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Bivariate Kumaraswamy models via modified symmetric FGM copulas: Properties and Applications in Insurance modeling

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Abstract

A copula is a useful tool for constructing bivariate and/or multivariate distributions. In this article, we consider a new modified class of (Farlie-Gumbel-Morgenstern) FGM bivariate copula for constructing several different bivariate Kumaraswamy type copulas and discuss their structural properties, including dependence structures. It is established that construction of bivariate distributions by this method allows for greater flexibility in the values of Spearman's correlation coefficient, ρ and Kendall's τ. For illustrative purposes, one representative data set is utilized to exhibit the applicability of these proposed bivariate copula models.

Key Words: Bivariate Kumaraswamy distribution, Copula based construction, Kendall's tau, Dependence structures

1 Introduction

Over the last decade or so, there has been a growing interest in constructing various bivariate distributions and study it's dependence structure. For an excellent survey on this, an interested reader is suggested to see Balakrishnan et al. (2009) and the references therein. Of late, copula based methods of construction have also gained a considerable amount of attention, mainly due to it's analytical tractability in the sense of discussing

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dependence structure between two dependent random variables. A copula is a multivariate distribution function whose marginals are uniform. It couples or links the marginal distribution to their joint distribution. In order to obtain a bivariate/multivariate distribution function, one needs to simply combine two (in the bivariate case) and/or several marginal distribution functions with any copula function. Consequently, for the purpose of statistical modeling, it is desirable to have a plethora of copulas at one's disposal. One of the most important parametric family of copulas is the Farlie- Gumbel- Morgenstern (FGM, henceforth) family defined as

$C(u,v) = uv[1+\theta(1-u)(1-v)]$, $(u,v) \in (0,1)$, where $\theta \in [-1,1]$. This family of copulas have the following properties.

- Symmetry: $C(u,v) = C(v,u)$, $\forall (u,v) \in [0,1]^2$, and have the lower and upper tail dependence coefficients equal to zero.
- It is positive quadrant dependent (PQD) for $\theta \in (0,1]$ and negative quadrant dependent (NQD) for $\theta \in [-1,0)$.

However, the major drawback of FGM copula is that the the values of Spearmans correlation coefficient (ρ) and Kendals (τ) are somewhat restricted ($\rho \in [-1/3, 1/3]$ and $\tau \in [-2/9, 2/9]$). To overcome this limited nature of dependence, several authors proposed extensions of this family, for example, Kotz et al. (2000). This fuels to work in this direction in the sense of considering a modified FGM class and use it as a pivot for constructing bivariate and multivariate Kumaraswamy models.

Kumaraswamy (1980) introduced a two parameter absolutely continuous distribution useful for double bounded random processes with hydrological applications. The Kumaraswamy distribution (hereafter the KW distribution) on the interval $(0,1)$, has its probability density function (pdf) and its cdf with two shape parameters $a > 0$ and $b > 0$

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defined by

$$f(x) = \delta\beta x^{\delta-1}(1-x^{\delta})^{\beta-1} I(0<x<1), \quad \text{and} \quad F(x) = 1-(1-x^{\delta})^{\beta} \tag{1.1}$$

If a random variable X has (1.1) as its density then we will write $X \sim KW(\delta, \beta)$. The density function in (1.1) has similar properties to those of the beta distribution. The Kumaraswamy pdf is unimodal, uniantimodal, increasing, decreasing or constant depending (similar to the beta distribution) on the values of the parameters. However, the construction of bivariate Kumaraswamy distributions has received limited attention. Barreto-Souza et al. (2013) introduced a bivariate Kumaraswamy distribution related to a Marshall- Olkin survival copula and discussed some structural properties of their bivariate Kumaraswamy distributions. Arnold et al. (2017 a, 2017 b) discussed some different strategies for constructing legitimate bivariate Kumaraswamy models via conditional specification, conditional survival function specification, and via Arnold-Ng bivariate beta construction approach. In a separate article, Arnold et al. (2017 c) discussed a wide variety of Arnold-Ng type bivariate and multivariate copulas for constructing several types of bivariate Kumaraswamy models. Again, in Ghosh et al. (2016), the authors discussed some copula based approach to construct several bivariate Kumaraswamy type models along with an application to a real life data set focusing on financial risk assessment. This article is a follow up paper of Ghosh et al. (2016), in which, we examine in details the utility of well known bivariate FGM copula by a slight modification to allow greater flexibility in modeling various types of data sets (in the sense of those results described in details in Rodriguez-Lallena et al. (2004)). In this article we start with a standard Kumaraswamy quantile function from two independent Kumaraswamy distributions (with two different sets of shape parameters) and construct the corresponding bivariate copula with different shape parameters. The rest of the article is organized as follows: In section 2, we define the modified FGM copula and discuss it?s several structural properties. In

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section 3, we consider two special classes of modified bivariate Kumaraswamy FGM type copulas for constructing bivariate Kumaraswamy distributions. In section 4, we establish some dependence structures for those developed bivariate Kumaraswamy FGM type copulas. In section 5, an outline of simulation from a copula density is provided. In section 7, we consider a real life data on insurance claims and apply our developed bivariate Kumaraswamy type copula models to illustrate their applicability. In section 7, some concluding remarks are presented.

2 Modified bivariate FGM copula

We consider the following modified version of the bivariate (Farlie-Gumbel-Morgenstern) FGM copula defined as

$$C(u, v) = uv \left[1 + \theta\Phi(u)\Psi(v)\right], \tag{2.1}$$

where $\Phi(u)$ and $\Psi(v)$ are two absolutely continuous functions on $(0, 1)$ with the following conditions, and $\theta \in [-1, 1]$.

- $\Phi(0) = \Psi(0) = \Phi(1) = \Psi(1) = 0$. This are known as boundary conditions.
- $|\frac{\partial\Phi(u)}{\partial u}| \leq 1$ for every $u \in [0, 1]$. Similarly for the other function $\Psi(v)$.

Theorem 1. The function in (2.1) is a valid copula provided, the functions $\Phi(u), \Psi(v)$ satisfies all the conditions stated above. For details on the proof (on a similar structure to this, see Rodriguez-Lallena et al.(2004)).

Proof. It is straightforward and hence omitted. First, we make a note of the following:

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- The associated bivariate copula density from (2.1) will be

$$c(u,v) = \frac{\partial C(u,v)}{\partial u \partial v} = 1 + \theta\Phi(u)\Psi(v)\left\{\left[1 + u\frac{\partial \Phi(u)}{\partial u}\right]\left[1 + v\frac{\partial \Phi(v)}{\partial v}\right]\right\}. \qquad (2.2)$$

- The conditional copula density of U given $V = v$, from (2.2), will be

$$c(u|v) = \frac{\partial C(u,v)}{\partial v} = u\left\{1 + \theta\Phi(u)\Psi(v)\,(1+v)\right\}. \qquad (2.3)$$

Similarly, one can find the conditional copula density of V given $U = u$.

It is noteworthy to mention that copulas are instrumental for understanding the dependence be- tween random variables. With them we can separate the underlying dependence from the marginal distributions. It is well known that a copula which characterizes dependence is invariant under strictly monotone transformations, subsequently a better global measure of dependence would also be invariant under such transformations. Among other dependence measures, Kendall's τ and Spearman's ρ are invariant under strictly increasing transformations, and, as we will see in the next, they can be expressed in terms of the associated copula.

- Kendall's τ: Kendall's τ measures the amount of concordance present in a bivariate distribution. Suppose that (X, Y) and $(\tilde{X}, \tilde{Y})$ are two pairs of random variables from a joint distribution function. We say that these pairs are concordant if large values of one tend to be associated with large values of the other, and small values of one tend to be associated with small values of the other. The pairs are called discordant if large goes with small or vice versa. Algebraically we have concordant pairs if $(X - \tilde{X})(Y - \tilde{Y}) > 0$ and discordant pairs if we reverse the inequality. The formal definition is:

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$$\tau(X,Y) = Prob\left\{\left(X-\tilde{X})(Y-\tilde{Y}) > 0\right) - \left(X-\tilde{X})(Y-\tilde{Y}) < 0\right)\right\},$$

where $(\tilde{X},\tilde{Y})$ is an independent copy of (X,Y).

Let X and Y be continuous random variables with copula C. Then Kendall's τ is given by

$$\tau(X,Y) = 4\iint_{[0,1]^2} C(u,v)dC(u,v) - 1. \tag{2.4}$$

- Spearman's ρ: For two random variables X and Y is equal to the linear correlation coefficient on the variables $F_1(X)$ and $F_2(Y)$ where F_1 and F_2 are the marginal distributions of X and Y respectively. Let X and Y be continuous random variables with copula C. Then Spearman's ρ_s is given by

$$\rho_s = 12\iint_{[0,1]^2} uvdC(u,v) - 3. \tag{2.5}$$

Alternatively, ρ_s can be written as $\rho_s = 12\int_0^1\int_0^1 [C(u,v) - uv]\, dudv$. Also, as mentioned earlier, one can equivalently show that $\rho_s(U,V) = \rho\left(F_1(X), F_2(V)\right).$

Proposition 1. Let (X,Y) be a random pair with copula $C(u,v)$ given by (2.1). Then the expression for the association coefficients are

- $\rho_\theta = \theta A(u,v)$, where $A(u,v) = 12\left[\int_0^1 u\Phi(u)du\right]\left[\int_0^1 v\Psi(v)dv\right]$

- Kendall's τ will be

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$$\begin{aligned}
\tau_\theta &= 1 + \int_0^1 \{u\Phi(u)\}\, du \int_0^1 \{v\Psi(v)\}\, dv \\
&+\theta \left(\int_0^1 \left[\Phi(u)\left\{u + u^2 \frac{\partial \Phi(u)}{\partial u}\right\}\right] du\right) \left(\int_0^1 \left[\Psi(v)\left\{v + v^2 \frac{\partial \Psi(v)}{\partial v}\right\}\right] dv\right) \\
&+\theta^2 \left(\int_0^1 \left[\Phi^2(u)\left\{u + u^2 \frac{\partial \Phi(u)}{\partial u}\right\}\right] du\right) \left(\int_0^1 \left[\Psi^2(v)\left\{v + v^2 \frac{\partial \Psi(v)}{\partial v}\right\}\right] dv\right).
\end{aligned}$$

Proof. The proof are almost similar in approach for the two coefficients. We only consider for the Spearmans ρ_θ. For our copula model in (2.1), the corresponding ρ_θ will be

$$\begin{aligned}
\rho_\theta &= 12 \int_0^1 \int_0^1 C(u,v) du dv - 3 \\
&= 12 \left[\int_0^1 v \left(\int_0^1 u(1 + \theta\Phi(u)\Psi(v)) du\right) dv\right] - 3. \qquad (2.6)
\end{aligned}$$

Next, consider the integral in parenthesis which, after some simplification, reduces to

$$\int_0^1 u(1 + \theta\Phi(u)\Psi(v)) du = \frac{1}{2} + \theta\Psi(v) \int_0^1 u\Phi(u) du. \qquad (2.7)$$

On substitution of (2.7) in (2.6), we get,

$$\begin{aligned}
\rho_\theta &= 12 \left[\int_0^1 v \left(\frac{1}{2} + \theta\Psi(v) \int_0^1 u\Phi(u) du\right)\right] dv - 3 \\
&= \theta A(u,v),
\end{aligned}$$

after simple algebraic operation. Hence the result. Similarly, one can get the expression for τ_θ.

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Next section, we will consider some specific choices of $\Phi(u)$ and $\Psi(v)$ to construct bivariate Kumaraswamy type copulas.

3 Bivariate FGM Type Kumaraswamy models

In this section, we discuss in detail two different types of bivariate FGM type copula models to construct bivariate Kumaraswamy distribution.

Modified bivariate F-G-M (Type I) Kumaraswamy model

Here, we consider the following functional form for both $\Phi(u)$ and $\Psi(v)$:

- $\Phi(u) = u(1 - u^{a_1})^{b_1}$, for $(a_1, b_1) > 0$.
- $\Psi(v) = v(1 - v^{a_2})^{b_2}$, for $(a_2, b_2) > 0$.

Note that this particular functional form does satisfy all the conditions stated earlier for $\Phi(u)$ and $\Psi(v)$. In that case, the corresponding bivariate copula will be given by

$$C(u, v) = uv \left[1 + \theta \left(u(1 - u^{a_1})^{b_1}\right) \left(v(1 - v^{a_2})^{b_2}\right)\right]. \tag{3.1}$$

Next, suppose $X_1 \sim KW(\lambda_1, \alpha_1)$ $X_2 \sim KW(\lambda_2, \alpha_2)$ and that they are independent. Then, using (3.1), a bivariate dependent FGM-Kumaraswamy (Type I) distribution will be of the following form (replacing u and v by the quantiles of X_1 and X_2 respectively):

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$$
\begin{aligned}
&F(x_1, x_2) \\
&= \left(1-(1-x_1^{\lambda_1})^{\alpha_1}\right)\left(1-(1-x_1^{\lambda_1})^{\alpha_1}\right) \\
&\times \left\{1+\theta\left(1-(1-x_1^{\lambda_1})^{\alpha_1}\right)\left(1-\left(1-(1-x_1^{\lambda_1})^{\alpha_1}\right)^{b_1}\right)\right. \\
&\times \left.\left(1-(1-x_2^{\lambda_2})^{\alpha_2}\right)\left(1-\left(1-(1-x_2^{\lambda_2})^{\alpha_2}\right)^{b_2}\right)\right\},
\end{aligned}
$$

for $(\lambda_1, \lambda_2, \alpha_1, \alpha_2) > 0$ and $0 < (x_1, x_2) < 1$.

Modified bivariate F-G-M (Type II) Kumaraswamy model

Here, we consider the following functional form for both $\Phi(u)$ and $\Psi(v)$:

- $\Phi(u) = u^{\delta_1}(1-u)^{1-\delta_1}$, for $\delta_1 > 0$.
- $\Psi(v) = v^{\delta_2}(1-v)^{1-\delta_2}$, for $\delta_2 > 0$.

Note that this particular functional form does satisfy all the conditions stated earlier for $\Phi(u)$ and $\Psi(v)$. In that case, the corresponding bivariate copula (henceforth, BK-FGM(Type II) copula) will be given by

$$
C(u, v) = uv\left[1+\theta u^{\delta_1} v^{\delta_2}(1-u)^{1-\delta_1}(1-v)^{1-\delta_2}\right]. \tag{3.2}
$$

In this case, like the previous one, a bivariate dependent FGM (Type II) Kumaraswamy distribution, arising from two independent Kumaraswamy variables, will be of the following form:

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$$
\begin{aligned}
&F(x_1, x_2) \\
&= \left(1-\left(1-x_1^{\lambda_1}\right)^{\alpha_1}\right)^{\delta_1}\left(1-\left(1-x_2^{\lambda_2}\right)^{\alpha_2}\right)^{\delta_2} \\
&\times\left[1+\theta\left(1-\left(1-x_1^{\lambda_1}\right)^{\alpha_1}\right)^{\delta_1}\left(\left(1-x_1^{\lambda_1}\right)^{\alpha_1(1-\delta_1)}\right)\right. \\
&\times\left.\left(1-\left(1-x_2^{\lambda_2}\right)^{\alpha_2}\right)^{\delta_2}\left(\left(1-x_2^{\lambda_2}\right)^{\alpha_2(1-\delta_2)}\right)\right].
\end{aligned}
$$

Modified bivariate FGM (Type III) Kumaraswamy model:

Here, we consider the following functional form for both $\Phi(u)$ and $\Psi(v)$:

- $\Phi(u) = u\left[\log\left(1+(1-u)\right)\right]$
- $\Psi(v) = v\left[\log\left(1+(1-v)\right)\right]$

Note that this particular functional form does satisfy all the conditions stated earlier for $\Phi(u)$ and $\Psi(v)$. In that case, the corresponding bivariate copula (henceforth, BK-FGM(Type III) copula) will be given by

$$
C(u, v) = uv\left[1+\theta uv\left\{\log\left(1+(1-u)\right)\log\left(1+(1-v)\right)\right\}\right]. \tag{3.3}
$$

In this case, one can also obtain a closed form expression for the associated distribution function.

Bivariate Kumaraswamy (Type-IV) copula

For the standard Kumaraswamy distribution with parameters (a, b), we have the pdf (probability density function), cdf(cumulative distribution function) and the inverse cdf are given by respectively

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$f_i(x_i) = abx_i^{a-1}(1-x_i^a)^{b-1}$, $F(x) = 1-(1-x_i^a)^b$ and $F_i^{-1}(u_i) = 1-(1-u_i^{1/b})^{1/a}$, $a > 0, \quad b > 0$.

Hence, the associated copula for suitable parameters a and b and having two given marginal distributions which are the standard Kumaraswamy distributions, has the following form:

$$\begin{aligned} C(u_1, u_2) &= u_1 \left(1-(1-u_2)^{1/b}\right)^{1/a} + u_2 \left(1-(1-u_1)^{1/b}\right)^{1/a} \\ &\quad - \left(1-(1-u_1)^{1/b}\right)^{1/a} \left(1-(1-u_2)^{1/b}\right)^{1/a}. \end{aligned} \tag{3.4}$$

For details on this see Ghosh et al. (2016)

For the sake of notational simplicity and for the remainder of the article, we call (3.4) as BK - copula (Type IV).

4 Some properties of the bivariate BK-FGM type copulas

Here, we have the following

- For the BK-FGM (Type I) copula, closed form expression for Kendall's τ is not available in closed form. Numerical integration has to be considered.

- Spearman's correlation coefficient will be

$$\rho_\theta = \theta \left(a_1 a_2\right)^{-1},$$

provided $\max(a_1, a_2) < 3$.

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For the BK-FGM (Type II) copula

- Kendall's τ will be

$$\begin{aligned}\tau_\theta &= B(\delta_1+2,2-\delta_1)B(\delta_1+3,2-\delta_1)+(\delta_1-1)[B(\delta_1+2,2-\delta_1)-B(\delta_1+1,1-\delta_1)]\\ &\quad+\delta_1 B(\delta_1+2,1-\delta_1)\left(\delta_1-\frac{1}{2}\right)-\frac{B(\delta_1+1,1-\delta_1)}{2},\end{aligned}$$

provided $\delta_1 < 1$.

- Corresponding Spearman's correlation coefficient will be

$$\rho_\theta = \theta\left(B(\delta_1+2,2-\delta_1)\right)^2,$$

provided $\delta_1 < 2$,

where $B(,)$ is Euler's beta function.

Note For BK-copula (Type III), closed form expressions for both the dependence measures are not available. In this case, one has to consider numerical integration.

For the BK-copula (Type IV)

- Kendall's τ will be

$$\tau = 4\left(1-\frac{\Gamma(1+1/a)\Gamma(1+b)}{\Gamma(1+1/a+b)}-\left(1-\frac{\Gamma(1+1/a)\Gamma(1+b)}{\Gamma(1+1/a+b)}\right)^2\right)-1.$$

(by straightforward integration)

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- Spearman's correlation coefficient will be

$$\rho_s = 12\left(1 - \frac{\Gamma(1+1/a)\Gamma(1+b)}{\Gamma(1+1/a+b)} - \left(1 - \frac{\Gamma(1+1/a)\Gamma(1+b)}{\Gamma(1+1/a+b)}\right)^2\right) - 3.$$

4.1 Dependence properties

In this section, we focus on the following properties:

Tail dependence property: The upper tail dependence coefficient (parameter) λ_U is the limit (if it exists) of the conditional probability that Y is greater than 100α th percentile of G given that X is greater than the 100α th percentile of F as α approaches 1.

$$\lambda_U = \lim \alpha \uparrow 1 P\left(Y > G^{-1}(\alpha) | X > F^{-1}(\alpha)\right)$$

. If $\lambda_U > 0$, then X and Y are upper tail dependent and asymptotically independent otherwise. Similarly, the lower tail dependence coefficient is defined as

$\lambda_L = \lim_{\alpha \downarrow 1} P\left(Y \leq G^{-1}(\alpha) | X \geq F^{-1}(\alpha)\right)$. Let, C be the copula of X and Y. Then, equivalently we can write $\lambda_L = \lim_{u \downarrow 0} \frac{C(u,u)}{u}$ and $\lambda_U = \lim_{u \uparrow 1} \frac{\tilde{C}(u,u)}{1-u}$.

Next, we consider the following.

- In our case (for the modified FGM bivariate Kumaraswamy (Type I)) copula model,

$$\begin{aligned} \lambda_L &= \lim_{u \to 0} \frac{C(u,u)}{u} \\ &= \lim_{u \to 0} u\left(1 + \theta\left(u^2(1-u^{a_1})^{b_1}(1-u^{a_2})^{b_2}\right)\right) \\ &= 0. \end{aligned} \tag{4.1}$$

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So, X and Y are asymptotically independent. The corresponding joint survival copula will be given by

$$\begin{aligned}\tilde{C}(u,u) &= 2u-1+C(1-u,1-u)\\ &= 2u-1+(1-u)^2\left(1+\theta\left((1-u)^2\left[1-((1-u)^{a_1})\right]^{b_1}\left[1-((1-u)^{a_2})\right]^{b_2}\right)\right).\end{aligned}$$

Again,

$$\begin{aligned}\lambda_U &= \lim_{u\to 1}\frac{\tilde{C}(u,u)}{1-u}\\ &= \lim_{u\to 1}\frac{2u-1}{1-u}+\lim_{u\to 1}(1-u\left(1+\theta\left((1-u)^{a_1+a_2}\left[1-((1-u)^{a_1})\right]^{b_1}\left[1-((1-u)^{a_2})\right]^{b_2}\right)\right)\\ &= \infty+0\\ &= \infty\end{aligned}$$

So, (X,Y) are not upper tail dependent.

- For the modified FGM bivariate Kumaraswamy (Type II)) copula model,

$$\begin{aligned}\lambda_L &= \lim_{u\to 0}\frac{C(u,u)}{u}\\ &= \lim_{u\to 0}u\left(1+\theta\left(u^{\delta_1+\delta_2}(1-u)^{2-\delta_1+\delta_2}\right)\right)\\ &= 0,\end{aligned} \tag{4.2}$$

provided $2>\delta_1+\delta_2$. Hence, it is asymptotically independent provided $2>\delta_1+\delta_2$.

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Again,

$$\lambda_U \quad = \lim_{u \to 1} \frac{\tilde{C}(u,u)}{1-u} \quad = \infty$$

Again, by similar argument as before, $\lambda_U \neq 0$, implying that (X, Y) are not upper tail dependent.

Similarly, one can establish these properties for the modified FGM bivariate Kumaraswamy (Type III) and (Type IV)) copula model.

Positive Quadrant Dependent (PQD) and Left-Tail decreasing (LTD) property:

According to Amblard et al.(2002),(Theorem 3), for $\theta > 0$ and (X, Y) a random pair with copula $C(u, v)$ as defined in (2), we have the following result:

- X and Y are PQD if and only if either $\forall\, u \in (0,1)$ and $\forall\, u \in (0,1)$, $\Phi(u)\,[\Psi(v)] \geq 0$ or $\Phi(u)\,[\Psi(v)] \leq 0$.
- X and Y are LTD if and only if $\frac{\Phi(u)}{u}$ and $\frac{\Psi(v)}{v}$ is monotone. Next, consider the following:

Proposition 1. The modified BK-FGM (Type I, Type II and Type III) copulas are PQD.

Proof. For the modified BK-FGM (Type I) copula, we have $\Phi(u) = u^{a_1}(1 - u^{a_1})^{b_1}$ and $\Psi(v) = v^{a_2}(1 - v^{a_2})^{b_2}$. Note that for any real $(a_1, a_2, b_1, b_2) > 0$, $\Phi(u) \geq 0$, for all $u \in (0,1)$ as well as $\Psi(v) \geq 0$, for all $v \in (0,1)$. Hence, (X, Y) are PQD.

Similarly, one can easily check the PQD property for the other two copula models.

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Proposition 2. The modified BK-FGM (Type I and Type III) copula exhibit LTD property while for modified BK-FGM (Type II) it is indeterministic.

Proof. For the modified BK-FGM (Type I) copula, consider the ratio $\frac{\Phi(u)}{u} = u^{a_1}(1 - u^{a_1})^{b_1}$. It is monotonically decreasing provided, $a_1 > 1$ and for any $b_1 > 0$, and it is true for any $u \in (0, 1)$. Similar results holds for the other ratio $\frac{\Psi(v)}{v}$, for any $v \in (0, 1)$. Hence, it is LTD for only $a_1 > 1$ and for any $b_1 > 0$, but not for any other possible choices of the constants a_1 and b_1.

Again, for the modified BK-FGM (Type III)copula, the ratio $\frac{\Phi(u)}{u} = \log\left(1 + (1 - u)\right)$. It is monotonically decreasing for any $u \in (0, 1)$. Similar results will hold for the other ratio $\frac{\Psi(v)}{v}$, for any $v \in (0, 1)$. Hence, it is LTD.

However, for the modified BK-FGM (Type II) copula, these ratios are not uniformly increasing and/or decreasing. That is why it is indeterministic in the sense that it could exhibit both PQD as well as LTD depending on specific choices (here, sub-intervals for $u, v \in (0, 1)$.

5 Simulation from a bivariate copula

There are several different methods (for example, acceptance-rejection sampling for bivariate cases, via transformation to a known bivariate distribution etc.) that are available to simulate/generate bivariate random samples from a bivariate copula. We can in principle, use the following result (Joe, 1997, page 146) to simulate random samples from our modified BK-FGM type copula as follows. Let us define, the conditional copula distribution function, (say, of V given $U = u$), $C_{2|1}(v|u) = \frac{\partial C(u,v)}{\partial u}$. Next, if U and W are independent $U(0, 1)$ random variables, then $(U, V) = U, C_{2|1}^{-1}(W|U)$ will have the distribution $C(u, v)$. This method, sometimes known as conditional distribution approach or iterative condi-

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tioning, is appealing because it involves only univariate simulation. In our case, we do have closed form expressions of $C_{2|1}(v|u)$ for both types of modified BK-FGM bivariate copula available. Consequently, we can easily apply this method. Needless to say, there are other distinct sampling procedures that are available also, for example, importance sampling, adaptive acceptance-rejection sampling etc., which is suitable for other class of copulas.

6 Application in risk management

In practice, several risk managers employ VaR as a tool of risk measurement. Briefly speaking, VaR is the maximal potential loss of a position or a portfolio on some investment horizon at a given confidence level. Because of the enormous literature, we only provide its definition. Let $\{P_t\}_{t=1}^n$ be the market values of an asset or a portfolio of assets over n periods, and $X_t = -\log\left(\frac{P_t}{P_{t-1}}\right)$ be the negative log return (loss) over the t-th period. Next, given a positive value α close to 0, the VaR of X at confidence level $(1-\alpha)$ is given by

$$VaR = inf\left\{x \in \mathbb{R} | P(X \le x) \ge 1-\alpha\right\}.$$

For a detailed study on the computation of VaR used in the pure copula method, an interested reader is suggested to see Zi-sheng et al. (2009) and the references therein. Here we will propose one idea based on bivariate Kumaraswamy copula (Type II). We list the steps as follows:

1. Simulate U, V and W independently from standard uniform distribution,
2. If $U \le \lambda_s$, for the given bivariate Kumaraswamy copula (Type II), (say, $C_{\rho_s,1,}$) take $(X,Y)^T = \left(F_1^{-1}(V), F_2^{-1}(C_{\rho_s,1,U}^{-1}(W))\right)^T$.

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3. If $U > \lambda_s$, for the given bivariate Kumaraswamy copula (Type II), (say, $C_{\rho_s,2},$) take $(X,Y)^T = \left(F_1^{-1}(V), F_2^{-1}(C^{-1}_{\rho_s,2,U}(W))\right)^T$.

Then the random vector (X,Y) has the joint distribution

$$\tilde{F}(x,y) = \lambda_s C_{\rho_s,1}(F_1(x), F_2(y)) + (1-\lambda_s) C_{\rho_s,2}(F_1(x), F_2(y)),$$

where $\lambda_s = \frac{\rho_{s,2}-\rho_s}{\rho_{s,2}-\rho_{s,1}}$, and its marginal distributions are F_1 and F_2, and linear correlation is ρ_s. After this, we consider the following formula $R = -\log\left(\lambda_1 \exp(X_1) + \lambda_2 \exp(X_2)\right)$ to generate the random number of the negative log returns of portfolios. Here λ_1 and λ_2 are the weights and must satisfy $\lambda_1+\lambda_2 = 1$. Then VaR_α, will be computed by calculating the $(1-\alpha)$-th quantile of R.

6.1 An application to insurance data

Here, we consider one application for the four proposed bivariate Kumaraswamy copula models to a heavily used data set, originally considered by Frees et al. (1998), Genest et al. (2005), Cook et al. (1981, 1986) as well as in Ghosh et al. (2016). This data set contains two variables:

- X_1: an indemnity payment
- X_2: an allocated loss adjustment expense (comprising lawyers' fees and claim investigation process).

This data set comprises of 1500 general liability claims. Several other authors among others, have used (for e.g., Klugman and Parsa (1999) and Chen and Fan (2005)) this data set to demonstrate copula-model selection and fitting in an insurance context. We conjecture that this data might well be explained by one or more bivariate Kumaraswamy

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copula models derived in this paper. For the sake of simplicity, we apply all four bivariate Kumaraswamy copula models to 1466 uncensored claims. As suggested by Genest et al. (2005), based on a comparative study on the numerical estimates of the dependence parameter (θ), this imposed restriction has a very little or no effect on it. For the uncensored sample, the observed value of Kendall's tau is 0.4328. In the table below, we provide results of the goodness-of-fit tests based on the statistics S_n, T_n, and $S_{\xi n}$ with $\xi = 0$. For a detailed description on each of these goodness-of-fit statistics, see Genest et al. (2005).

Table 1. Goodness of fit statistics for the insurance data.

BK copula	θ	S_n	T_n	S_{0n}	p-value (in %)	Critical value (c_{2n})
BK copula (Type I)	0.623	3.0755	2.643	1.036	45.3	0.422
BK copula (Type II)	1.233	2.189	3.547	0.427	0.18	0.163
BK copula (Type III)	1.026	0.147	0.564	0.117	78.3	0.795
BK copula (Type IV)	0.342	0.422	0.642	0.137	88.2	0.831

Here, the dependence parameter θ is estimated in each case through inversion of Kendall's τ. The critical values and p-values reported in Table 1 are based on $N = 30,000$ repetitions of the parametric bootstrap procedure discussed in Genest et al. (2005). From Table 2, it appears that bivariate Kumaraswamy (Type III and Type IV) copula provides a better fit as compared to other BK copula models.

7 Conclusion

In this paper, we consider a modified version of the FGM family of copulas and studied some important structural properties including the dependence structure. With this modified version, we consider the construction of bivariate Kumaraswamy distributions and discuss some of it's structural properties as well as it's applicability in modeling some insurance data. However, this modified FGM class is a class of symmetric copula. It is evident from (2.1), that depending upon suitable choices of $\Phi()$ and $\Psi()$ functions, satis-

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fying associated boundary conditions as mentioned earlier, one can generate a plethora of such copula models and subsequently develop a wide spectrum of bivariate Kumaraswamy distributions. Our future work would focus on the following:

- Extension to the multivariate case and study several associated properties. It is noteworthy to mention that, albeit complex nature of these type of models (involving several parameters), we expect that multivariate Kumaraswamy distribution construction via such type of copula models will be much more interesting and computationally will be more easy to handle.

- For modeling large losses, asymmetric copulas are more useful as compared to symmetric copulas. So, we will consider a family of asymmetric copulas as introduced in Nelsen (2006), chapter 4, which has the following form:

$$C(u, v) = uv + \theta a(u)b(v), \quad \theta \in [-1, 1],$$

  here a and b are functions defined on the interval $(0, 1)$. The associated several types of dependence measures will also be considered. Also, based on this, bivariate and subsequently multivariate Kumaraswamy distributions construction will be considered and then a com- parison study will be made with those bivariate and multivariate Kumaraswamy models constructed under symmetric class of copulas.

- Since, a convex combination of any two (or more) valid copulas is also a copula. We would be interested to study the role of such a mixture of copula in developing bivariate and sub- sequently a multivariate Kumaraswamy type distributions. For example, one may start with the following:

$$C^{mixture}(u, v) = \theta_1 C^{symmetric}(u, v) + (1 - \theta_1) C^{asymmetric}(u, v),$$

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for a $\theta_1 \in (0, 1]$.

- A natural multivariate extension of the above asymmetric copula would be

$$C(u_1, u_2, \cdots, u_p) = \prod_{i=1}^{p} u_1 + \theta \prod_{i=1}^{p} a_i(u_i),$$

with $(u_1, u_2, \cdots, u_p) \in [0, 1]^p, \quad \theta \in [-1, 1]$. A natural question would be what judicious choices of the functions $a_i()$, for $i = 1, 2, ..., p$ would result in a tractable model. Associated model inference will be a challenging task due to involvement of so many parameters. We plan to report all these findings in a separate article somewhere else.

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