(X) \mid X\geq x]\lambda(x), \quad x\in H,$$

is defined with some real function λ . Assume that $q, h\in C^1(H)$, $\lambda\in C^2(H)$ and F is twice continuously differentiable and strictly monotone function on the set H. Finally, assume that the equation $h\lambda=q$ has no real solution in the interior of H Then F is uniquely determined by the functions q, h and λ, particularly

$$F(x)=\int_a^x C\left|\frac{\lambda'(u)}{\lambda(u)h(u)-g(u)}\right|\exp(-s(u))\,du\ ,$$

where the function s is a solution of the differential equation $s'=\frac{\eta' h}{\eta h-q}$ and C is a constant, chosen to make $\int_H dF=1$.

We like to mention that this kind of characterization based on the ratio of truncated moments is stable in the sense of weak convergence, in particular, let us assume that there is a sequence $\{X_n\}$ of random variables with distribution functions $\{F_n\}$ such that the functions q_n , h_n and λ_n $(n\in \mathrm{N})$ satisfy the conditions of Theorem 4.1 and let $q_n\to g$, $h_n\to h$ for some continuously differentiable real functions q and h. Let, finally, X be a random variable with distribution F. Under the condition that $q_n(X)$ and $h_n(X)$ are uniformly integrable and the family $\{F_n\}$ is relatively compact, the sequence X_n converges to X in distribution if and only if λ_n converges to λ , where where the function s is a solution of the differential equation $s'=\frac{\lambda' h}{\lambda h-q}$ and C is a constant, chosen to make $\int_H dF=1\cdot$

$$\lambda(x)=\frac{E[q(X) \mid X\geq x]}{E[h(X) \mid X\geq x]}.$$

This stability theorem makes sure that the convergence of distribution functions is reflected by corresponding convergence of the functions q, h and λ , respectively. It guarantees, for instance, the 'convergence' of characterization of the Wald distribution to that of the Lévy-Smirnov distribution if $\alpha \to \infty$, as was pointed out in (2001).

A further consequence of the stability property of Theorem 3.1 is the application of this theorem to special tasks in statistical practice such as the estimation of the parameters of discrete distributions. For such purpose, the functions q , h and, specially, λ should be as simple as possible. Since the function triplet is not uniquely determined it is possible to choose λ as a linear function. Therefore, it is worth analyzing some special cases which helps to find new characterizations reflecting the relationship between individual continuous univariate distributions and appropriate in other areas of statistics.

Remark 3.1. In Theorem 3.1, the interval H need not be closed since the condition is only on the interior of H .

Proposition 3.1. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let

$$h(x) = \left\{1 + \lambda - \lambda\left[\frac{e^{\theta G(x;\xi)} - 1}{e^{\theta} - 1}\right]\right\}^{-1}$$

and

$$q(x) = h(x) e^{\theta G(x;\xi)} \ for x \in (0,\infty).$$

The pdf of X is (6) if and only if the function λ defined in Theorem 3.1 has the form

$$\lambda(x) = \frac{1}{2}\left[e^{\theta} + e^{\theta G(x;\xi)}\right], \quad x > 0.$$

Proof. Let X have density (6), then

$$(1 - F(x))\ \mathbf{E}\left[h(X) \ \mid X \ge x\right] = \frac{1}{\theta\left(e^{\theta} - 1\right)}\left[e^{\theta} - e^{\theta G(x;\xi)}\right], \quad x > 0 \ ,$$

and

$$(1 - F(x))\ \mathbf{E}\left[q(X) \ \mid X \ge x\right] = \frac{1}{2\theta\left(e^{\theta} - 1\right)}\left\{e^{2\theta} - e^{2\theta G(x;\xi)}\right\} \ , \quad x > 0 \ ,$$

and finally

$$\lambda(x) h(x) - q(x) = \frac{1}{2} h(x)\left\{e^{\theta} - e^{\theta G(x;\xi)}\right\} > 0 \ \text{for} \ x > 0 \ .$$

Conversely, if λ is given as above, then

$$s'(x) = \frac{\lambda'(x)\ h(x)}{\lambda(x)\ h(x) - q(x)} = \frac{\theta g(x;\xi) e^{\theta G(x;\xi)}}{e^{\theta} - e^{\theta G(x;\xi)}}, \quad x > 0 \ ,$$

and hence

$$s(x) = -\ln\left\{e^{\theta} - e^{\theta G(x;\xi)}\right\}, \quad x > 0.$$

Now, in view of Theorem 3.1, X has density (6).

Corollary 3.1. Let $X : \Omega \to (0,\infty)$ be a continuous random variable and let $h(x)$ be as in Proposition 3.1. The pdf of X is (6) if and only if there exist functions q and λ defined in Theorem 3.1 satisfying the differential equation

$$\frac{\lambda'(x)h(x)}{\lambda(x)h(x)-q(x)} = \frac{\theta g(x;\xi)e^{\theta G(x;\xi)}}{e^{\theta}-e^{\theta G(x;\xi)}}, \quad x>0.$$

Remarks 3.2. (a) The general solution of the differential equation in Corollary 3.1 is

$$\lambda(x) = \left\{e^{\theta}-e^{\theta G(x;\xi)}\right\}^{-1}\left[-\int \theta g(x;\xi)e^{\theta G(x;\xi)}\left(h(x)\right)^{-1} q(x)dx + D\right],$$

for $x>0$, where D is a constant. One set of appropriate functions is given in Proposition 3.1 with $D=\frac{1}{2}$.

(b) Clearly there are other triplets of functions (h,q,λ) satisfying the conditions of Theorem 3.1. We presented one such triplet in Proposition 3.1.

4. Maximum likelihood estimation

Several approaches for parameter estimation were proposed in the literature but the maximum likelihood method is the most commonly employed. Here, we consider the estimation of the unknown parameters of the GTP-G family from complete samples only by maximum likelihood. Let $x_1,\ldots,x_n$ be a random sample from this family with parameters λ,θ and ϕ . Let $\Theta = (\lambda,\theta,\phi^T)^T$ be the $p\times 1$ parameter vector. To obtain the MLE of Θ , the log-likelihood function is given by

$$L = L(\Theta) = n\log\theta - n\log\left(e^{\theta}-1\right) + \theta\sum_{i=0}^{n} G(x_i;\phi) + \sum_{i=0}^{n}\log g(x_i;\phi) + \sum_{i=0}^{n}\log q_i ,$$

Where $q_i = \left[1+\lambda-2\lambda\frac{e^{\theta G(x_i;\phi)}-1}{e^{\theta}-1}\right]$. The components of the score vector, $\mathbf{U}(\Theta) = \frac{\partial L}{\partial\Theta} = \left(\frac{\partial L}{\partial\lambda},\frac{\partial L}{\partial\theta},\frac{\partial L}{\partial\phi_k}\right)^T = \left(U_{\lambda},U_{\theta},U_{\phi_k}\right)^T$, are

$$U_{\lambda} = \sum_{i=1}^{n} q_i^{-1}\left\{1-2\left[e^{\theta G(x_i;\phi)}-1\right]/\left(e^{\theta}-1\right)\right\},$$

$$\begin{aligned} U_{\theta} = {} & (n/\theta) - \left[n/\left(e^{\theta}-1\right)\right] + \sum_{i=0}^{n} G(x_i;\phi) \\ & -\left[2\lambda/\left(e^{\theta}-1\right)\right]\sum_{i=0}^{n}\left[G(x_i;\phi)e^{\theta G(x_i;\phi)}/q_i\right] \\ & +\left[2\lambda e^{\theta}/\left(e^{\theta}-1\right)^2\right]\sum_{i=0}^{n}\left\{\left[e^{\theta G(x_i;\phi)}-1\right]/q_i\right\} \end{aligned}$$

and

$$\begin{aligned} U_{\phi_k} = {} & \theta\sum_{i=0}^{n} G'(x_i;\phi) + \sum_{i=0}^{n}\left[g'(x_i;\phi)/g(x_i;\phi)\right] \\ & -\left[2\lambda\theta/\left(e^{\theta}-1\right)\right]\sum_{i=0}^{n}\left\{\left[\left(e^{\theta G(x_i;\phi)}-1\right)G'(x_i;\phi)\right]/q_i\right\}, \end{aligned}$$

where $g'(x_i;\phi) = \partial g(x_i;\phi)/\partial\phi_k$ and $G'(x_i;\phi) = \partial G(x_i;\phi)/\partial\phi_k$.

5. Numerical results

In this section, we will assess the performance of the MLEs using two simulation studies and two real data applications.

5.1 Simulation study

In order to assess the performance of the MLEs, a small simulation study is performed using the statistical software R through the package (stats4), command mle. The number of Monte Carlo replications was $20,000$. For maximizing the log-likelihood function, we use the MaxBFGS subroutine with analytical derivatives. The evaluation of the estimates was performed based on the following quantities for each sample size: the empirical mean squared errors (MSEs) are calculated using the R package from the Monte Carlo replications. The MLEs are determined for each simulated data, say, $(\hat{\alpha}_i, \hat{\beta}_1, \hat{\theta}_1, \hat{\lambda}_i)$ for $i = 1, 2, \ldots, 20000$ and the biases and MSEs are computed by

$$bias_h(n) = \frac{1}{20000}\sum_{i=1}^{20000}(\hat{h}_i - h) \text{ and } MSE_h(n) = \frac{1}{20000}\sum_{i=1}^{20000}(\hat{h}_i - h)^2,$$

for $h = \{\alpha, \beta, \theta, \lambda\}$. We consider the sample sizes at $n = 100$, 200 and consider different values for the parameters. The empirical results are given in Table 2. The figures in Table 2 indicate that the estimates are quite stable and, more importantly, are close to the true values for these sample sizes. Furthermore, as the sample size increases, the MSEs decreases as expected.

For GTPW distribution we follow the same exact procedure as mentioned above, with $h = \{\alpha, \theta, \lambda\}$. The results show that the maximum likelihood estimation method performs well. In general, the biases and standard deviations of the parameters are reasonably small. The biases and standard deviations decreases as the sample size increases. The results from this simulation study suggest that the maximum likelihood method can be used to estimate the parameters of the GTPW and GTPLi.

Table 1: Bias and MSE of the estimates under the maximum likelihood method (GTPW).

Sample Size	Actual Value				Bias				MSE			
n	α	β	θ	λ	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\beta}$	$\hat{\theta}$	$\hat{\lambda}$
100	0.5	0.5	2	−1	−0.417	−0.419	0.355	−0.393	0.290	0.065	0.523	0.897
	0.5	0.5	3	−0.85	−0.778	0.323	−0.214	−0.342	0.203	0.071	0.115	0.324
	0.7	0.8	4	−0.5	0.489	−0.246	−0.622	0.482	0.163	0.864	0.037	0.0.953
	0.9	0.7	6	0.5	0.188	0.475	−0.509	0.056	0.189	0.067	0.87	0.786
	1	1.5	0.9	0.65	0.187	−0.498	−0.429	−0.545	0.198	0.143	0.266	0.356
	1.5	2	0.6	1	−0.081	−0.363	−0.405	−0.220	0.371	0.677	0.629	0.884
200	0.5	0.5	2	−1	0.041	−0.064	−0.052	0.009	0.109	0.015	0.063	0.189
	0.5	0.5	3	−0.85	0.030	−0.107	0.031	0.003	0.017	0.069	0.006	0.008
	0.7	0.8	4	−0.5	0.044	−0.073	−0.134	0.007	0.014	0.022	0.018	0.209
	0.9	0.7	6	0.5	0.028	−0.095	−0.051	−0.003	0.042	0.523	0.0193	0.041
	1	1.5	0.9	0.65	0.1258	0.022	−0.002	0.032	0.018	0.020	0.028	0.032
	1.5	2	0.6	1	0.0343	0.006	0.048	0.020	0.073	0.010	0.267	0.137
500	0.5	0.5	2	−1	−0.006	−0.008	−0.0165	−0.027	0.0094	0.0114	0.0111	0.0842
	0.5	0.5	3	−0.85	−0.001	−0.111	−0.035	0.002	0.004	0.010	0.022	0.018
	0.7	0.8	4	−0.5	−0.073	−0.052	0.046	−0.022	0.003	0.031	0.024	0.036
	0.9	0.7	6	0.5	−0.102	−0.020	−0.018	−0.023	0.008	0.005	0.010	0.021
	1	1.5	0.9	0.65	−0.008	−0.051	−0.016	0.002	0.027	0.007	0.012	0.015
	1.5	2	0.6	1	0.007	−0.069	0.066	−0.085	0.096	0.004	0.014	0.0128

Table 2: Bias and MSE of the estimates under the maximum likelihood method (GTPLi).

Sample Size	Actual Value			Bias			MSE		
n	α	θ	λ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\lambda}$
100	0.5	0.5	−1	−0.154	0.136	−0.125	0.099	0.181	0.377
	0.5	0.5	−0.85	0.198	0.2358	0.542	0.042	0.677	0.177
	0.7	0.8	−0.5	0.176	−0.186	0.193	0.103	0.199	1.581
	0.9	0.7	0.5	0.248	0.156	0.254	0.053	0.808	0.986
	1	1.5	0.65	0.694	0.112	0.163	0.453	0.638	0.390
	1.5	2	1	0.184	0.166	0.197	0.611	0.594	1.054
200	0.5	0.5	−1	0.0132	−0.0151	−0.0281	0.028	0.089	0.155
	0.5	0.5	−0.85	0.0187	0.1225	0.0893	0.011	0.251	0.072
	0.7	0.8	−0.5	0.0877	0.1057	0.0303	0.074	0.088	0.446
	0.9	0.7	0.5	0.1457	0.1076	0.0213	0.012	0.279	0.237
	1	1.5	0.65	0.0161	0.0260	0.1870	0.022	0.239	0.218
	1.5	2	1	0.0253	0.0084	0.1388	0.035	0.365	0.313
500	0.5	0.5	−1	0.002	−0.093	−0.0185	0.015	0.049	0.104
	0.5	0.5	−0.85	0.0137	0.104	0.0657	0.009	0.138	0.032
	0.7	0.8	−0.5	0.0345	0.007	0.0201	0.034	0.036	0.164
	0.9	0.7	0.5	0.091	0.083	0.0132	0.008	0.156	0.0015
	1	1.5	0.65	0.0161	0.0117	0.134	0.011	0.117	0.149
	1.5	2	1	0.0113	0.0032	0.1027	0.023	0.238	0.0014

5.2 Applications

Now, we provide two applications to real data to illustrate the flexibility of the GTPLi and GTPW models presented in Section 2. The goodness-of-fit statistics for these models are compared with other competitive models and the MLEs of the model parameters are determined.

Data set I: Relief times of twenty patients

The first data set (Gross and Clark, 1975) on the relief times of twenty patients receiving an analgesic is: 1.1, 1.4, 1.3, 1.7, 1.9, 1.8, 1.6, 2.2, 1.7, 2.7, 4.1, 1.8, 1.5, 1.2, 1.4, 3, 1.7, 2.3, 1.6, 2. This data set has been used by Afify et al. (2016) to fit the beta transmuted Lindley distribution. For these data, we compare the fits of the GTPLi distribution with the The Kumaraswamy Lindley (KwLi) (Cakmakyapan and Kadılar, 2014). beta Lindley (BLi) (Merovci and Sharma, 2014)), Lindley-Poisson (LiP) (Gui et al., 2014), power Lindley (PoLi) (Ghitany et al., 2013), TLi (Merovci, 2013) and Lindley (Li) models ($x > 0$ for all of them).

- The KwLi density given by

$$f(x)=\frac{ab\alpha^2(1+x)}{(1+\alpha)}\exp\left[-(\alpha x)\right]\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\exp\left[-(\alpha x)\right]\right]^{a-1}\left\{1-\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\exp\left[-(\alpha x)\right]\right]^{a}\right\}^{b-1}.$$

- The BLi density given by

$$f(x)=\frac{\alpha^2(1+x)}{B(a,b)(1+\alpha)}\exp\left[-(\alpha x)\right]\left[1-\frac{1+\alpha+\alpha x}{1+\alpha}\exp\left[-(\alpha x)\right]\right]^{a-1}\left\{\frac{1+\alpha+\alpha x}{1+\alpha}\exp\left[-(\alpha x)\right]\right\}^{b-1}.$$

- The LiP density given by

$$f(x)=\frac{\beta\alpha^2(1+x)}{(1+\alpha)\left(e^{\beta}-1\right)}\exp\left[-(\alpha x)\right]\exp\left\{\frac{\beta(1+\alpha+\alpha x)}{1+\alpha}\exp\left[-(\alpha x)\right]\right\}.$$

- The PoLi density given by

$$f(x)=\frac{\beta\alpha^2}{1+\alpha}(1+x^{\beta})\,x^{\beta-1}\exp\left(-\alpha x^{\beta}\right).$$

- The TLi density given by

$$f(x)=\frac{\alpha^2}{1+\alpha}(1+x)\exp\left(-\alpha x^{\beta}\right)\left[1-\lambda+2\lambda\frac{1+\alpha+\alpha x}{1+\alpha}\exp\left(-\alpha x^{\beta}\right)\right].$$

The parameters of the above densities are all positive real numbers except for the TLi distribution for which $|\lambda|\leq 1$.

Data set II: The nicotine data

The second data set works with nicotine measurements, made from several brands of cigarettes in 1998, collected by the Federal Trade Commission which is an independent agency of the US government. The free form data set can be found at http://pw1.netfom.com/rda vis2/smoke.html. This data set consists of 346 observations and it has been used by Afify et al. (2016) to fit the Marshall-Olkin additive Weibull distribution. We compare the fits of the GTPW distribution with other competitive models, namely: the beta transmuted Weibull (BTW) (Afify et al., 2016), transmuted exponentiated generalized Weibull (TEGW) (Yousof et al., 2015), Kumaraswamy Weibull (KwW) (Cordeiro et al., 2010), McDonald Weibull (McW) (Cordeiro et al., 2014), transmuted modified Weibull (TMW) (Khan and King, 2013), beta Weibull (BW) (Lee et al., 2007) and Weibull (W) distributions with corresponding densities (for $x>0$)

- The BTW density given by

$$f(x)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}\left\{1-\lambda+2\lambda\exp\left[-(\alpha x)^{\beta}\right]\right\}\left\{\left(1-\exp\left[-(\alpha x)^{\beta}\right]\right)\left(1+\lambda\exp\left[-(\alpha x)^{\beta}\right]\right)\right\}^{a-1}$$
$$\times\exp\left[-(\alpha x)^{\beta}\right]\left\{1-\left(1-\exp\left[-(\alpha x)^{\beta}\right]\right)\left(1+\lambda\exp\left[-(\alpha x)^{\beta}\right]\right)\right\}^{b-1}.$$

- The TEGW density given by

$$f(x)=ab\beta\alpha^{\beta}x^{\beta-1}e^{-a(\alpha x)^{\beta}}\left\{1-\exp\left[-a(\alpha x)^{\beta}\right]\right\}^{b-1}\left\{1+\lambda-2\lambda\left(1-\exp\left[-a(\alpha x)^{\beta}\right]\right)^{b}\right\}.$$

- The KwW density given by

$$f(x)=ab\beta\alpha^{\beta}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}^{a-1}\left\{1-\left(1-\exp\left[-(\alpha x)^{\beta}\right]\right)^{a}\right\}^{b-1}.$$

- The McW density given by

$$f(x)=\frac{\beta c\alpha^{\beta}}{B(a/c,b)}x^{\beta-1}e^{-(\alpha x)^{\beta}}\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}^{a-1}\left\{1-\left(1-\exp\left[-(\alpha x)^{\beta}\right]\right)^{c}\right\}^{b-1}.$$

- The TMW density given by

$$f(x)=\left(\alpha+\gamma\beta x^{\beta-1}\right)\exp\left[-\left(\alpha x+\gamma x^{\beta}\right)\right]\left\{1-\lambda+2\lambda e\exp\left[-\left(\alpha x+\gamma x^{\beta}\right)\right]\right\}.$$

- The BW density given by

$$f(x)=\frac{\beta\alpha^{\beta}}{B(a,b)}x^{\beta-1}\exp\left[-b(\alpha x)^{\beta}\right]\left\{1-\exp\left[-(\alpha x)^{\beta}\right]\right\}^{a-1}.$$

The parameters of the above densities are all positive real numbers except for the BT-W, TEGW and TMW distributions for which $|\lambda|\leq 1$.

In order to compare the fitted models, we consider some goodness-of-fit measures including the Akaike information criterion (AIC), consistent Akaike information criterion ($CAIC$), Hannan-Quinn information criterion ($HQIC$), Bayesian information criterion (BIC) and $-2\hat{L}$, where $\hat{L}$ is the maximized log-likelihood, $AIC=-2\hat{L}+2p$, $CAIC=-2\hat{L}+2pn/(n-p-1)$, $HQIC=-2\hat{L}+2p\log\left[\log(n)\right]$ and $BIC=-2\hat{L}+p\log(n)$, p is the number of parameters and n is the sample size. Moreover, we use the Anderson-Darling (A^{*}) and the Cramér-von Mises (W^{*}) statistics in order to compare the fits of the two new models with other nested and non-nested models. The statistics are widely used to determine how closely a specific cdf fits the empirical distribution of a given data set. These statistics are given by

$$A^{*}=\left(\frac{9}{4n^{2}}+\frac{3}{4n}+1\right)\left\{n+\frac{1}{n}\sum_{j=1}^{n}(2j-1)\log\left[z_{i}\left(1-z_{n-j+1}\right)\right]\right\}$$

and

$$W^* = \left(\frac{1}{2n}+1\right)\left\{\sum_{j=1}^{n}\left(z_i - \frac{2j-1}{2n}\right)^2 + \frac{1}{12n}\right\},$$

respectively, $z_i = F\left(y_j\right)$, where the y_j 's values are the ordered observations. The smaller these statistics are, the better the fit. Upper tail percentiles of the asymptotic distributions of these goodness-of-fit statistics were tabulated in [Ni]

Tables 3 and 5 list the values of $-2\hat{L}$, AIC , $CAIC$, $HQIC$, BIC , W^* and A^* whereas the MLEs and their corresponding standard errors (in parentheses) of the model parameters are given in Tables 4 and 6. These numerical results are obtained using the Mathcad program**.** In Table 3, we compare the fits of the GTPLi model with the KwLi, BLi, PoLi, LiP, TLi and Li models. We note that the GTPLi model has the lowest values for the $-2\hat{L}$, AIC , $CAIC$, $HQIC$, BIC , W^* and A^* statistics (for the relief times data) among the fitted models. So, the GTPLi model could be chosen as the best model. In Table 5, we compare the fits of the GTPW model with the BTW, TEGW, KwW, McW, TMW, BW and W models. The figures in this table reveal that the GTPW model has the lowest values for $-2\hat{L}$, AIC , $CAIC$, $HQIC$, BIC , W^* and A^* statistics (for the nicotine data) among all fitted models. So, the GTPW model can be chosen as the best model. It is quite clear from the values in Tables 3 and 5 that the GTPLi and GTPW models provide the best fits to these data sets. So, we claim that these new distributions can be better models than other competitive models.

Table 3: The statistics $-2\hat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^* and A^* for the relief times.

Model	Goodness-of-fit criteria						
	$-2\hat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
GTPLi	33.688	39.688	41.188	40.271	42.675	0.0809	0.47112
KwLi	34.791	40.791	42.291	41.374	43.778	0.08907	0.5313
BLi	34.884	40.884	42.384	41.467	43.871	0.08379	0.52035
PoLi	40.864	44.864	45.57	45.253	46.855	0.17517	1.05209
LiP	60.506	64.506	65.212	64.895	66.498	0.50191	9.12
Li	60.499	62.499	62.721	62.693	63.495	0.5031	9.15779
TLi	61.729	65.729	66.435	66.118	67.721	0.53509	10.74948

Table 4: MLEs and their standard errors (in parentheses) for the relief times.

Model	Estimates		
GTPLi	$\hat{\alpha}$= 2.7054 (0.448)	$\hat{\theta}$= 24.6249 (11.65)	$\hat{\lambda}$= −0.5946 (0.544)
KwLi	$\hat{\alpha}$= 1.609 (0.854)	$\hat{a}$= 9.2594 (7.884)	$\hat{b}$= 1.756 (1.701)
BLi	$\hat{\alpha}$= 1.505 (0.707)	$\hat{a}$= 9.3 (5.055)	$\hat{b}$= 1.717 (1.349)
PoLi	$\hat{\alpha}$= 0.3445 (0.1)	$\hat{\beta}$= 2.253 (0.307)	
LiP	$\hat{\alpha}$= 0.8197 (0.188)	$\hat{\beta}$= 0.0019 (1.063)	
TLi	$\hat{\alpha}$= 0.6653 (0.048)	$\hat{\lambda}$= 0.3587 (0.332)	
Li	$\hat{\alpha}$= 0.8161 (0.136)		

Table 5: The statistics $-2\widehat{\ell}$, AIC, $CAIC$, $HQIC$, BIC, W^* and A^* for nicotine data.

Model	Goodness-of-fit criteria						
	$-2\widehat{\ell}$	AIC	$CAIC$	$HQIC$	BIC	W^*	A^*
GTPW	214.94	222.94	223.057	229.066	238.325	0.36131	1.96785
TMW	217.219	225.219	225.336	231.345	240.604	0.37111	2.07974
BTW	216.977	226.977	227.154	234.636	246.21	0.39678	2.2383
TEGW	218.86	228.86	229.037	236.519	248.093	0.43241	2.4457
W	226.581	230.581	230.616	233.644	238.274	0.55744	3.20719
BW	225.173	233.173	233.29	239.30	248.559	0.49664	2.89774
KwW	255.044	263.044	263.162	269.171	278.43	0.99745	5.82509
McW	302.714	312.714	312.89	320.372	331.946	1.61799	9.5949

Table 6: MLEs and their standard errors (in parentheses) for nicotine data.

Model	Estimates				
BTW	$\widehat{\alpha}= 1.0107$ (0.243)	$\widehat{\beta}= 2.5907$ (0.41)	$\widehat{a}= 0.6672$ (0.175)	$\widehat{b}= 1.3249$ (1.177)	$\widehat{\lambda}= -0.8589$ (0.137)
TEGW	$\widehat{\alpha}= 1.3007$ (2.994)	$\widehat{\beta}= 2.634$ (0.316)	$\widehat{a}= 0.6402$ (3.881)	$\widehat{b}= 0.7332$ (0.183)	$\widehat{\lambda}= -0.6755$ (0.156)
McW	$\widehat{\alpha}= 1.3078$ (0.596)	$\widehat{\beta}= 0.5317$ (0.079)	$\widehat{a}= 16.858$ (4.219)	$\widehat{b}= 10.1043$ (3.995)	$\widehat{c}= 1.1644$ (0.793)
GTPW	$\widehat{\alpha}= 1.5519$ (0.172)	$\widehat{\beta}= 1.8427$ (0.203)	$\widehat{\theta}= 2.3072$ (0.098)	$\widehat{\lambda}= -0.4258$ (0.285)	
TMW	$\widehat{\alpha}= 0.3255$ (0.315)	$\widehat{\beta}= 2.5962$ (0.244)	$\widehat{\gamma}= 1.2691$ (0.22)	$\widehat{\lambda}= -0.7616$ (0.242)	
KwW	$\widehat{\alpha}= 2.5072$ (1.191)	$\widehat{\beta}= 0.4839$ (0.076)	$\widehat{a}= 11.8142$ (3.707)	$\widehat{b}= 18.7953$ (6.212)	
BW	$\widehat{\alpha}= 0.6686$ (0.578)	$\widehat{\beta}= 3.1645$ (0.426)	$\widehat{a}= 0.7784$ (0.163)	$\widehat{b}= 3.0922$ (8.174)	
W	$\widehat{\alpha}= 1.0477$ (0.022)	$\widehat{\beta}= 2.7208$ (0.114)			

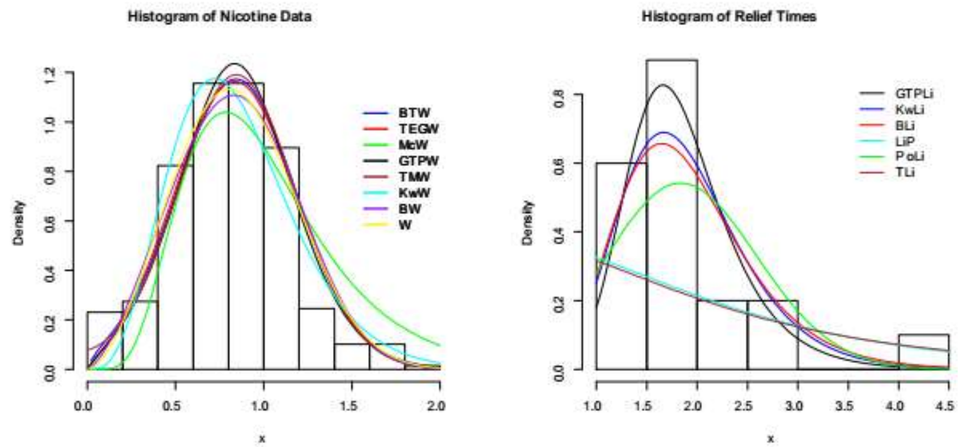

Figure 3: Plots of the estimated pdfs (left panel) for nicotine data and (right panel) for relief times data

6. Conclusion

In this work, we introduce a new class of continuous distributions called the generalized Poisson (GTP-G) family which extends the quadratic rank transmutation map. We provide some special models for the new family. Some of its mathematical properties including Rényi and q -entropies, order statistics and characterizations are derived. The estimations of the model parameters are performed by maximum likelihood method. The flexibility of the proposed family is illustrated by means of two applications to real data sets. The GTP-G family can be viewed as a mixture representation of the exponentiated G densities. The new family can also be viewed as a suitable model for fitting the symmetric, the right-skewed, and bimodal data. The generalized transmuted Poisson Lindley is much better than the Kumaraswamy Lindley, beta Lindley, Lindley-Poisson, power Lindley, Transmuted Lindley and Lindley models in modeling the relief times data as well as the generalized transmuted Poisson Weibull is much better than the beta transmuted Weibull, transmuted exponentiated generalized Weibull, Kumaraswamy Weibull, McDonald Weibull, transmuted modified Weibull, beta Weibull and Weibull models in modeling the nicotine data.

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