Alpha-Power Transformed Lindley Distribution: Properties and Associated Inference

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Abstract

The Lindley distribution has been generalized by many authors in recent years. A new two-parameter distribution is introduced, called Alpha Power Transformed Lindley (αPTL) distribution that provides better fit than the Lindley distribution and some of the well known distributions. The new model includes the Lindley distribution as a special case. Several properties of the proposed distribution, including explicit expressions for the ordinary moments, incomplete and conditional moments, mean residual lifetime, mean deviations, L-moments, moment generating function, cumulant generating function, characteristic function, Bonferroni and Lorenz curves, entropies, stress-strength reliability, stochastic ordering, statistics and distribution of sums, differences, ratios and products are derived. The new distribution can have decreasing and increasing failure rates function depending on its parameters. The estimates of the model parameters are obtained by the method of maximum likelihood estimation. Also, we obtain the confidence intervals of the model parameters. A simulation study is carried out to examine the bias and mean squared error of the maximum likelihood estimators of the parameters. Finally, two earthquakes data sets have been analyzed to show how the proposed models work in practice.

Keywords: Lindley distribution, Moments, Stress-strength reliability, Maximum likelihood estimation.

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1 Introduction

In recent years, researchers proposed various ways of generating new continuous distributions in lifetime data analysis to enhance its capability to fit diverse lifetime data which have a high

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degree of skewness and kurtosis. These extended distributions provide greater flexibility in modelling certain applications and data in practice. Due to the computational and analytical facilities available in programming softwares such as R, Maple and Mathematica, it is easy to tackle the problems involved in computing special functions in these extended distributions. A detailed survey of methods for generating distributions were discussed by Lee et al. (2013) and Jones (2015). Most of these distributions are special cases of the T-X class defined by Alzaatreh et al. (2013). This class of distributions extends some recent families such as the beta-G pioneered by Eugene et al. (2002), the gamma-G defined by Zografos and Balakrishnan (2009), the Kw-G family proposed by Cordeiro and Castro (2011) and the Weibull-G introduced by Bourguignon et al. (2014) and so on.

The one parameter Lindley distribution was originally introduced by Lindley (1958) in the context of Bayesian statistics, as a counter example of fiducial statistics. Ghitany et al. (2008) provided a comprehensive account of the statistical properties of the Lindley distribution and established that it performs better than the well-known exponential distribution in many ways. Lindley distribution has only one scale parameter and capable of modeling the data with monotonic increasing failure rate. To increase the flexibility for modeling purposes it will be useful to consider further alternatives of this distribution. Since the shape parameter plays a vital role in describing the various behavior of the distribution, many generalizations of the Lindley distribution have been attempted by researchers under different scenarios. Notable among these generalizations which we are aware of are : Sankaran (1970) discussed the discrete Poisson-Lindley distribution by compounding the Poission distribution and the Lindley distribution. Ghitany et al. (2008) investigated the properties of the zero-truncated Poisson-Lindley distribution. Zakerzadeh and Dolati (2009) introduced and analyzed a three-parameter generalization of the Lindley distribution. Weighted Lindley (WEL) distribution due to Ghitany et al. (2011). Nadarajah et al. (2011) proposed a generalized Lindley distribution (GL) and provided comprehensive account of the mathematical properties of the distribution. Bakouch et al. (2012) extended the Lindley distribution by exponentiation. Exponential Poisson Lindley (EPL) distribution due to Barreto-Souza and Bakouch (2013). Power Lindley (PL) distribution due to Ghitany et al. (2013). Shanker et al. (2013) introduced a two-parameter Lindley distribution of which the one-parameter Lindley distribution is a particular case, for modeling waiting and survival times data. A new weighted Lindley distribution (WL) due to Asgharzadeh et al. (2016).

Many authors have discussed the situations where the data shows decreasing, increasing and upside-down bathtub (UBT) shape hazard rates. For example: Proschan (1963) found that the air-conditioning systems of planes follows decreasing failure rate. Kus (2007) analyzed earthquakes in the last century in North Anatolia fault zone and found that decreasing failure rate distribution fits well. Woosley and Cossman (2007) observed that drugs during clinical development have increasing hazard rates; Saidane et al. (2010) observed that the demand interval in spare parts inventory systems have increasing hazard rates; Koutras (2011) also observed that software degradation times have increasing hazard rates; Lai (2013) investigated the optimum

number of minimal repairs for systems under increasing hazard rates. Efron (1988) analyzed the data set in the context head and neck cancer, in which the hazard rate initially increased, attained a maximum and then decreased before it stabilized owing to a therapy. Bennette (1983) analyzed lung cancer trial data which showed that failure rates were unimodal in nature. Langlands et al. (1997) have studied the breast carcinoma data and found that the mortality reached a peak after some finite period, and then declined gradually.
The aim of this note is to derive a new distribution from the Lindley distribution by $\alpha-$ power transformation as suggested by Mahdavi and Kundu (2016), called Alpha Power Transformed Lindley (αPTL) distribution. This concept of generalization is well established in the statistical literature, see Dey et al. (2017a, 2017b). The proposed distribution encompasses the behavior of and provides better fits than some well known lifetime distributions, such as exponential, Lindley and Weibull distributions. We are motivated to introduce the distribution because (i) it is capable of modeling decreasing, increasing and upside-down bathtub shaped hazard rates; (ii) it can be viewed as a suitable model for fitting the skewed data which may not be properly fitted by other common distributions and can also be used in a variety of problems in different areas such as earthquakes analysis; and (iii) two real data applications show that it compares well with other competing lifetime distributions in modeling earthquakes data.

The rest of the paper is organized as follows. In Sections 2 and 3, we introduce the alpha-power transformed Lindley distribution, and discuss some properties of this family of distributions. In Section 4, maximum likelihood estimators of the unknown parameters are obtained and a small simulation study is also conducted. The analysis of two earthquakes real data sets have been presented in Section 5. Finally, in Section 6, we conclude the paper.

2 Model Description

If $F(x)$ is an absolute continuous distribution function with the probability density function (pdf) $f(x)$, then $F_{APT}(x)$ is also an absolute continuous distribution function with the pdf:

$$f_{ATP}(x) = \begin{cases} \frac{\log\alpha\ f(x)\ \alpha^{F(x)}}{(\alpha-1)} & \text{if } \alpha > 0,\ \alpha \neq 1 \\ f(x) & \text{if } \alpha = 1 \end{cases} \tag{1}$$

It is clear that for $\alpha \neq 1$, $f_{APT}(x)$ is a weighted version of $f(x)$, where the weight function is

$$w(x) = \alpha^{F(x)}. \tag{2}$$

Thus, $f_{APT}(x)$ can be written as

$$f_{APT}(x) = \frac{w(x;\alpha)\ f(x)}{E(w(x;\alpha))} \tag{3}$$

where $w(x;\alpha)$ is non-negative and

$$E[w(x;\alpha)] = \int_{-\infty}^{\infty} w((x;\alpha)\ f(x)dx < \infty \tag{4}$$

Applications of a weighted distribution to biased samples in various areas including medicine, ecology, reliability, and branching processes can be seen in Rao (1965), Patil and Rao (1978), and Gupta and Kundu (2009) and the references therein. In this case the weight function $w(x;\alpha)$ can be increasing or decreasing depending on whether $\alpha > 1$ or $\alpha < 1$.

3 Alpha-Power Transformed Lindley Distribution

Let X be Lindley random variable with parameter $\theta > 0$. Recall that the pdf and cdf associated to X are respectively given by

$$f_\theta(x) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x}, \quad x \geq 0 \tag{5}$$

and

$$F_\theta(x) = 1 - \left(\frac{1+\theta+\theta x}{\theta+1}\right) e^{-\theta x}, \quad x \geq 0, \tag{6}$$

We now introduce the notion of Alpha-Power Transformed Lindley Distribution.
A random variable X follows a Alpha-Power Transformed Lindley Distribution, denoted $X \sim \alpha PTL(\alpha, \theta)$, if its pdf and cdf are given by

$$f_{\alpha PT,\theta}(x) = \begin{cases} \frac{\ln(\alpha) f_\theta(x) \alpha^{F_\theta(x)}}{\alpha-1} & \text{if} \quad \alpha > 0,\ \alpha \neq 1, \\ \\ f_\theta(x) \quad \text{if} \quad \alpha = 1. \end{cases}$$

and

$$F_{\alpha PT,\theta}(x) = \begin{cases} \frac{\alpha^{F_\theta(x)} - 1}{\alpha-1} & \text{if} \quad \alpha > 0,\ \alpha \neq 1, \\ \\ F_\theta(x) \quad \text{if} \quad \alpha = 1, \end{cases}$$

where f_θ and F_θ are the pdf and the cdf of Lindley distribution described in (5) and (6).
A random variable X is said to have αPTL distribution if its probability density function (*pdf*) is of the form

$$f_{APT}(x;\alpha,\theta) = \begin{cases} \frac{\log\alpha}{\alpha-1}(\frac{\theta^2}{\theta+1})(1+x)\ e^{-\theta x}\alpha^{[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}}, & \text{if} \quad x > 0;\ \ \alpha,\ \theta > 0,\ \alpha \neq 1 \\ (\frac{\theta^2}{\theta+1})(1+x)\ e^{-\theta x} & \text{if} \quad x > 0;\ \ \alpha,\ \theta > 0,\ \alpha = 1 \end{cases} \tag{7}$$

The corresponding *cdf*, survival function and hazard rate functions are, respectively, given by

$$F_{APT}(x;\alpha,\theta) = \begin{cases} \frac{\alpha^{[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}} - 1}{\alpha-1}, & \text{if} \quad x > 0;\ \ \alpha,\ \theta > 0,\ \alpha \neq 1 \\ [1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x} & \text{if} \quad x > 0;\ \ \alpha,\ \theta > 0,\ \alpha = 1 \end{cases} \tag{8}$$

$$S_{APT}(x;\alpha,\theta) = \begin{cases} 1 - \frac{\alpha^{[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}} - 1}{\alpha - 1}, & \text{if} \quad x > 0; \ \ \alpha, \ \theta > 0, \ \alpha \neq 1 \\ 1 - [1 - (\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x} & \text{if} \quad x > 0; \ \ \alpha, \ \theta > 0, \ \alpha = 1 \end{cases} \tag{9}$$

and

$$h_{APT}(x;\alpha,\theta) = \begin{cases} \frac{\log\alpha(\frac{\theta^2}{\theta+1})(1+x)\ e^{-\theta x}\alpha^{[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}}}{\alpha-\alpha^{[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}}}, & \text{if} \quad x > 0; \ \ \alpha, \theta > 0, \ \alpha \neq 1 \\ \frac{(\frac{\theta^2}{\theta+1})(1+x)\ e^{-\theta x}}{1-[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}} & \text{if} \quad x > 0; \ \ \alpha, \theta > 0, \ \alpha = 1 \end{cases} \tag{10}$$

Hereafter, a random variable X that follows the distribution in (9) is denoted by $X \sim \alpha PTL(\alpha, \theta)$. Figure 1 shows the various curves for the pdf and the hazard rate function, respectively, of αPTL distribution with various values of α and θ. Figure 1 shows that the hazard function h(x) of αPTL distribution can be decreasing, increasing or upside down bathtub (UBT) shapes. One of the advantages of the αPTL distribution over the Lindley distribution is that the latter cannot model phenomenon showing an upside-down bathtub and decreasing shape failure rates.

Result1 : $\alpha PTL(\alpha, \theta)$ distribution has the following mixture representation for $\alpha > 1$. $\frac{\log(\alpha)}{(\alpha-1)}$ is a decreasing function from 1 to 0, as α varies from 1 to ∞. If $X \sim \alpha PTL(\alpha, \theta)$, then it can be represented as follows:

$$X = \begin{cases} X_1 & \text{with probability} \ \ (\frac{\log\alpha}{\alpha-1}) \\ X_2 & \text{with probability} \ \ 1 - (\frac{\log\alpha}{\alpha-1}) \end{cases} \tag{11}$$

where X_1 and X_2 have the following *pdf*s

$$f(X_1) = \frac{\theta^2}{\theta+1}(1+x)e^{-\theta x} \tag{12}$$

$$f(X_2) = \left[\frac{\log\alpha}{(\alpha-1-\log\alpha)}\right]\left(\frac{\theta^2}{\theta+1}\right)(1+x)e^{-\theta x}\left[\alpha^{[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}} - 1\right] \tag{13}$$

respectively. It is clear from the representation (7) that as α approaches 1, X behaves like an Lindley distribution, and α increases, it behaves like X_2.
Result 2 : If $f(x)$ is a decreasing function, and $\alpha \leq 1$, then $f_{APT}(x)$ is a decreasing function.
Proof: The result easily follows by taking $\log f_{APT}(x)$, and by using the fact that sum of two decreasing functions is a decreasing function.

3.1 On the shape of the density and hazard function

Result 3: If the parent pdf $f(x)$ is log- concave then the density of α-power transformed Lindley pdf will also be log-concave for any choices of $\alpha < 1$ and when $\alpha > 1$, this will hold iff $\log\alpha < \frac{\frac{\partial \log f(x)}{\partial x}}{f(x)}$.

Proof: From Equation (1) on the paper, we can write

$$\ln\left(f_{\alpha PT,\theta}(x)\right) = \ln\frac{\ln(\alpha)}{\alpha-1} + \ln f(x) + F(x)\ln(\alpha).$$

Therefore,

$$\frac{\partial^2 \ln\left(f_{\alpha PT,\theta}(x)\right)}{\partial x^2} = \frac{f"(x)}{f(x)} + f'(x)\left[\ln(\alpha) - \frac{1}{f(x)}\frac{\partial \log f(x)}{\partial x}\right]. \tag{14}$$

Note that, from (1), it is quite obvious that when $\alpha < 1$, the second order derivative is < 0 and for any $\alpha > 1$, it is negative provided $\log\alpha < \frac{\frac{\partial \log f(x)}{\partial x}}{f(x)}$. Hence, the proof.

Result 4: The hazard rate function is increasing for $\alpha > 1$ and $\theta > 2$ while for $\alpha > 1$ and $\theta < 1$ the hazard rate function is decreasing.

Proof: First of all, for notational simplicity let us write $\alpha^{[1-(\frac{1+\theta+\theta x}{\theta+1})]e^{-\theta x}} = u(x)$.
Therefore, $\frac{\partial u(x)}{\partial x} = u'(x) = u(x)\left(\frac{\log\alpha\theta}{\theta+1}\right)e^{-\theta x}(\theta-1)$.
Also, we write $C_1 = \frac{\log\alpha}{\alpha-1}(\frac{\theta^2}{\theta+1})$.
Then, we may rewrite our density function as

$$f(x) = C_1(1+x)e^{-\theta x}u(x).$$

Therefore,

$$f''(x) = C_1 e^{-\theta x}u(x)\left[1 + (1+x)\frac{u'(x)}{u(x)} - \theta(1+x)\right].$$

Hence, $\eta(x) = -\frac{f'(x)}{f(x)} = -\frac{1}{1+x} - \frac{u'(x)}{u(x)} + \theta$. Also,

$$\eta'(x) = \frac{1}{(1+x)^2} + \frac{(u'(x))^2 - u(x)u''(x)}{u^2(x)}. \tag{15}$$

Again,

$$u''(x) = \left(\frac{\log\alpha\theta}{\theta+1}\right)e^{-\theta x}(\theta-1)\left[\frac{u'(x)}{u(x)} - \theta\right].$$

Therefore, the second term in (2), reduces to

$$\frac{(u'(x))^2 - u(x)u''(x)}{u^2(x)} = (2-\theta)\left\{\left(\frac{\log\alpha\theta}{\theta+1}\right)e^{-\theta x}(\theta-1)\right\}^2 + \left(\frac{\log\alpha\theta^2}{\theta+1}\right)e^{-\theta x}(\theta-1).$$

Therefore, from (2), we may write,

$$\eta'(x) = \frac{1}{(1+x)^2} + (2-\theta)\left\{\left(\frac{\log\alpha\theta}{\theta+1}\right)e^{-\theta x}(\theta-1)\right\}^2 + \left(\frac{\log\alpha\theta^2}{\theta+1}\right)e^{-\theta x}(\theta-1). \tag{16}$$

Note that, for $\alpha > 1$ and $\theta > 2$, $\eta'(x) > 0$, therefore from Glaser (1980), hazard function is increasing. Again, for $\alpha > 1$ and $\theta < 1$, from Theorem (b) of Glaser (1980), we can say that the hazard function decreasing. Hence, the proof.

3.2 Moments

Now, we present an infinite sum representation for the nth moment $\mu_n' = E[X^n]$, and consequently obtain the first four moments and variance for the αPTL distribution. Let X be a random variable with the probability density function (7). Calculating moments of X requires the following

Lemma 1. *Let $f(x)$ and $F(x)$ be given by* (7) *and* (8), *respectively. For $a > 0$, $b > 0$, $c > 0$ and $\delta > 0$, let*

$$K(a,b,c,\delta) = \int_0^{\infty} x^c(1+x)\ a^{\left[1-\frac{1+b+bx}{1+b}\right]e^{-bx}} e^{-\delta x} dx.$$

We have

$$K(a,b,c,\delta) = \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l+1}\binom{p}{k}\binom{k}{l}\binom{l+1}{m}\frac{(-1)^k(loga)^p\ b^k}{p!(1+b)^k}\frac{\Gamma(c+l+1)}{(pb+\delta)^{c+l+1}}.$$

Proof. Using the power series expansion, (7), one can write

$$\begin{aligned}
K(a,b,c,\delta) &= \sum_{p=0}^{\infty}\frac{(loga)^p}{p!}\int_0^{\infty} x^c(1+x)\ \left[1-\frac{1+b+bx}{1+b}\right]^p e^{-(pb+\delta)x}dx \\
&= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\frac{(-1)^k(loga)^p\ b^k}{p!(1+b)^k}\binom{p}{k}\binom{k}{l}\int_0^{\infty} x^c(1+x)^{l+1}e^{-(pb+\delta)x}dx \\
&= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l+1}\frac{(-1)^k(loga)^p\ b^k}{p!(1+b)^k}\binom{p}{k}\binom{k}{l}\binom{l+1}{m}\int_0^{\infty} x^{c+l}e^{-(pb+\delta)x}dx.
\end{aligned}$$

The result of the lemma follows by the definition of the gamma function. □

It follows from Lemma 1 that

$$E(X^n) = \frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{\theta+1}\right)K(\alpha,\theta,n,\theta). \tag{17}$$

In particular, the first four moments of X are

$$E(X) = \frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{\theta+1}\right)K(\alpha,\theta,1,\theta),$$

$$E(X^2) = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{\theta + 1}\right) K(\alpha, \theta, 2, \theta),$$

$$E(X^3) = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{\theta + 1}\right) K(\alpha, \theta, 3, \theta)$$

and

$$E(X^4) = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{\theta + 1}\right) K(\alpha, \theta, 4, \theta).$$

The expression (17) can be readily computed numerically using standard statistical software. In numerical applications, a large natural number N can be used in the sums instead of infinity. Several quantities of X (central moments, variance, skewness and kurtosis) can be derived using (17).
The central moments μ_r and cumulants k_r of X can be determined from (17) as

$$\mu_r = \sum_{k=0}^{r} (-1)^k \begin{pmatrix} r \\ k \end{pmatrix} \mu_1'^{r} \mu_{r-k}',$$

and

$$k_r = \mu_r' - \sum_{k=1}^{r-1} \begin{pmatrix} r-1 \\ k-1 \end{pmatrix} k_r \mu_{r-k}',$$

where $k_1 = \mu_1'$. Thus $k_2 = \mu_2' - (\mu_1')^2$, $k_3 = \mu_3' - 3\mu_2'\mu_1' + 2(\mu')_1^3$, $k_4 = \mu_4' - 4\mu_3'\mu_1' - 3(\mu_2')^2 + 12\mu_2'(\mu_1')^2 - 6(\mu_1')^4$, etc. The skewness $\gamma_1 = k_3/k_2^{3/2}$ and kurtosis $\gamma_2 = k_4/k_2^2$ can be calculated from the second, third and fourth standardized cumulants.

Using Equation (17), we obtain the values of mean, median, variance, skewness and kurtosis of the αPTL distribution. These values are displayed in Table 1 for various values of α and θ. It can be noticed from Table 1 that for fixed θ the mean, median, and the variance of the αPTL distribution are increasing functions of α, while the skewness and the kurtosis are decreasing function of α. Also, it is observed that for fixed α, the mean, median, variance, skewness and the kurtosis are increasing function of θ.

Incomplete moments of a distribution are used in measuring inequality: for example, the Lorenz and Bonferroni curves and Gini measures of inequality depend on the incomplete moments. These curves are very useful in economics, demography, insurance, engineering and medicine.
The nth incomplete moment of αPTL distribution is defined by

$$\begin{aligned} m_n(y) &= E\left[X^n | x < y\right] = \int_0^y x^n f(x) dx \\ &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l+1} \frac{(-1)^k (log\alpha)^p \, \theta^k}{p!(1+\theta)^k} \begin{pmatrix} p \\ k \end{pmatrix} \begin{pmatrix} k \\ l \end{pmatrix} \begin{pmatrix} l+1 \\ m \end{pmatrix} \\ &\times \frac{\gamma[n+l+1, \; \theta(p+1)y]}{[\theta(p+1)]^{n+l+1}}, \end{aligned}$$

where $\gamma(a, \; x) = \int_0^x t^{a-1} exp(-t) dt$ denote the complementry incomplete gamma function.

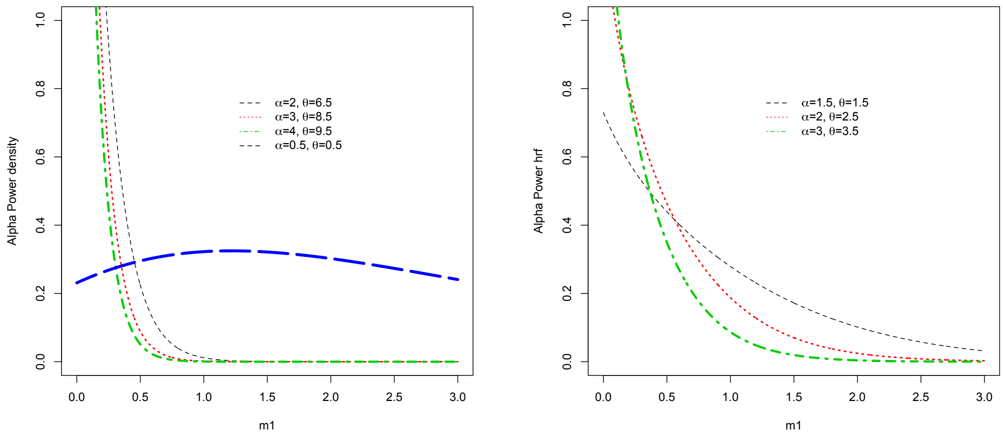

Figure 1: Plot of the Alpha Power Lindley density and hazard rate functions with various values of parameters

Table 1: Mean, median, variance, skewness, and kurtosis of the α-power transformed Lindley distribution for some values of θ and α .

θ	α	Mean	Median	Variance	Skewness	Kurtosis
0.5	0.5	0.4352	0.2970	0.4182	2.8162	17.4352
	2.5	0.7945	0.5762	0.7814	2.0592	11.4302
	5	1.5069	0.6987	1.3471	1.8748	9.1407
	15	1.8402	1.1852	1.6316	1.6952	8.6289
2.5	0.5	1.2954	0.9987	1.1865	1.5964	7.0623
	2.5	1.6732	1.4682	1.5629	1.1936	6.1629
	5	1.9975	1.7891	1.8767	0.9973	5.1372
	15	2.2068	2.0965	2.3056	0.8946	5.0201
5	0.5	1.5926	1.4125	1.4625	1.1058	6.8712
	2.5	2.1582	2.0158	2.0582	1.0121	5.1621
	5	2.4316	2.3624	2.2056	0.9562	5.0182
	15	2.7989	2.5821	2.4168	0.8672	5.0012
15	0.5	2.3462	2.2782	2.2926	0.9863	4.9960
	2.5	3.1604	3.0187	3.0018	0.6054	4.1256
	5	3.4250	3.2162	3.1253	0.6011	4.0621
	15	3.6925	3.6289	3.3197	0.5982	4.1850

3.3 Conditional moments

For lifetime models, it is also of interest to know what $E(X^n|X > x)$ is. Calculating these moments requires the following

Lemma 2. *Let* $f(x)$ *and* $F(x)$ *be given by* (7) *and* (8), *respectively. For* $a > 0$, $b > 0$, $c > 0$, *and* $\delta > 0$, *let*

$$L(a, b, c, \delta, t) = \int_t^\infty x^c (1 + x)\ a^{\left[1-\frac{1+b+bx}{1+b}\right]e^{-bx}} e^{-\delta x} dx.$$

We have

$$\begin{aligned} L(a, b, c, \delta) &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l+1} \binom{p}{k}\binom{k}{l}\binom{l+1}{m} \\ &\times \frac{(-1)^k (loga)^a\ b^k}{p!(1+b)^k} \frac{\Gamma[c+l+1, (pb+\delta)t]}{(pb+\delta)^{c+l+1}}, \end{aligned} \tag{18}$$

where $\Gamma(a,\ x) = \int_x^\infty t^{a-1} exp(-t)dt$ *denote the complementry incomplete gamma function. If* c *is an integer then* (18) *can be simplified to*

$$\begin{aligned} L(a, b, c, \delta) &= \sum_{p=0}^{\infty}\sum_{k=0}^{\infty}\sum_{l=0}^{k}\sum_{m=0}^{l+1} \binom{p}{k}\binom{k}{l}\binom{l+1}{m} \frac{(-1)^k (loga)^a\ b^k}{p!(1+b)^k} \\ &\times \frac{(c+l)! e^{-(pb+\delta)t}}{(pb+\delta)^{c+l+1}} \sum_{q=0}^{c+l} \frac{(pb+\delta)^q}{q!}. \end{aligned}$$

Proof. The proof of (18) is similar to the proof of Lemma 4.1, but using the definition of the complementary gamma function. The final relation follows by using the fact

$$\Gamma(a, x) = (a-1)! exp(-x) \sum_{i=0}^{a-1} \frac{x^i}{i!}.$$

□

Using Lemma 2, it is easily seen that

$$E(X^n|X > x) = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{1+\theta}\right)\frac{1}{[1 - V(x)]} L(\alpha, \theta, n, \theta, x), \tag{19}$$

where

$$V(x) = \frac{\alpha^{\left[1-\frac{1+\theta+\theta x}{1+\theta}\right]e^{-\theta x}} - 1}{\alpha - 1}. \tag{20}$$

In particular,

$$E(X|X > x) = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{1+\theta}\right)\frac{1}{[1 - V(x)]} L(\alpha, \theta, 1, \theta, x),$$

$$E(X^2|X>x)=\frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{1+\theta}\right)\frac{1}{[1-V(x)]}L(\alpha,\theta,2,\theta,x),$$

$$E(X^3|X>x)=\frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{1+\theta}\right)\frac{1}{[1-V(x)]}L(\alpha,\theta,3,\theta,x)$$

and

$$E(X^4|X>x)=\frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{1+\theta}\right)\frac{1}{[1-V(x)]}L(\alpha,\theta,4,\theta,x).$$

An application of the conditional moments is the mean residual life (MRL). MRL function is the expected remaining life, $X-x$, given that the item has survived to time x. Thus, in life testing situations, the expected additional lifetime given that a component has survived until time x is called the (MRL). The MRL function in terms of the first conditional moment as

$$m_X(x)=E(X-x|X>x)=\frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{1+\theta}\right)\frac{1}{[1-V(x)]}L(\alpha,\theta,1,\theta,x)-x.$$

where $L(\alpha,\theta,1,\theta,x)$ can be obtained from (18) where $n=1$.
Another application of the conditional moments is the mean deviations about the mean and the median. They are used to measure the dispersion and the spread in a population from the center. If we denote the median by M, then the mean deviations about the mean and the median can be calculated as

$$\delta_\mu \quad = \quad \int_0^\infty |x-\mu|\, f(x)dx = 2\mu F(\mu)-2\mu+2\frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{1+\theta}\right)L(\alpha,\theta,1,\theta,\mu)$$

and

$$\delta_M \quad = \quad \int_0^\infty |x-M|\, f(x)dx = 2\frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{1+\theta}\right)L(\alpha,\theta,1,\theta,M)-\mu$$

respectively. Where $L(\alpha,\theta,1,\theta,\mu)$ and $L(\alpha,\theta,1,\theta,M)$ can obtained from (9). Also, $F(\mu)$ and $F(M)$ are easily calculated from (8).

3.4 L moments

Some other important measures useful for lifetime models are the L moments due to Hoskings (1990). It can be shown using Lemma 1 that the kth L moment is

$$L_r=\sum_{j=0}^{r-1}(-1)^{r-1-j}\left(\begin{array}{c} r-1 \\ j \end{array}\right)\left(\begin{array}{c} r-1+j \\ j \end{array}\right)\lambda_j,$$

where

$$\lambda_r=\frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{1+\theta}\right)K(\alpha(r+1),\ \theta,\ 1,\ \theta).$$

In particular

$$L_1 = \lambda_0,$$

$$L_2 = 2\lambda_1 - \lambda_0,$$

$$L_3 = 6\lambda_2 - 6\lambda_1 + \lambda_0$$

and

$$L_4 = 20\lambda_3 - 30\lambda_2 + 12\lambda_1 - \lambda_0,$$

where

$$\lambda_0 = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{1+\theta}\right) K(\alpha,\ \theta,\ 1,\ \theta),$$

$$\lambda_1 = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{1+\theta}\right) K(2\alpha,\ \theta,\ 1,\ \theta),$$

$$\lambda_2 = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{1+\theta}\right) K(3\alpha,\ \theta,\ 1,\ \theta)$$

and

$$\lambda_3 = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{1+\theta}\right) K(4\alpha,\ \theta,\ 1,\ \theta).$$

The L moments have several advantages over ordinary moments: for example, they apply for any distribution having finite mean; no higher-order moments need be finite.

3.5 MGF, CHF and CGF

Let X denote a random variable with the probability density function (7). It follows from Lemma 1 that the moment generating function of X, $M(t) = E[e^{(tx)}]$, is given by

$$M(t) = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{\theta+1}\right) K(\alpha,\ \theta,\ 0,\ \theta - t),$$

for $t < \theta$. So, the characteristic function of X, $\phi(t) = E[e^{(itX)}]$, and the cumulant generating function of X, $K(t) = log\phi(t)$, are given by

$$\phi(t) = \frac{log\alpha}{\alpha - 1}\left(\frac{\theta^2}{\theta+1}\right) K(\alpha,\ \theta,\ 0,\ \theta - it),$$

and

$$K(t) = log\left(\frac{log\alpha}{\alpha - 1}\right) + log\left(\frac{\theta^2}{\theta+1}\right) + logK(\alpha,\ \theta,\ 0,\ \theta - it),$$

respectively, where $i = \sqrt{-1}$.

3.6 Bonferroni curve and Lorenz curve

Bonferroni curve was proposed by Bonferroni (1930) and Lorenz curve by Lorenz (1905). The two curves are used to measure the inequality of the distribution of a variable. Also Gini index is another method for measuring inequality of the distribution of variable. They are applicable in the field of economics but also in other areas like reliability, medical and demography. These index are defined as:

$$B(p) = \frac{1}{p\mu}\int_0^q xf(x)dx$$

and

$$L(p) = \frac{1}{\mu}\int_0^q xf(x)dx,$$

respectively, where $\mu = E(X)$ and $q = F^{-1}(p)$. which can be computed numerically. If X has the probability density function, (7), then, by Lemma 2, one can calculate Bonferroni of the αPTL distribution is

$$B(p) \quad = \quad \frac{1}{p\mu}\left[\mu - \frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{\theta+1}\right)L(\alpha,\ \theta,\ 1,\ \theta, q)\right].$$

Traditionally, applications of the Lorenz curve and the Gini index are in income modeling and related areas, see Kleiber and Kotz (2003). The list of applications is too long to cite, but these concepts have also received applications in other areas such as hierarchy theory for digraphs (Egghe (2002)); depression and cognition (Maldonado et al. (2007)); disease risk to optimize health benefits under cost constraints (Gail (2009)); seasonal variation of environmental radon gas (Groves-Kirkby et al. (2009)); statistical non uniformity of sediment transport rate (Radice (2009)).

The Lorenz curve of the αPTL distribution is

$$L(p) \quad = \quad \frac{1}{\mu}\left[\mu - \frac{log\alpha}{\alpha-1}\left(\frac{\theta^2}{\theta+1}\right)L(\alpha,\ \theta,\ 1,\ \theta, q)\right].$$

The area between the line $L(F(x)) = F(x)$ and the Lorenz curve, known as the area of concentration, may be regarded as a measure of inequality of income, so it is important in insurance, economics and other fields like reliability and medicine.

3.7 Entropies

Entropy is used to measure the randomness of systems and it is widely used in areas like physics, molecular imaging of tumors and sparse kernel density estimation. If X has the probability distribution function $f(\cdot)$ Rényi entropy (Rényi (1961)) defined by

$$H_\delta(x) \quad = \quad \frac{1}{1-\delta}log\left(\int_0^\infty f^\delta(x)dx\right), \qquad \delta > 0, \qquad \delta \neq 1. \tag{21}$$

Some recent applications of the Rényi entropy have been considered such as sparse kernel density estimations (Han et al. (2011)); high-resolution scalar quantization (Kreitmeier and Linder (2011)); estimation of the number of components of a multicomponent non stationary signal (Sucic et al. (2011)); identification of cardiac autonomic neuropathy in diabetes (Jelinek et al. (2007)); and signal segmentation in time-frequency plane (Popescu and Aiordachioaie (2013)). Suppose X has the probability density function (7). Then, one can calculate

$$
\begin{aligned}
\int_0^\infty f^\delta(x)dx &= \left(\frac{\theta^2 log\alpha}{(\alpha-1)(1+\theta)}\right)^\delta \int_0^\infty (1+x)^\delta \alpha^{\delta\left[1-\frac{1+\theta+\theta x}{\theta+1}\right]e^{-\theta x}} e^{-\delta\theta x} dx \\
&= \left(\frac{\theta^2 log\alpha}{(\alpha-1)(1+\theta)}\right)^\delta \sum_{p=0}^\infty \frac{(\delta log\alpha)^p}{p!} \int_0^\infty (1+x)^\delta \left[1-\frac{1+\theta+\theta x}{\theta+1}\right]^p e^{-(p+\delta)\theta x} dx \\
&= \left(\frac{\theta^2 log\alpha}{(\alpha-1)(1+\theta)}\right)^\delta \sum_{p=0}^\infty \sum_{k=0}^\infty \frac{(-1)^k(\delta log\alpha)^p}{p!(1+\theta)^k} \begin{pmatrix} p \\ k \end{pmatrix} \\
&\times \int_0^\infty (1+x)^\delta (1+\theta+\theta x)^k exp\{-(p+\delta)\theta x\} dx \\
&= \left(\frac{\theta^2 log\alpha}{(\alpha-1)(1+\theta)}\right)^\delta \sum_{p=0}^\infty \sum_{k=0}^\infty \frac{(-1)^k(\delta log\alpha)^p}{p!(1+\theta)^k} \begin{pmatrix} p \\ k \end{pmatrix} \sum_{l=0}^k \begin{pmatrix} k \\ l \end{pmatrix} \theta^k \\
&\times \int_0^\infty (1+x)^{\delta+l} exp\{-(p+\delta)\theta x\} dx \\
&= \left(\frac{\theta^2 log\alpha}{(\alpha-1)(1+\theta)}\right)^\delta \sum_{p=0}^\infty \sum_{k=0}^\infty \frac{(-1)^k(\delta log\alpha)^p}{p!(1+\theta)^k} \begin{pmatrix} p \\ k \end{pmatrix} \sum_{l=0}^k \begin{pmatrix} k \\ l \end{pmatrix} \theta^k \\
&\times \ exp\{(p+\delta)\theta\} \int_1^\infty y^{\delta+l} exp\{-(p+\delta)\theta y\} dy \\
&= \left(\frac{\theta^2 log\alpha}{(\alpha-1)(1+\theta)}\right)^\delta \sum_{p=0}^\infty \sum_{k=0}^\infty \frac{(-1)^k(\delta log\alpha)^p}{p!(1+\theta)^k} \begin{pmatrix} p \\ k \end{pmatrix} \sum_{l=0}^k \begin{pmatrix} k \\ l \end{pmatrix} \theta^k \\
&\times \ exp\{(p+\delta)\theta\}\{(l+\delta)\theta\}^{-p-\delta-1}\ \Gamma(l+\beta+1,\ (p+\delta)\theta),
\end{aligned}
$$

the final step follows by the definition the complementary incomplete gamma function. So, one obtains the Rényi entropy as

$$
\begin{aligned}
H_\delta(x) &= \frac{\delta}{1-\delta} log\left(\frac{\theta^2 log\alpha}{(\alpha-1)(1+\theta)}\right) \\
&+ \frac{1}{1-\delta} log\Big\{\sum_{p=0}^\infty \sum_{k=0}^\infty \frac{(-1)^k(\delta log\alpha)^p}{p!(1+\theta)^k} \begin{pmatrix} p \\ k \end{pmatrix} \sum_{l=0}^k \begin{pmatrix} k \\ l \end{pmatrix} \theta^k \\
&\times \ exp\{(p+\delta)\theta\}\{(l+\delta)\theta\}^{-p-\delta-1}\ \Gamma(l+\beta+1,\ (p+\delta)\theta)\Big\}. \qquad (22)
\end{aligned}
$$

Shannon entropy (Shannon (1951)) defined by $E[-logf(x)]$ is the particular case of (21) for $\delta \uparrow 1$. Limiting $\delta \uparrow 1$ in (22) and using L'Hospital's rule, one obtains after considerable algebraic manipulation that

$$
\begin{aligned}
E[-logf(x)] &= -log\left(\frac{\theta^2\ log\alpha}{(\alpha-1)(1+\theta)}\right)+(p+1)\theta E(X)+\sum_{k=1}^{\infty}\frac{(-1)^k}{k}E(X^k)\\
&- \ log\left(\sum_{p=0}^{\infty}\frac{(log\alpha)^p}{p!}\right)+p\sum_{l=1}^{\infty}\frac{\theta^l}{l(1+\theta)^l}E(X^l).
\end{aligned}
$$

Finally, consider the cumulative residual entropy (Rao *et al.* 2004)defined by

$$\Im_c=\int Pr(X>x)logPr(X>x)dx. \tag{23}$$

Using the series expansions,

$$(1-x)^{n-1}=\sum_{p=0}^{\infty}(-1)^p\left(\begin{array}{c}n-1\\ p\end{array}\right)x^p \tag{24}$$

and

$$log(1-x)=-\sum_{p=1}^{\infty}\frac{x^p}{p},$$

one calculate (23) as

$$
\begin{aligned}
\Im_c &= \sum_{i=1}^{\infty}\frac{1}{i}\int_0^{\infty}\left[\frac{\alpha^{\left[1-\frac{1+\theta+\theta x}{1+\theta}\right]e^{-\theta x}}-1}{\alpha-1}\right]^i\left[1-\frac{\alpha^{\left[1-\frac{1+\theta+\theta x}{1+\theta}\right]e^{-\theta x}}-1}{\alpha-1}\right]dx\\
&= \sum_{i=1}^{\infty}\frac{1}{i}\left\{\int_0^{\infty}\left[\frac{\alpha^{\left[1-\frac{1+\theta+\theta x}{1+\theta}\right]e^{-\theta x}}-1}{\alpha-1}\right]^i-\left[\frac{\alpha^{\left[1-\frac{1+\theta+\theta x}{1+\theta}\right]e^{-\theta x}}-1}{\alpha-1}\right]^{i+1}\right\}dx\\
&= \sum_{i=1}^{\infty}\frac{1}{i(\alpha-1)^i}\sum_{k=0}^{i}(-1)^{i-k}\left(\begin{array}{c}i\\ k\end{array}\right)\int_0^{\infty}\alpha^{i\left[1-\frac{1+\theta+\theta x}{1+\theta}\right]e^{-\theta x}}dx\\
&- \sum_{i=1}^{\infty}\frac{1}{i(\alpha-1)^i}\sum_{k=0}^{i+1}(-1)^{i+1-k}\left(\begin{array}{c}i+1\\ k\end{array}\right)\int_0^{\infty}\alpha^{(i+1)\left[1-\frac{1+\theta+\theta x}{1+\theta}\right]e^{-\theta x}}dx\\
&= \sum_{i=1}^{\infty}\frac{1}{i(\alpha-1)^i}\sum_{l=0}^{\infty}\sum_{m=0}^{l}\sum_{q=0}^{m}\left(\begin{array}{c}l\\ m\end{array}\right)\left(\begin{array}{c}m\\ q\end{array}\right)\frac{(-1)^m\theta^{q\theta}e^{l\theta}\Gamma(q+1,\ l\theta)}{(\theta+1)^m\ l}\\
&\times \left\{\sum_{k=0}^{i}(-1)^{i-k}\left(\begin{array}{c}i\\ k\end{array}\right)\frac{(ilog\alpha)^l}{(l\theta)^q}-\sum_{k=0}^{i+1}(-1)^{i+1-k}\left(\begin{array}{c}i+1\\ k\end{array}\right)\frac{[(i+1)log\alpha]^l}{(l\theta)^{q+1}}\right\}.
\end{aligned}
$$

3.8 Stochastic ordering

Ordering of distributions, particularly among lifetime distributions play an important role in the statistical literature. Johnson et al. (1995) have a major section on ordering of different lifetime distributions. Here we consider four different stochastic orders, namely, the usual, the hazard rate, the mean residual life, and the likelihood ratio order for two independent αPTL

random variables under a restricted parameter space. It may be recalled that if a family has a likelihood ratio ordering, it has the monotone likelihood ratio property. This implies there exists a uniformly most powerful test for any one sided hypothesis when the other parameters are known. If X and Y are independent random variables with CDFs F_X and F_Y respectively, then X is said to be smaller than Y in the

- stochastic order ($X \leq_{st} (Y)$) if $F_X(x) \geq F_Y(x)$ for all x
- hazard rate order ($X \leq_{hr} (Y)$) if $h_X(x) \geq h_Y(x)$ for all x
- mean residual life order ($X \leq_{mrl} (Y)$) if $m_X(x) \geq m_Y(x)$ for all x
- likelihood ratio order ($X \leq_{lr} (Y)$) if $\frac{f_X(x)}{f_Y(x)}$ decreases in x.

The following results due to Shaked and Shanthikumar (1994) are well known for establishing stochastic ordering of distributions.

$$\begin{array}{ccc} X \leq_{lr} Y \Rightarrow X & \leq_{hr} & Y \Rightarrow X \leq_{mlr} Y \\ & \Downarrow & \\ & X \leq_{st} Y & \end{array}$$

The αPTL distribution is ordered with respect to the strongest 'likelihood ratio' ordering as shown in the following theorem. It shows the flexibility of two parameter αPTL distribution.

Theorem 1: Let $X \sim \alpha PTL(\alpha_1, \theta_1)$ and $Y \sim \alpha PTL(\alpha_2, \theta_2)$. If $\alpha_1 = \alpha_1 = \alpha$ and $\theta_1 \geq \theta_2$, then $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$.

Proof. The likelihood ratio is

$$\frac{f_X(x)}{f_Y(x)} = \frac{\theta_2^2(\alpha_1 - 1)(\theta_1 + 1)log\alpha_2(1 + x)\ e^{-\theta_2 x}\alpha_2^{\left[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}\right]e^{-\theta_2 x}}}{\theta_1^2(\alpha_2 - 1)(\theta_2 + 1)log\alpha_1(1 + x)\ e^{-\theta_1 x}\alpha_1^{\left[1-\frac{1+\theta_1+\theta_1 x}{1+\theta_1}\right]e^{-\theta_1 x}}}$$

thus,

$$\begin{aligned} \frac{d}{dx}log\frac{f_X(x)}{f_Y(x)} &= (\theta_1 - \theta_2) + \theta_2 e^{-\theta_2 x}\left[1 - \frac{1+\theta_2+\theta_2 x}{1+\theta_2}\right]log\alpha + \frac{\theta_2}{1+\theta_2}e^{-\theta_2 x}log\alpha_2 \\ &- \theta_1 e^{-\theta_1 x}\left[1 - \frac{1+\theta_1+\theta_1 x}{1+\theta_1}\right]log\alpha - \frac{\theta_1}{1+\theta_1}e^{-\theta_1 x}log\alpha_1. \end{aligned}$$

Now if $\alpha_1 = \alpha_2 = \alpha$ and $\theta_1 \geq \theta_2$ then $\frac{d}{dx}log\frac{f_X(x)}{f_Y(x)} \leq 0$, which implies that $X \leq_{lr} Y$ and hence $X \leq_{lr} Y$, $X \leq_{hr} Y$, $X \leq_{mrl} Y$ and $X \leq_{st} Y$. □

3.9 Order statistics

Suppose $X_1, X_2, \ldots, X_n$ is a random sample from (7). Let $X_{1:n}, X_{2:n}, \ldots, X_{n:n}$ denote the corresponding order statistics. It is well known that the probability density function and the cumulative distribution function of the r^th order statistic, say $Y = X_{r:n}$, are given by

$$\begin{aligned} f_Y(y) &= \frac{n!}{(r-1)!(n-r)!} F^r(y)[1-F^r(y)]^{n-r} f(y) \\ &= \frac{n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r} (-1)^u \begin{pmatrix} n-r \\ u \end{pmatrix} F^{r-1+u}(y) f(y) \end{aligned}$$

and

$$F_Y(y) = \sum_{j=r}^{n} \begin{pmatrix} n \\ j \end{pmatrix} F^j(y)[1-F(y)]^{n-j} = \sum_{j=r}^{n}\sum_{u=0}^{n-j} \begin{pmatrix} n \\ j \end{pmatrix} \begin{pmatrix} n-j \\ u \end{pmatrix} (-1)^u F^{j+u}(y),$$

respectively, for $r = 1, 2, \ldots, n$, It follows from (7) and (8) that

$$f_Y(y) = \frac{\frac{log\alpha}{(\alpha-1)} \left(\frac{\theta^2}{1+\theta}\right) n!}{(r-1)!(n-r)!} \sum_{u=0}^{n-r} (-1)^u \begin{pmatrix} n-r \\ u \end{pmatrix} (1+y)\ e^{-\theta y} \alpha^{[1-\frac{1+\theta+\theta y}{1+\theta}]e^{-\theta y}}\ [V(y)]^{r-1+u}$$

and

$$F_Y(y) = \sum_{j=r}^{n}\sum_{u=0}^{n-j} \begin{pmatrix} n \\ j \end{pmatrix} \begin{pmatrix} n-j \\ u \end{pmatrix} (-1)^u [V(y)]^{j+u},$$

where $V(\ .\ )$ is given by (24). The q^{th} moment of Y can be expressed as

$$\begin{aligned} E\left(Y^q\right) &= \frac{log\alpha\ n!}{(r-1)!(n-1)!} \sum_{u=0}^{n-r}\sum_{v=0}^{r-1+u} \begin{pmatrix} n-r \\ u \end{pmatrix} \begin{pmatrix} r-1+u \\ v \end{pmatrix} \frac{(-1)^{u+r-1+v}}{(\alpha-1)^{r+u+1}} \\ &\times \sum_{k=0}^{\infty}\sum_{l=0}^{k} \begin{pmatrix} k \\ l \end{pmatrix} \frac{(-1)^l[(1+v)log\alpha]^k}{k!} \sum_{m=0}^{l}\sum_{p=0}^{m+1} \begin{pmatrix} l \\ m \end{pmatrix} \begin{pmatrix} m+1 \\ p \end{pmatrix} \\ &\times \frac{\theta^{m+2}\ \Gamma(p+q+1)}{(1+\theta)^{l+1}[(k+1)\theta]^{p+q+1}}, \end{aligned}$$

for $q \geq 1$.

In order to derive the asymptotic distribution of the sample minima $X_{1:n}$, we consider Theorem 8.3.6 of Arnold et al.(2008). Observe that, since $G^{-1}(0) = 0$, it follows from the theorem that the asymptotic distribution of the sample minima $X_{1:n}$ is not of Fréchet type. The asymptotic distribution of $X_{1:n}$ will be of Weibull type with parameter $\delta > 0$ if

$$\lim_{\varepsilon \to 0_+} \frac{G(\varepsilon x)}{G(\varepsilon)} = x^\delta, \quad \text{for all} \quad x > 0.$$

By using L'Hôpital's rule, it follows that

$$\begin{aligned}
\lim_{\varepsilon\to 0_+}\frac{G(\varepsilon x)}{G(\varepsilon)} &= x\lim_{\varepsilon\to 0_+}\frac{g(\varepsilon x)}{g(\varepsilon)}\\
&= x\lim_{\varepsilon\to 0_+}\frac{\frac{\log\alpha}{\alpha-1}(\frac{\theta^2}{\theta+1})(1+x\varepsilon)\ e^{-\theta x\varepsilon}\alpha^{[1-(\frac{1+\theta+\theta x\varepsilon}{\theta+1})]e^{-\theta x\varepsilon}}}{\frac{\log\alpha}{\alpha-1}(\frac{\theta^2}{\theta+1})(1+\varepsilon)\ e^{-\theta\varepsilon}\alpha^{[1-(\frac{1+\theta+\theta\varepsilon}{\theta+1})]e^{-\theta\varepsilon}}}\\
&= x
\end{aligned}$$

Hence we obtain that the asymptotic distribution of the sample minima $X_{1:n}$ is of the Weibull type with shape parameter 1.

The asymptotic distribution of the sample maxima $X_{n:n}$, can be viewed as $G_n(x)$, where $G_n(x) = 1 - G_1(-x)$, where $G_1(.)$ is the c.d.f of $X_{1:n}$.

3.10 Stress Strength Reliability

Let X be the strength of a system which is subjected to a stress Y, and if X follows $\alpha PTL(\alpha_1,\theta_1)$ and Y follows $\alpha PTL(\alpha_2,\theta_2)$, provided X and Y are statistically independent random variables, then $R = P(Y < X)$. It has many applications especially in engineering concepts such as structures, deterioration of rocket motors, static fatigue of ceramic components, fatigue failure of aircraft structures, and the aging of concrete pressure vessels. In the area of stress strength models there has been a large amount of work as regards estimation of the reliability R when X_1 and X_2 are independent random variables belonging to the same univariate family of distributions and its algebraic form has been worked out for the majority of the well-known standard distributions. Here, we derive the reliability R when X_1 and X_2 are independent variables distributed according to (7) with parameters (α_1,θ_1) and (α_2,θ_2) respectively.

Several representations can be derived for the reliability R. Firstly, note from (7) and (8) that

$$\begin{aligned}
R &= \frac{log\alpha}{(\alpha_1-1)}\left(\frac{\theta^2}{1+\theta}\right)\int_0^\infty(1+x)\ e^{-\theta_1 x}\ \alpha_1^{[1-\frac{1+\theta_1+\theta_1 x}{1+\theta_1}]e^{-\theta_1 x}}\left[\frac{\alpha_2^{[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}]e^{-\theta_2 x}}-1}{\alpha_2-1}\right]dx\\
&= \frac{log\alpha}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta^2}{1+\theta}\right)\int_0^\infty(1+x)\ e^{-\theta_1 x}\ \alpha_1^{[1-\frac{1+\theta_1+\theta_1 x}{1+\theta_1}]e^{-\theta_1 x}}\ \alpha_2^{[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}]e^{-\theta_2 x}}dx\\
&- \frac{log\alpha}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta^2}{1+\theta}\right)\int_0^\infty(1+x)\ e^{-\theta_1 x}\ \alpha_1^{[1-\frac{1+\theta_1+\theta_1 x}{1+\theta_1}]e^{-\theta_1 x}}dx\\
&= \frac{log\alpha}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta^2}{1+\theta}\right)\{I_1-I_2\},
\end{aligned}$$

where

$$I_1 = \int_0^\infty(1+x)\ e^{-\theta_1 x}\ \alpha_1^{[1-\frac{1+\theta_1+\theta_1 x}{1+\theta_1}]e^{-\theta_1 x}}\ \alpha_2^{[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}]e^{-\theta_2 x}}dx.$$

Applying the power series expansion, one can obtain the representation

$$
\begin{aligned}
I_1 &= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(log\alpha_1)^i\ (log\alpha_2)^j}{i!\ j!\ (1+\theta_1)^i\ (1+\theta_2)^j}\int_0^{\infty}(1+x)\ [1+\theta_1+\theta_1 x]^i \\
&\times\ [1+\theta_2+\theta_2 x]^j\ e^{-\theta_1(i+1)x-j\theta_2 x}dx. \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\binom{i}{k}\binom{j}{l}\frac{(log\alpha_1)^i\ (log\alpha_2)^j\ \theta_1^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l} \\
&\times\ \int_0^{\infty}(1+x)\ x^{k+l}e^{-\theta_1(i+1)x-j\theta_2 x}dx \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\binom{i}{k}\binom{j}{l}\frac{(log\alpha_1)^i\ (log\alpha_2)^j\ \theta_1^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l} \\
&\times\ \left[\int_0^{\infty} x^{k+l}e^{-\theta_1(i+1)x-j\theta_2 x}dx+\int_0^{\infty} x^{k+l+1}e^{-\theta_1(i+1)x-j\theta_2 x}dx\right] \\
&= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\binom{i}{k}\binom{j}{l}\frac{(log\alpha_1)^i\ (log\alpha_2)^j\ \theta_1^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l} \\
&\times\ \frac{(k+l)!}{[\theta_1(i+1)+\theta_2 j]^{k+l+1}}\left[1+\frac{k+l+1}{\theta_1(i+1)+\theta_2 j}\right],
\end{aligned}
$$

where the final step follows by the definition of the gamma function.
Similarly, we obtain

$$
I_2 = \sum_{i=0}^{\infty}\sum_{j=0}^{i}\binom{i}{j}\frac{(log\alpha_1)^i\ \theta_1^j}{i!\ (1+\theta_1)^j}\frac{j!}{[\theta_1(i+1)]^{j+1}}\left[1+\frac{j+1}{\theta_1(i+1)}\right].
$$

3.11 Distribution of sums, differences, ratios and products

In this subsection, we give expressions for the distribution of sums, differences, products and ratios of independent random variables distributed according to the αPTL distribution.

3.11.1 Distribution of Sums

Let X_1 and X_2 be two independent random variables such that $X_1 \sim \alpha PTL(\alpha_1;\theta_1)$ and $X_2 \sim \alpha PTL(\alpha_2;\theta_2)$. Then, the *pdf* of the random variable $S = X_1 + X_2$ is given by

$$
\begin{aligned}
f_{X_1+X_2}(s) &= \frac{log\alpha_1\ log\alpha_2}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\int_0^s(1+s-x)\ e^{-\theta_1(s-x)} \\
&\times\ \alpha_1^{\left[1-\frac{1+\theta_1+\theta_1(s-x)}{1+\theta_1}\right]e^{-\theta_1(s-x)}}\ (1+x)\ e^{-\theta_2 x}\ \alpha_2^{\left[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}\right]e^{-\theta_2 x}}dx \\
&= \frac{1}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{(log\alpha_1)^{i+1}(log\alpha_2)^{j+1}}{i!\ j!\ (1+\theta_1)^i\ (1+\theta_2)^j} \\
&\times\ \int_0^s(1+s-x)\ e^{-\theta_1(s-x)}[1+\theta_1+\theta_1(s-x)]^i\ e^{-i\theta_1(s-x)} \\
&\times\ (1+x)\ e^{-\theta_2 x}[1+\theta_2+\theta_2 x]^j\ e^{-j\theta_2 x}dx
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\begin{pmatrix} i \\ k \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix} e^{-(i+1)\theta_1 s} \\
&\times \frac{(log\alpha_1)^{i+1}(log\alpha_2)^{j+1}\ \theta_1^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l}\int_0^s (1+s-x)\ (1+x)\ x^l\ (s-x)^k e^{-\theta_2(j+1)x+\theta_1(i+1)x}dx \\
&= \frac{1}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\begin{pmatrix} i \\ k \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix} e^{-(i+1)\theta_1 s} \\
&\times \frac{(log\alpha_1)^{i+1}(log\alpha_2)^{j+1}\ \theta_1^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l}\Big\{\int_0^s\ (s-x)^k\ (1+x)\ x^l e^{-\theta_2(j+1)x+\theta_1(i+1)x}dx \\
&+ \int_0^s\ (s-x)^{k+1}\ (1+x)\ x^l e^{-\theta_2(j+1)x+\theta_1(i+1)x}dx\Big\} \\
&= \frac{1}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\begin{pmatrix} i \\ k \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix} e^{-(i+1)\theta_1 s} \\
&\times \frac{(log\alpha_1)^{i+1}(log\alpha_2)^{j+1}\ \theta_1^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l}\left[\sum_{m=0}^{k}(-1)^k\begin{pmatrix} k \\ m \end{pmatrix}s^{k-m}+\sum_{m=0}^{k+1}(-1)^{k+1}\begin{pmatrix} k+1 \\ m \end{pmatrix}s^{k+1-m}\right] \\
&\times \left[\frac{\gamma(m+l+1,\ (\theta_2(j+1)-(i+1)\theta_2))s}{[\theta_2(j+1)-(i+1)\theta_1]^{m+l+1}}+\frac{\gamma(m+l+2,\ (\theta_2(j+1)-(i+1)\theta_2))s}{[\theta_2(j+1)-(i+1)\theta_1]^{m+l+2}}\right].
\end{aligned}
$$

where the final step follows from the definition of the incomplete gamma function.

3.11.2 Distribution of differences

Let X_1 and X_2 be two independent random variables such that $X_1 \sim \alpha PTL(\alpha_1;\theta_1)$ and $X_2 \sim \alpha PTL(\alpha_2;\theta_2)$. Then, the *pdf* of the random variable $U = X_1 - X_2$ is given by

$$
\begin{aligned}
f_{X_1-X_2}(u) &= \frac{log\alpha_1\ log\alpha_2}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\int_0^{\infty}(1+u+x)\ e^{-\theta_1(u+x)} \\
&\times\ \alpha_1^{\left[1-\frac{1+\theta_1+\theta_1(u+x)}{1+\theta_1}\right]e^{-\theta_1(u+x)}}\ (1+x)\ e^{-\theta_2 x}\ \alpha_2^{\left[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}\right]e^{-\theta_2 x}}\ dx \\
&= \frac{1}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\begin{pmatrix} i \\ k \end{pmatrix}\begin{pmatrix} j \\ l \end{pmatrix} e^{-(i+1)\theta_1 u} \\
&\times \frac{(log\alpha_1)^{i+1}(log\alpha_2)^{j+1}\ \theta_1^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l}\left[\sum_{m=0}^{k}\begin{pmatrix} k \\ m \end{pmatrix}u^{k-m}+\sum_{m=0}^{k+1}\begin{pmatrix} k+1 \\ m \end{pmatrix}u^{k+1-m}\right] \\
&\times \left[\frac{(m+l!)}{[\theta_2(j+1)+(i+1)\theta_1]^{m+l+1}}+\frac{(m+l+2)!}{[\theta_2(j+1)+(i+1)\theta_1]^{m+l+2}}\right],
\end{aligned}
$$

where the final step follows from the definition of the gamma function.

3.11.3 Distribution of ratio

Let X_1 and X_2 be two independent random variables such that $X_1 \sim \alpha APTL(\alpha_1;\theta_1)$ and $X_2 \sim \alpha PTL(\alpha_1;\theta_1)$. Then, the *pdf* of the random variable $R = \frac{X_1}{X_2}$ is given by

$$\begin{aligned}
f_{X_1/X_2}(r) &= \frac{log\alpha_1\ log\alpha_2}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\int_0^\infty x(1+rx)\ e^{-\theta_1 rx} \\
&\times\ \alpha_1^{\left[1-\frac{1+\theta_1+\theta_1 rx}{1+\theta_1}\right]e^{-\theta_1 rx}}\ (1+x)\ e^{-\theta_2 x}\ \alpha_2^{\left[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}\right]e^{-\theta_2 x}}\ dx \\
&= \frac{1}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\left(\begin{array}{c} i\\ k\end{array}\right)\left(\begin{array}{c} j\\ l\end{array}\right) \\
&\times\ \frac{(log\alpha_1)^{i+1}(log\alpha_2)^{j+1}\ (\theta_1 r)^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l}\Big[\frac{\Gamma(k+l+2)}{[(i+1)\theta_1 r+(j+1)\theta_2]^{k+l+2}} \\
&+\ (r+1)\frac{\Gamma(k+l+3)}{[(i+1)\theta_1 r+(j+1)\theta_2]^{k+l+3}}+r\frac{\Gamma(k+l+4)}{[(i+1)\theta_1 r+(j+1)\theta_2]^{k+l+4}}\Big],
\end{aligned}$$

where the final step follows from the definition of the gamma function.

3.11.4 Distribution of product

Let X_1 and X_2 be two independent random variables such that $X_1 \sim \alpha PTL(\alpha_1;\theta_1)$ and $X_2 \sim \alpha PTL(\alpha_1;\theta_1)$. Then, the *pdf* of the random variable $P = X_1.X_2$ is given by

$$\begin{aligned}
f_{X_1.X_2}(p) &= \frac{log\alpha_1\ log\alpha_2}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\int_0^\infty \frac{1}{x}\left(1+\frac{p}{x}\right)\ e^{-\frac{\theta_1 p}{x}} \\
&\times\ \alpha_1^{\left[1-\frac{1+\theta_1+\frac{\theta_1 p}{x}}{1+\theta_1}\right]e^{-\frac{\theta_1 p}{x}}}\ (1+x)\ e^{-\theta_2 x}\ \alpha_2^{\left[1-\frac{1+\theta_2+\theta_2 x}{1+\theta_2}\right]e^{-\theta_2 x}}\ dx \\
&= \frac{1}{(\alpha_1-1)(\alpha_2-1)}\left(\frac{\theta_1^2}{1+\theta_1}\right)\left(\frac{\theta_2^2}{1+\theta_2}\right)\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\sum_{k=0}^{i}\sum_{l=0}^{j}\left(\begin{array}{c} i\\ k\end{array}\right)\left(\begin{array}{c} j\\ l\end{array}\right) \\
&\times\ \frac{(log\alpha_1)^{i+1}(log\alpha_2)^{j+1}\ (\theta_1 p)^k\ \theta_2^l}{i!\ j!\ (1+\theta_1)^k\ (1+\theta_2)^l} \\
&\times\ \Big[(p+1)\left\{\frac{(j+1)\theta_2}{(i+1)\theta_1 p}\right\}^{(l-k)/2} K_{l-k}\left(2\sqrt{2\theta_1\theta_2 p(i+1)(j+1)}\right), \\
&+\ \left\{\frac{(j+1)\theta_2}{(i+1)\theta_1 p}\right\}^{(l-k+1)/2} K_{l-k+1}\left(2\sqrt{2\theta_1\theta_2 p(i+1)(j+1)}\right) \\
&+\ p\left\{\frac{(j+1)\theta_2}{(i+1)\theta_1 p}\right\}^{(l-k-1)/2} K_{l-k-1}\left(2\sqrt{2\theta_1\theta_2 p(i+1)(j+1)}\right),
\end{aligned}$$

where the final step follows from equation (3.471.10) in Gradshteyn and Ryzhik (2014), and $K_\upsilon(.)$ denotes the modified Bessel function of the second kind defined by

$$K_\upsilon(x) = \frac{\sqrt{\pi}x^\upsilon}{2^\upsilon\Gamma\left(\upsilon+\frac{1}{2}\right)}\int_1^\infty exp(-xt)(t^2-1)^{\upsilon-\frac{1}{2}}dt.$$

4 Parameter Estimation

Among several available methods for estimating the parameters of a model under the classical approach, the maximum likelihood method is the most commonly used procedure. Here, we consider the estimation of the unknown parameters of the αPTL distribution from complete information of samples only by maximum likelihood method. One may consider the method of moments, however it is too complicated in our case. For a random sample of size n, $x_1, \ldots, x_n$ drawn from the density in (7) with parameters α, θ, the associated log-likelihood function will be

$$\begin{aligned}\ell = \ell(\alpha,\theta) &= n\left\{\log(\log\alpha) - \log(\alpha-1) + 2\log\theta - \log(\theta+1)\right\} \\ &\quad + \sum_{i=1}^{n}\log(1+x_i) - \theta\sum_{i=1}^{n}x_i + \log\alpha\sum_{i=1}^{n}\left[1-(\frac{1+\theta+\theta x_i}{\theta+1})\right]e^{-\theta x_i}.\end{aligned}$$

Therefore, the maximum likelihood equations are given by

$$\frac{\partial\ell}{\partial\alpha} = \frac{n}{\alpha(\log\alpha)} - \frac{1}{\alpha-1} + \frac{1}{\alpha}\sum_{i=1}^{n}\left[1-(\frac{1+\theta+\theta x_i}{\theta+1})\right]e^{-\theta x_i} \tag{25}$$

and

$$\frac{\partial\ell}{\partial\theta} = \frac{2n}{\theta} - \frac{n}{\theta+1} - \sum_{i=1}^{n}x_i + \log\alpha\sum_{i=1}^{n}\left[\frac{x_i e^{-\theta x_i}}{\theta+1}\left(\theta x_i - \frac{1}{\theta+1}\right)\right]. \tag{26}$$

Equating (25) and (26) to zero and solving simultaneously, one would obtain the maximum likelihood estimates of α and θ.

To estimate the model parameters, numerical iterative techniques must be used to solve these equations. We may investigate the global maxima of the log-likelihood by setting different starting values for the parameters. The information matrix will be required for interval estimation. The elements of the 2×2 total observed information matrix (since expected values are difficult to calculate) , $J(\vec{\gamma}) = J_{r,s}(\vec{\theta})$ (for $r, s = \alpha, \theta$) can be obtained from the authors under request, where $\vec{\gamma} = (\alpha, \theta)$. The asymptotic distribution of $(\hat{\vec{\gamma}} - \vec{\gamma})$ is $N_2(\vec{0}, K(\gamma)^{-1})$, under the regularity conditions, where $K(\theta) = E\left(J(\vec{\gamma})\right)$, is the expected information matrix, and $J(\hat{\vec{\gamma}})^{-1}$ is the observed information matrix. The multivariate normal $N_2(\vec{0}, K(\gamma)^{-1})$ distribution can be used to construct approximate confidence intervals for the individual parameters.

4.1 Simulation study

In order to assess the performance of the MLEs, a small simulation study is performed using the statistical software R through the package (stats4), command mle. The number of Monte Carlo replications was 10000. For maximizing the log-likelihood function, we use the MaxBFGS subroutine with analytical derivatives. The evaluation of the estimates was performed based on the following quantities for each sample size: the empirical mean squared errors (MSEs) are calculated using the R package from the Monte Carlo replications. The MLEs are determined for each simulated data, say, $(\hat{\alpha}_i, \hat{\theta}_i)$ for $i = 1, 2, \cdots, 10000$ and the biases and MSEs are computed by

$$bias_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{h}_i - h),$$

and

$$MSE_h(n) = \frac{1}{10000} \sum_{i=1}^{10000} (\hat{h}_i - h)^2,$$

for $h = \alpha, \theta$. We consider the sample sizes at $n = 25, 50, 100$ and 200 and consider different values for the parameters . The empirical results are given in Table 2. The figures in Table 2 indicate that the estimates are quite stable and, more importantly, are close to the true values for the these sample sizes. Furthermore, as the sample size increases, the MSEs decreases as expected.

Table 2: Bias, MSE and confidence intervals (CIs) of the estimates under the maximum likelihood method.

Sample Size	Actual Value		Bias		MSE		95% CI	
n	α	θ	$\hat{\alpha}$	$\hat{\theta}$	$\hat{\alpha}$	$\hat{\theta}$	α	θ
25	1.5	0.5	0.6731	−0.7922	0.3461	1.6341	(0.032, 2.126)	(0.018, 0.878)
	2	0.5	−0.8931	0.5468	0.1147	0.8923	(0.565, 3.231)	(0.022, 0.921)
	2.5	0.8	0.6232	−0.5829	0.1345	0.1673	(0.923, 3.014)	(0.096, 1.237)
	3.0	0.7	0.2136	1.0329	0.1638	0.1822	(1.131, 4.229)	(0.074, 1.062)
50	1.5	0.5	−0.4173	−0.4198	0.0538	0.972	(0.118, 1.983)	(0.045, 0.734)
	2	0.5	−0.7738	0.3239	0.0978	0.626	(0.678, 2.245)	(0.028, 0.742)
	2.5	0.8	0.4891	−0.2460	0.1065	0.167	(1.167, 2.893)	(0.024, 1.092)
	3.0	0.7	0.1883	0.9799	0.0442	0.113	(1.275, 4.006)	(0.089, 0.951)
100	1.5	0.5	.0717	0.361	0.0248	0.3132	(0.583, 1.754)	(0.078, 0.678)
	2	0.5	0.5182	0.4321	0.0457	0.5781	(0.886, 2.185)	(0.093, 0.653)
	2.5	0.8	0.3165	0.1594	0.0441	0.1587	(1.327, 2.704)	(0.081, 1.0326)
	3.0	0.7	0.1375	−0.0493	0.0203	0.0955	(1.682, 3.450)	(0.124, 0.874)
200	1.5	0.5	−0.04609	−0.02898	0.0111	0.0842	(0.788, 1.697)	(0.182, 0.633)
	2	0.5	−0.0512	−0.1110	0.0229	0.0182	(1.234, 2.053)	(0.142, 0.569)
	2.5	0.8	−0.0730	−0.0527	0.0241	0.0364	(1.564, 2.603)	(0.112, 0.957)
	3.0	0.7	−0.1023	−0.0208	0.0102	0.0217	(1.789, 3.044)	(0.283, 0.571)

5 Applications

In this section, we fit the αPTL distribution to two real data sets and compare with the Lindley (Equation (5)), exponential (E), exponentiated logistic (EL), and Weibull distributions, whose densities are given by

$$f_{EL}(x;\beta_1,\delta_1)=\frac{1}{-\log\delta_1}\frac{\beta_1(1-\delta_1)\exp\left(-\beta_1 x\right)}{1-(1-\delta_1)\exp\left(-\beta_1 x\right)},\quad \beta_1>0,\quad \delta_1\in(0,1),$$

$$f_W(x;\beta_2,\gamma)=\gamma\beta_2^{\gamma}x^{\gamma-1}\exp\left(-\beta_2 x\right),\quad (\beta_2,\gamma)>0,$$

$$f_E(x;\eta)=\eta\, e^{-\eta x},\quad \eta>0,$$

for $x>0$, respectively. We use the R-package Adequacy Model. The first data set are given by Kus (2007) and represents the period between successive earthquakes in the last century in North Anatolia fault zone: $1163, 501, 2039, 4863, 3258, 616, 217, 143, 323, 398, 9, 182, 159, 67, 633, 2117, 756,$

$896, 461, 3709, 409, 8592, 1821, 979$. In the second data set, we consider the time intervals (in days) of the most desolated earthquakes in Iran. These data were analyzed by Tahmasbi and Rezaei (2008) and consist of 13 observations: $136, 1187, 117, 944, 24, 70, 716, 1126, 378, 166, 152, 264, 275$. These two data sets were also separately analyzed by Barreto- Souza et al. (2011).

Tables 3 and 4 give us the estimated values of the parameters, Kolmogorov-Smirnov (K-S) statistics and its respective p-values for the first and second data sets, respectively. We observe that all the distributions in both tables show a reasonably good fit for the given data sets and, in each case, we do not reject the hypothesis that the data come from distribution considered at any usual significance level. However, from the K-S test, the αPTL distribution provides a better fit than the other distributions that are present in the literature and considered here; under the αPTL model, the K-S statistics take the smallest values for both data sets analyzed and consequently the associated p-values are the largest.

Table 3: Estimates of the parameters, K-S statistics and associated p-values for the first data set.

Distribution	Lindley	EL	Weibull	αPTL	Exp (E)
Parameter Estimates	$\hat{\theta} = 1.237$	$\hat{\beta}_1 = 0.0004$	$\hat{\beta}_2 = 0.0008$	$\hat{\alpha} = 1.034$	
		$\hat{\delta}_1 = 0.1259$	$\hat{\gamma} = 0.7855$	$\hat{\theta} = 0.698$	$\hat{\eta} = 1.982$
K-S	0.1172	0.0886	0.1004	0.0628	0.2347
K-S p-value	0.4311	0.9831	0.9490	0.9879	0.2391

Table 4: Estimates of the parameters, K-S statistics and associated p-values for the second data set

Distribution	Lindley	EL	Weibull	αPTL	Exp (E)
Parameter Estimates	$\hat{\theta} = 1.489$	$\hat{\beta}_1 = 0.0022$	$\hat{\beta}_2 = 0.0023$	$\hat{\alpha} = 2.485$	
		$\hat{\delta}_1 = 0.7906$	$\hat{\gamma} = 1.0367$	$\hat{\theta} = 1.0324$	$\hat{\eta} = 2.683$
K-S	0.2785	0.1307	0.1527	0.0529	0.2912
K-S p-value	0.2348	0.9585	0.8783	0.9536	0.1854

6 Concluding remarks

In this paper, we proposed a new two-parameter family of distributions, so-called the αPTL distribution. The proposed αPTL distribution has one shape parameter and one scale parameter. The new αPTL model can be used as an alternative to the Lindley, Weibull and exponentiated logistic distributions and is expected that in some situations it might work better (in terms of model fitting) than the models stated, although it cannot be always guaranteed. In this paper, the αPTL distribution shows its ability to earthquakes data. Finally, we hope that the αPTL

distribution attract wider sets of applications such as medical, engineering and social sciences etc.

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