Polarization entropy of the two-photon system

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Abstract. We consider the bipartite state of a two-photon polarization system and obtain the exact analytical expression for the von Neumann entropy in the particular case of polarization when the entropy depends on only five parameters rather than the full fifteen parameters. We investigate and graphically illustrate the dependence of the entropy on these five parameters, in particular, the existence of exotic to nonexotic statetransitions in single- and two-photon correlations.

1. General

Inspired by the pioneering experiments on quantum entanglement by Aspect*et al*. [1,2], thetheory of two-photon polarization phenomena were developed at theInstitute for Physical Research of the Armenian Academy of Sciences in the latter part of the 1980's[3-5].In these works, all possible states of polarization of the two-photon system were classified and analyzedand, thus, related toall possible outcomes of polarization measurements of the two-photon system.All such properties were studied in the case of two-photon emission from an atom in an external resonant field [4]. In addition, symmetry properties of a two-photon system were analysed with respect to transformations of both space rotations and inversion [6].

The role of the von Neumann entropy [7] has gained in importance in recent years owing to the extensive development of the physics of entangled quantum states. The entropy plays a fundamental role as a quantitative measure of entanglement [8,9]. Accordingly, any exact result of the von Neumann entropy for a correlated quantum system is of extreme importance. In this paper, we consider the bipartite system of a pair of photons and exactly calculate the general expression for the von Neumann entropy for correlated, polarized states [5,10].

The generalized conditional entropy has been analysed for this system [11] for which knowledge of the eigenvalues of the density matrix is not required.

Correlation transfer from one-photon to two-photon systems, not in any restricted subspace, but in the complete space of the polarization degree of freedom has been studied [12]. Three protocols for directly measuring the concurrence of two-photon polarization-entangled states, including pure states and mixed states has been considered [13]. An experimentally realizable scheme for manipulating the entanglement of an arbitrary state of two polarization entangled qubits has been introduced, where the von Neumann entropy provides a convenient and useful measure of the purity of the state [14].

The paper is arranged as follows. In Sec. 2, we review the general properties of the normalized polarization density matrix for two photons. In Sec. 3, we obtain the exact, general expression for the entropy, which is used in Sec. 4 to study three relatively simpler cases.

2. Von Neumann entropy for two-photon system

The general form of the normalized polarization density matrix for two photons [5,10] is given by

$$\hat{\rho}^{(1,2)} = (1/4)\left[\tilde{I}^{(1)} \otimes \tilde{I}^{(2)} + \boldsymbol{\xi}^{(1)}\hat{\boldsymbol{\sigma}}^{(1)} \otimes \tilde{I}^{(2)} + \tilde{I}^{(1)} \otimes \boldsymbol{\xi}^{(2)}\hat{\boldsymbol{\sigma}}^{(2)} + \zeta_{ij}\hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_j^{(2)} \right], \tag{1}$$

where $\tilde{I}^{(1,2)}$ and $\hat{\boldsymbol{\sigma}}^{(1,2)}$ are 2×2 Pauli vector matrices acting in the polarization space of photons and the real dimension less quantities $\boldsymbol{\xi}^{(1,2)}, \zeta_{ij}(i,j=1,2,3)$ are functions of the photon momenta and of the parameters of the emitting system. The vectors $\boldsymbol{\xi}^{(1,2)}$ are the Stokes vectors of photons 1, 2 respectively, while the parameter ζ_{ij} describes the two-photon polarization correlation. In the case of no photon entanglement, one has that

$$\zeta_{ij} = \xi_i^{(1)}\xi_j^{(2)}. \tag{2}$$

The Stokes parameters $\boldsymbol{\xi}^{(1,2)}$ and ζ_{ij} satisfy the inequalities [5]

$$\left| \xi_i^{(1)} + \xi_j^{(2)} \right| - 1 \le \zeta_{ij} \le \left| \xi_i^{(1)} - \xi_j^{(2)} \right| + 1 \tag{3.a}$$

$$\xi^{(1)^2} + \xi^{(2)^2} + \zeta_{ij}^2 \le 3. \tag{3.b}$$

The reduced density matrix for photon 1, viz., $\hat{\rho}^{(1)}$, is obtained by taking the trace of (1) over the quantum states of photon 2, which gives us the Stokes matrix of photon 1,

$$\hat{\rho}^{(1)} = Tr_2\hat{\rho}^{(1,2)} = (1/2)\left[\tilde{I}^{(1)} + \boldsymbol{\xi}^{(1)}\hat{\boldsymbol{\sigma}}^{(1)} \right]. \tag{4.a}$$

Conversely, by taking a trace of (1) over the quantum states of photon 1, one obtains the Stokes matrix of photon 2, viz. $\hat{\rho}^{(2)}$,

$$\hat{\rho}^{(1)} = Tr_{r_1}\hat{\rho}^{(1,2)} = (1/2)\left[\tilde{I}^{(2)} + \boldsymbol{\xi}^{(2)}\hat{\boldsymbol{\sigma}}^{(2)} \right]. \tag{4.b}$$

The goal of our paper is to calculate the von Neumann entropy [1]

$$S = -Tr\left[\hat{\rho}^{(1,2)} \ln \hat{\rho}^{(1,2)} \right] \tag{5}$$

of the two photon system in a mixed quantum state, that is, when

$$\hat{\rho}^{(1,2)^2} \ne \hat{\rho}^{(1,2)}.$$

Araki and Lieb [15] have proven that

$$\left| S_1 - S_2 \right| \le S \le S_1 + S_2, \tag{6}$$

where S_1, S_2 are the reduced von Neumann entropies of photons 1,2

$$S_\alpha = \ln 2 - (1/2) \ln \left\{ \left[1 - \xi^{(\alpha)}\right]^{1-\xi^{(\alpha)}} \left[1 + \xi^{(\alpha)}\right]^{1+\xi^{(\alpha)}} \right\}, (\alpha = 1, 2). \tag{7}$$

The entropy S_α is a monotonically decreasing function of the Stokes parameter for $0 \le \xi^{(\alpha)} \le 1$ and achieves its maximum value $\ln 2$ for the completely unpolarized state when $\xi^{(\alpha)} = 0$ and its minimum value zero when the photon is in a pure polarized state, viz. $\xi^{(\alpha)} = 1$.

3. Entropy for two-photon polarization density matrix

The arbitrary polarization state of the photon pair (1) is described by fifteen real parameters [5] and owing to the Araki-Lieb inequality (6), the maximum entropy attainable is

$$S_{\max} = 2 \ln 2 \tag{8}$$

when all 15parameters are set equal to zero. That is to say, no correlations of any kind, neither individually nor for the pair of photons.

The eigenvalue equation of the matrix (1) is

$$\lambda^4 - \lambda^3 + c_2 \lambda^2 - c_1 \lambda + c_0 = 0, \tag{9}$$

where the coefficients are defined as follows

$$c_2 = (1/8)p,$$

$$c_1 = (1/16)\left[p - 2(1 + \det \hat{\zeta} - \xi_i^{(1)} \zeta_{ij} \xi_j^{(2)}) \right], \tag{10}$$

$$c_0 = \det \hat{\rho}^{(1,2)}.$$

In (10), p is a nonnegative number (see (3.b)) which is called the *purity* of the state [16]

$$p = 3 - \xi^{(1)^2} - \xi^{(2)^2} - \sum_{i,j} \zeta_{ij}^2 \tag{11}$$

and describes the "distance" of the mixed state of the system from the pure state where $p = 0$ [10, 5].The four non-negative solutions $\lambda_j (j = 1,...,4)$ yield the von Neumann entropy

$$S = -\lambda_j \ln \lambda_j. \tag{12}$$

The roots of the quartic equation (9), with coefficients given by (10), are extremely awkward to write and thus the exact expression for the entropy (12) would not be too useful. Accordingly, we consider special cases of (9) that lead to the quartic form (9) being reduced to the product of two quadratic polynomials.

4. Photon correlations

We consider the special case of the density matrix (1) which contains the main properties of the two-photon system and which is easier to handle mathematically than the general case that includes all 15 real parameters. Namely, we consider the case where the 15 real parameters are reduced to actually 5.

$$\hat{\rho}^{(1,2)} = (1/4)\left[\tilde{I}^{(1)} \otimes \tilde{I}^{(2)} + \xi_3^{(1)} \hat{\sigma}_3^{(1)} \otimes \tilde{I}^{(2)} + \tilde{I}^{(1)} \otimes \xi_3^{(2)} \hat{\sigma}_3^{(2)} + \zeta_{ii} \hat{\sigma}_i^{(1)} \otimes \hat{\sigma}_j^{(2)} \right], \quad (13)$$

where the 5 parameters satisfy inequalities (3)

$$-1 \le \zeta_{11}, \zeta_{22} \le 1, \tag{14.a}$$

$$\left| \xi_3^{(1)} + \xi_3^{(2)} \right| - 1 \le \zeta_{33} \le \left| \xi_3^{(1)} - \xi_3^{(2)} \right| + 1, \tag{14.b}$$

$$-1 \le \xi_3^{(1)}, \xi_3^{(2)} \le 1, \tag{14.c}$$

and

$$\xi_3^{(1)^2} + \xi_3^{(2)^2} + \zeta_{ii}^2 \le 3. \tag{14.d}$$

The density matrix (13) describes all possible polarization states of the pair of photons, that is, from the completely unpolarised state to that of the pure polarized state as defined in [5]. The eigenvalue equation (9) factors into the product of two quadratic polynomials and the eigenvalues are given by

$$\lambda_{1,2} = (1/4)\left[1 + \zeta_{33} \pm x_+ \right], \tag{15.a}$$

$$\lambda_{3,4} = (1/4)\left[1 - \zeta_{33} \pm x_- \right], \tag{15.b}$$

where

$$x_\nu = \left[\left(\xi_3^{(1)} + \nu \xi_3^{(2)} \right)^2 + \left(\zeta_{11} - \nu \zeta_{22} \right)^2 \right]^{1/2}, \nu = \pm. \tag{15.c}$$

One obtains for the entropy (12) the expression

$$S = S_{\max} - (1/4) \sum_\nu \ln \left\{ \left[1 + \nu \zeta_{33} + x_\nu \right]^{1 + \nu \zeta_{33} + x_\nu} \left[1 + \nu \zeta_{33} - x_\nu \right]^{1 + \nu \zeta_{33} - x_\nu} \right\} \tag{16}$$

Entropy (16) depends on the three quantities ζ_{33}, x_+, x_-. The requirement that the eigenvalues (15) be real, positive quantities in order to give rise to a real valued entropy implies that

$$-1 \le \zeta_{33} \le 1 \tag{17}$$

and

$$-1 \mp \zeta_{33} \leq x_{\pm} \leq 1 \pm \zeta_{33}. \tag{18}$$

Note, that the general expression for the entropy (16) is a function of the three variables ζ_{33}, x_{+}, x_{-}. For definiteness, we consider $\zeta_{33} \geq 0$. The entropy (16) assumes its maximum value $2\ln 2$ when

$$\zeta_{33} = x_{+} = x_{-} = 0 \tag{19}$$

(completely unpolarised state) and it assumes the value of zero (pure state of two photon polarization) when

$$\zeta_{33} = 1, x_{+} = 2, x_{-} = 0. \tag{20}$$

The requirement that the eigenvalues (15) be positive quantities is also satisfied provided

$$1 \leq p/2 + \zeta_{33}^2, \tag{21.a}$$

$$|q| \leq p/2 + \zeta_{33}^2 - 1, \tag{21.b}$$

where q is defined by

$$q = \xi_3^{(1)} \xi_3^{(2)} - \zeta_{33} - \zeta_{11} \zeta_{22}. \tag{21.c}$$

The parameter q also describes the purity of the two-photon polarization state given by the density matrix (13), which is equal to zero for pure polarized states of the pair of photons [5].

4.1 Two-photon correlations

Consider the correlated density matrix

$$\hat{\rho}^{(1,2)} = (1/4)\left[\tilde{I}^{(1)} \otimes \tilde{I}^{(2)} + \zeta \left(\hat{\sigma}_1^{(1)} \otimes \hat{\sigma}_1^{(2)} - \hat{\sigma}_2^{(1)} \otimes \hat{\sigma}_2^{(2)} \right) + \zeta_{33} \hat{\sigma}_3^{(1)} \otimes \hat{\sigma}_3^{(2)} \right], \tag{22}$$

where the positivity of the eigenvalues requires that

$$-1 \leq \zeta_{33} \leq 1 \tag{23.a}$$

and

$$-1 - \zeta_{33} \leq 2\zeta \leq 1 + \zeta_{33}. \tag{23.b}$$

The von Neumann entropy (16) is

$$S(\zeta, \zeta_{33}) = S_{\max} - (1/2)\ln(1 - \zeta_{33})^{1-\zeta_{33}} -$$
$$-(1/4)\ln\left[(1 + \zeta_{33} + 2\zeta)^{1+\zeta_{33}+2\zeta} (1 + \zeta_{33} - 2\zeta)^{1+\zeta_{33}-2\zeta} \right] \tag{24}$$

The plot of the entropy (24) is shown in Fig.1 where it is evaluated for different values of ζ_{33}.
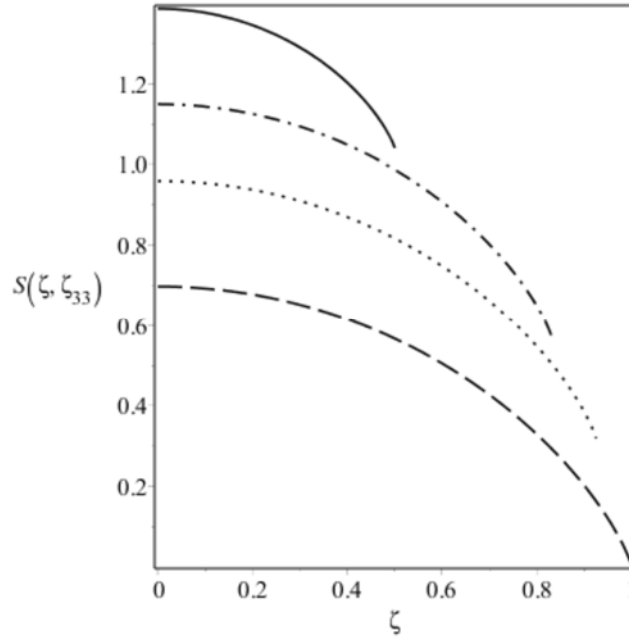
Fig. 1: Plot of the entropy (24) for $\xi_3^{(1)} = \xi_3^{(2)} = 0$, $\zeta_{11} = -\zeta_{22} = \zeta$, which gives $x_+ = 2|\zeta|$ and $x_- = 0$. Inequality (18) becomes $0 \le \zeta_{33} \le 1$ and $0 \le 2\zeta \le 1 + \zeta_{33}$. The values for ζ_{33} are as follows: $\zeta_{33} = 0$ (solid), $\zeta_{33} = 0.66$ (dash-dot), $\zeta_{33} = 0.85$ (dot), and $\zeta_{33} = 1.0$ (single - photon entropy (7)) (dash).

Note that the range of values for ζ is restricted by the reality condition for the von Neumann entropy by $0 \le 2\zeta \le 1 + \zeta_{33}$. The decrease of the entropy for increasing values of ζ_{33} is to be expected since decreasing the value of ζ_{33} means that the state of the two photons becomes more chaotic.

4.2 Exotic to non-exotic transition in single- and two-photon correlations

The previous example and the one to follow below given by (26) are in agreement with our common understanding of entropy as a measure of the disorder of a system. In both these cases, the entropy is a monotonically decreasing function of the polarization. However, the system in this second example, gives rise to a very different behaviour of the entropy as a function of the polarization.

Suppose that two of the eigenvalues in (15) are set equal to zero, viz. $\lambda_2 = \lambda_4 = 0$, which is satisfied for, $\zeta_{11} = \zeta_{22} = \sqrt{\left(1 - \xi_3^{(1)}\right)\left(1 - \xi_3^{(2)}\right)}$ and $\zeta_{33} = \xi_3^{(1)} + \xi_3^{(2)} - 1$. The density matrix (13) becomes

$$
\begin{aligned}
\hat{\rho}^{(1,2)} = (1/4)\Big\{ &\tilde{I}^{(1)} \otimes \tilde{I}^{(2)} + \xi_3^{(1)} \hat{\sigma}_3^{(1)} \otimes \tilde{I}^{(2)} + \tilde{I}^{(1)} \otimes \xi_3^{(2)} \hat{\sigma}_3^{(2)} + \\
&+ \sqrt{\left(1 - \xi_3^{(1)}\right)\left(1 - \xi_3^{(2)}\right)} \left[\hat{\sigma}_1^{(1)} \otimes \hat{\sigma}_1^{(2)} + \hat{\sigma}_2^{(1)} \otimes \hat{\sigma}_2^{(2)} \right] + \left(\xi_3^{(1)} + \xi_3^{(2)} - 1 \right) \hat{\sigma}_3^{(1)} \otimes \hat{\sigma}_3^{(2)} \Big\}
\end{aligned}
\tag{25}
$$

Entropy (16) for the density matrix (25) is

$$S(\xi_3^{(1)}, \xi_3^{(2)}) = -\frac{\xi_3^{(1)} + \xi_3^{(2)}}{2} \ln \frac{\xi_3^{(1)} + \xi_3^{(2)}}{2} -$$

$$-\left(1 - \frac{\xi_3^{(1)} + \xi_3^{(2)}}{2}\right) \ln \left(1 - \frac{\xi_3^{(1)} + \xi_3^{(2)}}{2}\right) = S_1\left(\xi_3^{(1)} + \xi_3^{(2)}\right). \tag{26}$$

Note that the dependence of entropy (26) on the Stokes parameters $\xi_3^{(1)}$ and $\xi_3^{(2)}$ is via their sum, while such is not the case for the density matrix (25).

The case $\xi_3^{(1)} = \xi_3^{(2)} = \xi$, shown in Fig.2 by the solid plot, gives for the entropy (26)

$$S(\xi) = -\xi \ln \xi - (1 - \xi) \ln(1 - \xi), \tag{27}$$

which is actually the binary entropy function $h(\xi)$ introduced by Wooters [2] in the definition of the entanglement of formation. Also, one obtains the reduced single-photon entropy (7), dash plot in Fig.2, for $\xi_3^{(1)} + \xi_3^{(2)} = 1 - \xi$.

Figure 2 shows the reduced single-photon entropy (7), dash graph, together with the two-photon entropy (26) as a function of the Stokes parameter for photon 1, $\xi_3^{(1)}$, for various fixed values of the Stokes parameter for photon 2, $0 \le \xi_3^{(2)} < 1$. The points of intersection between the dash plot and the other plots represent the transition points whereby $S\left(\xi_3^{(1)} + \xi_3^{(2)}\right)$ as a function of $\xi_3^{(1)}$, goes from the region where $S_1 > S$ (exotic states) to the region where $S > S_1$ (non-exotic states), that is, where the quantum conditional entropy [17] changes sign from negative to positive.
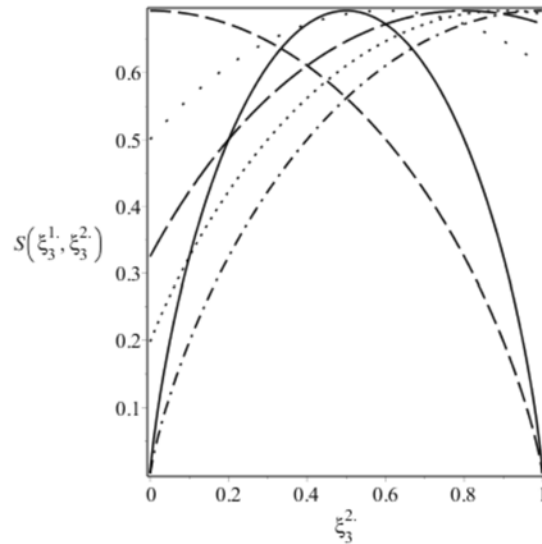

Fig. 2: Plot of the two-photon entropy (26) as a function of $\xi_3^{(1)}$ for different values of $\xi_3^{(2)}$. The values are as follows: for $\xi_3^{(1)} = \xi_3^{(2)}$ (solid), $\xi_3^{(2)} = 0$ (dash-dot), $\xi_3^{(2)} = 0.1$ (dot), $\xi_3^{(2)} = 0.2$ (long-dash), $\xi_3^{(2)} = 0.4$ (space-dot), and $\xi_3^{(2)} = 1 - \xi - \xi_3^{(1)}$ (dash). The dash plot corresponds to the single-photon entropy of (7) which intersects the other graphs at $\xi_3^{(1)} = \left(1 - \xi_3^{(2)}\right)/2$.

4.3 Stokes parameters and normal single- andtwo-photon correlations

In order to find the role of the polarization states of individual photons in the entropy of the two-photon system, we consider the case where the Stokes parameters $\xi_3^{(1)} = \xi_3^{(2)} \equiv \xi$ are nonzero and the photon correlation parameters $\zeta_{11} = -\zeta_{22} \equiv \zeta$, and $\zeta_{33} = 1$. The latter choice of parameters will allow the pure state to be realized when $\zeta = 0$ for $\xi = 0$ in the entropy (28) given below. The corresponding density matrix and entropy are

$$\hat{\rho}^{(1,2)} = (1/4)\Big[\tilde{I}^{(1)} \otimes \tilde{I}^{(2)} + \xi\big(\hat{\sigma}_3^{(1)} \otimes \hat{I}^{(2)} - \hat{I}^{(1)} \otimes \hat{\sigma}_3^{(2)}\big) +$$
$$+ \zeta\big(\hat{\sigma}_1^{(1)} \otimes \hat{\sigma}_1^{(2)} - \hat{\sigma}_2^{(1)} \otimes \hat{\sigma}_2^{(2)}\big) + \hat{\sigma}_3^{(1)} \otimes \hat{\sigma}_3^{(2)}\Big], \tag{27}$$

and

$$S = S_{\max} - (1/4)\ln\Big[\big(2+x_+\big)^{2+x_+}\big(2-x_+\big)^{2-x_+}\Big], \tag{28}$$

where

$$x_+ = 2\sqrt{\xi^2 + \zeta^2} \le 2, 0 \le \xi \le 1. \tag{29}$$

As can be seen in Fig.3, the two-photon entropy mimics that of the single photon entropy, that is, the single photon entropy (7) decreases with increasing values of ξ while the two-photon entropy (28) decreases for given value of ξ with increasing values of ζ.

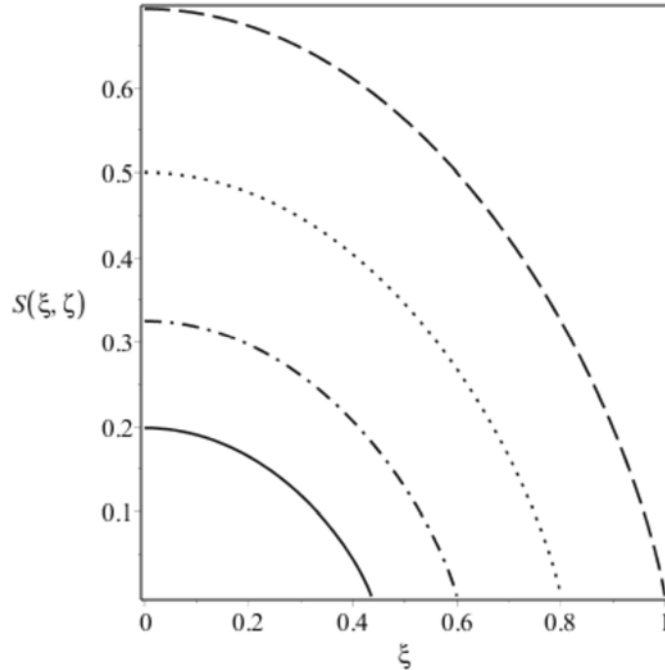
Fig.3: Plot of the entropy (28) for $\xi_3^{(1)} = \xi_3^{(2)} = \xi, \zeta_{11} = -\zeta_{22} = \zeta$ and$\zeta_{33} = 1$, which gives $x_+ = 2\sqrt{\xi^2 + \zeta^2} \le 2$ and $x_- = 0$. The values for ζ are as follows: $\zeta = 0$ (dash) (single photon entropy (7)), $\zeta = 0.6$ (dot), $\zeta = 0.8$ (dashdot), and $\zeta = 0.9$ (solid).

5   Conclusions

Thus, we have obtained exact results for the joint von Neumann entropy for the polarization states of a two-photon system governed by a five-parameter polarization density matrix and studied the sign of the quantum conditional entropy. We find that the quantum conditional entropy may assume positive or negative values. The latter indicates the presence of exotic states. We believe that these results may be of interest in the general area of quantum computation and quantum information theories.

References

[1]A.Aspect, P.Grangier and G.Roger, Phys. Rev. Lett., **49**, (1982) 94.
[2]A.Aspect, J.Dalibard and G.Roger, Phys. Rev. Lett., **49**, (1982) 1804.
[3]V.E.Mkrtchian, V.O.Chaltykyan, Izv.AN Arm.SSR Fiz. **22** (1987) 241.
[4]V.E.Mkrtchian, V.O.Chaltykyan, Izv.AN Arm.SSR Fiz. **22** (1987) 297.
[5] V.E. Mkrtchian, V.O. Chaltykian, Opt. Comm. **63** (1987) 239.
[6]V.E.Mkrtchian, V.O.Chaltykyan, Proc.Arm.Nat.Acad.Sci. **29** (1994) 55.
[7] J.von Neumann, *Mathematische Grundlagen der Quantenmechanik* (Springer, 1968).
[8] W.K.Wooters, Phys.Rev.Lett.**80**(1998) 2245.
[9] L.Amico, R. Fazio, A.Osterloh, V.Vedral, Rev.Mod.Phys**. 80** (2008) 517.
[10]U.Fano, Rev.Mod.Phys. **55** (1983) 855.
[11]N.Gigena, R.Rossignoli, J.Phys.A: Math.Theor. **47** (2014) 015302.
[12] G.Kulkarni, V.Subrahmanyam, A.K.Jha, Phys.Rev.A **93** (2016) 063842.
[13] L.H.Zhang, Q.Yang, M.Yang, W.Song, Z.-L.Cao, Phys.Rev.A **88**(2013) 062342.
[14] R.T.Thew, W.J.Munro, Phys.Rev.A **64** (2001) 022320.
[15] H.Araki, E.H.Lieb, Commun. Math.Phys. **18** (1970) 160.
[16] J.Gemmer, M.Michel, and G.Mahler, *Quantum thermodynamics*, (Springer, 2009).
[17] M.A.Nielsen and I.Chuang, *Quantum Computation and Quantum Information* (Cambridge, UK, 2000).