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A NEW CLASS OF KUMARASWAMY MIXTURE DISTRIBUTION FOR INCOME MODELING

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Abstract

Pareto distribution and their close relatives and generalizations provide very flexible families of heavy-tailed distributions which may be used to model income distributions as well as a wide variety of other social and economic distributions. Based on Kumaraswamy distribution, we describe a new distribution Kumaraswamy-Pareto (IV) distribution (hereafter called as KW-P(IV) distribution). It includes as special sub-models the Pareto and exponentiated Pareto distributions. Some structural properties of the proposed distribution are studied including explicit expressions for the moments and mean deviations. We provide the density function of the order statistics, moments and also the asymptotic distribution of the smallest order statistic. The method of maximum likelihood and quantile method of estimation are used for estimating the model parameters. A real data set is used to compare the new model with widely known distributions.

1. Introduction

The modeling and analysis of lifetimes is indeed an important aspect of

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statistical work in a wide variety of scientific and technological fields. In this article, we propose the Kumaraswamy-Pareto (IV) distribution for the first time, and study some of its structural properties. The Pareto distribution is a simple model for nonnegative data with a power law probability tail. In Vilfredo Pareto's economic book, he discussed that the number of persons in a population whose incomes exceed x is often well approximated by $Cx^{-\alpha}$, for some real C and some positive α. Accumulating experience rapidly pointed out that it is only in the upper tail of the income distributions that Pareto-like behavior can be expected. Among various types of Pareto models, Pareto (IV) merits more attention. The Pareto (IV) family was suggested by Arnold and Laguna [4], as well as Ord [19] and Cronin [7, 8]. It is to be noted that most of the distribution theory regarding the Pareto (IV) distribution can be obtained by using available results for the Burr distributions as mentioned in Kotz et al. [15]. However, there are also various types of characterization of the Pareto (IV) distribution in the literature. Among them one could consider earlier works by Dubey [9] and Harris and Singapurwalla [14], as they all arrived at the Pareto (IV) distribution by mixture of Weibull random variables. Later Singh and Maddala [20] had proposed the Pareto (IV) model using an argument involving decreasing failure rates. One striking feature of the Pareto (IV) model is that it is the most general family of Pareto distributions with of course maximum number of parameters involved for a Paretian family of densities.

The paper by Kumaraswamy [16] proposed a new probability distribution for double bounded random processes with hydrological applications. The Kumaraswamy's distribution appears to have received considerable interest in hydrology and related areas. We start with the Kumaraswamy's distribution (called from now on the KW distribution), having the probability density function (pdf) and the (cumulative distribution function) cdf with two shape parameters $a > 0$ and $b > 0$ defined by

$$f(x) = abx^{a-1}(1 - x^a)^{b-1} \quad \text{and} \quad F(x) = 1 - (1 - x^a)^b, \tag{1}$$

for $x \in (0, 1)$.

The density function in (1) has many of the same properties as the beta distribution but has some advantages in terms of tractability. For example, the KW densities are also unimodal, increasing, decreasing or constant depending in the same way as the beta distribution on the values of its parameters. He highlighted several advantages of the KW distribution over the beta distribution: the normalizing constant is very simple; simple explicit formulae for the distribution and quantile functions which do not involve any special functions; a simple formula for random variate generation; explicit formulae for *L*-moments and simpler formulae for moments of order statistics. Consider starting from a parent continuous distribution function $G(x)$. A natural way of generating families of distributions on some other support from a simple starting parent distribution with pdf $g(x) = \frac{dG(x)}{dx}$ is to apply the quantile function to a family of distributions on the interval $(0, 1)$. From an arbitrary parent cdf $G(x)$, the cdf $F(x)$ of the KW-G distribution (Cordeiro and Castro [6]) is defined by

$$F(x) = 1 - (1 - G(x)^a)^b, \tag{2}$$

where $a > 0$ and $b > 0$ are two additional parameters whose role is to introduce skewness and to vary tail weights. Because of its tractable distribution function (2), the KW-G distribution can be used quite effectively even if the data are censored. Correspondingly, the density function of this family of distributions has a very simple form

$$f(x) = abg(x)G(x)^{a-1}(1 - G(x)^a)^{b-1}. \tag{3}$$

Nadarajah et al. [18] studied some mathematical properties of the above density in (3). The new density (3) has an advantage over the class of generalized beta distributions due to Eugene et al. [10], since it does not involve any special function. For each continuous name distribution (here name denotes the name of the parent distribution), we can associate the KW-name distribution with two extra parameters *a* and *b* from the cdf $G(x)$ and pdf $g(x)$ of the name distribution whose density function is defined by formula (3). In this paper, we consider a particular class of $G(\cdot)$, namely the Pareto (IV) (hereafter P(IV) in short) model and on substitution in (3), we get Kumaraswamy-Pareto (IV) (hereafter will be called as KW-P(IV)) distribution. It is noteworthy to mention that other generalizations of Pareto distribution exist in the literature such as the Kumaraswamy-Pareto (Bourguignon et al. [5]).

It is interesting to mention that the KW-P(IV) is a generalization of Pareto (IV) distribution with the property that it can be left-skewed, right skewed, and symmetric. This provides more flexibility to the KW-P(IV) distribution in comparison to Pareto (IV) distribution in modeling different data sets. The property of left-skewness is not shared by many generalizations of Pareto distributions. However, the gamma-Pareto (IV) and Weibull-Pareto (Alzaatreh et al. [2]) distributions exhibit this property. We provide a motivation for this new distribution in the reliability context. Suppose that a system is formed by b independent components which are exposed to an outside stress. Next, consider that each of the b components is made up of a subcomponent. The system fails if any of the b components fail and that each component fail if all of the a subcomponents fail. Let $Y_{j1}, Y_{j2}, ..., Y_{ja}$ denote the lifetimes of the subcomponents within the jth component, for $j = 1(1)b$, having a common cdf $F(x)$. Further suppose that Y_j denote the lifetime of the jth component, for $j = 1(1)b$ and let Y denote the lifetime of the entire system. The cdf of Y will be

$$\begin{aligned}
P(Y \leq y) &= 1 - P(Y_1 > y, Y_2 > y, ..., Y_b > y) \\
&= 1 - (P(Y_1 > y))^b \\
&= 1 - (1 - P(Y_1 \leq y))^b \\
&= 1 - (1 - P(Y_{11} \leq y, Y_{12} \leq y, ..., Y_{1a} \leq y))^b \\
&= 1 - (1 - P^a(Y_{11} \leq y))^b \\
&= 1 - (1 - G^a(x))^b.
\end{aligned}$$

If X follows a Pareto (IV) distribution with parameters α, δ and θ with the cdf $F(x) = 1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}$, $x > 0$, then (3) reduces to

$$g(x) = \frac{ab\alpha}{\delta\theta}\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}-1}\left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}$$

$$\times\left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}\left(1 - \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{b-1}, \ x > 0. \quad (4)$$

From (4), the cdf of the KW-P(IV) can be written as

$$G(x) = 1 - \left(1 - \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{b}. \quad (5)$$

A random variable X with the pdf $g(x)$ in (4) is said to follow the KW-P(IV) distribution with parameters α, θ, δ, a and b. When $a = 1$ and $b = 1$, then the KW-P(IV) reduces to the Pareto (IV) distribution with parameters α, δ and θ.

Again, using the generalized binomial expansion (Abramowitz and Stegun [1]) (4) can be written as

$$g(x) = \frac{ab\alpha}{\delta\theta}\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}-1}\left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}$$

$$\times\left(\sum_{i=0}^{\infty}(-1)^{i}\binom{b-1}{i}\left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a(i+1)-1}\right)$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\frac{ab\alpha}{\delta\theta}\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}-1}\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha(j+1)+1)}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}$$

$$\times\, ab\,\frac{\alpha(j+1)}{\delta\theta(j+1)}\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}-1}\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha(j+1)+1)}$$

$$= \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^{i+j}\binom{b-1}{i}\binom{a(i+1)-1}{j}\frac{ab}{(j+1)}P(IV)(\alpha(j+1),\delta,\theta), \quad (6)$$

with $(\alpha, \beta, \theta, a, b) \in \mathbb{R}^{+}$ and provided *a* and *b* are integers.

From (6), one can observe that the KW-P(IV) density can be written as an infinite sum of Pareto(IV) distribution with parameters $\alpha(j+1)$, δ and θ. Therefore, some mathematical properties can be obtained directly from those properties of the P(IV) distribution.

The remainder of this paper is organized in the following way: In Section 2, we study various properties of the KW-P(IV) including the limiting behavior, transformation, quantiles. In Section 3, the moments and the mean deviations from the mean and median are studied. In Section 4, we study the reliability parameter in the context of two independent KW-P(IV) with different choices for the parameters α and *a* and *b* with fixed θ, δ. Section 5 deals with order statistics, and also the limiting distribution of the sample minima and the sample maxima for a random sample of size *n* drawn from the KW-P(IV) distribution. In Section 6, we will consider the estimation of the parameters using the method of maximum likelihood. In Section 7, we will consider the sample quantiles to estimate the parameters. In Section 8, we will consider a simulation study. One real life data set is used to illustrate the application of KW-P(IV) in Section 9.

2. Properties of the KW-P(IV) Distribution

We provide below a characterization of the KW-P(IV) distribution which establishes the relation between KW-P(IV) and uniform distribution.

Lemma 1. (Transformation): *If a random variable U follows a uniform* $(0, 1)$ *distribution, then* $X = \theta((1-(1-(1-U)^{1/b})^{1/a})^{-1/\alpha} - 1)^{\delta}$ *follows the KW-P(IV) with parameters* $\alpha, \delta, \theta, a$ *and b.*

Proof. The result follows immediately using the transformation technique.

The hazard function associated with the KW-P(IV) distribution is

$$h_g(x) = \frac{g(x)}{1 - G(x)}$$

$$= \frac{\frac{ab}{\alpha\theta}\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}-1}\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}\left(1-\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}}{\left(1-\left(1-\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)}, \; x > 0.$$

The limiting behaviors of the KW-P(IV) pdf and the hazard function are given in the following theorem.

Theorem 1. *The limit of the KW-P(IV) density function and the hazard function as* $x \to \infty$ *is* 0, *and the limit as* $x \to 0^+$ *is* 0.

Proof. The fact $x > 0$ and $\int_0^\infty g(x)dx = 1$, imply that $\lim_{x\to\infty} g(x) = 0.$ To show $\lim_{x\to\infty} h_g(x) = 0,$ L'Hôpital's rule implies $\lim_{x\to\infty} h_g(x) = -\lim_{x\to\infty} \frac{\partial}{\partial x}(\log g(x))$ which can be shown that it equals to zero. Similarly the other part of the theorem can be proved. □

In Figures 1 and 2, various graphs of $g(x)$, and $h_g(x)$ are provided for different parameter values. The plots indicate that the KW-P(IV) can be approximately symmetric, right-skewed or left-skewed. Also, the KW-P(IV) hazard function can be a decreasing failure rate or upside down bathtub shapes.

Proof. For any real number $x \in \mathbb{R}^+$, $\alpha_1 > \alpha_2$, $\delta_1 > \delta_2$, $a_1 < a_2$ and $b_1 < b_2$, we have

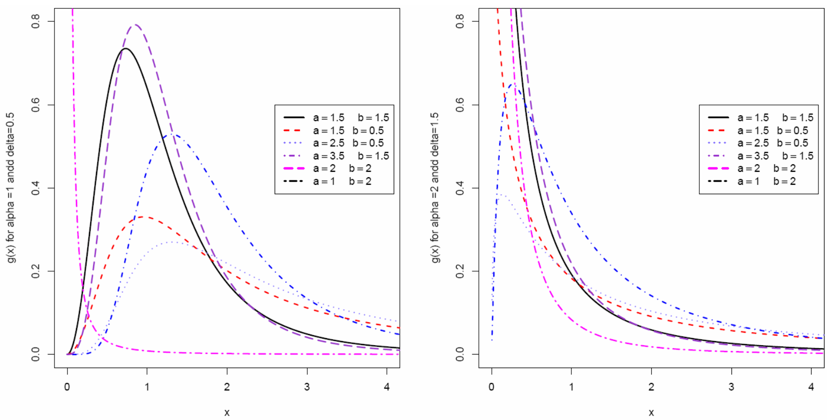

Figure 1. Graphs of the KW-P(IV) p.d.f for various choices of α, δ and θ, *a* and *b*.

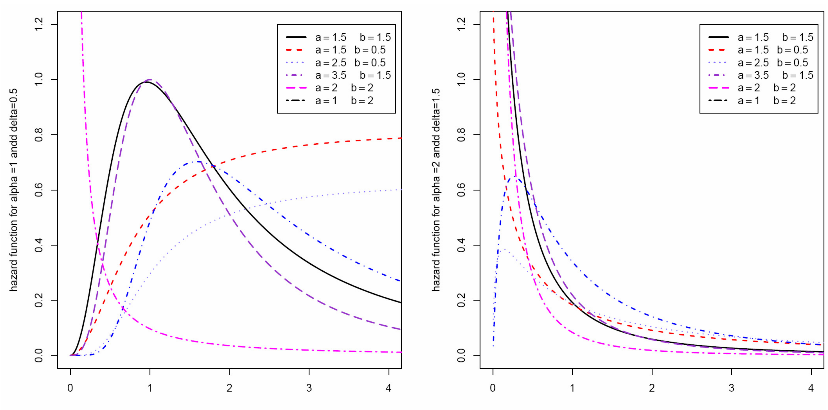

Figure 2. Graphs of the KW-P(IV) hazard function for various choices of α, δ and θ, *a* and *b*.

A direct application of Lemma 1 is described in Lemma 2.

Lemma 2. *The quantile function for the KW-P(IV) distribution is given by*

$$Q(\lambda) = \theta((1-(1-(1-\lambda)^{1/b})^{1/a})^{-1/\alpha} - 1)^{\delta}, \quad where \ 0 \le \lambda \le 1. \tag{7}$$

Proof. The result follows immediately by using $G(Q(\lambda)) = \lambda$ in (5) and then solving it for $Q(\lambda)$. □

Setting $\lambda = 0.25, 0.5$ and 0.75, the first, second and third quantiles of KW-P(IV) can be obtained.

3. Moments and Mean Deviations

We know that if $Y \sim Pareto(IV)(\alpha, \theta, \delta)$, then the rth order raw moment is given by $E(Y^r) = \theta^r B(r\delta + 1, \alpha + 1 - r\delta)$. Hence, from (6), the rth order raw moment (for any $r \ge 1$) for KW-P(IV) distribution will be

$$E(X^r) = ab\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j\binom{b-1}{i}$$

$$\times\binom{a(i+1)-1}{j}\frac{1}{(j+1)}B(r\delta+1, \alpha(j+1)+1-r\delta). \tag{8}$$

3.1. Mean deviations

The deviation from the mean and the deviation from the median are used to measure the dispersion and the spread in a population from the center. If we denote the median by M, then the mean deviation from the mean, $D(\mu)$, and the mean deviation from the median, $D(M)$, can be written as

$$D(\mu) = 2\mu G(\mu) - 2\int_0^{\mu} xg(x)dx = 2\mu G(\mu) - 2E(X(\mu)), \tag{9}$$

$$D(M) = \mu - 2\int_0^{M} xg(x)dx = \mu - 2E(X(M)), \tag{10}$$

where $E(X(a))$ represents the incomplete moment for X.

So from (8), the *r*th order incomplete moment for KW-P(IV) distribution would be

$$E(X(t)^r) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty}(-1)^j \binom{b-1}{i}\binom{a(i+1)-1}{j}$$

$$\times \frac{ab}{(j+1)}\theta^r I_{(t/\theta)^{1/\delta}}(r\delta+1,\ \alpha+1-r\delta), \qquad (11)$$

where $I_x(m, n) = \int_0^x u^{m-1}(1+u)^{-m-n}$ is the incomplete beta integral of the second kind.

Hence, on substitution $r = 1$ and $t = \mu$ and $t = M$ in (10), one can get explicit expression for (9) and (10).

4. Reliability Parameter

The reliability parameter *R* is defined as $R = P(X > Y)$, where *X* and *Y* are independent random variables. Numerous applications of the reliability parameter have appeared in the literature such as the area of classical stress-strength model and the break down of a system having two components. Other applications of the reliability parameter can be found in Hall [13] and Weerahandi and Johnson [21].

If *X* and *Y* are two continuous and independent random variables with the cdfs $F_1(x)$ and $F_2(y)$ and their pdfs $f_1(x)$ and $f_2(y)$, respectively, then the reliability parameter *R* can be written as

$$R = P(X > Y) = \int_{-\infty}^{\infty} F_2(t)\, f_1(t)\, dt.$$

Theorem 2. *Suppose that* $X \sim KW - P(IV)(\alpha_1, \theta, \delta, a_1, b_1)$ *and* $Y \sim KW - P(IV)(\alpha_2, \theta, \delta, a_2, b_2)$. *Then*

$$P(X > Y) = 1 - \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}(-1)^{j_1+j_2+j_3} \times \binom{b_2}{j_1}\binom{j_1 a_2}{j_2}\binom{j_2\alpha_1/\alpha_2}{j_3} B\left(b_1, \frac{j_3}{a_1}+1\right).$$

Proof. From (4) and (5), we have

$$P(X > Y) = \int_0^{\infty}\left\{1 - \left(1 - \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha_2}\right)^{a_2}\right)^{b_2}\right\}$$

$$\times \frac{a_1 b_1 \alpha_1}{\delta\theta}\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}-1}\left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha_1+1)}$$

$$\times \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha_1}\right)^{a_1-1}\left(1 - \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha_1}\right)^{a_1}\right)^{b_1-1} dx. \quad (12)$$

On using the substitution $1 - \left(1 - \left(1 + \left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha_1}\right)^{a_1} = u,$ (12) reduces to

$$P(X > Y)$$

$$= 1 - \int_0^1 u^{b_1-1}(1 - (1 - (1 - (1 - u)^{1/a_1})^{\alpha_1/\alpha_2})^{a_2})^{b_2}\, du$$

$$= 1 - \int_0^1 u^{b_1-1}\left(\sum_{j_1=0}^{\infty}(-1)^{j_1}\binom{b_2}{j_1}((1 - (1 - u)^{1/a_1})^{\alpha_1/\alpha_2})^{j_1 a_2}\right) du$$

$$= 1 - \sum_{j_1=0}^{\infty}\sum_{j_2=0}^{\infty}\sum_{j_3=0}^{\infty}(-1)^{j_1+j_2+j_3}\binom{b_2}{j_1}\binom{j_1 a_2}{j_2}\binom{j_2\alpha_1/\alpha_2}{j_3} B\left(b_1, \frac{j_3}{a_1}+1\right),$$

on successively applying generalized binomial series expansion. □

5. Order Statistics for the KW-P(IV) Distribution

In this section, we consider the expression for the general *r*th order statistic, the asymptotic distribution of the sample minima and the sample maxima when a random sample of size *n* is drawn from the KW-P(IV) distribution. The density function of the *r*th order statistic, $X_{r:n}$, for a random sample of size *n* drawn from (4), is given by

$$f_{X_{r:n}}(x) = \frac{1}{B(r,\ n-r+1)}(G(x))^{r-1}(1-G(x))^{n-r}g(x)$$

$$= \frac{g(x)}{B(r,\ n-r+1)}$$

$$\times\sum_{\ell=0}^{n-r}(-1)^{\ell}\binom{n-r}{\ell}\left\{1-\left(1-\left(1-\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{b}\right\}^{r+\ell-1}, \quad 0 < x < \infty.$$

The pdf of $X_{r:n}$ can be written as

$$f_{X_{r:n}}(x) = \frac{1}{B(r,\ n-r+1)}g(x)\sum_{\ell=0}^{n-r}(-1)^{\ell}\binom{n-r}{\ell}$$

$$\times\left\{\sum_{j=0}^{\infty}(-1)^{j}\binom{b}{j}\left(1-\left(1-\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{j}\right\}^{r+\ell-1}$$

$$= \frac{g(x)}{B(r,\ n-r+1)}\sum_{\ell=0}^{n-r}\sum_{k_1}^{\infty}$$

$$\cdots\sum_{k_{r+\ell-1}=0}^{\infty}(-1)^{\ell+s_k}\binom{n-r}{\ell}\left(1-\left(1-\left(1+\left(\frac{x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{s_k}p_k$$

$$= \frac{b}{B(r,\, n-r+1)} \sum_{\ell=0}^{n-r} \sum_{k_1}^{\infty}$$

$$\cdots \sum_{k_{r+\ell-1}=0}^{\infty} (-1)^{\ell+s_k} \binom{n-r}{\ell} \frac{p_k}{s_k+b}\, g(x\,|\,\alpha,\, \delta,\, \theta,\, a,\, s_k+b), \quad (13)$$

where $s_k = \sum_{i=1}^{r+\ell-1} k_i$ and $p_k = \prod_{i=1}^{r+\ell-1} \binom{b}{k_i}$. Also, $g(x\,|\,\alpha,\, \delta,\, \theta,\, a,\, s_k+b)$ is the KW-P(IV) density function with parameters $(\alpha,\, \delta,\, \theta,\, a,\, s_k+b)$. From (13), it is interesting to note that the pdf of the rth order statistic $X_{r:n}$ can be expressed as an infinite sum of the KW-P(IV) pdfs.

Note: We can derive in general mth order raw moments of $X_{r:n}$ using (8). So for any $m \geq 1$

$$E(X_{r:n}^m) = \frac{b}{B(r, n-r+1)} \sum_{j=0}^{n-r} \sum_{k_1}^{\infty} \cdots \sum_{k_{r+\ell-1}=0}^{\infty} (-1)^{\ell+s_k} \binom{n-r}{\ell} \frac{p_k}{s_k+b}$$

$$\times \left(a(s_k+b) \sum_{i=0}^{\infty} \sum_{j=0}^{\infty} (-1)^j \binom{s_k+b-1}{i} \right.$$

$$\left. \times \binom{a(i+1)-1}{j} \frac{1}{(j+1)} B(r\delta+1,\, \alpha(j+1)+1-r\delta) \right).$$

Next, let us consider the asymptotic distributions of the sample minima $X_{1:n}$. In order to derive the asymptotic distribution of the sample minima $X_{1:n}$, we consider Theorem 8.3.6 of Arnold et al. [3]. Observe that, since $G^{-1}(0) = 0$, it follows from the theorem that the asymptotic distribution of the sample minima $X_{1:n}$ is not of Fréchet type. The asymptotic distribution of $X_{1:n}$ will be of Weibull type with parameter $\delta > 0$ if

$$\lim_{\varepsilon \to 0_+} \frac{G(\varepsilon x)}{G(\varepsilon)} = x^{\delta}, \quad \text{for all } x > 0.$$

By using L'Hôpital's rule, it follows that

$$\lim_{\varepsilon\to 0_+}\frac{G(\varepsilon x)}{G(\varepsilon)} = x\lim_{\varepsilon\to 0_+}\frac{g(\varepsilon x)}{g(\varepsilon)}$$

$$= x^{1/\delta}\lim_{\varepsilon\to 0_+}\frac{\left(1+\left(\frac{\varepsilon x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}\left(1-\left(1+\left(\frac{\varepsilon x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}\left(1-\left(1-\left(1+\left(\frac{\varepsilon x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{b-1}}{\left(1+\left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}\left(1-\left(1+\left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}\left(1-\left(1-\left(1+\left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{b-1}}.$$

Now, since $\lim_{\varepsilon\to 0_+}\frac{\left(1-\left(1-\left(1+\left(\frac{\varepsilon x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{b-1}}{\left(1-\left(1-\left(1+\left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{b-1}} = 1,$ and

$$\lim_{\varepsilon\to 0_+}\frac{\left(1-\left(1+\left(\frac{\varepsilon x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}}{\left(1-\left(1+\left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}} = 1,$$

also

$$\lim_{\varepsilon\to 0_+}\frac{\left(1+\left(\frac{\varepsilon x}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}}{\left(1+\left(\frac{\varepsilon}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}} = 1,\ \lim_{\varepsilon\to 0_+}\frac{G(\varepsilon x)}{G(\varepsilon)} = x^{1/\delta}.$$

Hence, we obtain that the asymptotic distribution of the sample minima $X_{1:n}$ is of the Weibull type with shape parameter $\frac{1}{\delta}$.

Similarly, one can obtain the asymptotic distribution of the sample maximum.

6. Maximum Likelihood Estimation

In this case, the log-likelihood function is given by

$$\log L = n \log \alpha - n \log \delta - n \log \theta + \left(\frac{1}{\delta} - 1\right) \sum_{i=1}^{n} \log\left(\frac{x_i}{\theta}\right)$$

$$- (\alpha + 1) \sum_{i=1}^{n} \log\left(1 + \left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)$$

$$+ (a - 1) \sum_{i=1}^{n} \log\left(1 - \left(1 + \left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)$$

$$+ (b - 1) \sum_{i=1}^{n} \log\left(1 - \left(1 - \left(1 + \left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right). \qquad (14)$$

The likelihood equation for all the parameters is obtained by differentiating the likelihood function. So that we have the following:

$$\frac{\partial}{\partial a} \log L = \frac{n}{a} + \sum_{i=1}^{n} \log\left(1 - \left(1 + \left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)$$

$$- (b - 1) \sum_{i=1}^{n} \left(1 - \left(1 - \left(1 + \left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{-1}$$

$$\times \log\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}, \tag{15}$$

$$\frac{\partial}{\partial b}\log L = \frac{n}{b}+\sum_{i=1}^{n}\log\left(1-\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right), \tag{16}$$

$$\frac{\partial}{\partial \delta}\log L = -\frac{n}{\delta}-\frac{1}{\delta^2}\sum_{i=1}^{n}\log\left(\frac{x_i}{\theta}\right)-\frac{(\alpha+1)}{\delta^2}$$

$$\times\sum_{i=1}^{n}\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-1}\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\log\left(\frac{x_i}{\theta}\right)$$

$$-\frac{\alpha(a-1)}{\delta^2}\sum_{i=1}^{n}\left(\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{-1}\right.$$

$$\left.\times\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\log\left(\frac{x_i}{\theta}\right)\right)$$

$$-\frac{a\alpha(b-1)}{\delta^2}\sum_{i=1}^{n}\left(\left(1-\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{-1}\right.$$

$$\left.\times\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\log\left(\frac{x_i}{\theta}\right)\right), \tag{17}$$

$$\frac{\partial}{\partial\theta}\log L = -\frac{n}{\theta} - \frac{n(1-\delta)}{\delta\theta} + \frac{\alpha+1}{\delta\theta^{1/\delta+1}}\sum_{i=1}^{n}\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-1} x_i^{1/\delta}$$

$$-\frac{(a-1)\alpha}{\delta\theta^{1/\delta+1}}\sum_{i=1}^{n} x_i^{1/\delta}\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{-1}\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}$$

$$+\frac{a\alpha(b-1)}{\delta\theta^{1/\delta+1}}\sum_{i=1}^{n}\left(\left(1-\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{-1}\right.$$

$$\left.\times\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a-1}\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-(\alpha+1)}\right), \tag{18}$$

$$\frac{\partial}{\partial\alpha}\log L = -\frac{n}{\alpha} - \sum_{i=1}^{n}\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{-1}$$

$$\times\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\log\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)$$

$$+(b-1)\left(1-\left(1-\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\right)^{a}\right)^{-1}$$

$$\times\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right)^{-\alpha}\log\left(1+\left(\frac{x_i}{\theta}\right)^{\frac{1}{\delta}}\right). \tag{19}$$

Setting equations (15)-(19) to zero, one can get the MLE estimates of the parameters for a, b, δ, θ, and α, respectively.

7. Estimation Based on the Sample Quantiles

On substitution $\lambda = \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{3}{4}$ in (7) we get the following equations:

- $\theta\left(\left(1-\left(1-\left(1-\frac{1}{2}\right)^{1/b}\right)^{1/a}\right)^{-1/\alpha}-1\right)^{\delta} = Q\left(\frac{1}{2}\right).$

- $\theta\left(\left(1-\left(1-\left(1-\frac{1}{4}\right)^{1/b}\right)^{1/a}\right)^{-1/\alpha}-1\right)^{\delta} = Q\left(\frac{1}{4}\right).$

- $\theta\left(\left(1-\left(1-\left(1-\frac{1}{8}\right)^{1/b}\right)^{1/a}\right)^{-1/\alpha}-1\right)^{\delta} = Q\left(\frac{1}{8}\right).$

- $\theta\left(\left(1-\left(1-\left(1-\frac{1}{16}\right)^{1/b}\right)^{1/a}\right)^{-1/\alpha}-1\right)^{\delta} = Q\left(\frac{1}{16}\right).$

- $\theta\left(\left(1-\left(1-\left(1-\frac{3}{4}\right)^{1/b}\right)^{1/a}\right)^{-1/\alpha}-1\right)^{\delta} = Q\left(\frac{3}{4}\right).$

The above five non-linear equations needs to be solved for a, b, δ, θ, and α, respectively.

8. Simulation Study

We consider a random sample of size $n = 50, 100$ and 200 from our density corresponding to particular choices of the parameters as follows: $\alpha = 2$, $a = 0.4$, $b = 0.5$, $\delta = 1.5$, $\theta = 1$. Below we provide the bias and standard deviation for the estimates of all the parameters under both the methods of estimation. Table 1 provides the bias and standard error under the method of maximum likelihood, Table 2 provides the bias and standard error under the quantile method of moments. We consider 10000 simulations for drawing random samples each of size $n = 50$, $n = 100$, and $n = 200$ drawn from our density, respectively.

8.1. Comment on the simulation study

In simulations and real life data applications described later on, we maximized the log-likelihood function using SAS PROC NLMIXED. For each maximization, the SAS PROC NLMIXED function was executed for a wide range of initial values, and the maximum likelihood estimates were determined as the ones that corresponds to the largest of the maxima. From the simulation study for various choices of sample sizes $(n = 50, 100, 200)$, we observe is that for sample size $n = 50$, the estimates of the parameters using the quantile method are not good when compared with those obtained using the maximum likelihood method. Moreover, for the estimation of α under the maximum likelihood method has a negative bias when the sample sizes are 50 and 100. Furthermore for the quantile estimation after trying various different choices for the λ's for different parameter settings, the choices as mentioned in Section 5, were found to be efficient overall. So, overall we cannot make a general recommendation. In terms of the relative performance of the three estimation strategies, one is not always better than the other. Some further simulation studies are suggested here to see whether anomaly of the results obtained is artifactual or not.

Table 1. Bias and standard deviation of the parameter estimates using maximum likelihood

Sample size	Bias($\hat{\alpha}$)	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\delta}$)	Bias($\hat{\theta}$)	S.E($\hat{\alpha}$)	S.E($\hat{a}$)	S.E($\hat{b}$)	S.E($\hat{\delta}$)	S.E($\hat{\theta}$)
50	-0.098	0.0194	0.0032	0.0371	0.0161	0.0396	0.1098	0.05606	0.0148	0.0028
100	-0.087	0.0114	0.0017	0.0149	0.0118	0.0272	0.0532	0.0383	0.0109	0.0013
200	0.015	0.0089	0.0004	0.0065	0.0107	0.0154	0.0412	0.0203	0.0095	0.0012

Table 2. Bias and standard deviation of the parameter estimates using quantile method

Sample size	Bias($\hat{\alpha}$)	Bias($\hat{a}$)	Bias($\hat{b}$)	Bias($\hat{\delta}$)	Bias($\hat{\theta}$)	S.E($\hat{\alpha}$)	S.E($\hat{a}$)	S.E($\hat{b}$)	S.E($\hat{\delta}$)	S.E($\hat{\theta}$)
50	0.0179	0.0505	0.0681	0.0877	0.0169	0.0513	0.1644	0.0849	0.0068	0.0181
100	0.0114	0.0408	0.0590	0.0679	0.0131	0.0492	0.0904	0.0668	0.0040	0.0082
200	0.0077	0.0294	0.0331	0.0419	0.0109	0.0434	0.0308	0.0489	0.0338	0.0076

9. Application

In this section, one data set is fitted to the KW-P(IV) distribution. The data set is from Mahmoudi [17] and it represents the fatigue life of 6061-T6 aluminum coupons cut parallel with the direction of rolling and oscillated at 18 cycles per second. The maximum likelihood estimates, the log-likelihood value, the AIC (Akaike Information Criterion), the Kolmogorov-Smirnov test statistic (K-S), and the *p*-value for the K-S statistics for the fitted distributions is reported in Table 3. Mahmoudi [17] proposed a five-parameter beta generalized Pareto distribution and fitted the data in Table 3 and compared the result with beta-Pareto and other known distributions. The results of fitting beta generalized Pareto and beta-Pareto from Mahmoudi [17] are reported in Table 3 along with the results of fitting the Pareto (IV) and KW-P(IV) distributions to the data. The KS value from Table 3 indicate that the beta KW-P(IV) distribution provide the best fit follows by the generalized Pareto distribution while the value of AIC implies that the beta generalized Pareto, the beta-Pareto and the KW-Pareto (IV) provide equally adequate fit to the data. Figure 3 displays the empirical and the fitted cumulative distribution functions. This figure supports the results in Table 3.

Table 3. Parameter estimates for the fatigue life of 6061-T6 aluminum coupons data

Distribution	Pareto (IV)	Beta-Pareto	Beta-generalized Pareto	KW-P(IV)
Parameter estimates	$\hat{\delta} = 0.0253$ $\hat{\gamma} = 0.1234$	$\hat{\alpha} = 485.470$ $\hat{\beta} = 162.060$ $\hat{k} = 0.3943$ $\hat{\theta} = 3.910$	$\hat{\alpha} = 12.112$ $\hat{\beta} = 1.702$ $\hat{\mu} = 40.564$ $\hat{k} = 0.273$ $\hat{\theta} = 54.837$	$\hat{\alpha} = 4.2628$ $\hat{\delta} = 0.0637$ $\hat{a} = 1.0934$ $\hat{b} = 2.1176$ $\hat{\theta} = 1.092$
Log likelihood	-754.19	-458.65	-457.85	-443.28
AIC	1512.38	925.30	925.70	917.16
K-S	0.5827	0.091	0.070	0.048
K-S p-value	0.000	0.376	0.700	0.653

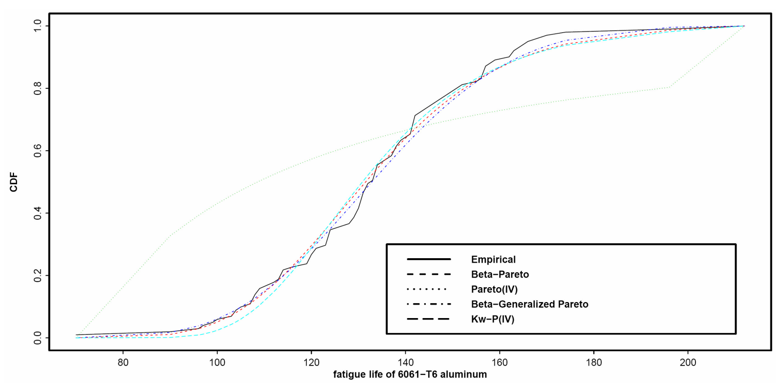

Figure 3. CDF for fitted distributions of the fatigue life of 6061-T6 Aluminium data.

10. Conclusion

A special case of the Kumaraswamy-*G* family of distributions, the

Kumaraswamy-Pareto (IV) distribution is defined and studied. Various properties of the Kumaraswamy-Pareto (IV) distribution are studied, including moments, hazard function, reliability parameter. The new model includes as special sub-models the Pareto(IV) and exponentiated Pareto (EP) distributions (Gupta et al. [11]). It is observed that the distribution can be symmetric, positively skewed and negatively skewed with a broader class of monotone hazard rates. An application to a real data set shows that the fit of the new model is superior to the fits of its main sub-models. Estimation of the model parameters under the bayesian paradigm is currently underway and will be reported in a separate article elsewhere.

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