ON β-FAVORABILITY OF THE STRONG CHOQUET GAME

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Abstract. In the main result, partially answering a question of Telgársky, the following is proven: if X is a 1st countable R_0-space, then player β (i.e. the EMPTY player) has a winning strategy in the strong Choquet game on X if and only if X contains a nonempty W_δ-subspace which is of the 1st category in itself.

1. Introduction

Various aspects, and applications of the so-called *strong Choquet game* $Ch(X)$ have been thoroughly studied in the literature (cf. [BLR], [CP], [Ch], [De1], [De2], [De3], [DM], [GT], [Ma], [NZ], [PZ1], [PZ2], [Por], [Te1], [Te2], [Zs1], [Zs2]). In the game, introduced by *Choquet* [Ch], two players, α and β, take turn in choosing objects in a topological space X: β starts, and always chooses an open set V and a point $x \in V$, then α chooses an open set U such that $x \in U \subseteq V$. After countably many rounds α wins the game if the intersection of the chosen open sets is nonempty, otherwise, β wins. Choquet proved, that in a metrizable space X, α has a *strategy*, depending on all the previous moves of the opponent, which wins every run of the game, if and only if, X is completely metrizable; Choquet actually proved that this is equivalent to α having a *tactic* in $Ch(X)$, i.e. a strategy depending on the very last move of the opponent. It turns out, that in a non-metrizable setting, a winning strategy for α does not always guarantee a winning tactic for α ([HZ, Example 2.7] with [De2] shows this, the completely regular example of [De3] is also of this kind). However, winning tactics, and strategies for α coincide in T_3-spaces with a base of countable order [BLR] (BCO, in short - see section 2 for definitions), or in 2nd countable T_1-spaces [DM].

In this paper we will be interested in β's chances of winning every run of the game, regardless of α's choices, i.e. when $Ch(X)$ is β-*favorable*. We will not have to worry about a winning tactic vs. strategy for β in $Ch(X)$,

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2010 *Mathematics Subject Classification.* Primary 91A44; Secondary 54E52, 54B20.

Key words and phrases. strong Choquet game, Banach-Mazur game, (hereditarily) Baire space, sieve, W_δ-set, Vietoris topology, Tychonoff plank, R_0-space.

since one implies the other [GT, Corollary 3]. The classical result about β-favorability of the strong Choquet game - independently obtained by *Debs* [De1, Theorem 4.1], and *Telgársky* [Te1, Theorem 1.2] - claims that *in a metrizable space X, $Ch(X)$ is β-favorable if and only if X is not hereditarily Baire* (i.e. when X has a nonempty closed non-Baire subspace), or equivalently by Hurewicz' theorem, iff X *contains the rationals as a closed (resp. G_δ) subspace*. Since the main goal of Debs' research in [De1] was to generalize Hurewicz' theorem to 1st countable T_3-spaces (see [vD] for an alternative proof), the following had not been specifically stated, but had been established in [De1]:

Debs' Theorem. *Let X be a T_3, 1st countable, perfect space (i.e. the closed sets are G_δ). Then the following are equivalent:*

(i) *$Ch(X)$ is β-favorable,*
(ii) *X is not hereditarily Baire.*

It is not hard to extend Debs' Theorem for any R_0-space with a BCO, although a new argument is necessary, since without regularity we cannot rely on embedding the rationals as a closed subspace to produce non-hereditary Baireness. As a byproduct, we prove Debs' Theorem in any 1st countable perfect space, with no additional separation axioms. To achieve these generalizations, we use so-called W_δ-subsets [CCN], introduced by *Wicke* and *Worrell* (they called them "sets of interior condensation" [WW1]). While studying β-favorability of the strong Choquet game in [Te1], *Telgársky* noticed that if X contains a nonempty W_δ-subset of the 1st category in itself, then $Ch(X)$ is β-favorable, and asked whether the converse is also true:

Telgársky's Problem. *Is it true that the following are equivalent:*

(i) *$Ch(X)$ is β-favorable,*
(ii) *X contains a nonempty W_δ-subset of the 1st category in itself?*

In our main result (Theorem 3.6) we show that this is indeed the case in 1st countable R_0-spaces. Finally, using hyperspaces with the Vietoris topology, we construct examples that demonstrate the limitations of the conditions from our generalizations of Debs' Theorem.

2. Preliminaries

Unless otherwise noted, all spaces are topological. As usual, ω denotes the non-negative integers, every $k \geq 1$ will be viewed as the set of predecessors $k = \{0, \dots, k-1\}$; ω_1 is the first uncountable ordinal. Let $\mathcal{B}$ be a base for a topological space X, and denote

$$\mathcal{E} = \mathcal{E}(X) = \mathcal{E}(X, \mathcal{B}) = \{(x, U) \in X \times \mathcal{B} : x \in U\}.$$

In the *strong Choquet game* $Ch(X)$ players β and α alternate in choosing $(x_n, V_n) \in \mathcal{E}$ and $U_n \in \mathcal{B}$, respectively, with β choosing first, so that for each $n < \omega$, $x_n \in U_n \subseteq V_n$, and $V_{n+1} \subseteq U_n$. The play

$$(x_0, V_0), U_0, \ldots, (x_n, V_n), U_n, \ldots$$

is won by α, if $\bigcap_n U_n (= \bigcap_n V_n) \neq \emptyset$; otherwise, β wins.

A *strategy* in $Ch(X)$ for α (resp. β) is a function $\sigma : \mathcal{E}^{<\omega} \to \mathcal{B}$ (resp. $\sigma : \mathcal{B}^{<\omega} \to \mathcal{E}$) such that

$$x_n \in \sigma((x_0, V_0), \ldots, (x_n, V_n)) \subseteq V_n \quad \text{for all } ((x_0, V_0), \ldots, (x_n, V_n)) \in \mathcal{E}^{<\omega}$$

(resp. $\sigma(\emptyset) = (x_0, V_0)$ and $V_n \subseteq U_{n-1}$, where $\sigma(U_0, \ldots, U_{n-1}) = (x_n, V_n)$, for all $(U_0, \ldots, U_{n-1}) \in \mathcal{B}^n$, $n \geq 1$). A strategy σ for α (resp. β) is a *winning strategy* (w.s. in short), if α (resp. β) wins every run of $Ch(X)$ compatible with σ, i.e. such that $\sigma((x_0, V_0), \ldots, (x_n, V_n)) = U_n$ for all $n < \omega$ (resp. $\sigma(\emptyset) = (x_0, V_0)$ and $\sigma(U_0, \ldots, U_{n-1}) = (x_n, V_n)$ for all $n \geq 1$). We will say that $Ch(X)$ *is α-, β-favorable*, respectively, provided α, resp. β has a w.s. in $Ch(X)$.

The *Banach-Mazur game* $BM(X)$ [HMC] (also called the *Choquet game* [Ke]) is played similarly to $Ch(X)$, the only difference is that both β, α choose open sets from a fixed π-base. Winning strategies, α-, and β-favorability of $BM(X)$ can be defined analogously to $Ch(X)$. We will only need the fact that in an arbitrary topological space X, $BM(X)$ is β-favorable iff X is not a *Baire space*, i.e. X has a nonempty open 1st category subspace [Ke].

A topological space X is an *R_0-space* [Da] (also called *essentially T_1* [WW1]), provided for any $x, y \in X$, $\overline{\{x\}}, \overline{\{y\}}$ are either disjoint, or equal; equivalently, if each open subset U of X contains the closure of each point of U. We will say that X has a *base of countable order* (BCO), provided there is a sequence $(\mathcal{B}_n)$ of bases for X such that whenever $x \in B_n \in \mathcal{B}_n$, and (B_n) is decreasing, then $\{B_n : n \in \omega\}$ is a base at x [Gr]. This definition mimics the definition of a *development* $(\mathcal{B}_n)$, in which we do not require (B_n) to be decreasing; a space with a development is *developable*, and a developable T_3-space is a *Moore space.* The term "base of countable order" is justified, because in R_0-spaces having a BCO is equivalent to the existence of a single base $\mathcal{B}$ for X such that whenever (B_n) is a strictly decreasing sequence of elements of $\mathcal{B}$ containing some $x \in X$, (B_n) forms a base of neighborhoods at x [WW1, Theorem 2]. Developable spaces have a BCO, but these notions are not equivalent: ω_1 with the order topology is not developable, but has a BCO (see [WW1] for more on these properties).

Let $Y \subseteq X$. A *sieve* of Y (cf. [CCN], [Gr]) in X is a pair (G, T), where $(T, <)$ is a tree of height ω with levels $T_0, T_1, \dots$, and G is a function on T with X-open values such that

- $\{G(t) : t \in T_0\}$ is a cover of Y,
- $Y \cap G(t) = \bigcup\{Y \cap G(t') : t' \in T_{n+1}, t' > t\}$ for each n, and $t \in T_n$,
- $t \leq t' \Rightarrow G(t) \supseteq G(t')$ for each $t, t' \in T$.

We will say that Y is a *W_δ-set* in X, if Y has a sieve (G, T) in X such that $\bigcap_n G(t_n) \subseteq Y$ for each branch (t_n) of T. A G_δ-set is also a W_δ-set. A Tychonoff space is *sieve complete* iff it is a W_δ-subspace of a compact space iff it is a continuous open image of a Čech-complete space [WW2, Theorem 4]; in particular, sieve complete spaces are of the 2nd category.

Lemma 2.1.

(i) *If in a space X the closed sets are W_δ, then X is an R_0-space.*
(ii) *If X has a BCO, then the closed subsets of X are W_δ.*

Proof. (i) Let U be open, and $x \in U$. Assume there is some $y \in \overline{\{x\}} \setminus U$, and let (G, T) be a sieve for $X \setminus U$ witnessing that $X \setminus U$ is a W_δ-set . Then there is a branch (t_n) of T with $y \in \bigcap_n G(t_n)$, hence, $x \in \bigcap_n G(t_n) \subseteq X \setminus U$, a contradiction.

(ii) Let $(\mathcal{B}_n)$ be a sequence of bases from the definition of a BCO, and Y a nonempty closed subset of X. Define $T_0 = \{t \in \mathcal{B}_0 : t \cap Y \neq \emptyset\}$. Assuming that T_n has been defined, let the successors of $t \in T_n$ be all those members of $\mathcal{B}_{n+1}$, that are included in t, and hit Y. Let G be the identity mapping on $T = \bigcup_n T_n$. Then (G, T) is a sieve of Y in X. Now, if we had a branch (t_n) in T so that $\bigcap_n G(t_n) \nsubseteq Y$, then there would be an $x \in \bigcap_n G(t_n) \setminus Y$, which is impossible, since $(G(t_n))$ is a base of neighborhoods at x, and $X \setminus Y$ is an open neighborhood of x. □

Proposition 2.2. *Let Y be a W_δ-subset of X. If $Ch(Y)$ is β-favorable, then so is $Ch(X)$.*

Proof. Let (G, T) be a sieve of Y in X, and σ_Y a w.s. for β in $Ch(Y)$. Well-order T, and for each Y-open U fix an X-open U' such that $U' \cap Y = U$.

We will define a strategy σ_X for β in $Ch(X)$: if $\sigma_Y(\emptyset) = (y_0, B_0) \in \mathcal{E}(Y)$, define $\sigma_X(\emptyset) = (y_0, B_0')$. Let A_0 be an X-open set such that $y_0 \in A_0 \subseteq B_0'$. Then $y_0 \in Y \cap A_0 \subseteq B_0$, so we can get $\sigma_Y(Y \cap A_0) = (y_1, B_1) \in \mathcal{E}(Y)$, and find the first t_0 in T_0 with $y_1 \in G(t_0)$. Define $\sigma_X(A_0) = (y_1, M_1)$, where $M_1 = B'_1 \cap G(t_0) \cap A_0$.

Assume that for some $n \geq 1$ and all $1 \leq k \leq n$, $\sigma_X(A_0, \dots, A_{k-1}) = (y_k, M_k) \in \mathcal{E}(X)$ has been defined where $M_k = B'_k \cap G(t_{k-1}) \cap A_{k-1}$ for some $t_{k-1} \in T_{k-1}$ with $t_0 < \dots < t_{k-1}$, and $\sigma_Y(Y \cap A_0, \dots, Y \cap A_{k-1}) = (y_k, B_k) \in \mathcal{E}(Y)$.

If A_n is an X-open set with $y_n \in A_n \subseteq M_n$, then $y_n \in Y \cap A_n \subseteq Y \cap B'_n = B_n$, so we can get $\sigma_Y(Y \cap A_0, \dots, Y \cap A_n) = (y_{n+1}, B_{n+1}) \in \mathcal{E}(Y)$ and find the first $t_n \in T_n$ with $t_n > t_{n-1}$ such that $y_{n+1} \in G(t_n)$. Put $M_{n+1} = B'_{n+1} \cap G(t_n) \cap A_n$, and define $\sigma_X(A_0, \dots, A_n) = (y_{n+1}, M_{n+1})$.

To show that σ_X is a w.s. for β, consider a run $(y_0, M_0), A_0, \dots, (y_n, M_n)$, $A_n, \dots$ of $Ch(X)$ compatible with σ_X, i.e. $M_0 = B'_0$ and $(y_n, M_n) = \sigma_X(A_0, \dots, A_{n-1})$ for all $n \geq 1$. Then

$$(y_0, B_0), Y \cap A_0, \dots, (y_n, B_n), Y \cap A_n, \dots$$

is a run of $Ch(Y)$ compatible with σ_Y, so $\bigcap_n B_n = \emptyset$. On the other side, $M_n \subseteq G(t_{n-1})$, so $\bigcap_{n \geq 1} M_n \subseteq \bigcap_{n \geq 1} G(t_{n-1}) \subseteq Y$, hence, $\bigcap_{n \geq 1} M_n \subseteq Y \cap \bigcap_{n \geq 1} B'_n = \bigcap_{n \geq 1} B_n = \emptyset$, and β wins this run of $Ch(X)$. □

Corollary 2.3. *Let X be a topological space, where the closed sets are W_δ. If X is not hereditarily Baire, then $Ch(X)$ is β-favorable.*

Denote by $CL(X)$ the set of all nonempty closed subsets of a T_1-space X, and for any $S \subseteq X$ put

$$S^- = \{A \in CL(X) : A \cap S \neq \emptyset\}, \text{ and } S^+ = \{A \in CL(X) : A \subseteq S\}.$$

The *Vietoris topology* [Mi] τ_V on $CL(X)$ has subbase elements of the form U^- and U^+, where $\emptyset \neq U \subseteq X$ is open. The space $(CL(X), \tau_V)$ is T_2 iff X is T_3, and $(CL(X), \tau_V)$ is compact iff X is compact [Mi]. If A is an open (resp. closed) subspace of X, then $CL(A)$ is an open (resp. closed) subspace of $CL(X)$; X embeds as a subspace in $CL(X)$ (it embeds as a closed subspace iff X is T_2). We will use that $(CL(\omega), \tau_V)$ is 1st countable, and zero-dimensional, since for each $A \in CL(\omega)$, $\{A^+ \cap \bigcap_{n \in F} \{n\}^- : F \subseteq A \text{ finite}\}$ forms a countable clopen base of neighborhoods at A.

3. β-FAVORABILITY OF THE STRONG CHOQUET GAME

The following is a consequence of a result of *Debs* [De1, Proposition 2.7]:

Theorem 3.1. *Let X be a 1st countable T_3-space. If $Ch(X)$ is β-favorable, then X contains a closed copy of the rationals.*

Theorem 3.2. *The following are equivalent*

(i) *$Ch(X)$ is β-favorable,*
(ii) *X is not hereditarily Baire.*
(iii) *X contains a closed copy of the rationals,*
(iv) *X contains a W_δ copy of the rationals,*

in any of the following cases:

(1) *X is a 1st countable, T_3-space, where the closed sets are W_δ,*
(2) *X is a T_3-space with a BCO.*

Proof. By Lemma 2.1(ii), (2) implies (1), so only consider (1): (ii)⇔(iii) holds in any 1st countable, T_3-space (cf. [vD], or [De1, Corollary 3.7]), (i)⇒(iii) is Theorem 3.1, (iii)⇒(iv) is trivial, and to see (iv)⇒(i), let $Y \subset X$ be a nonempty W_δ copy of the rationals, then $BM(Y)$ is β-favorable, and so is $Ch(Y)$; thus, $Ch(X)$ is β-favorable by Proposition 2.2. □

Corollary 3.3. *The following are equivalent*

(i) *$Ch(X)$ is β-favorable,*
(ii) *X is not hereditarily Baire.*
(iii) *X contains a closed copy of the rationals,*
(iv) *X contains a G_δ copy of the rationals,*
(v) *X contains a W_δ copy of the rationals,*

in any of the following cases:

(1) *X is a 1st countable, T_3 perfect space,*
(2) *X is a Moore space.*

The following example shows that in the previous two theorems we cannot use regularity and 1st countability alone (contrary to what Theorem 3.1 would suggest):

Example 3.4. *The space $(CL(\omega), \tau_V)$ is 1st countable, zero-dimensional, it contains a closed copy of the rationals, but $Ch(CL(\omega))$ is α-favorable.*

Proof. Observe that $\{\omega \setminus F : F \subset \omega \text{ finite}\}$ is a countable, dense-in-itself, regular, and closed subspace of $(CL(\omega), \tau_V)$, so the rationals embed in $(CL(\omega), \tau_V)$ as a closed subspace (see also [Pop, Example 6]); α-favorability of $Ch(CL(\omega), \tau_V)$ follows from [PZ2, Theorem 4.1] (see also [Zs2]), and the rest is well-known [Mi]. □

Proposition 3.5. *If X is not countably compact, then $(CL(X), \tau_V)$ contains a closed copy of the rationals.*

Proof. If X contains a closed copy of ω, then $CL(\omega)$ embeds as a closed subspace of $(CL(X), \tau_V)$, and Example 3.4 applies. □

Our main theorem reads as follows:

Theorem 3.6. *Let X be a 1st countable R_0-space. Then the following are equivalent:*

(i) *$Ch(X)$ is β-favorable,*

(ii) *X contains a nonempty G_δ-subset of the 1st category in itself,*

(iii) *X contains a nonempty W_δ-subset of the 1st category in itself.*

Proof. (i)⇒(ii): Fix a decreasing neighborhood base $\{N_n(x) : n \in \omega\}$ at each $x \in X$. Let σ be a w.s. for β in $Ch(X)$. If $(x_0, V_0), U_0, \dots, (x_n, V_n), U_n, \dots$ is a run compatible with σ, we can assume that

$$\overline{\{x_k\}} \neq \overline{\{x_{n+1}\}} \quad \text{for all } k \leq n; \tag{1}$$

otherwise, just take the first $m > n+1$ for which $x_m \notin \overline{\{x_n : k \leq n\}}$ and redefine $\sigma(U_0, \dots, U_n) = (x_m, V_m)$ (such an m exists, since σ is a w.s. for β). For each $s \in \omega^{<\omega}$ define, by induction on the length of s, open sets U_s, V_s, and $x_s \in V_s$ as follows: put $U_\emptyset = X$, $(x_\emptyset, V_\emptyset) = \sigma(U_\emptyset)$, and $U_{(0)} = V_\emptyset$.

Assume that we have constructed x_s, U_s, V_s for each $s \in \omega^k$ $(k < \omega)$ with $(x_s, V_s) = \sigma(U_{s|0}, \dots, U_{s|k-1}, U_s)$, where $s|i$ is the restriction of s to $i < k$; moreover, $U_{r^\frown 0} = V_r$ whenever $r \in \omega^{k-1}$ $(k \geq 1)$, and for all $n < \omega$,

$$U_{r^\frown(n+1)} \subseteq N_n(x_r).$$

Put $U_{s^\frown 0} = V_s$, and for $n \geq 1$, define $U_{s^\frown n} = U_{s^\frown(n-1)} \cap N_n(x_s)$, and denote $(x_{s^\frown n}, V_{s^\frown n}) = \sigma(U_{s|0}, \dots, U_s, U_{s^\frown n})$. It follows from the construction, that for each $s \in \omega^{<\omega}$,

$$(U_{s^\frown n})_n \quad \text{is a decreasing base of neighborhoods at } x_s. \tag{2}$$

Claim 1. *The set $Q = \{\overline{\{x_s\}} : s \in \omega^{<\omega}\}$ is of the 1st category in itself.*

We just need to show that each $\overline{\{x_s\}}$ is nowhere dense in Q: if $x \in U \cap \overline{\{x_s\}}$ for some X-open U, then by R_0-ness, $\overline{\{x_s\}} = \overline{\{x\}} \subseteq U$, and by (1),(2), we can find an $x_{s'} \in U$ with $\overline{\{x_s\}} \cap \overline{\{x_{s'}\}} = \emptyset$; thus, $Q \cap (U \setminus \overline{\{x_s\}}) \subseteq Q \cap U$ is a nonempty Q-open neighborhood of x missing $\overline{\{x_s\}}$.

Claim 2. *Q is a G_δ-subspace of X.*

Indeed, for each $n < \omega$, denote

$$G_n = \bigcup \{U_{s^\frown n} : s \in \omega^{<\omega}\}.$$

Since, by R_0-ness, $\overline{\{x_s\}} \in U_{s^\frown n}$ for every $s \in \omega^{<\omega}$, and $n < \omega$, we have $Q \subseteq \bigcap_n G_n$. On the other hand, assume $x \in \bigcap_n G_n \setminus Q$. We will define a finite-splitting subtree $T = \bigcup_{k<\omega} T_k$ of $\omega^{<\omega}$ with levels T_k, and a function $m : T \to \omega$ so that for all $k \geq 1$,

(3) $T_k = \{t \in \omega^k : \exists s \in \omega^{<\omega} : \ s|k = t, s|(k-1) \in T_{k-1}$ and $x \in U_{s^\frown(n_{k-1}+1)}\}$ is nonempty and finite,

(4) $n_{k-1} = \max\{m(t) : t \in \bigcup_{i<k} T_i\}$,

(5) $x \notin \bigcup\{U_{t^\frown(m(t)+1)} : t \in \bigcup_{i<k} T_i\}$.

First, put $T_0 = \{\emptyset\}$. Since $x \notin Q$, there is some $n_0 = m(\emptyset) < \omega$ with $x \in U_{(n_0)}$ and $x \notin U_{(n_0+1)}$ (otherwise by (2), $\overline{\{x\}} = \overline{\{x_\emptyset\}}$). Then, as $x \in G_{n_0+1}$, there must be some $s \in \omega^{<\omega}$ with $|s| \geq 1$ so that $x \in U_{s^\frown(n_0+1)} \subseteq V_s \subseteq V_{(s(0))}$. Note that for such s, $s|1 = s(0) \leq n_0$, otherwise, $x \in V_{(s(0))} \subseteq U_{(s(0))} \subseteq U_{(n_0+1)}$. It follows that the set

$$T_1 = \{t \in \omega^1 : \exists s \in \omega^{<\omega}(s|1 = t \text{ and } x \in U_{s^\frown(n_0+1)})\}$$

is nonempty and finite, and (3),(4),(5) are satisfied for $k = 1$.

By induction, assume that (3),(4),(5) have been demonstrated for some $k = j \geq 1$. Then for each $t \in T_j$, we can find $m(t) < \omega$ so that $x \notin U_{t^\frown(m(t)+1)}$, and $x \in U_{t^\frown m(t)}$ (otherwise by (2), $\overline{\{x\}} = \overline{\{x_t\}}$), which implies (5) for $k = j+1$.

Define $n_j = \max\{m(t) : t \in \bigcup_{i<j+1} T_i\}$. Since $x \in G_{n_j+1}$, it follows by (5) for $k = j+1$, that there is some $s \in \omega^{<\omega}$ with $|s| \geq j+1$ so that $x \in U_{s^\frown(n_j+1)} \subseteq V_s \subseteq V_{s|(j+1)}$. Note that $t = s|j \in T_j$, since $x \in U_{s^\frown(n_j+1)} \subseteq U_{s^\frown(n_{j-1}+1)}$. Moreover, $s(j) \leq n_j$, since otherwise,

$$x \in V_{s|(j+1)} \subseteq U_{s|(j+1)} \subseteq U_{t^\frown(n_j+1)} \subseteq U_{t^\frown(m(t)+1)}.$$

It follows that the set

$$T_{j+1} = \{t \in \omega^{j+1} : \exists s \in \omega^{<\omega}(s|(j+1) = t, s|j \in T_j \text{ and } x \in U_{s^\frown(n_j+1)})\}$$

is nonempty and finite. This completes the induction.

Since T is finite-splitting, by König's lemma, T has an infinite branch, so we have some $z \in \omega^\omega$ with $z|k \in T_k$ for all $k < \omega$. It follows that, given a k, there is some $s \in \omega^{<\omega}$ with $z|k = s|k$ and $x \in U_{s^\frown(n_{k-1}+1)} \subseteq V_s \subseteq V_{s|k} = V_{z|k}$. This is impossible however, since

$$(x_{z|0}, V_{z|0}), U_{z|1}, (x_{z|1}, V_{z|1}) \ldots, U_{z|k}, (x_{z|k}, V_{z|k}), \ldots$$

is a run of $Ch(X)$ compatible with σ; thus, $\bigcap_k V_{z|k} = \emptyset$. This contradiction yields that $\bigcap_n G_n \setminus Q = \emptyset$, and as a consequence, Q is a G_δ-subset of X.

(ii)⇒(iii), and (iii)⇒(i) are clear. □

Corollary 3.7. *Let X be a 1st countable T_1-space. Then the following are equivalent:*

(i) *$Ch(X)$ is β-favorable,*

(ii) *X contains a countable 1st category G_δ-subspace,*

(iii) *X contains a countable 1st category W_δ-subspace.*

Corollary 3.8. *Let X be a 1st countable R_0-space. If X is hereditarily Baire, then $Ch(X)$ is not β-favorable.*

Corollary 3.9. *The following are equivalent:*

(i) *$Ch(X)$ is β-favorable,*

(ii) *X is not hereditarily Baire,*

in any of the following cases:

(1) *X is a 1st countable, where the closed sets are W_δ,*

(2) *X is a space with a BCO,*

(3) *X is a 1st countable perfect space,*

(4) *X is a developable space.*

Our last example shows, that Corollary 3.7 may fail for non-1st countable spaces:

Example 3.10. *There exists a Hausdorff non-1st countable space X such that $Ch(X)$ is β-favorable, but all nonempty countable W_δ-subsets of X are of the 2nd category in themselves.*

Proof. Let $P = (\omega_1 + 1) \times (\omega + 1) \setminus \{(\omega_1, \omega)\}$ be the Tychonoff plank, and $X = CL(P)$ with the Vietoris topology. Then X is Hausdorff, since P is regular; moreover, X is not 1st countable, since neither is P. It was shown in [PZ2, Example 4.4] that $Ch(X)$ is β-favorable (a different proof follows from Remark 3.11).

Claim. *The nonempty countable W_δ-subsets of X are of the 2nd category in themselves.*

Let $\mathcal{M}$ be a countable W_δ-subset of X, and (G, T) a sieve for $\mathcal{M}$ in X witnessing that $\mathcal{M}$ is a W_δ-set. Denote by π the projection map from P onto $\omega_1 + 1$. There are two cases:

Case 1: $s_M = \sup \pi(M) < \omega_1$ for each $M \in \mathcal{M}$. Then $\lambda = \sup\{s_M : M \in \mathcal{M}\} < \omega_1$, and $P_0 = (\lambda + 1) \times (\omega + 1)$ is a clopen subspace of P. Moreover, $X_0 = CL(P_0)$ is a clopen subspace of X, and $\mathcal{M}$ is a W_δ-subset of X_0. Since P_0 is compact, so is X_0, thus, $\mathcal{M}$ is sieve complete, and consequently, of the 2nd category in itself.

Case 2: $s_M = \omega_1$ for some $M \in \mathcal{M}$. Let (t_n) be a branch in T so that $M \in G(t_n)$ for each n, and without loss of generality, assume that each $G(t_n)$ is a τ_V-basic element, i.e. $G(t_n) = G_n^+ \cap \bigcap_{i<m_n} U(x_{n,i})^-$, where $m_n \geq 1$, G_n is open in P, and $U(x_{n,i}) \subseteq G_n$ is a basic (compact) neighborhood of $x_{n,i} \in P$.

Since $(G(t_n))_n$ is decreasing, given n and $i < m_n$, there is $j < m_{n+1}$ such that $U(x_{n+1,j}) \subseteq U(x_{n,i})$, so we can assume that $m_{n+1} > m_n$, and for all $i < m_n$, $U(x_{n+1,i}) \subseteq U(x_{n,i})$. Fix $n < \omega$, and $i < m_n$. Then $\bigcap_{p \geq n} U(x_{p,i})$ is a nonempty compact set, moreover, we can choose $z_{n,i} \in \bigcap_{p \geq n} U(x_{p,i})$ with $\pi(z_{n,j}) < \omega_1$. Define $Z = \overline{\{z_{n,i} : n < \omega, i < m_n\}}$; then $\nu_0 = \sup \pi(Z) < \omega_1$. We have two subcases:

- *M is uncountable*: then $S = M \setminus [0, \nu_0] \times [0, \omega]$ is uncountable, and for all $s \in S$ we have $Z \cup \{s\} \in \bigcap_n G(t_n) \subseteq \mathcal{M}$, a contradiction;
- *M is countable*: then there is $k \in \omega$ with $(\omega_1, k) \in M \subset \bigcap_n G_n$, so there is $\nu_0 < c_n < \omega_1$ with $(c_n, \omega_1] \times \{k\} \subset G_n$ for all n; denote $c = \sup\{c_n : n < \omega\}$. Then for all $c < r < \omega_1$ we have $Z \cup \{(r, k)\} \in \bigcap_n G(t_n) \subseteq \mathcal{M}$, a contradiction. $\square$

Remark 3.11. In the previous example X, the nonempty countable W_δ's are of the 2nd category in themselves, however, *there exists an uncountable 1st category in itself G_δ-subset in X*, indicating that Telgársky's question might still have a positive answer. To see this, let

$$\mathcal{Z}_n = \{A \in X : |A \cap (\{\omega_1\} \times \omega)| = \omega \text{ and } A \cap (\omega_1 \times [n, \omega]) = \emptyset\},$$

and put $\mathcal{Z} = \bigcup_n \mathcal{Z}_n$. Then

- $\mathcal{Z}_n$ is nowhere dense in $\mathcal{Z}$ for each n: indeed, let $A \in \mathcal{Z}_n$, and $\mathcal{U} = U^+ \cap \bigcap_{i \leq k} ([w_i, y_i] \times \{i\})^-$ be a τ_V-open neighborhood of A, where $U \subseteq P$ open, $w_i \leq y_i \leq \omega_1$. Choose some $(\omega_1, j) \in A$ with $j > n$. Then $(\omega_1, j) \in U$, so there is $w < \omega_1$ with $[w, \omega_1] \times \{j\} \subset U$; pick a successor $e > w$ and put $A_0 = A \cup \{(e, j)\}$. It follows that $A_0 \in \mathcal{Z}_{j+1} \cap \mathcal{U} \cap ([w, \omega_1] \times \{j\})^-$, and $\mathcal{Z} \cap (\mathcal{U} \cap [w, \omega_1] \times \{j\}^-) \subset \mathcal{U} \setminus \mathcal{Z}_n$.
- $\mathcal{Z}$ is a G_δ-subset of X: let

$$\mathcal{G}_m = \bigcup_{F \in [\omega]^m} ((\omega_1 + 1) \times \omega)^+ \cap \bigcap_{f \in F} ((\omega_1 + 1) \times \{f\})^-.$$

Fix m, and $A \in \mathcal{Z}$. Let $F_0 = \{k \in \omega : A \cap \omega_1 \times \{k\} \neq \emptyset\}$, and $n = |F_0|$. If $n < m$, pick $F_1 \subset \omega \setminus F_0$ of size $m - n$ so that $(\omega_1, j) \in A$ for all $j \in F_1$. Then $F = F_0 \cup F_1 \in [\omega]^m$. If $n \geq m$, take a subset $F \subseteq F_0$ of size m. Then in both cases, $A \in ((\omega_1 + 1) \times \omega)^+ \cap \bigcap_{f \in F} ((\omega_1 + 1) \times \{f\})^-$, so $A \in \mathcal{G}_m$. Conversely, let $A \in \bigcap_m \mathcal{G}_m$. Then there is an infinite set $I \subseteq \omega$, such that $A \cap (\omega_1 + 1) \times \{i\} \neq \emptyset$ for each $i \in I$. Notice that $\{i : A \cap \omega_1 \times \{i\} \neq \emptyset\}$ is finite, otherwise, A has a cluster point in $\omega_1 \times \{\omega\}$, which is impossible, since $A \subset (\omega_1 + 1) \times \omega$. It follows, that $A \in \mathcal{Z}$.

Remark 3.12. The previous remark implies, that X is not hereditarily Baire, since $\overline{\mathcal{Z}}$ is of the 1st category in itself; moreover, since P is not countably compact, X contains a closed copy of the rationals by Proposition 3.5, but no W_δ copy of the rationals by Example 3.10. This further shows how Theorem 3.2 breaks down in general.

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