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The quantitative effects of deviations from the assumption of an exponential component failure distribution are considered by examining the difference in predicted system safe-times based on the exponential assumption and predicted system safe-times based on the Weibull family of component failure distributions. For series and parallel systems of n = 1(1)10 independent identical components, maximal regions of robustness are identified as subfamilies of the Weibull family. These subfamilies are such that no more than a prespecified error in predicted safe-times will result from the exponential assumption provided the actual component failure distribution is a member of the identified subfamily of the Weibull.

THE ROBUSTNESS OF RELIABILITY PREDICTIONS
BASED ON THE EXPONENTIAL COMPONENT
FAILURE DISTRIBUTION

by

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CHAPTER I

INTRODUCTION

1.1 Background

The reliability of a component at a specified time, t, denoted by $R(t)$, is defined as the probability that the component survives at least until time t. The reliability depends on the failure distribution which governs the component failure and, in most situations, the exact form of the component failure distribution is unknown. Traditionally, the exponential component failure distribution has been the most widely used component failure distribution. This is largely due to two reasons. First, the calculations involving the exponential component failure distribution are relatively simple to perform, thereby making it desirable to use from a mathematical viewpoint. Second, a large body of theoretical results have been derived about the exponential component failure distribution, thus reinforcing its popularity.

The use of the exponential component failure distribution as the assumed component failure distribution results in the equivalent assumption that the failure rate of the component is constant. That is, given that a component has survived for t_0 units of time, the conditional probability that the component survives for a given period of time A is independent of t_0. Obviously, in the real world, most components do not have a constant failure rate. A majority have an increasing failure rate; in other words, the component wears out (or is more likely to fail) with the passing of time. There are numerous examples for increasing failure rates in appliances, automobiles, and most consumer goods. A minority of components with non-constant failure rates have a decreasing failure rate. For example, after having passed through an initial stage, a component might then have a better chance of survival. All components which experience high infant mortality fall in this category. Examples of components of this type are the light bulb and the transitor, which after having survived the initial critical surge of current, seem to improve with age.

1.2 Statement of the Problem

The purpose of this research is to investigate the errors in calculations of the reliability of a component, or system of components, that result from the assumption of an exponential component failure distribution when, in fact, this assumption is invalid. Before the problem can be considered from a mathematical point of view, it is necessary to state explicitly what constitutes a deviation from the assumption of an exponential component failure distribution. This is accomplished by designating a general family of alternative component failure distributions and limiting all deviations from assumptions to be contained within this family.

The family which is specified as the one containing all deviations from the assumption of an exponential component failure distribution is the Weibull family of component failure distributions. The Weibull family allows for both increasing and decreasing failure rates and includes the exponential component failure family as one of its members.

Because of this, the Weibull family has wide applicability when one is confronted with a possible non-constant failure rate. Another reason for its choice is the fact that, next to the exponential family, the Weibull family is the most extensively used family of component failure distributions, and, as in the case of the exponential component failure distribution, many theoretical results are available.

With the designation of the Weibull family as the alternative family of component failure distributions, permissible deviations can now be identified. The problem can now be more precisely stated as an investigation of the effect of using a member of the exponential family of component failure distributions in the calculations of the reliability of a component or system of components when in actuality, the failure distribution of the component or system is some other member of the Weibull family.

In attempting to give a precise definition to the "error in calculations of the reliability" one encounters two alternatives. The reliability of the component can be computed for a fixed time t using both the exponential and some other Weibull component failure distributions. Then the differences in these two reliabilities can be compared and designated as the error that results from using the exponential when some other Weibull is the true component failure distribution.

Alternately, a desired reliability can be specified, such as 95% probability of survival, and the maximum time during which the component may be safely used with this assurance of survival may be computed. This maximum time will be denoted as the 95% safe-time.

Then 95% safe-times can be computed using both the exponential and the Weibull component failure distributions, and the difference in the two times can be compared and designated as the error resulting from the use of the exponential when the Weibull is the actual component failure distribution.

In essence the first-method specifies error as a difference in probabilities of survival and the second method specifies error as a difference in safe-times. The first definition of error was adopted by Posten [3] and Powers and Posten [4] and errors that resulted from the use of the exponential assumption were investigated. The present research considers the second definition of error and compares the results with those obtained in [3] and [4].

In the context of the previously developed terminology, the purpose of the research may now be stated as identifying a subfamily of the Weibull family such that no more than a prespecified small error will result from the use of the exponential component failure distribution in calculations of reliability when, in fact, the true failure distribution of the component is a member of the identified Weibull subfamily. In other words, this research will investigate the extend to which the failure rate can be non-constant yet assumed constant without resulting in substantial error. This research will investigate series and parallel systems of components. It will be assumed, throughout, that the components in the systems will have independent, identical failure distributions.

CHAPTER II

MATHEMATICAL PRELIMINARIES

2.1 The Exponential Family

2.1.1 The Exponential Component Failure Distribution

The exponential distribution, which assumes constant failure rate, has a density function of the form

$$f(t) = 1/\theta \; e^{-t/\theta} \qquad t > 0;\ \theta > 0.$$
$$= 0 \qquad \text{otherwise.}$$

In this case,

$$E[T] = \theta \tag{2.1}$$

and is called the mean-time-to-failure.

2.1.2 Reliability of a Component with Exponential Failure Distribution

The reliability function, $R(t)$, for a component is defined as the probability that the component survives at least until time t. Let T denote the time at which the component fails. If the component has failure density function $f(t)$, then the reliability function is

$$R(t) = P(T \geq t) = 1 - P(T < t) = 1 - F(t)$$

where

$$F(t) = \int_0^t f(s)\, ds$$

is the cumulative distribution function of T. For the exponential distribution, the reliability function is

$$R(t) = 1 - \int_0^t 1/\theta \; e^{-s/\theta}\, ds = e^{-t/\theta} \tag{2.2}$$

2.1.3 Failure Rates

The failure rate function for a component with failure density function, $f(t)$, is defined as

$$r(t) = f(t) / R(t)$$

where $R(t)$ represents the component's reliability function. "This function has a useful probabilistic interpretation; namely, $r(t)\,dt$ represents the probability that an object of age t will fail in the interval $[t, t + dt]$." [2, p.10] In other words, given that an object has lasted at least until time t, the conditional probability that the object fails in the interval $[t, t + dt]$ is approximately $r(t)$. The failure rate function, in the case of a component with an exponential failure distribution, becomes

$$r(t) = 1/\theta e^{-t/\theta} / e^{-t/\theta} = 1/\theta,$$

which is constant. It can be shown that the exponential component failure distribution is characterized by the property of constant failure rate.

2.2 The Weibull Family

2.2.1 The Weibull Component Failure Distribution

The density function for the Weibull distribution is of the form

$$f(t;\alpha,\beta) = \frac{\alpha t^{\alpha-1}}{\beta} e^{-t^{\alpha}/\beta} \quad t > 0;\ \alpha > 0,\ \beta > 0.$$

The mean-time-to-failure is

$$E[T] = \beta^{1/\alpha}\Gamma(1 + 1/\alpha). \tag{2.3}$$

The parameter α and β are, respectively, the shape and scale parameters of the Weibull density function.

2.2.2 Reliability of a Component with Weibull Failure Distribution

The reliability function for a component with a Weibull failure distribution is

$$R(t) = 1 - \int_0^t \frac{\alpha s^{\alpha-1}}{\beta} e^{-s^\alpha/\beta} ds = e^{-t^\alpha/\beta} \tag{2.4}$$

2.2.3 Failure Rates

The failure rate function for the Weibull family of component failure distributions is

$$r(t) = \frac{\alpha t^{\alpha-1}}{\beta} e^{-t^\alpha/\beta} \Big/ e^{-t^\alpha/\beta} = \frac{\alpha t^{\alpha-1}}{\beta}$$

which is dependent on the time, t. The parameter, α, allows for a variety of failure rates to be associated with components having a Weibull failure distribution. If $\alpha < 1$, $\alpha t^{\alpha-1}/\beta$ decreases as t increases, thus reflecting a decreasing failure rate. If $\alpha > 1$, $\alpha t^{\alpha-1}/\beta$ increases as t increases, which indicates the component has an increasing failure rate. The special case when $\alpha = 1$ reduces the failure rate function to

$$r(t) = \frac{1}{\beta}$$

which represents the constant failure rate associated with the exponential component failure distribution.

2.3 The Exponential Family as a Subfamily of the Weibull Family: A Standardization of the Problem.

When faced with the problem of selecting the appropriate component failure distribution, it seems reasonable to assume that the mean-time-to-failure will be inherently fixed by the users particular situation. That is, even though there may be some question as to which component failure distribution to use, the mean-time-to-failure should not be altered by that choice. Therefore, it seems reasonable to assume throughout this research that a component will have the same mean-time-to-failure regardless of whether the exponential or some other Weibull component failure distribution is used in determining the reliability of the component. This result in requiring the mean of the Weibull distribution (2.3) to be equal to that of the exponential (2.1). Therefore, it is required that

$$\theta = \beta^{1/\alpha}\Gamma(1 + \frac{1}{\alpha})$$

or

$$\beta = \theta^{\alpha}\Gamma^{-\alpha}(1 + \frac{1}{\alpha}) \ .$$

Making this substitution in (2.4) results in the reliability function

$$R(t) = e^{-t^{\alpha}\Gamma^{\alpha}(1 + \frac{1}{\alpha}) / \theta^{\alpha}}$$

for the Weibull distribution. Without loss of generality, a change of scale will be effected by letting $\theta = 1$ resulting in a mean-time-to-failure of 1 for both the Weibull and exponential distributions. Throughout the rest of this research, all distributions will be assumed to have mean-time-to-failure one unless otherwise stated.

Thus, the reliability function for the exponential (2.2) and Weibull (2.4) become, respectively,

$$R(t) = e^{-t} \tag{2.5}$$

and

$$R(t) = e^{-t^{\alpha}\Gamma^{\alpha}(1 + \frac{1}{\alpha})} \tag{2.6}$$

It may again be observed that the exponential failure family is a subfamily of the Weibull resulting from setting $\theta = 1$, provided that the two component failure distributions are required to have the same mean. With this in mind, the reliability function for the exponential (2.5) will be denoted as $R_1(t)$ and the reliability function for the Weibull (2.6) will be denoted by $R_\alpha(t)$.

A graph of the Weibull reliability function (2.6) for selected values of α appears in Figure 1.

2.4. <u>Error in Reliability Calculations</u>

The two previously mentioned definitions of the error in calculations of reliability of a component or system of components resulting from the use of the exponential component failure distribution when some other Weibull is the correct component failure distribution can now be more precisely defined. The previously investigated method of Posten [3] and Powers and Posten [4] defined error as the difference in the calculations of reliability using both component failure distributions, for a fixed time, t. As indicated in Figure 2, this error is represented by the broken vertical line for a fixed $t = t^*$. Symbolically, this error is represented as $R_1(t^*) - R_\alpha(t^*)$. This approach puts emphasis on the error in the probability of survival, for a fixed time, t.

Figure 1

Graph of function $R(t) = e^{-t^{\alpha}\Gamma^{\alpha}(1 + \frac{1}{\alpha})}$

for selected values of α

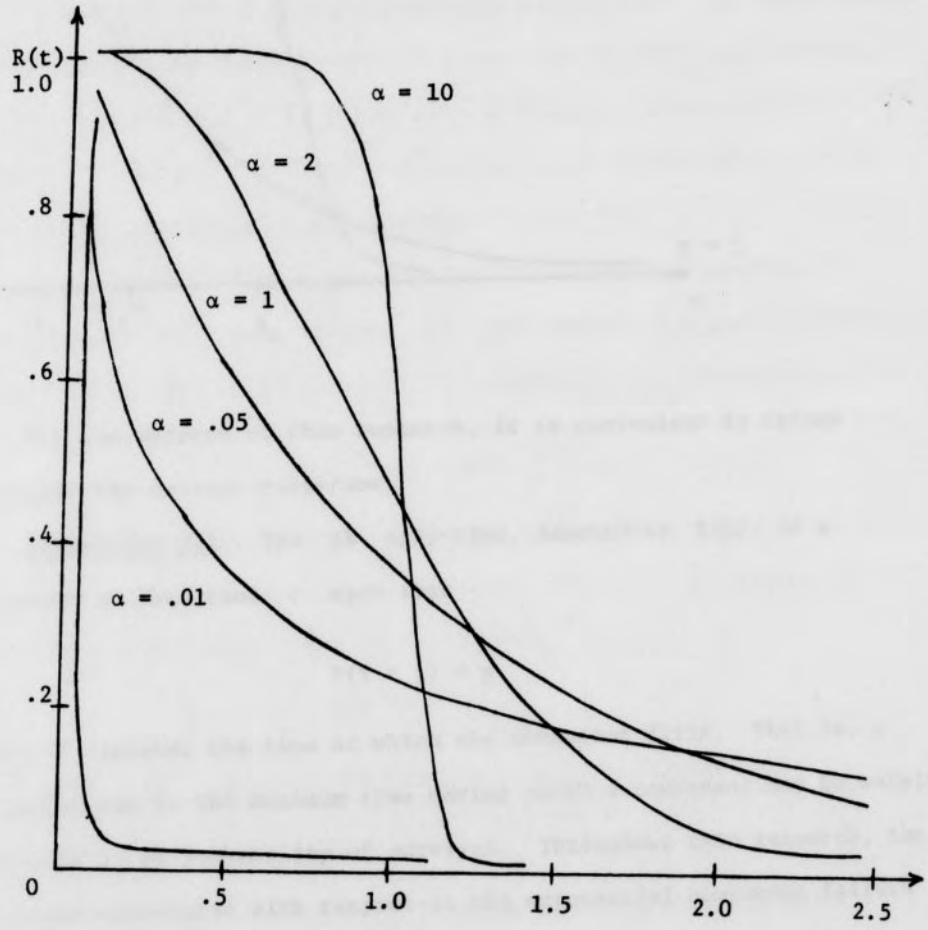
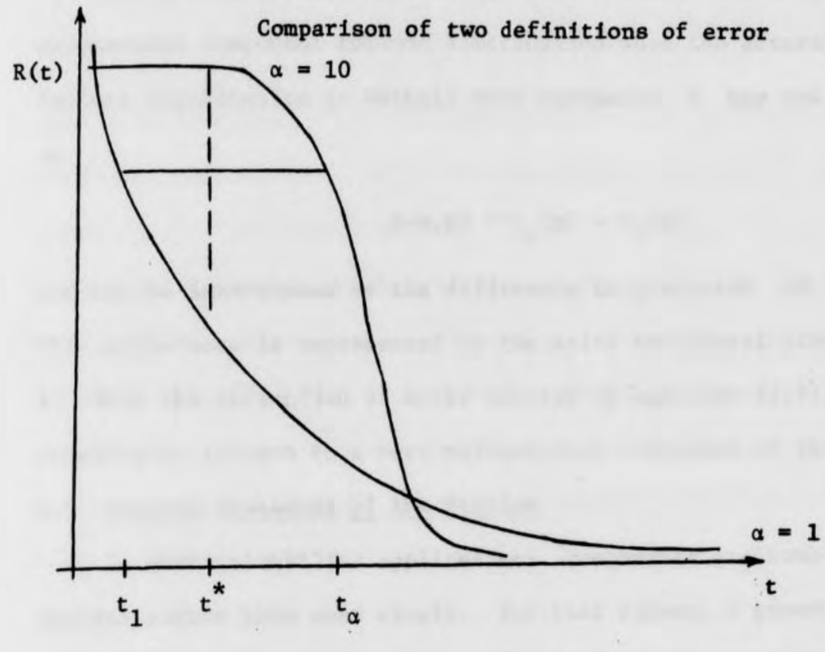


Figure 2

Comparison of two definitions of error



For the purpose of this research, it is convenient to define precisely the percent safe-time.

<u>Definition 2.1</u>. The p% safe-time, denoted by $t(p)$, of a component is that time t such that

$$P(T > t) = p$$

where T denotes the time at which the component fails. That is, a p% safe-time is the maximum time during which a component may be safely used with a p% probability of survival. Throughout this research, the safe-time calculated with respect to the exponential component failure distribution will be denoted as $t_1(p)$ and the safe-time calculated using the Weibull component failure distribution will be denoted by $t_\alpha(p)$.

For a single component, the error resulting from assuming an exponential component failure distribution when the actural component failure distribution is Weibull with parameter α may now be defined as

$$\Delta(\alpha,p) = t_{\alpha}(p) - t_{1}(p) \qquad (2.7)$$

and may be interpreted as the difference in predicted p% safe-times. This difference is represented by the solid horizontal line in Figure 2. With the definition of error adopted in equation (2.7), it is possible to proceed to a more mathematical statement of the problem.

2.5 General Statement of the Problem

In most reliability applications, components are combined in systems rather than used singly. For that reason, a general definition of the reliability of a system of n components may be introduced as

$$R^{(n)}(t) = P(T^{(n)} > t)$$

where $T^{(n)}$ denotes the failure time of the system rather than a single component. In an analogous manner, the p% safe-time of a system may be defined as the t such that

$$P(T^{(n)} > t) = p$$

where again $T^{(n)}$ denotes the failure time of the system of n components and is denoted by $t^{(n)}(p)$.

The error that results from assuming the components in the system have exponential component failure distribution when, in fact, the component failure distribution is Weibull with parameter α may be denoted by

$$\Delta^{(n)}(\alpha,p) = t_\alpha{}^{(n)}(p) - t_1{}^{(n)}(p) \tag{2.8}$$

Equation (2.8) represents the difference in p% system safe-times computed, respectively, using the assumption of Weibull and exponential component failure distributions.

Within this framework, the present research will identify a range of α values for which $|\Delta^{(n)}(\alpha,p)|$ is less than some prespecified maximum error, ϵ. This in turn will identify a subfamily of the Weibull family of component failure distributions such that the p% system safe-time may be computed using the exponential assumption and no more than a prespecified error will result so long as the true component failure distribution is a member of the identified subfamily. Such a subfamily will be denoted as a region of robustness of level ϵ and is defined as any set of α-values, S, so that $|\Delta^{(n)}(\alpha,p)| \leq \epsilon$ for all $\alpha \in S$.

It is also desirable to define and obtain the set of <u>all</u> α-values such that $|\Delta^{(n)}(\alpha,p)| \leq \epsilon$. This largest set of α-values is the maximal region of robustness of level ϵ and is given by

$$R^{(n)}(\epsilon) = \{\alpha\colon |\Delta^{(n)}(\alpha,p)| \leq \epsilon\}. \tag{2.9}$$

The size of $R^{(n)}(\epsilon)$ may be interpreted as a measure of the sensitivity of reliability calculations to the assumption of a constant failure rate and provides insight into how careful one must be in making such an assumption.

CHAPTER III

SERIES SYSTEMS

3.1 Introduction

In this chapter, maximal regions of robustness, $R^{(n)}(\epsilon)$, as defined in (2.9), will be identified for series systems of independent, identical components. The design of a series system is such that the system will fail if any one component in it fails. That is, the system will continue to function only if all components continue to function. Therefore, for a system of n components in series, the probability that the system survives at least until time t is equal to the probability that all components survive at least until time t. Utilizing the assumption of independence of the component failures within the system, this latter probability is simply the product of each component's probability of survival and since all components are identical, $P(T > t)$ is the same for each component. Thus

$$P(T^{(n)} > t) = [P(T > t)]^n \tag{3.1}$$

where $T^{(n)}$ represents the failure time of the system and T denotes an individual component's failure time.

In terms of reliability functions, equation (3.1) may be denoted as

$$R^{(n)}(t) = [R(t)]^n \tag{3.2}$$

where $R^{(n)}(t)$ represents the system's reliability function and $R(t)$ represents an individual component's reliability function. Therefore for a series system of n independent identical components, each with an exponential component failure distribution, the system reliability function $R_1^{(n)}(t)$ is obtained by substituting (2.5) in equation (3.2). Thus

$$R_1^{(n)}(t) = [R_1(t)]^n = e^{-nt}.$$

Similarly, by substituting equation (2.6) in equation (3.2) the system reliability function for a series system of n independent, identical components, each with a Weibull component failure distribution, is found to be

$$R_\alpha^{(n)}(t) = [R_\alpha(t)]^n = e^{-nt^\alpha\Gamma^\alpha(1 + \frac{1}{\alpha})}.$$

For a series system, the p% safe-time, defined in Defintion 2.1, is obtained by finding $t(p)$ as the value of t which solves the equation

$$R^{(n)}(t) = p.$$

Therefore, given a series system in which component failures are assumed to be exponential, $t_1(p)$ is found by solving

$$e^{-nt} = p \quad \text{for} \quad t.$$

Thus

$$t_1(p) = -\ln p/n. \tag{3.3}$$

In a similar manner, when component failures are assumed to be Weibull for a series system, then $t\ (p)$ is found by solving

$$e^{-nt^\alpha\Gamma^\alpha(1 + \frac{1}{\alpha})} = p \quad \text{for} \quad t$$

which results in

$$t_\alpha(p) = (- \ell n\ p)^{1/\alpha} \Big/ (n)^{1/\alpha}\ \Gamma(1 + \tfrac{1}{\alpha}). \qquad (3.4)$$

Substitution of $t_\alpha(p)$ and $t_1(p)$ obtained in equations (3.4) and (3.3), respectively, in (2.9) yeilds

$$\Delta^{(n)}(\alpha,p) = \frac{(- \ell n\ p)^{1/\alpha}}{(n)^{1/\alpha}\Gamma(1 + \frac{1}{\alpha})} - \frac{- \ell n\ p}{n} \qquad (3.5)$$

which is the difference in the p% series safe-times calculated using the Weibull and exponential component failure distributions. Equation (3.5) may be considered as the error that results from assuming each component in a series system has an exponential component failure distribution when, if fact, the components failures are Weibull with parameter α. With this in mind, the defintion of maximal regions of robustness for series systems may now be stated.

<u>Definition 3.1</u>. For a fixed probability p, the maximal region of robustness of level ϵ for a series system of n components is defined as

$$R^{(n)}(\epsilon) = \{\alpha : |\ \Delta^{(n)}(\alpha,p)\ | \le \epsilon\}$$

where $\Delta^{(n)}(\alpha,p)$ is defined in (3.5).

3.2 Regions of Robustness for Series Systems

The ultimate goal of this chapter is the identification, for a series system of n components, of maximal regions of robustness of level ϵ for pre-specified levels of reliability, p. This is equivalent to finding all values of α such that, for fixed probability p and error ϵ,

$$\left|\ \Delta^{(n)}(\alpha,p)\ \right| = \left| \frac{(-\ \ell n\ p)^{1/\alpha}}{(n)^{1/\alpha}\Gamma(1+\frac{1}{\alpha})} - \frac{-\ \ell n\ p}{n} \right| \le \epsilon \quad (3.6)$$

To facilitate identification of the maximal regions of robustness, it is desirable to rewrite equation (3.6) in an equivalent form without absolute value signs as

$$-\ \epsilon \le \Delta^{(n)}(\alpha,p) \le \epsilon$$

If it can be shown that $\Delta^{(n)}(\alpha,p)$ is a monotone increasing function of α for fixed n and p, and if it is possible to find values of α, α_1 and α_2, such that

$$\Delta^{(n)}(\alpha_1,p) = -\ \epsilon \text{ and } \Delta^{(n)}(\alpha_2,p) = \epsilon \quad (3.7)$$

then for $\alpha_1 \le \alpha \le \alpha_2$

$$-\ \epsilon \le \Delta^{(n)}(\alpha,p) \le \epsilon.$$

Hence, the maximal region of robustness of level ϵ would be

$$R^{(n)}(\epsilon) = \{\alpha:\ \left|\ \Delta^{(n)}(\alpha,p)\ \right| \le \epsilon\} = \{\alpha: -\epsilon \le \Delta^{(n)}(\alpha,p) \le \epsilon\}$$

$$= \{\alpha:\alpha_1 \le \alpha \le \alpha_2\}.$$

The following theorem establishes the desired monotonicity property of $\Delta^{(n)}(\alpha,p)$.

<u>Theorem 3.1</u> For fixed n and p, the function $\Delta^{(n)}(\alpha,p)$ is a monotone increasing function of α provided $p > .57037$.

<u>Proof</u>: To show that $\Delta^{(n)}(\alpha,p)$ is a monotone increasing function it is necessary only to show that the partial derivative of $\Delta^{(n)}(\alpha,p)$

with respect to α is positive for all values of α. The partial derivative of $\Delta^{(n)}(\alpha,p)$ with respect to α is

$$\frac{\partial\Delta^{(n)}(\alpha,p)}{\partial\alpha} = \frac{(n)^{-1/\alpha}(-\ell n p)^{1/\alpha}}{-\alpha^2\Gamma(1+\frac{1}{\alpha})}\left[\ell n(-\ell n\ p) - \psi(1+\frac{1}{\alpha}) - (\ell n\ n)\right]$$

where $\psi(x)$ is defined by $\psi(x) = \frac{d\ \ell n\Gamma(x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}$. This is equivalent to

$$\frac{\partial\Delta^{(n)}(\alpha,p)}{\partial\alpha} = k\left[\ell n(-\ell n\ p) - \psi(1+\frac{1}{\alpha}) - (\ell n\ n)\right]$$

where the constant k is defined by $k = (n)^{-1/\alpha}(-\ell n\ p)^{1/\alpha}/-\alpha^2\Gamma(1+\frac{1}{\alpha})$ and is negative for all α. Therefore if it can be shown that

$$\left[\ell n(-\ell n\ p) - \Psi(1+\frac{1}{\alpha}) - (\ell n\ n)\right] < 0,$$

then the partial derivative of $\Delta^{(n)}(\alpha,p)$ with respect to α will be positive and the proof of the monotonicity of $\Delta^{(n)}(\alpha,p)$ will be complete.

The inequality

$$\left[\ell n(-\ell n\ p) - \psi(1+\frac{1}{\alpha}) - (\ell n\ n)\right] \le \left[\ell n(-\ell n\ p) - \psi(1+\frac{1}{\alpha})\right]$$

is true since $n \ge 1$ and therefore $\ell n\ n \ge 0$ for all n. Therefore, it is sufficient to establish the inequality

$$\left[\ell n(-\ell n\ p) - \psi(1+\frac{1}{\alpha})\right] < 0$$

or equivantly, the inequality

$$\psi(1+\frac{1}{\alpha}) > \ell n(-\ell n\ p). \tag{3.8}$$

Since $1 + \frac{1}{\alpha} > 1$ for all α and since $\psi(1 + \frac{1}{\alpha})$ is an increasing function of $1/\alpha$ [1, p 258-259], then

$$\psi(1 + \frac{1}{\alpha}) > \psi(1) = -.5772.$$

Thus the inequlaity (3.8) holds provided $\ell n(-\ell n\ p) < -.5772$. But this latter inequality is equivalent to $p > e^{-e^{-.5772}} = .57037$. Hence, the partial derivative of $\Delta^{(n)}(\alpha,p)$ with respect to α is positive for all values of α and for values of $p > .57037$, thereby establishing the monotone increasing nature of $\Delta^{(n)}(\alpha,p)$.

Theorem 3.2 For fixed n and p, $\Delta^{(n)}(1,p) = 0$.

Proof: The proof follows directly from the substitution of $\alpha = 1$ into the definition of $\Delta^{(n)}(\alpha,p)$ as found in equation (3.5).

Theorem 3.3 For fixed n and $p > .57037$ $| \Delta^{(n)}(\alpha,p) |$ monotonically increases from 0 as α deviates from unity in either direction.

Proof: Since $\Delta^{(n)}(1,p) = 0$ by Theorem 3.2 and since $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing function of α by Theorem 3.1, the proof is immediate.

Theorem 3.4 For fixed n and $p > .57037$, if $\alpha_1 < \alpha_2$ all such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$ and $\Delta^{(n)}(\alpha_2,p) = \epsilon$, then $\alpha_1 < 1 < \alpha_2$ and $| \Delta^{(n)}(\alpha,p) | \leq \epsilon$ for all $\alpha \in [\alpha_1,\alpha_2]$.

Proof: The proof is immediate from Theorem 3.2 and Theorem 3.3.

Finally, the maximal regions of robustness for series systems of n components may be identified.

Theorem 3.5 For $p > .57037$, $R^{(n)}(\epsilon) = \{\alpha: \alpha_1 \leq \alpha \leq \alpha_2\}$ where $\Delta^{(n)}(\alpha_1,p) = -\epsilon$ and $\Delta^{(n)}(\alpha_2,p) = \epsilon$.

Proof: The proof follows from Theorem 3.4 and the definition of $R^{(n)}(\epsilon)$.

For certain values of n and p, an improvement in Theorem 3.5 may be obtained and the maximal region of robustness identified as $R^{(n)}(\epsilon) = \{\alpha: 0 < \alpha \le \alpha_2\}$ where $\Delta^{(n)}(\alpha_2,p) = \epsilon$. The following two theorems establish the necessary results.

Theorem 3.6 For fixed $n \ge 1$ and $p \ge e^{-1}$ $\lim_{\alpha\to 0} \Delta^{(n)}(\alpha,p) = \frac{\ell np}{n}$.

Proof:
$$\lim_{\alpha\to 0} \Delta^{(n)}(\alpha,p) = \lim_{\alpha\to 0}\left[\frac{(-\ell n\ p)^{1/\alpha}}{(n)^{1/\alpha}\Gamma(1+\frac{1}{\alpha})}\right] - \left(\frac{-\ell n\ p}{n}\right)$$
$$= \lim_{\alpha\to 0} \frac{(-\ell n\ p)^{1/\alpha}}{(n)^{1/\alpha}\Gamma(1+\frac{1}{\alpha})} - \lim_{\alpha\to 0} \frac{-\ell n\ p}{n}$$
$$= \lim_{\alpha\to 0} \frac{\ell n\ p}{n}^{1/\alpha} \cdot \lim_{\alpha\to 0} \frac{1}{\Gamma(1+\frac{1}{\alpha})} - \left(-\frac{\ell n\ p}{n}\right)$$

Since $n \ge 1$ and $p \ge .90$, $-\frac{\ell np}{n} < 1$ and $\lim_{\alpha\to 0} -\frac{\ell n\ p}{n}^{1/\alpha} = 0$.

Also $\lim_{\alpha\to 0} (1 + \frac{1}{\alpha}) = \infty$. Therefore
$$\lim_{\alpha\to 0} \Delta^{(n)}(\alpha,p) = \frac{\ell n\ p}{n}\ .$$

Theorem 3.7 For fixed n and p such that $\frac{\ell n\ p}{n} > -\epsilon$ and $\epsilon < 1$, $R^{(n)}(\epsilon) = \{\alpha: 0 < \alpha \le \alpha_2\}$ where $\Delta^{(n)}(\alpha_2,p) = \epsilon$.

Proof: If $\ell n\ p/n > -\epsilon$, and $\epsilon < 1$ then $\lim_{\alpha\to 0} \Delta^{(n)}(\alpha,p) > -\epsilon$ by Theorem 3.6. Since $\Delta^{(n)}(\alpha,p)$ is a monotone increasing function of α for fixed n and p, $\Delta^{(n)}(\alpha,p) > -\epsilon$ for all $\alpha < 0$. Further, $\Delta^{(n)}(1,p) = 0$. Therefore, $-\epsilon < \Delta^{(n)}(\alpha,p) \le 0$ for all $\alpha \in (0,1]$. Hence $|\ \Delta^{(n)}(\alpha,p)\ | \le \epsilon$ for all $\alpha \in (0,1]$ and by definition, $R^{(n)}(\epsilon) = \{\alpha: 0 < \alpha \le \alpha_2\}$ where $\Delta^{(n)}(\alpha_2,p) = \epsilon$.

The function $\ell n\ p/n$ is tabulated in Appendix I for $n = 1(1)10$ and $p = .90, .95, .975, .99, .999$.

3.3 Identification of Maximal Regions of Robustness for Series Systems

As a result of Theorem 3.5, it is necessary only to find α_1 and α_2 that satisfy $\Delta^{(n)}(\alpha_1,p) = -\epsilon$ and $\Delta^{(n)}(\alpha_2,p) = \epsilon$ and the resulting interval $[\alpha_1,\alpha_2]$ then satisfies a maximal region of robustness of level ϵ. That is, the maximum error that results from the assumption of exponential component failure when the actual component failure distribution is Weibull with parameter α does not exceed ϵ provided $\alpha \in [\alpha_1,\alpha_2]$.

The problem of determining α_1 and α_2 is complicated by the fact that it does not appear that the equations

$$\Delta^{(n)}(\alpha_1,p) = -\epsilon \quad \text{and} \quad \Delta^{(n)}(\alpha_2,p) = \epsilon$$

can be solved explicitly for α_1 and α_2. Thus, numerical methods were used for the determination of α_1 and α_2. For fixed n, p, and ϵ, α_1 and α_2 were determined by finding approximate solutions to equations (3.7) using the half-interval (binary search) techniques. Iteration was continued until α_1 and α_2 were determined to six decimal places and $\Delta^{(n)}(\alpha_1,p)$ and $\Delta^{(n)}(\alpha_2,p)$ differ from $-\epsilon$ and ϵ, respectively, by less than 10^{-6}.

For series systesm of $n = 1(1)10$ components, and fixed $p = .90, .95, .975,. 99, .999$, maximal regions of robusntess of level $\epsilon = .10, .05, .01, .001$ were identified using the numerical procedure described above. These regions of robustness are presented in Tables 1 - 5. The zeros entries should be interpreted to mean $0 < \alpha < \alpha_1$

and are present because the cooresponding values of n and p satisfy the assumption of Theorem 3.7 and when the value of $\ell n\ p/n$ in Appendix I is compared with $-\epsilon$, it is observed that $\ell n\ p/n > -\epsilon$.

Table 1

Maximal Regions of Robustness for Series Systems with p = .90, n = 1(1)10, and ϵ = .10, .05, .01, .001

$n \backslash \epsilon$	.10		.05		.01		.001	
	α_1	α_2	α_1	α_2	α_1	α_2	α_1	α_2
1	.4973	1.3480	.8101	1.1737	.9642	1.0353	.9964	1.0035
2	0	1.4862	.5470	1.2537	.9415	1.0547	.9943	1.0056
3	0	1.5872	0	1.3156	.9190	1.0716	.9924	1.0075
4	0	1.6671	0	1.3665	.8956	1.0867	.9906	1.0093
5	0	1.7335	0	1.4098	.8706	1.1005	.9888	1.0110
6	0	1.7902	0	1.4475	.8429	1.1132	.9871	1.0126
7	0	1.8399	0	1.4811	.8112	1.1251	.9854	1.0141
8	0	1.8841	0	1.5112	.7727	1.1362	.9837	1.0157
9	0	1.9239	0	1.5386	.7214	1.1466	.9820	1.0171
10	0	1.9601	0	1.5638	.6333	1.1565	.9804	1.0186

Table 2

Maximal Regions of Robustness for Series Systems with p = .95, n = 1(1)10, and ϵ = .10, .05, .01, .001

$n \backslash \epsilon$	.10 α_1	.10 α_2	.05 α_1	.05 α_2	.01 α_1	.01 α_2	.001 α_1	.001 α_2
1	0	1.4924	.4989	1.2574	.9403	1.0557	.9942	1.0057
2	0	1.6749	0	1.3715	.8930	1.0883	.9904	1.0095
3	0	1.7987	0	1.4532	.8381	1.1152	.9868	1.0128
4	0	1.8930	0	1.5174	.7631	1.1385	.9833	1.0160
5	0	1.9694	0	1.5703	.5857	1.1591	.9799	1.0190
6	0	2.0336	0	1.6154	0	1.1777	.9766	1.0218
7	0	2.0891	0	1.6547	0	1.1945	.9733	1.0246
8	0	2.1379	0	1.6896	0	1.2101	.9700	1.0273
9	0	2.1816	0	1.7210	0	1.2244	.9667	1.0298
10	0	2.2211	0	1.7495	0	1.2378	.9634	1.0323

Table 3

Maximal Regions of Robustness for Series Systems with p = .975, n = 1(1)10, and ϵ = .10, .05, .01, .001

$n \backslash \epsilon$	.10		.05		.01		.001	
	α_1	α_2	α_1	α_2	α_1	α_2	α_1	α_2
1	0	1.6786	0	1.3739	.8917	1.0890	.9903	1.0095
2	0	1.8974	0	1.5203	.7580	1.1396	.9831	1.0161
3	0	2.0382	0	1.6186	0	1.1790	.9763	1.0220
4	0	2.1427	0	1.6930	0	1.2116	.9697	1.0275
5	0	2.2259	0	1.7530	0	1.2394	.9630	1.0327
6	0	2.2952	0	1.8034	0	1.2638	.9562	1.0375
7	0	2.3545	0	1.8467	0	1.2855	.9493	1.0422
8	0	2.4064	0	1.8849	0	1.3050	.9423	1.0466
9	0	2.4525	0	1.9189	0	1.3228	.9350	1.0509
10	0	2.4525	0	1.9189	0	1.3228	.9350	1.0509

Table 4

Maximal Regions of Robustness for Series Systems with p = .99,
n = 1(1)10, and ϵ = .10, .05, .01, .001

$n \backslash \epsilon$	.10 α_1	.10 α_2	.05 α_1	.05 α_2	.01 α_1	.01 α_2	.001 α_1	.001 α_2
1	0	1.9765	0	1.5753	.4998	1.1611	.9796	1.0193
2	0	2.2288	0	1.7551	0	1.2404	.9627	1.0329
3	0	2.3842	0	1.8686	0	1.2966	.9454	1.0447
4	0	2.4970	0	1.9519	0	1.3403	.9269	1.0553
5	0	2.5858	0	2.0179	0	1.3762	.9064	1.0650
6	0	2.6591	0	2.0726	0	1.4067	.8830	1.0740
7	0	2.7214	0	2.1193	0	1.4332	.8548	1.0824
8	0	2.7757	0	2.1600	0	1.4567	.8185	1.0902
9	0	2.8238	0	2.1962	0	1.4778	.7641	1.0976
10	0	2.8670	0	2.2287	0	1.4969	.5869	1.1046

Table 5

Maximal Regions of Robustness for Series Systems with p = .999, n = 1(1)10, and ϵ = .10, .05, .01, .001

$n \backslash \epsilon$	.10		.05		.01		.001	
	α_1	α_2	α_1	α_2	α_1	α_2	α_1	α_2
1	0	2.8688	0	2.2301	0	1.4977	.5000	1.1049
2	0	3.1557	0	2.4471	0	1.6285	0	1.1597
3	0	3.3253	0	2.5760	0	1.7082	0	1.1983
4	0	3.4462	0	2.6680	0	1.7658	0	1.2284
5	0	3.5403	0	2.7397	0	1.8109	0	1.2530
6	0	3.6174	0	2.7985	0	1.8480	0	1.2739
7	0	3.6826	0	2.8483	0	1.8796	0	1.2920
8	0	3.7392	0	2.8915	0	1.9070	0	1.3080
9	0	3.7891	0	2.9296	0	1.9313	0	1.3223
10	0	3.8338	0	2.9638	0	1.9531	0	1.3353

3.4 Interpretation and use of tables

For illustrative purposes, a specific example from Tables 1-5 will be discussed in detail. As an example, let $n = 5$, $p = .95$ and $\epsilon = .05$. The corresponding entry from Table 2 is $[\alpha_1, \alpha_2] = (0, 1.5703]$. Thus the maximal region of robustness of level .05 is all α such that $0 < \alpha \leq 1.5703$. This may be interpreted as follows: The 95% safe-time for a series system of 5 components computed under the assumption of exponential component failures will not differ from the actual 95% safe-time by more than .05 provided the actual component failure distribution is Weibull with parameter α with α between 0 and 1.5703. Another way of viewing the interpretation is to say that the exponential component failure distribution may be used in place of the more complicated Weibull provided the actual component failure distribution has parameter α between 0 and 1.5703, and the computed 95% safe-time will have no more than 5% error.

An additional comment is needed to clarify the interpretation of 5% error in safe-times. Since it has been required that the means of the exponential and Weibull both be equal to unity, the region of robustness

$$| \Delta^{(n)}(\alpha,p) | \leq \epsilon$$

may be interpreted as

$$| \Delta^{(n)}(\alpha,p) | \leq \epsilon \cdot 1 = \epsilon \mu \cdot$$

so that ϵ represents the error expressed as a precent of the mean-time-to-failure of a component in the system. That is, for $\epsilon = .05$, $R^{(n)}(\epsilon)$ will specify values of α such that the absolute difference in 95% safe-times will not exceed 5% of the component mean-time-to-failure. Thus, if one were working with components with mean-time-to-failure of 1000 hours, the .05 region of robustness will provide values of α such that the absolute difference in 95% safe-times will not exceed $(.05) \times (1000) = 50$ hours.

To justify the preceding interpretation of percent errors in safe-times, it is desirable to state the following theorem.

Theorem 3.8: If T_1 has an exponential distribution with parameter and T_α has a Weibull distribution with parameters α, β and $E[T_1] = E[T_2] = \mu$, then

$$\Delta^{(n)}(\alpha,p) = \left[\frac{(-\ln p)^{1/\alpha}}{(n)^{1/\alpha}\Gamma(1+\frac{1}{\alpha})} - \frac{-\ln p}{n} \right] \mu.$$

Proof: The proof follows immediately from the development of $\Delta^{(n)}(\alpha,p)$.

CHAPTER IV

PARALLEL SYSTEMS

4.1 Introduction

Regions of robustness, $R(\epsilon)$, as specified in (2.10) are now investigated for independent, identical components which are combined in parallel systems. In this case, the system will continue to operate unless all components in the system fail. The probability that the parallel system survives at least until time t is found by utilizing the following basic theorem of probability: given a set A and its complement A',

$$P(A) = 1 - P(A'). \tag{4.1}$$

Thus, the probability that a parallel system survives at least until time t is one minus the probability that the system (or equivantly all components) fail before time t.

Utilizing the assumption of independent, identical components within the system, the probability that all components fail before time t is the product of each component's probability of failure before time t. By using (4.1) this is equal to one minus the product of each component's probability of survival at least until time t. Thus

$$P[T^{(n)} > t] = 1 - [P(T < t)]^n = 1 - [1 - P(T > t)]^n, \tag{4.2}$$

where $T^{(n)}$ represents the failure time of the system of n components and T denotes the failure time of each component.

Equation (4.2), rewritten in terms of reliability functions, is

$$R^{(n)}(t) = 1 - [1 - R(t)]^n, \tag{4.3}$$

where the system's reliability function is represented by $R^{(n)}(t)$ and each individual component's reliability is $R(t)$. Within the framework of this investigation, the system reliability function for a parallel system of n independent, identical components, each with an exponential component failure distribution, is denoted as $R_1^{(n)}(t)$ and by substituting equation (2.5) in (4.3) is

$$R_1^{(n)}(t) = 1 - [1 - R_1(t)]^n = 1 - [1 - e^{-t}]^n.$$

In an analogous manner, the system reliability function for a parallel system of n independent, identical components, each with a Weibull component failure distribution with mean-time-to-failure one, is denoted as $R_\alpha^{(n)}(t)$ and by substituting (2.6) in (4.3) is found to be

$$R_\alpha^{(n)}(t) = 1 - [1 - R_\alpha(t)]^n = 1 - [1 - e^{-t^\alpha \Gamma^\alpha (1+\frac{1}{\alpha})}]^n .$$

As in the case of series systems, the $p\%$ safe-time for a parallel system is obtained by finding the value of t such that

$$R^{(n)}(t) = p.$$

When exponential component failure is assumed for components in a parallel system, then $t_1(p)$ is found by solving

$$1 - [1 - e^{-t}]^n = p \quad \text{for } t$$

which yields

$$t_1(p) = - \ln(1 = (1 - p)^{1/n}). \tag{4.4}$$

Similarly, when Weibull component failure is assumed for each component in the system, $t_\alpha(p)$ is found by solving

$$1 - [1 - e^{-t^\alpha \Gamma^\alpha (1+\frac{1}{\alpha})}]^n = p \quad \text{for} \quad t.$$

Thus

$$t_\alpha(p) = [-\ln(1-(1-p)^{1/n})]^{1/\alpha} \Big/ \Gamma(1+\tfrac{1}{\alpha}). \qquad (4.5)$$

Substitution of $t_\alpha(p)$ and $t_1(p)$ given in equations (4.5) and (4.4) respectively in (2.9) results in the difference in p% parallel system safe-times

$$\Delta^{(n)}(\alpha,p) = [-\ln(1-(1-p)^{1/n})]^{1/\alpha} \Big/ \Gamma(1+\tfrac{1}{\alpha}) - (-\ln(1-(1-p)^{1/n})). \qquad (4.6)$$

Thus equation (4.6) is the error that results from the assumption of exponential component failure distribution for each component in a parallel system of n components when, in fact, the Weibull failure component distribution with parameter α is the correct failure distribution. In this context, maximal regions of robustness for parallel systems may now be defined.

Definition 4.1 For fixed n and p, the maximal region of robustness of level ϵ for a parallel system is defined as

$$R^{(n)}(\epsilon) = \{\alpha: |\Delta^{(n)}(\alpha,p)| \le \epsilon\}$$

where $\Delta^{(n)}(\alpha,p)$ is defined in equation (4.6).

4.2 Regions of Robustness for Parallel Systems

As in the series system case, it is now possible to state the purpose of this chapter as the identification of maximal regions of robustness of level ϵ for pre-specified levels of reliability for a parallel system of n components. This is equivalent to the identification of all values of α so that, for a fixed n and p and a specified error ϵ,

$$|\Delta^{(n)}(\alpha,p)| = \left| [-\ell n(1-(1-p)^{1/n})]^{1/\alpha}/\Gamma(1+\frac{1}{\alpha}) - (-\ell n(1-(1-p)^{1/n})) \right| \leq \epsilon. \tag{4.7}$$

An equivalent form of equation (4.7) without absolute value signs is desired to aid in the identification of maximal regions of robustness. This equivalent representation is

$$-\epsilon \leq \Delta^{(n)}(\alpha,p) \leq \epsilon$$

where $\Delta^{(n)}(\alpha,p)$ is defined in equation (4.6).

However, before the maximal regions of robustness may be determined, it is necessary to investigate properties of $\Delta^{(n)}(\alpha,p)$.

Theorem 4.1 For fixed n and p, $\Delta^{(n)}(\alpha,p)$ is monotonically increasing for all α such that $\psi(1 + \frac{1}{\alpha}) > \ell n[-\ell n(1 - (1 - p)^{1/n}]$, monotonically decreasing for all α such that $\psi(1 + \frac{1}{\alpha}) < \ell n[-\ell n(1 - (1 - p)^{1/n})]$. and has extrema at all α such that $\psi(1 + \frac{1}{\alpha}) = \ell n[-\ell n(1 - (1 - p)^{1/n})]$.

Proof: The partial derivative of $\Delta^{(n)}(\alpha,p)$ with respect to α is

$$\frac{\partial\Delta^{(n)}(\alpha,p)}{\partial\alpha} = \frac{[-\ell n(1-(1-p)^{1/n})]^{1/\alpha}}{-\partial^2\Gamma(1+\frac{1}{\alpha})} [\ell n[- n(1-(1-p)^{1/n})] - \psi(1+\frac{1}{\alpha})] \tag{4.8}$$

where $\psi(x) = \frac{d\ \ell n\ (x)}{dx} = \frac{\Gamma'(x)}{\Gamma(x)}$. Equation (4.10) may be rewritten as

$$\frac{\partial \Delta^{(n)}(\alpha,p)}{\partial \alpha} = f(\alpha,n,p) \cdot g(\alpha,n,p)$$

where

$$f(\alpha,n,p) = \frac{[-\ \ell n(1 - (1 - p)^{1/n})]^{1/\alpha}}{-\alpha^2 \Gamma(1 + \frac{1}{\alpha})}$$

and

$$g(\alpha,n,p) = \ell n[-\ \ell n(1 - (1 - p)^{1/n})] - \psi(1 + \frac{1}{\alpha}).$$

Since $f(\alpha,n,p) < 0$ for all $\alpha,n,p > 0$, $\partial \Delta^{(n)}(\alpha,p) / \partial \alpha$ is positive if $g(\alpha,n,p) < 0$, is zero if $g(\alpha,n,p) = 0$, and is negative if $g(\alpha,n,p) > 0$. Thus, $\Delta^{(n)}(\alpha,p)$ is monotonically increasing if $\psi(1 + \frac{1}{\alpha}) > \ell n[-\ \ell n(1 - (1 - p)^{1/n})]$, is monotonically decreasing if $\psi(1 + \frac{1}{\alpha}) < \ell n[-\ \ell n(1 - (1 - p)^{1/n})]$, and since $\psi(x)$ is monotonic, $\Delta^{(n)}(\alpha,p)$ has extrema at all α such that $\psi(1 + \frac{1}{\alpha}) = \ell n[-\ \ell n(1 - (1 - p)^{1/n})]$.

Theorem 4.2 For fixed n and p, $\Delta^{(n)}(1,p) = 0$.

Proof: The proof follows directly from a substitution of $\alpha = 1$ in equation (4.6).

Theorem 4.3 For fixed n and p such that $\ell n[-\ \ell n(1 - (1 - p)^{1/n})] > \psi(1) = -\ .5772$, $\Delta^{(n)}(\alpha,p)$ has exactly one extrema at $\alpha = \alpha_0$, where α_0 is the unique solution of $\psi(1 + \frac{1}{\alpha}) = \ell n[-\ \ell n(1 - (1 - p)^{1/n})]$. Also, $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing function of α for all $\alpha < \alpha_0$ and is a monotonically decreasing function of α for all $\alpha > \alpha_0$.

<u>Proof</u>: For fixed n and p such that $\ln[-\ln(1-(1-p)^{1/n})] > \psi(1)$, $\ln[-\ln(1-(1-p)^{1/n}]$ is a constant and is crossed exactly once by $\psi(1+\frac{1}{\alpha})$. This follows directly from the monotonicity of $\psi(x)$ [1,p 258-259]. Since $\psi(x)$ is a monotonically increasing function of x, $\psi(1+\frac{1}{\alpha})$ is a monotonically decreasing function of α. Further, $\lim\limits_{\alpha\to\infty} (1+\frac{1}{\alpha}) = \psi(1) = -.5772$ and $\lim\limits_{\alpha\to 0} \psi(1+\frac{1}{\alpha}) = \infty$. Thus, as α increases, $\psi(1+\frac{1}{\alpha})$ decreases monotonically from ∞ to $-.5772$ and crosses $\ln[-\ln(1-(1-p)^{1/n})]$ eactly once if n and p are such that $\ln[-\ln(1-(1-p)^{1/n})] > -.5772$. Thus $\psi(1+\frac{1}{\alpha}) = \ln[-\ln(1-(1-p)^{1/n})]$ has exactly one solution, denoted as α_0. By Theorem 4.1, α_0 is the unique extremum of $\Delta^{(n)}(\alpha,p)$. Additionally, by Theorem 4.1, since $\psi(1+\frac{1}{\alpha}) > \ln[-\ln(1-(-p)^{1/n})]$ for $\alpha < \alpha_0$, $\Delta^{(n)}(\alpha,p)$ is monotonically increasing for $\alpha < \alpha_0$ and since $\psi(1+\frac{1}{\alpha}) < \ln[-\ln(1-(1-p)^{1/n})]$ for $\alpha > \alpha_0$, $\Delta^{(n)}(\alpha,p)$ is monotonically decreasing for $\alpha < \alpha_0$.

<u>Theorem 4.4</u> For fixed n and p such that $\ln[-\ln(1-(1-p)^{1/n})] < \psi(1) = -.5772$, $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing fuction of α for all $\alpha > 0$.

<u>Proof</u>: As seen in the proof of Theorem 4.3, $\psi(1+\frac{1}{\alpha}) > \psi(1)$ for all $\alpha > 0$. Thus, if n and p are such that $\ln[-\ln(1-(1-p)^{1/n})] < \psi(1)$, then $\ln[-\ln(1-(1-p)^{1/n})] < \psi(1+\frac{1}{\alpha})$ for all $\alpha > 0$ and by Theorem 4.1, $\Delta^{(n)}(\alpha,p)$ is monotonically increasing for all $\alpha > 0$.

For fixed $n = 1(1)10$ and $p = .90, .95, .975, .99, .999$, the function $\ell n[-\ell n(1-(1-p)^{1/n})]$ is tabulated in Table 6. In addition to the value of the function, an asterick appears for those values of n and p such that $\ell n[-\ell n(1-(1-p)^{1/n})] < \psi(1)$. Since the equation $\psi(1+\frac{1}{\alpha}) = \ell n[-\ell n(1-(1-p)^{1/n})]$ does not appear to have an explicit solution in α, numerical methods were utilized to obtain α_0. The binary search (half interval) method was employed to determine α_0 to seven decimal places. This value of α_0 was then substituted in equation (4.6) to obtain $\Delta^{(n)}(\alpha_0,p)$.

For values of n and p which satisfy the assumption of Theorem 4.3 (see Table 6) the values of α_0 and $\Delta^{(n)}(\alpha_0,p)$ are given in Table 7. The lack of an entry for a specific n and p indicates that the n and p satisfy the assumptions of Theorem 4.4 and $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing function of α for all $\alpha > 0$.

Theorem 4.5 For fixed n and p such that $\ell n[-\ell n(1-(1-p)^{1/n}] < \psi(1) = -.5772$, if $\alpha_1 < \alpha_2$ are such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$ and $\Delta^{(n)}(\alpha_2,p) = \epsilon$, then the maximal region of robustness is

Table 6

Values of $\ell n[-\ \ell n(1 - (1 - p)^{1/n})]$ for n = 1(1)10 and p = .90, .95, .975, .99, .999.

n \ p	.90	.95	.975	.99	.999
1	-2.2504*	-2.9702*	-3.6762*	-4.6001*	-6.9073*
2	- .9672*	-1.3740*	-1.7596*	-2.2504*	-3.4379*
3	- .4717	- .7776*	-1.0617*	-1.4162*	-2.2504*
4	- .1908	- .4458	- .6795*	- .9672*	-1.6306*
5	- .0032	- .2270	- .4301	- .6779*	-1.2404*
6	- .1341	- .0684	- .2508	- .4717	- .9672*
7	- .2405	.0536	- .1137	- .3151	- .7627*
8	.3263	.1516	- .0042	- .1908	- .6021*
9	.3977	.2326	.0860	- .0888	- .4717
10	.4584	.3012	.1622	- .0032	- .3631

Table 7

Values of α_0 and $\Delta^{(n)}(\alpha_0,p)$ for n = 1(1)10

and p = .90, .95, .975, .99, .999.*

n \ p	.90	.95	.975	.99	.999
1					
2					
3	14.8722 .3796				
4	3.5567 .2262	11.7967 .3652			
5	2.1816 .1307	3.9941 .2455	10.4640 .3565		
6	1.6410 .0695	2.5436 .1628	4.3351 .2583	14.8722 .3796	
7	1.3498 .0313	1.9287 .1041	2.8553 .1858	5.5658 .2932	
8	1.1666 .0097	1.5871 .0625	2.1866 .1312	3.5567 .2262	
9	1.0400 .0007	1.3686 .0338	1.8042 .0897	2.6768 .1731	14.8722 .3796
10	.9469 .0016	1.2163 .0150	1.5557 .0584	2.1816 .1307	6.9680 .3195

*The value of α_0 appears sbove the value of $\Delta^{(n)}(\alpha_0,p)$

given by

$$R^{(n)}(\epsilon) = \{\alpha\colon \alpha_1 \le \alpha \le \alpha_2\}.$$

Proof: For fixed n and p satisfying the assumption of Theorem 4.4, $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing function of α, and since $\Delta^{(n)}(1,p) = 0$, $|\ \Delta^{(n)}(\alpha,p)\ |$ montonically increases from zero as α deviates from unity in either direction. Therefore, if $\alpha_1 < \alpha_2$ are such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$ and $\Delta^{(n)}(\alpha_2,p) = \epsilon$, then $|\ \Delta^{(n)}(\alpha,p)\ | \le \epsilon$ for all $\alpha \in [\alpha_1,\alpha_2]$ and $|\ \Delta^{(n)}(\alpha,p)\ | > \epsilon$ for $\alpha \notin [\alpha_1,\alpha_2]$. Hence by definition $R^{(n)}(\epsilon) = \{\alpha\colon \alpha_1 \le \alpha \le \alpha_2\}$.

As in the investigation of series systems, there exists certain values of n and p such that $\Delta^{(n)}(\alpha,p) > -\epsilon$ for all $\alpha \in (0,1]$ and an α_1, as defined in Theorem 4.5, does not exist. In these instances, an improved statement of Theorem 4.5 is possible. The following theorems provide the desired result.

Theorem 4.6 For fixed n and p, $\lim\limits_{\alpha\to 0} \Delta^{(n)}(\alpha,p) = \ell n(1 - (1 - p)^{1/n})$.

Proof:
$$\lim_{\alpha\to 0} \frac{[-\ell n(1 - (1 - p)^{1/n})]^{1/\alpha}}{\Gamma(1 + \frac{1}{\alpha})} - [-\ell n(1 - (1 - p)^{1/n})]$$

$$= \lim_{x\to\infty} \frac{[-\ell n(1 - (1 - p)^{1/n})]^{x}}{\Gamma(1 + x)} - [-\ell n(1 - (1 - p)^{1/n})]$$

$$= \lim_{x\to\infty} \frac{a^x}{\Gamma(1 + x)} - a$$

where a is the constant $-\ell n(1 - (1 - p)^{1/n})$. As shown in Appendix II, $\lim\limits_{x\to\infty} a^x/\Gamma(1 + x) = 0$. Therefore, $\lim\limits_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) = -a = \ell n(1 - (1 -p)^{1/n})$.

<u>Theorem 4.7</u> For fixed n and p such that $\ell n(1 - (1 - p)^{1/n}) > -\epsilon$ and $\ell n[-\ell n(1 - (1 - p)^{1/n})] < \psi(1) = -.5772$

$$R^{(n)}(\epsilon) = \{\alpha: 0 < \alpha \le \alpha_2\}.$$

where α_2 is such that $\Delta^{(n)}(\alpha_2,p) = \epsilon$.

<u>Proof</u>: Note that the inequalities $\ell n(1 - (1 - p)^{1/n}) > -\epsilon$ and $\ell n[-\ell n(1\ (1 - p)^{1/n})] < \psi(1)$ are equivalent to $\ell n(1 - (1 - p)^{1/n}) < \min(\epsilon, e^{\psi(1)})$.

If $\ell n(1 - (1 - p)^{1/n}) > -\epsilon$, then $\lim\limits_{\alpha\to 0} \Delta^{(n)}(\alpha,p) > -\epsilon$ by Theorem 4.6. Since for fixed n and p such that $\ell n[-\ell n(1 - (1 - p)^{1/n})] < \psi(1)$, $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing function of α, $\Delta^{(n)}(\alpha,p) > -\epsilon$ for all $\alpha > 0$. Further $\Delta^{(n)}(1,p) = 0$. Therefore, $-\epsilon < \Delta^{(n)}(\alpha,p) \le 0$ for all $\alpha \in (0,1]$. Hence $|\Delta^{(n)}(\alpha,p)| \le \epsilon$ for all $\alpha \in (0,1]$ and by definition, $R^{(n)}(\epsilon) = \{\alpha: 0 < \alpha \le \alpha_2\}$ where $\Delta^{(n)}(\alpha_2,p) = \epsilon$.

The function $\ell n(1 - (1 - p)^{1/n})$ is tablulated in Appendix III for $n = 1(1)10$ and $p = .90, .95, .975, .99, .999$. Before proceeding to the identification of maximal regions of robustness, a final property of $\Delta^{(n)}(\alpha,p)$ will be established.

<u>Theorem 4.8</u> For fixed n and p.

$$\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) = 1 - [-\ell n(1 - (1 - p)^{1/n}]$$

<u>Proof</u>:

$$\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) = \lim_{\alpha\to\infty} \frac{[-\ell n(1 - (1 - p^{1/n})]^{1/\alpha}}{\Gamma(1 + \frac{1}{\alpha})} - [-\ell n(1 - (1 - p)^{1/n})]$$

$$= \frac{\lim\limits_{\alpha\to\infty} [-\ell n(1 - (1 - p)^{1/n})]^{1/\alpha}}{\lim\limits_{\alpha\to\infty} \Gamma(1 + \frac{1}{\alpha})} - \lim_{\alpha\to\infty} [-\ell n(1 - (1 - p)^{1/n})].$$

Since $\lim_{\alpha\to\infty} [- \ell n(1 - (1 - p)^{1/n})]^{1/\alpha} = 1$ and since $\lim_{\alpha\to\infty} \Gamma(1 + \frac{1}{\alpha}) = 1$, then

$$\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) = 1 - [- \ell n(1 - (1 - p)^{1/n})].$$

The function $1 - [- \ell n(1 - (1 - p)^{1/n})]$ is tabulated in Table 8 for $n = 1(1)10$ and $p = .90, .95, .975, .99, .999$. With the above mentioned properties of $\Delta^{(n)}(\alpha,p)$ it is now possible to proceed with the identification of maximal regions of robustness for parallel systems.

4.3 Identification of Maximal Regions of Robustness for Parallel Systems

As a result of Theorems 4.3 and 4.4, the monotonicity of $\Delta^{(n)}(\alpha,p)$ depends on the value of $\ell n[- \ell n(1 - (1 - p)^{1/n})]$. The function $\ell n[- \ell n(1 - (1 - p)^{1/n})]$ is tabluated in Table 6 for $n = 1(1)10$ and $p = .90, .95, .975, .99, .999$. An asterick appears with the entries corresponding to values of n and p such that $\ell n[- \ell n(1 - (1 - p)^{1/n})] < \psi(1)$. For these values of n and p, $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing function of α for all α by Theorem 4.3. For all other values of n and p, $\Delta^{(n)}(\alpha,p)$ will have a maximum at $\alpha = \alpha_0$, where α_0 is the solution of $\psi(1 + \frac{1}{\alpha}) = \ell n[- \ell n(1 - (1 - p)^{1/n})]$. For these values of n and p, $\Delta^{(n)}(\alpha,p)$ increases for $\alpha < \alpha_0$, is maximum at $\alpha = \alpha_0$, and decreases for $\alpha > \alpha_0$ by Theorem 4.3. The values of α_0 and $\Delta^{(n)}(\alpha_0,p)$ are given in Table 7.

Table 8

Values of $1 - [-\ell n(1 - (1 - p)^{1/n})]$ for n = 1(1)10

and p = ,90, .95, .975, .99, .999

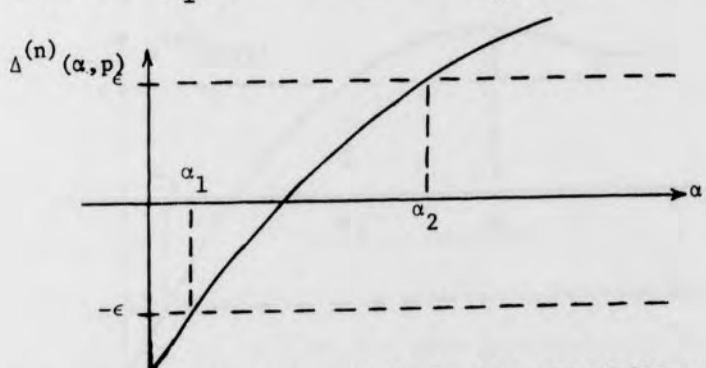
n \ p	.90	.95	.975	.99	.999
1	.8946	.9487	.9749	.9899	.9990
2	.6199	.7469	.8279	.8946	.9680
3	.3761	.5405	.6541	.7574	.8946
4	.1737	.3597	.4931	.6199	.8042
5	.0032	.2031	.3496	.4923	.7107
6	-.1435	.0662	.2219	.3761	.6199
7	-.2718	-.0551	.1075	.2703	.5336
8	-.3859	-.1637	.0042	.1739	.4523
9	-.4884	-.2619	-.0898	.0850	.3761
10	-.5815	-.3514	-.1760	.0032	.3045

Since the identification of maximal regions of robustness depends on the behavior of $\Delta^{(n)}(\alpha,p)$ and this function's behaviour in turn depends on n and p, it is beneficial to break the problem of identifying maximal regions of robustness into several cases. Individual graphs will be present with each case to illustrate the general behavior of $\Delta^{(n)}(\alpha,p)$.

Case I: If n and p are such that $\ell n[-\ell n(1-(1-p)^{1/n})] < \psi(1)$, $\Delta^{(n)}(\alpha,p)$ is a monotonically increasing function of α for all $\alpha > 0$, and the maximal region of robustness is

$$R^{(n)}(\epsilon) = \{\alpha: \alpha_1 \le \alpha \le \alpha_2\}. \tag{4.9}$$

where $\Delta^{(n)}(\alpha_1,p) = -\epsilon$ and $\Delta^{(n)}(\alpha_2,p) = \epsilon$.



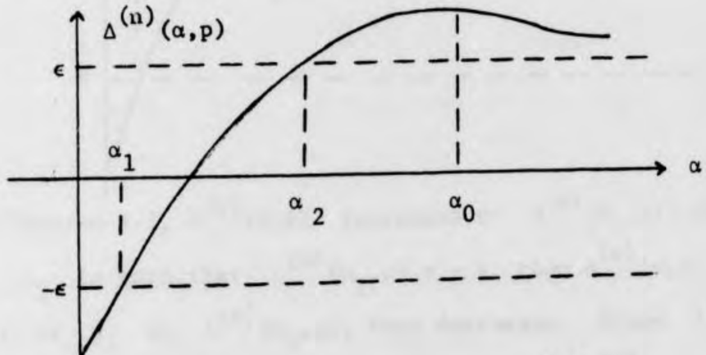
The region of roubstness (4.9) in this case follows directly from the assumptions of the case and Theorem 4.4 and 4.5. Only those entries in Table 6 which have an asterick fall into this case. If no α_1 exists such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$, then by Theorem 4.7, α_1 is zero.

For fixed n and p such that $\ell n[-\ell n(1-(1-p)^{1/n})] > \psi(1)$, a further investigation of the behavior of $\Delta^{(n)}(\alpha,p)$ is necessary. Specifically, it is necessary to determine α_0, the corresponding value of $\Delta^{(n)}(\alpha,p)$ evaluated at α_0, and the $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p)$. The first two values are tabulated in Table 7 and the latter in Table 8. The investigation of these values yields five subdivisions to the general case when $\ell n[-\ell n(1-(1-p)^{1/n})] > \psi(1)$ and $\Delta^{(n)}(\alpha,p)$ is not monotone for all $\alpha > 0$. Although theorectically Theorem 4.7 must also be considered, for the given values of n, ϵ and p, an α_1 always exist and is greater than zero.

<u>Case II - A</u>: If $\Delta^{(n)}(\alpha_0,p) > \epsilon$ and $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) > \epsilon$, the maximal region of robustness of level ϵ is

$$R^{(n)}(\epsilon) = \{\alpha:\ \alpha_1 \le \alpha \le \alpha_2\}. \qquad (4.10)$$

where $\alpha_1 < \alpha_2$ and $|\ \Delta^{(n)}(\alpha_i,p)\ | = \epsilon \quad i = 1,2.$



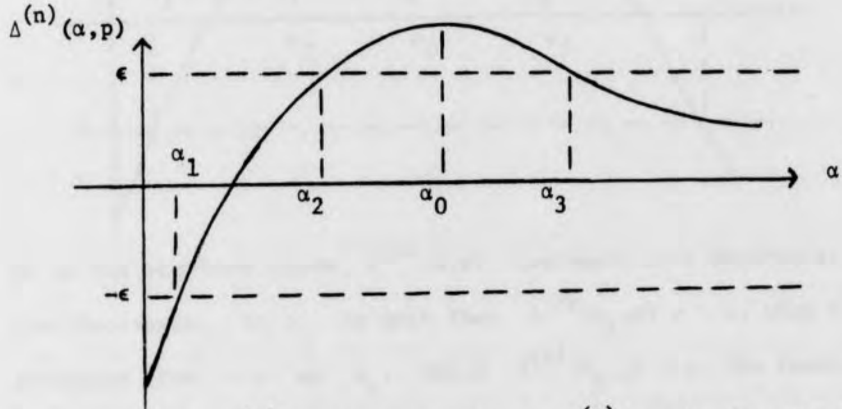
By Theorem 4.3, $\Delta^{(n)}(\alpha,p)$ increases to $\Delta^{(n)}(\alpha_0,p)$, then decreases. Since $\Delta^{(n)}(\alpha_0,p) > \epsilon$, the function rises above ϵ, then decreases. If α is such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$, then $\Delta^{(n)}(\alpha,p)$ increases from $-\epsilon$ at α_1, crosses ϵ at α_2, is maximum at α_0, then decreases. However, since $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) > \epsilon$, the function never again crosses ϵ. Therefore, the solution of $\Delta^{(n)}(\alpha,p) = \epsilon$ is at some $\alpha < \alpha_0$ and is unique.

This solution is defined to be α_2. Thus all α such that $\alpha \in [\alpha_1, \alpha_2]$ satisfy the condition (4.7) and (4.10) is the maximal region of robustness of level ϵ.

Case II - B: If $\Delta^{(n)}(\alpha_0,p) > \epsilon$ and $|\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p)| < \epsilon$, the maximal region of robustness of level ϵ is

$$R^{(n)}(\epsilon) = \{\alpha\colon \alpha_1 \le \alpha \le \alpha_2\} \cup \{\alpha\colon \alpha_3 \le \alpha < \infty\} \qquad (4.11)$$

where $\alpha_1 < \alpha_2 < \alpha_3$ and $|\Delta^{(n)}(\alpha_i,p)| = \epsilon \quad i = 1,3$

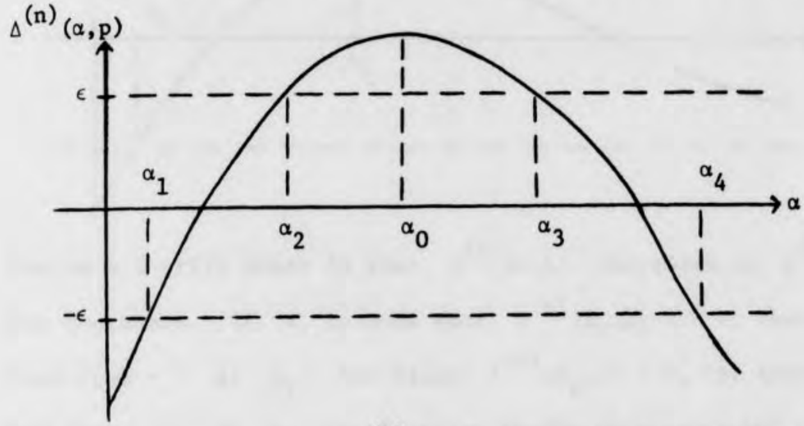


By Theorem 4.3, $\Delta^{(n)}(\alpha,p)$ increases to $\Delta^{(n)}(\alpha_0,p)$, then decreases. If α_1 is such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$, then $\Delta^{(n)}(\alpha,p)$ increases from $-\epsilon$ at α_1 to $\Delta^{(n)}(\alpha_0,p)$, then decreases. Since $\Delta^{(n)}(\alpha_0,p) > \epsilon$, there is at least one solution $\alpha_2 < \alpha_0$ to $\Delta^{(n)}(\alpha,p) = \epsilon$. However, since $|\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p)| < \epsilon$, the function, after having reached $\Delta^{(n)}(\alpha_0,p)$, then decreases below ϵ, so that there exists an $\alpha_3 > \alpha_0$ such that $\Delta^{(n)}(\alpha,p) = \epsilon$. Thus, $|\Delta^{(n)}(\alpha,p)| \le \epsilon$ if $\alpha \in [\alpha_1, \alpha_2]$ or if $\alpha > \alpha_3$, and the maximal region of robustness of level ϵ (4.11) is the union of two intervals, $[\alpha_1,\alpha_2]$ and $[\alpha_3,\infty)$.

<u>Case II - C</u>: If $\Delta^{(n)}(\alpha_0,p) > \epsilon$ and $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) < -\epsilon$, the maximal region of robustness of level ϵ is

$$R^{(n)}(\epsilon) = \{\alpha: \alpha_1 \le \alpha \le \alpha_2\} \cup \{\alpha: \alpha_3 \le \alpha \le \alpha_4\} \qquad (4.12)$$

where $\alpha_1 < \alpha_2 < \alpha_3 < \alpha_4$ and $|\Delta^{(n)}(\alpha_i,p)| = \epsilon$ $i = 1,4$.

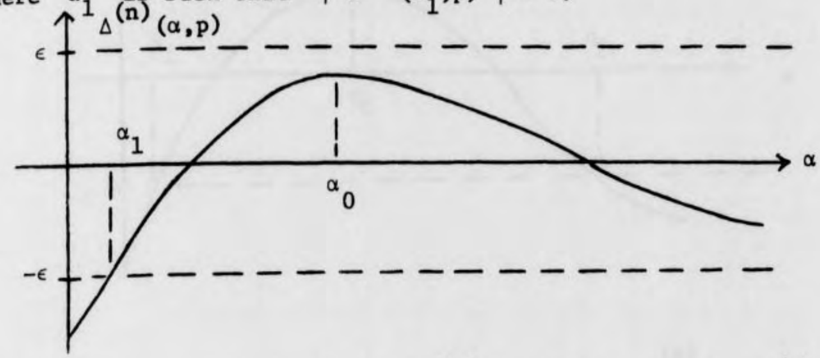


As in the previous cases, $\Delta^{(n)}(\alpha,p)$ increases to a maximum at α_0 then decreases. If α_1 is such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$, then $\Delta^{(n)}(\alpha,p)$ increases from $-\epsilon$ at α_1. Since $\Delta^{(n)}(\alpha_0,p) > \epsilon$, the function goes above ϵ to a maximum at α_0 and since $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) < -\epsilon$, the function falls back below ϵ after passing through α_0. Therefore, there exists an $\alpha_2 < \alpha_0$ and $\alpha_3 > \alpha_0$ such that $\Delta^{(n)}(\alpha_2,p) = \Delta^{(n)}(\alpha_3,p) = \epsilon$, with α_3 indicating the left endpoint of a second interval. Since $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) < -\epsilon$ the function again falls below $-\epsilon$, thereby indicating a second solution α_4 to $\Delta^{(n)}(\alpha,p) = -\epsilon$. Thus the maximal region of robustness of level (4.12) is the union of two intervlas, $[\alpha_1,\alpha_2]$ and $[\alpha_3,\alpha_4]$.

<u>Case II - D</u>: If $\Delta^{(n)}(\alpha_0,p) < \epsilon$ and $|\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p)| < \epsilon$, the maximal region of robustness of level ϵ is

$$R^{(n)}(\epsilon) : \{\alpha: \alpha_1 \leq \alpha < \infty\}. \tag{4.13}$$

where α_1 is such that $| \Delta^{(n)}(\alpha_1,p) | = \epsilon$.

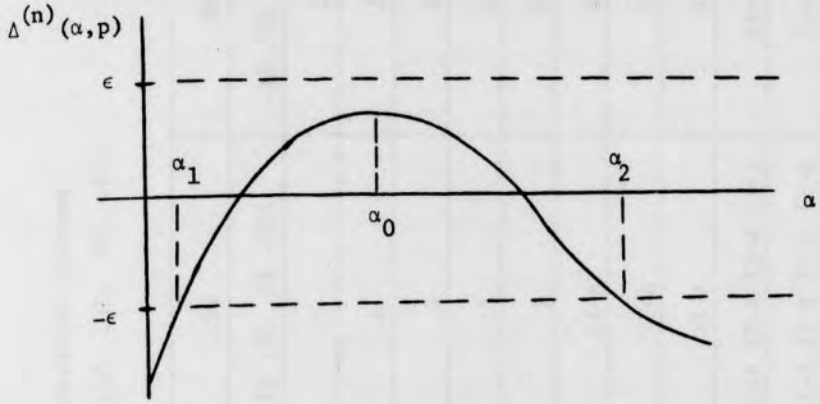


Theorem 4.3 still holds in that $\Delta^{(n)}(\alpha,p)$ increases to $\Delta^{(n)}(\alpha_0,p)$ then decreases. If α, is such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$, then the function rises from $-\epsilon$ at α_1. But since $\Delta^{(n)}(\alpha_0,p) < \epsilon$, the function never goes above ϵ. At α_0, the function begins decreasing but since $|\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p)| < \epsilon$, it never goes below $-\epsilon$, thereby always remaining in the bounds for the maximal region of robustness of level ϵ as $\alpha \to \infty$. Thus, the maximal region of robustness of level ϵ (4.13) is $[\alpha_1,\infty)$.

<u>Case II - E</u>: If $\Delta^{(n)}(\alpha_0,p) < \epsilon$ and $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) < -\epsilon$, the maximal region of robustness of level ϵ is

$$R^{(n)}(\epsilon) = \{\alpha: \alpha_1 \leq \alpha \leq \alpha_2\}. \tag{4.14}$$

where $\alpha_1 < \alpha_2$ and $| \Delta^{(n)}(\alpha_i,p) | = \epsilon \quad i = 1,2.$



For this final case, Theorem 4.3 still applies. $\Delta^{(n)}(\alpha,p)$ increases to $\Delta^{(n)}(\alpha_0,p)$, then decreases. Since $\Delta^{(n)}(\alpha_0,p) < \epsilon$, $\Delta^{(n)}(\alpha,p)$ increases, but does not go above ϵ and then decreases after having passing through $\Delta^{(n)}(\alpha_0,p)$. If $\alpha_1 < \alpha_0$ is such that $\Delta^{(n)}(\alpha_1,p) = -\epsilon$, then $\Delta^{(n)}(\alpha,p)$ increases from $-\epsilon$ at α_1 to $\Delta^{(n)}(\alpha_0,p) < \epsilon$, then decreases. Since $\lim_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) < -\epsilon$, $\Delta^{(n)}(\alpha,p)$ decreases to below $-\epsilon$ for some $\alpha > \alpha_0$. Thus there exists an $\alpha_2 > \alpha_0$ such that $\Delta^{(n)}(\alpha_2,p) = -\epsilon$, and the maximal region of robustness of level ϵ (4.14) is $[\alpha_1,\alpha_2]$.

For each $n = 1(1)10$ and $p = .90, .95, .975, .99, .999$, and $\epsilon = .10, .05, .01, .001$, a step-by-step classification of the behavior of $\Delta^{(n)}(\alpha,p)$ into cases I, II-A, II-B, II-C, II-D, and II-E was performed. This process involves a simultaneous reference to Tables 6-8 for each n and p. The results of this classification is shown in Table 9.

Table 9

Classification of Maximal Regions of Robustness for Parallel Systems
with n = 1(1)10, p = .90, .95, .975, .99. 999, and ϵ = .10, .05, .01, .001.

p	.90	.95	.975	.99	.999
n	.10 .05 .01 .001	.10 .05 .01 .001	.10 .01 .01 .001	.10 .05 .01 .001	.10 .05 .01 .001
1	I	I	I	I	I
2	I	I	I	I	I
3	II-A	I	I	I	I
4	II-A	II-A	I	I	I
5	II-B II-B II-B II-A	II-A	II-A	I	I
6	II-E II-C II-C II-C	II-B II-A II-A II-A	II-A	II-A	I
7	II-E II-E II-C II-C	II-B II-C II-C II-C	II-A	II-A	I
8	II-E II-E II-E II-C	II-E II-C II-C II-C	II-B II-B II-B II-A	II-A	I
9	II-E	II-E II-E II-C II-C	II-D II-C II-C II-C	II-B II-A II-A II-A	II-A
10	II-E II-E II-E II-C	II-E II-E II-C II-C	II-E II-C II-C II-C	II-B II-B II-B II-A	II-A

As an example of the classification process, consider $n = 5$, $p = .95$ and $\epsilon = .05$. By reference to Table 6, it is seen that $\Delta^{(n)}(\alpha,p)$ is not monotone, since for $p = .95$ and $n = 5$, $\ell n[-\ell n(1-(1-p)^{1/n})] = -.227$, which is greater than $\psi(1) = -.5772$. Therefore, Case I is eliminated and reference is then made to Table 7 for the values of α_0 and $\Delta^{(n)}(\alpha_0,p)$. For $n = 5$ and $p = .95$, $\alpha_0 = 3.9941$ and $\Delta^{(n)}(\alpha_0,p) = .24548$. In this example $\Delta^{(n)}(\alpha_0,p) > \epsilon = .05$ thereby eliminating Cases II-D and II-E. Finally by reference to Table 8, $\lim\limits_{\alpha\to\infty} \Delta^{(n)}(\alpha,p) = .203$, which is greater than $\epsilon = .05$, thus determining Case II-A as the correct classification.

The numerical process of the actual identification of regions of robustness for parallel systems is complicated by the lack of monotonicity of $\Delta^{(n)}(\alpha,p)$ in α for all n and p. The process involves the determination of α-values which are solutions to the equation $|\Delta^{(n)}(\alpha,p)| = \epsilon$. In contrast to the series system, the number and location of these solutions depend on n, p and ϵ. Thus, for a fixed n, p and ϵ, it is necessary to determine the nature of the region of robustness from Table 9. Cases I, II-A, and II-E require the identification of two α-values $\alpha_1 < 1 < \alpha_2$ such that $|\Delta^{(n)}(\alpha_1,p)| = |\Delta^{(n)}(\alpha_2,p)| = \epsilon$. Case II-B requires the identification of three α-values, $\alpha_1 < 1 < \alpha_2 < \alpha_0 < \alpha_3$ such that $|\Delta^{(n)}(\alpha_i,p)| = \epsilon$, $i = 1,2,3$. Case II-C requires the identification of four α-values $\alpha_1 < 1 < \alpha_2 < \alpha_0 < \alpha_3 < \alpha_4$ such that $|\Delta^{(n)}(\alpha_i,p)| = \quad$, $i = 1,2,3,4$. Finally, Case II-D requires the identification of a single α-value $\alpha_1 < 1$ such that $|\Delta^{(n)}(\alpha_1,p)| = \epsilon$.

For $n = 1(1)10$ and $p = .90, .95, .975, .99, .999$, maximal regions of robustness of levels $\epsilon = .10, .05, .01, .001$ were obtained by the following process: For each n, p and ϵ combination reference was made to Table 9 to determine the number and location of solutions to the equation $| \Delta^{(n)}(\alpha,p) | = \epsilon$. Thus, as in the series system case, the half interval (binary search) technique was used to obtain the solutions α_i of $| \Delta^{(n)}(\alpha_i,p) | = \epsilon$ with α_i determined to six decimal places. The resulting maximal regions of robustness are given in Tables 10-14.

Table 10

Maximal Regions of Robustness for Parallel Systems with

$p = .90$, $n = 1(1)10$, and $\epsilon = .10, .05, .01, .001$.

n \ ε	.10	.05	.01	.001
1	[.4972,1.3480]	[.8101,1.1737]	[.9642,1.0353]	[.9964,1.0035]
2	[.8266,1.2128]	[.9097,1.1000]	[.9813,1.0191]	[.9981.1.0019]
3	[.8449,1.2166]	[.9170,1.0978]	[.9824,1.0182]	[.9982,1.0018]
4	[.8392,1.2668]	[.9117,1.1127]	[.9807,1.0202]	[.9980,1.0020]
5	[.8232,1.4176]∪ [4.3721, ∞)	[.8966,1.1462]∪ [10.9706, ∞)	[.9773,1.0245]∪ [82.7260, ∞)	[.9977,1.0024]
6	[.7998,15.5032]	[.8810,1.2353]∪ [2.3834,6.6714]	[.9714,1.0323]∪ [3.5450,4.3004]	[.9970,1.0030]∪ [3.8630,3.9381]
7	[.7695,3.9935]	[.8544,2.8352]	[.9611,1.0492]∪ [1.8547,2.1647]	[.9957,1.0044]∪ [1.9953,2.0261]
8	[.7332,2.4793]	[.8180,1.9227]	[.9414,1.5121]	[.9927,1.0077]∪ [1.3771,1.4043]
9	[.6924,1.9072]	[.7713,1.5406]	[.8995,1.2254]	[.9793,1.1077]
10	[.6499,1.6215]	[.7180,1.3496]	[.8251,1.1049]	[.8858,.9172] [.9783,1.0158]

Table 11

Maximal Regions of Robustness for Parallel Systems with

p = .95, n = 1(1)10, and ϵ = ,10, .05, .01, .001.

n \ ϵ	.10	.05	.01	.001
1	(0,1.4924]	[.4989,1.2574]	[.9403,1.0557]	[.9942,1.0057]
2	[.7889,1.2380]	[.8929,1.1139]	[.9781,1.0221]	[.9978,1.0022]
3	[.8371,1.2082]	[.9144,1.0967]	[.9821,.10184]	[.9982,1.0018]
4	[.8450,1.2188]	[.9169,1.0984]	[.9823,1.0183]	[.9982,1.0018]
5	[.8409,1.2553]	[.9130,1.1094]	[.9811,1.0198]	[.9981,1.0019]
6	[.8304,1.3346]∪ [13.5186, ∞)	[.9050,1.1301]	[.9788,1.0225]	[.9978,1.0022]
7	[.8151,1.6453]∪ [2.3180, ∞)	[.8934,1.1684]∪ [4.7305,123.1183]	[.9754,1.0269]∪ [8.5618,12.9134]	[.9974,1.0026]∪ [10.1416,10.5627]
8	[.7957,10.6082]	[.8776,1.2658]∪ [2.0976,5.4884]	[.9702,1.0340]∪ [3.1157,3.7259]	[.9969,1.0032]∪ [3.3759,3.4367]
9	[.7722,4.2314]	[.8570,2.9649]	[.9622,1.0471]∪ [1.9300,2.2518]	[.9959,1.0042]∪ [2.9759,2.1070]
10	[.7451,2.7864]	[.8305,2.1204]	[.9492,1.0827]∪ [1.3832,1.6595]	[.9940,1.0062]∪ [1.5219,1.5484]

Table 12

Maximal Regions of Robustness for Parallel Systems with

$p = .975$, $n = 1(1)10$, and $\epsilon = .10, .05, .01, .001$.

$n \setminus \epsilon$	.10	.05	.01	.001
1	(0,1.6786]	(0,1.3739]	[.8917,1.0890]	[.9903,1.0095]
2	[.7252,1.2775]	[.8666,1.1351]	[.9734,1.0267]	[.9973,1.0027]
3	[.8198,1.2169]	[.9067,1.1024]	[.9807,1.0197]	[.9981,1.0019]
4	[.8408,1.2083]	[.9159,1.0961]	[.9823,1.0181]	[.9982,1.0018]
5	[.8450,1.2202]	[.9168,1.0988]	[.9823,1.0.83]	[.9982,1.0018]
6	[.8418,1.2490]	[.9137,1.1075]	[.9813,1.0195]	[.9981,1.0019]
7	[.8342,1.3021]	[.9079,1.1222]	[.9797,1.0215]	[.9979,1.0021]
8	[.8233,1.4157] $\cup$ [4.4254, ∞)	[.8997,1.1459] $\cup$ 11.2189, ∞)	[.9773,1.0244]$\cup$ [97.1254, ∞)	[.9977,1.0024]
9	[.8096, ∞)	[.8890,1.1875] $\cup$ [3.4698,15.6704]	[.9740,1.0287] $\cup$ [5.5288,7.2474]	[.9973,1.0027] $\cup$ [6.2129,6.3827]
10	[.7930,8.8903]	[.8754,1.2926] $\cup$ [1.9366,4.9503]	[.9694,1.0351] $\cup$ [2.8979,3.4426]	[.9968,1.0033] $\cup$ [3.1319,3.1862]

Table 13

Maximal Regions of Robustness for Parallel Systems with

$p = .99$, $n = 1(1)10$, and $\epsilon = .10, .05, .01, .001$.

n \ ε	.10	.05	.01	.001
1	(0,1.9765]	(0,1.5753]	[.4998,1.1611]	[.9796,1.0193]
2	[.4973,1.3480]	[.8101,1.1737]	[.9642,1.0353]	[.9964,1.0035]
3	[.7836,1.2416]	[.8906,1.1158]	[.9777,1.0225]	[.9978,1.0022]
4	[.8266,1.2128]	[.9097,1.0999]	[.9813,1.0191]	[.9981,1.0019]
5	[.8409,1.2083]	[.9159,1.0961]	[.9823,1.0181]	[.9982,1.0018]
6	[.8449,1.2166]	[.9170,1.0978	[.9824,1.0182]	[.9982,1.0018]
7	[.8437,1.2335]	[.9153,1.1034]	[.9818,1.0189]	[.9981,1.0019]
8	[.8392,1.2668]	[.9117,1.1127]	[.9807,1.0202]	[.9980,1.0020]
9	[.8322,1.3184] ∪ [31.0462, ∞)	[.9064,1.1263]	[.9792,1.0220]	[.9979,1.0021]
10	[.8232,1.4176] [4.3721, ∞)	[.8996,1.1462] ∪ 10.9706, ∞)	[.9773,1.0245] ∪ [82.7260, ∞)	[.9977,1.0024]

Table 14

Maximal Regions of Robustness for Parallel Systems with

$p = .999$, $n = 1(1)10$, and $\epsilon = .10, .05, .01, .001$.

$n \backslash \epsilon$	.10	.05	.01	.001
1	(0,2.8688]	(0,2.2301]	(0,1.4977]	[.5000,1.1049]
2	(0,1.6112]	(0,1.3307]	[.9126,1.0760]	[.9919,1.0080]
3	[.4973,1.3480]	[.8101,1.1737]	[.9642,1.0353]	[.9964,1.0035]
4	[.7508,1.2627]	[.8768,1.1271]	[.9752,1.0250]	[.9975,1.0025]
5	[.8038,1.2277]	[.8995,1.1083]	[.9794,1.0209]	[.9979,1.0021]
6	[.8266,1.2128]	[.9097,1.0999]	[.9813,1.0191]	[.9981,1.0019]
7	[.8377,1.2081]	[.9146,1.0966]	[.9821,1.0183]	[.9982,1.0018]
8	[.8430,1.2099]	[.9167,1.0961]	[.9824,1.0181]	[.9982,1.0018]
9	[.8449,1.2166]	[.9170,1.0978]	[.9824,1.0182]	[.9982,1.0018]
10	[.8445,1.2280]	[.9161,1.1012]	[.9820,1.0186]	[.9982,1.0018]

4.4 Interpretation and Use of Tables

As in the case of series systems, it is desirable to discuss in detail for parallel systems the interpretation of entries in Tables 10-14. However, since the solution to the problem of finding maximal regions of robustness for parallel systems yielded several different types of regions, it is desirable to examine more than one example to fully explain Tables 10-14.

For example, let $n = 5$, $p = .95$ and $\epsilon = .05$. This corresponds to an entry in Table 11 which is $[\alpha_1, \alpha_2] = [.9130, 1.1094]$. There the maximal region of robustness of level .05 is all α such that $.9130 \leq \alpha \leq 1.1094$. The interpretation of this region is as follows: The 95% safe-time for a parallel system of five components computed under the assumption of an exponential component failure distribution will not differ from the actural 95% safe-time by more than .05 provided the actual component failure distribution is Weibull with parameter α with α between .9130 and 1.1094.

To illustrate a more complicated entry, let $n = 8$, $p = .975$. and $\epsilon = .01$. The corresponding entry from Table 12 is $[\alpha_1, \alpha_2] \cup [\alpha_3, \infty] = [.9773, 1.0244] \cup [97.1254, \infty)$. This may be interpreted as follows: the exponential component failure distribution may be used in place of the more complicated Weibull provided the actual component failure distribution has parameter α between .9773 and 1.0244 or between 97.1254 and infinity, and the computed 97.5% safe-time will have no more than 1% error.

It should be noted that, as in the case of series systems, ϵ actually represents the error expressed as a precent of the mean-time-to-failure, since it has been required that the means of the two distribtuions be equal to one. That is, for $n = 8$, $p = .975$ and $\epsilon = .01$, $R^{(n)}(\epsilon)$ will specify values of α such that the absolute difference in 97.5% safe-times will not exceed 1% of the component mean-time-to-failure.

CHAPTER V

SUMMARY AND CONCLUSIONS

For series systems of n components, certain trends can be observed about the solution of $R^{(n)}(\epsilon)$ when one variable is allowed to change while holding the remaining two variables constant. For a fixed number of components n and a fixed p% safe-time, the maximal region of robustness decreases in size as the level of the error decreases. In other words, as less error is allowed, a smaller range of α-values will satisfy the defintion of $R^{(n)}(\epsilon)$. Secondly, for a fixed n and fixed level of error ϵ, the maximal region of robustness increases as p increases. Thus, for fixed n and ϵ, as more reliability is required for a series system, the range of α-values which will satisfy $R^{(n)}(\epsilon)$ increases. Finally, for a fixed level of error ϵ and a fixed p% safe-time, the maximal region of robustness increases as the number of components increases. That is, for fixed ϵ and p, the more components the system contains, the larger $R^{(n)}(\epsilon)$ will be.

Little can be said about trends in solutions to $R^{(n)}(\epsilon)$ for parallel systems of n components. This is consistent with the higher level of difficulty encountered in the proofs of Chapter IV, and indicates a stronger interaction between the number of components n and the p% safe-times. The only trend observable is that for a fixed number of components n and a fixed p% safe-time the maximal region of robustness decreases as the level of error ϵ

decreases. It should be noted that this trend also appears in the series system solutions of $R^{(n)}(\epsilon)$.

The maximal regions of robustness obtained, using the definition of error adopted for this research, followed different trends than those reported in Posten [3] and Powers and Posten [4]. Specifically, Posten observed that regions of robustness for series systems decreased as n increased for fixed ϵ. For a fixed n and for all $p = .90$, .95, .975, .99, .999, the regions, for series system, obtained in this research are larger than those reported in Posten [3].

Regions of robustness for parallel systems reported in Powers and Posten [4] behaved in a somewhat more erratic manner than those obtained for series systems in Posten [3]. The behavior of parallel systems obtained in Powers and Posten depended on n in that for fixed ϵ, the size of the region first decreased as n increased, but at some n (depending on ϵ) the size then began to decrease. However, these regions for parallel systems were less dependent on n than the regions for parallel systems obtained in the present research. In both [3] and [4], all regions were of the form $[\alpha_1,\alpha_2]$. Since the regions of robustness for parallel systems obtained in the present research are not always of the form $[\alpha_1,\alpha_2]$ and since the regions behave very erratically as functions of n and p, it is not informative to compare them with the parallel regions obtained in Powers and Posten [4].

In Chapters III and IV regions of robustness for series and parallel systems were identified as sets of α-values such that no more than a prespecified error would result from the assumption of an exponential component failure distribution provided the true component failure distribution is Weibull with parameter α in the identified sets. The degree to which a user need concern himself with making the assumption of an exponential component failure distribution depends on the size of the region of robustness applicable to his specific n and p and the amount of error ϵ, he can tolerate. Thus a statement on the sensitivity of calculations of p% safe-times to derivations from constant failure rate depends heavily on the individual situation. The value of this research is in the identification of the regions which provide a user with the necessary information to apply to his particular need. That is, a user may refer to the tables presented in this research and decide on the degree to which he need concern himself with assumptions of constant failure rate for each individual problem he encounters. For one problem the sensitivity may be much more central than for another, but this research provides vital information to make such assessments.

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APPENDIX I

Values of ℓn p/n for n = 1(1)10 and

p = .90, .95, .975, .99, .999

n \ p	.90	.95	.975	.99	.999
1	-.1054	-.0513	-.0253	-.0101	-.0010
2	-.0527	-.0257	-.0127	-.0050	-.0005
3	-.0351	-.0171	-.0084	-.0034	-.0003
4	-.0264	-.0128	-.0063	-.0025	-.0003
5	-.0211	-.0103	-.0051	-.0020	-.0002
6	-.0176	-.0086	-.0042	-.0017	-.0002
7	-.0151	-.0073	-.0036	-.0014	-.0001
8	-.0132	-.0064	-.0032	-.0013	-.0001
9	-.0117	-.0057	-.0028	-.0011	-.0001
10	-.0105	-.0051	-.0025	-.0025	-.0001

APPENDIX II

Theorem: $\lim\limits_{x\to\infty} a^x \mid \Gamma(x) = 0.$

Proof: Since $a^x \mid \Gamma(x)$ is a continuous function of x,

$\lim\limits_{x\to\infty} a^x \mid \Gamma(x) = \lim\limits_{\to\infty} a^n \mid \Gamma(n)$. By the ratio test,

$$S_{n+1} \mid S_n = \frac{\dfrac{a^{n+1}}{\Gamma(n+1)}}{\dfrac{a^n}{\Gamma(n)}} = \frac{a}{n}$$

. Thus as

$n\to\infty$, $S_{n+1} \mid S_n \to 0.$

APPENDIX III

Values of $\ln[1 - (1 - p)^{1/n}]$ for n = 1(1)10 and

p = .90, .95, .975, .99, .999

n \ p	.90	.95	.975	.99	.999
1	-.1054	-.0513	-.0253	-.0101	-.0010
2	-.3801	-.2531	-.1721	-.1054	-.0321
3	-.6240	-.4595	-.3459	-.2426	-.1054
4	-.8263	-.6403	-.5069	-.3801	-.1958
5	-.9968	-.7969	-.6504	-.5077	-.2893
6	-1.1435	-.9338	-.7781	-.6239	-.3801
7	-1.2718	-1.0551	-.8925	-.7297	-.4664
8	-1.3859	-1.1637	-.9958	-.8263	-.5477
9	-1.4884	-1.2619	-1.0898	-.9150	-.6239
10	-1.5815	-1.3514	-1.1760	-.9968	-.6955