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"Every graph of order three or more is reconstructible." Frank Harary restated one of the most famous unsolved problems in graph theory. In the early 1900's, while one was working on his doctoral dissertation, two mathematicians made a conjecture about the reconstructibility of graphs. This came to be known as the Reconstruction Conjecture or the Kelly-Ulam Conjecture. The conjecture states:

Let G and H be graphs with

$$V(G) = \{v_1, v_2, \cdots, v_n\}, V(H) = \{u_1, u_2, \cdots, u_n\}, n \geq 3.$$

If $G - v_i \simeq H - u_i \ \forall i = 1, \cdots, n$, then $G \simeq H$.

Much progress has been made toward showing that this statement is true for all graphs. This paper will discuss some of that progress, including some of the families of graphs which we know that the conjecture is true. Another big field of interest about the Reconstruction Conjecture is the information that is retained by a graph when we begin looking at its vertex-deleted subgraphs. Many graph theorists believe that this may show us more about the conjecture as a whole.

While working on a possible proof to the Reconstruction Conjecture, many mathematicians began to think about different approaches. One approach that was fairly common was to relate the Reconstruction Conjecture to edges of a graph instead of the vertices. People realized that when deleting only one edge of a graph, then logically more information about the original graph would be retained.

ON THE RECONSTRUCTION CONJECTURE

by

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Approved by

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Committee Chair

This thesis is dedicated to

My daughters Kirsten and Caitlyn,
the two people who were always there no matter what.

My fiance Casey,
my rock, my strength and my sanity.

My mother Pamela,
my inspiration.

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

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CHAPTER I
PRELIMINARIES

In 1957 Paul Kelly wrote his doctoral dissertation under the supervision of Stanislaw Ulam. His thesis proved that the conjecture which we have come to know as the Reconstruction Conjecture was true for trees. Three years later, Ulam published a statement of the reconstruction conjecture, but he knew about the ideas that became the Reconstruction Conjecture as early as 1929. Ulam had spent many years collecting problems that were posed by fellow graduate students and professors during his years in graduate school in Poland. This has created some difficulties in trying to determine who should have credit for posing this still unsolved problem in graph theory. The commonly accepted solution to this dispute is to call the conjecture the Kelly-Ulam conjecture [9].

First we are going to look at the original statement of the Reconstruction Conjecture. The reader should keep in mind that this version of the conjecture is put in here for purely historical purposes; we will never do any computations with this version.

Theorem 1 (Ulam's Statement of the Reconstruction Conjecture)**.** *[1]*

Suppose that in two sets A and B, each containing n elements, there is defined a distance function ρ for every pair of distinct points, with values either 1 or 2, and $\rho(p,p) = 0$. Assume that for every subset of $n-1$ points of A, there exists an isometric system of $n-1$ points of B, and that the number of distinct subsets isometric to any given subset of $n-1$ points is the same in A and in B. Are A and B isometric?

There are many restatements of Ulam's original conjecture, each dealing with another way of talking about sets. The version of this conjecture that we will work with is the application of it to graphs, where the set of vertices are the sets A and B from Ulam' original conjecture, stated below.

Theorem 2 (The Kelly - Ulam Conjecture). *[1]*

Let G and H be graphs with $V(G) = \{v_1, \cdots, v_n\}$ and $V(H) = \{u_1, \cdots, u_n\}$ for $n \geq 3$. If $G - v_i \simeq H - u_i$, $\forall i = 1, 2, \cdots, n$ then $G \simeq H$.

This is the form of the conjecture which Kelly had proven for two trees T and S in his doctoral dissertation.

Throughout the past years there has been a substantial amount of work toward trying to decide if the conjecture that Kelly and Ulam developed is true. There has been remarkable progress made and there are many things that we now know about what kinds of graphs can and cannot be reconstructed. We do know, at this point, that there exist classes of graphs for which the Reconstruction Conjecture is always true, such as regular graphs. There are also classes of graphs for which the Reconstruction Conjecture is always false, one such example being tournaments. The other thing that has been accomplished thus far is the recognition of several properties that are recoverable about any graph from its vertex-deleted subgraphs. These properties do not depend on the reconstructibility of the graph itself, but are simply based on the information that every graph retains when we look at those subgraphs.

Several mathematicians have found other ways to restate the Reconstruction Conjecture. One of the most useful restatements was formulated by Frank Harary.

Theorem 3 (Harary's Restatement of the Reconstruction Conjecture). *[1]*

If G is a simple graph with $n \geq 3$ vertices and if the n subgraphs $G - v_i$ are *given, then the entire graph G can be reconstructed, uniquely up to isomorphism, from these vertex-deleted subgraphs.*

The Kelly-Ulam version, which simply talks about the existence of an isomorphism, and the Harary version, which deals with determining the structure of the graph G, are logically equivalent, though they appear quite different at the first glance. So, while working on an overall proof of this problem it is acceptable to work toward a solution of any statement of the problem which is logically equivalent to either one of these.

CHAPTER II

NOTATION AND BASIC INFORMATION

A *graph* is an ordered pair $G = (V, E)$ such that V is a set of vertices and E is an unordered set of pairs of vertices in V called edges. Given a graph G and a vertex of G, v, we say that the *degree of* v, denoted $\deg v$, is the number of edges which are incident to v. A *directed graph* is a graph where for (u_1, v_1), $(v_1, u_1) \in E$, $(u_1, v_1) \neq (v_1, u_1)$. Clearly this makes it so that there can be more than one edge connecting any two vertices of $V(G)$. A *multigraph* is a graph G where the elements of E need not be distinct, so that, for example, for $V = \{v_1, \cdots, v_4\}$, we can have $E = \{(v_1, v_2), (v_1, v_2), (v_1, v_2)\}$. We will denote the edge that connects v_i and v_j by $e_{i,j}$. In a multigraph, if the set E has any elements that start and end at the same vertex, then that edge is referred to as a *loop*. A *simple graph* will be a graph which is not a multigraph and is not a directed graph. Much of our work involves examining graphs where we have deleted one vertex and all of the edges which are incident to that vertex. We will refer to these graphs as *vertex-deleted subgraphs* [3].

A graph G is considered to be *reconstructible* if we can recover the unique graph, up to isomorphism, from the vertex-deleted subgraphs. We also note that a graph G is referred to as *labeled* if its vertices are associated with distinct labels in a one to one manner. For the purposes of the Reconstruction Conjecture, we assume that our graph is not labeled. If we were to look at labeled graphs, then the conjecture would trivially be true since we could easily see the connections between any two vertices simply by looking at the vertex-deleted subgraph where neither of these vertices have been deleted [1].

We will denote the set of vertices of a graph G as $V(G)$ where $V(G) = \{v_i | i \in \mathbb{N}\}$. We will also let $\bar{G}$ denote the complement of G, a graph where if v_i and v_j are connected in G by an edge $e_{i,j}$ then they are not connected in $\bar{G}$ and if v_i and v_j are not connected in G by any edge, then they are connected by an edge $e_{i,j}$ in $\bar{G}$ [4].

There are some facts about all graphs that we can recover from the vertex-deleted subgraphs, such as the number of vertices. Any fact that we can do this with is called a *recoverable* fact.

In a connected graph G, we say that v is a *cut-vertex* if the graph G - v is disconnected. In a disconnected graph G, v is a *cut-vertex* if it is a cut-vertex in any of the components of G. [1]

In a graph G a $u - v$ *walk* is an ordered subset of $V(G)$ such that the first vertex is u, the last vertex is v, and given any two consecutive vertices v_i, v_j in a simple graph, the edge $e_{i,j} \in E(G)$. A *path* in a graph G is a walk in G which has no repeated vertices, except possibly for the end vertices. Then we say that a *cycle* of a graph G is a path of G which starts and ends at the same vertex. Moreover, a graph G is *unicyclic* if it contains exactly one cycle. A *cycle* of a graph G is a walk in G which has no repeated edges. [1]

Another type of graph that we are interested in is a tree. A graph is a *tree* if it is a connected graph with no cycles. An edge $e_{i,j}$ in a connected graph G is called a *bridge* if $G - e_{i,j}$ is disconnected and in a disconnected graph, the edge $e_{i,j}$ is a *bridge* if it is a bridge in one of the components of G. It is a well-known theorem that in a tree, every edge of the graph is a bridge. [6]

A pair of graphs G and H are called *hypomorphic* if there exists a bijection $\sigma : V(G) \rightarrow V(H)$ such that $G - v \simeq H - \sigma(v) \, \forall v \in V(G)$. We refer to the function σ as a *hypomorphism* of G onto H. [3]

One thing that needs to be noted is that the Reconstruction Conjecture is only stated for graphs of order three or more. If we were to look at two graphs of order two, then we can already show that the Reconstruction Conjecture is false.

Example. [1]

Let H be the graph consisting of just two vertices and G be a path or order two.

$G =$ 1 ——— 2 $H =$ 1   2

Then clearly $G - v_1 \simeq H - u_1$ and $G - v_2 \simeq H - u_2$. However, G and H are not isomorphic to each other. □

Now we will turn our attention to look at results that relate some of the information that we will need.

Theorem 4. *[[1]] Let G be a non-trivial connected graph and let $u \in V(G)$. If v is a vertex of G such that the length of a $u - v$ path in G is maximal, then v is not a cut-vertex in G.*

Proof by Contradiction. Assume that v is a cut-vertex. So, there exists a vertex w of $G - v$ such that w is in a different component of $G - v$ than u. Then because v is a cut-vertex in G, every $u - w$ path contains v. Therefore, there is a $u - w$ path which has length longer than any $u - v$ path. However since there is a $u - v$ path which is maximal, this is a contradiction. So it must be that v is not a cut-vertex. □

Corollary 1. *[1] If G is a non-trivial connected graph, then G contains at least two vertices that are not cut vertices.*

Proof. Let $u, v \in V(G)$ such that the length of a $u - v$ path is maximal among all paths in G. Then by Theorem 4, u and v are not cut vertices. □

Corollary 2. *[1] If G is a graph of order $n \geq 3$ with q edges and G has no isolated vertices, then*

a. if n is odd, $q \geq \frac{n+1}{2}$

b. if n is even, $q \geq \frac{n}{2}$

Proof by Induction. Start with $n = 3$.

Since there are no isolated vertices, then each vertex must have an edge which is incident to it. By inspection, the least number of edges that you could have is 2.

$$2 \geq \frac{3+1}{2} = 2.$$

So the inequality in (a) holds for $n = 3$.

Next show that if the inequalities hold for n, then they hold for $n + 1$.

Case 1, n is even

Let q_i be the number of edges in a graph of order i. Assume that $q_n \geq \frac{n}{2}$.

We need to show that $q_{n+1} \geq \frac{(n+1)+1}{2}$. To get a graph of order $n+1$ for any graph of order n, we have to add a vertex. Since we have assumed that there are no isolated vertices, we also have to add at least one edge to connect that vertex to another vertex already in the graph. Then it follows that

$$q_{n+1} \geq q_n + 1 \geq \left(\frac{n}{2}\right) + 1 = \frac{n+2}{2} = \frac{(n+1)+1}{2}.$$

So from this we get that

$$q_{n+1} \geq \frac{(n+1)+1}{2} \geq \frac{n+1}{2}.$$

Case 2, n is odd

Assume that $q_n \geq \frac{n+1}{2}$. We need to show that $q_{n+1} \geq \frac{n+1}{2}$. Since there are no isolated vertices in G, then by the same argument as in case 1,

$$q_{n+1} \geq q_n + 1 \geq \frac{n+1}{2}.$$

So from this we get that

$$q_{n+1} \geq \frac{n+1}{2}.$$

So by induction, the corollary is true $\forall n \in \mathbb{N}$. □

Theorem 5. *[1] In a graph G with vertices u and v, if there is a $u - v$ walk, then there is a $u - v$ path.*

Proof by Construction. Let G be a graph with vertices u and v. Assume that there is a $u - v$ walk. If this walk is a path then we are done. So assume that our $u - v$ walk, call it W, is not a path. Then by the definitions of walk and path, we know that there are repeated vertices in W. So, list W by the vertices, $W = \{v_1, v_2, \cdots, v_r\}$.

(1) Since W is not a path, then there are $i, j \in \{1, \cdots, r\}$ such that $v_i = v_j$. Without loss of generality, assume that $i < j$. Then in W, we can delete $\{v_i, v_{i+1}, \cdots, v_j\}$. Let $W - \{v_i, v_{i+1}, \cdots, v_j\}$ be denoted W'.

If W' is a path then we are done. If not, then we can repeat the process described in (1) until the resulting walk is a path. □

CHAPTER III
SOME RECOVERABLE FACTS

This chapter uses [1] and [3] as the primary references unless otherwise noted.

One of the most obviously recoverable facts about G is its order, the number of vertices in G. Since we have all of the vertex-deleted subgraphs $G - v_i$, then we will have exactly one $G - v_i$ for each of the vertices $v_i \in G$. From this we can see that the number of the $G - v_i$ that we have will be the same as the order of G. Since we now know that the order of a graph G is recoverable, let n represent the order of a graph.

Another fact about G that is recoverable is the total number of edges, q. When looking at the graphs $G - v_i$, we notice that each of the edges in G appears in $n - 2$ of the subgraphs. Specifically, the two $G - v_i$ that the edge would not be in are the two $G - v_i$ where the vertices incident to that specific edge deleted. Let q_i be the number of edges in each of the $G - v_i$, $i = 1, \cdots, n$, then the total number of edges is

$$q = \sum \frac{q_i}{n-2}.$$

We are also able to tell the degree of each $v_i \in G$. Now that we know that there are q edges in the whole graph G and from the way that we have constructed the subgraphs G_i, we can see that in each of these G_i the only edges that are missing are going to be the ones that have v_i as an endpoint. Therefore,

$$\deg v_i = q - q_i, \forall i = 1, \cdots, n.$$

When looking at the $G-v_i$, we can easily count the number of loops in each of the subgraphs. Then, by the definition of a loop, each loop would appear in $n-1$ of the $G-v_i$. Now if we add up the number of loops in all of the $G-v_i$ and divide the result by $n-1$, it follows that this quotient is the number of loops in the original G. Then clearly the number of loops in G is reconstructible.

Theorem 6. *If G is a graph with $V(G)=\{v_1,\cdots,v_n\}$, for $n\geq 3$, and $\forall i=1,\cdots,n$, $G-v_i$ is the subgraph with the vertex v_i and the edges incident to it deleted. Then G is connected if and only if at least two of the $G-v_i$ are connected.*

Proof. Let G be connected. By Theorem 4, we know that G contains two vertices which are not cut vertices.

Assume that there exist two vertices of G, u,v, such that both $G-u$ and $G-v$ are connected. This means that in $G-u$, v is connected to each $v_i, i\geq 3$ and in $G-\{v,u\}$ is connected to each $v_i, i\geq 3$. So we know that there exists a vertex $w\in V(G)$ such that there is a $u-w$ path in $G-v$ and there is a $v-w$ path in $G-u$. Then we know that there exists a $u-v$ walk in G. Therefore, by previous work, we know that there exists a $u-v$ path in G. So, G is connected.

Assume that G is a connected graph. That means, by Corollary 1, there are two vertices u and v in G such that u and v are not cut vertices. Then by the definition of cut vertices, we get that $G-u$ and $G-v$ are connected. □

By Theorem 6, it is obvious that if we have all the vertex-deleted subgraphs, $G-v_i$. So we are able to tell if a graph is connected or disconnected. Therefore, the connectivity of a graph is another fact about G that is reconstructible from its vertex-deleted subgraphs.

If G is connected, then by looking at the $G-v_i$, we can also tell how many cut vertices are in the original graph G. We already know that we can tell whether

G is connected or not. Thus for a connected G, look at all of the $G - v_i$ which are disconnected. Each vertex v_i that created a disconnected $G - v_i$ is a cut-vertex. If G is disconnected, we can look at the number of components in each of the $G - v_i$. Since we cannot have a graph where all the vertices are cut vertices, then there must be at least one $G - v_i$ which has the same number of components as G, this will be the $G - v_i$ with the least number of components. Then for every $G - v_i$ which has more components, v_i is a cut-vertex. So whether G is connected or disconnected is irrelevent. In either case we can determine the number of cut vertices.

Theorem 7. *Let G be a graph with n vertices and q edges, denoted an (n, q) graph. Then G is unicyclic if and only if G is connected and $n = q$.*

Proof. Let G be an (n, q) unicyclic graph and let $e_{i,j}$ be an edge of the cycle of G. The $(n, q-1)$ graph $G - e_{i,j}$ is a path and is connected. Since the number of edges in a connected tree is equal to the number of vertices minus one, we get that $n - 1 = q - 1$. Obviously if G is unicyclic, it is connected by the definition of unicyclic. Also if $n - 1 = q - 1$ then it follows that $n = q$.

Let G be a connected (n, q) graph such that $n = q$. Since $n = q$, G is not a tree. Then because G is not a tree, not every edge of G is a bridge. Since there is some edge of G that is not a bridge, call it $e_{i,j}$, then $e_{i,j}$ must be an edge on a cycle. Hence, $G - e_{i,j}$ is connected and has $n - 1$ edges. Therefore, $G - e_{i,j}$ is a tree. Since, $e_{i,j}$ was an edge on the only cycle in G it follows that G is unicyclic. □

So, since we know that we can determine if a graph G is disconnected or connected and because we can reconstruct the n and q, then we can determine from Theorem 7 whether a graph G is unicyclic or not.

Theorem 8. *Let G be a graph in which every vertex has degree at least 2. Then G contains a cycle.*

Proof. Let v_0 be a vertex of G. Since it has degree at least 2, then we can pick one edge of G which has v_0 as one end-vertex of an edge, $e_{0,1}$, in G and label the other incident vertex as v_1. Since the degree of v_1 is at least 2, we can pick another edge, $e_{1,2}$, which is incident to v_1. Then the second vertex which is incident to this edge would be labeled now as v_2. We can continue this process until eventually we will pick a vertex that has already been chosen before. Thus, the choice of vertices from the first time we chose that vertex until the second time will be a cycle of G. We are guaranteed that this will happen since every vertex is incident to at least two edges, making it impossible to pick all the edges without repeating a vertex. □

We say that a graph G is *Eulerian* if it contains a cycle which uses every edge of G exactly one time. Now we will look at a few theorems that help us in deciding if a graph is Eulerian. The ultimate goal is to prove that one of the recoverable facts is whether or not a graph is Eulerian.

Theorem 9. *A connected graph G is Eulerian if and only if every vertex has even degree.*

Proof. Let G be a connected graph. Assume that G is Eulerian. Let C be an Eulerian cycle in G. Put u as the starting and ending vertex of C. Let v be a vertex of G such that $v \neq u$; then because G is connected, v is a vertex on C. Now, every time v is on the cycle, it must be that there was one edge of G used to get to v and a different edge of G used to leave v since C uses every edge of G exactly once. Next, every time that v appears in C, we add two to our count for the number of edges which have used v thus far. So, v has even degree. Since C began and ended at u, the beginning and ending edges of C add two our continuous count of the number of edges which have used u. Any other time that u appears in C, the same idea as we used with v applies, we add two to the continuous count of the number of edges which have used u. Thus, the degree of u is even.

Now, let G be a connected graph such that every vertex has even degree. We prove that G is Eulerian by induction on the number of edges in G. First since G is connected, if G has no edges, then G must contain just the starting and ending vertex, u. Therefore, trivially, G is Eulerian.

Next assume that if G has $1, \cdots, n$ edges, for $n > 0$, then G is Eulerian. We need to show that if G has $n+1$ edges, then G is Eulerian. Since G is connected, then no vertex of G has degree zero. By Theorem 8, because every vertex has even degree, G contains a cycle. Call the cycle K_1. If this cycle contains all the edges of G we are done. If K_1 does not contain every edge of G, we delete every edge of G that is contained in K_1. Then we get a subgraph of G which has only the unused edges of G, call it G_1. We should note that G_1 can be disconnected and that G_1 has the same vertex set as G. All the vertices of G_1 have even degree still since the edges we removed took away two edges from every vertex that had any edges removed from it. Clearly, G_1 has less than $n+1$ edges since we have removed edges to obtain it. Next we can apply the induction hypothesis and each of the components of G_1 must be Eulerian. Then because we obtained G_1 by deleting edges of G, each component of G_1 must have at least one vertex in common with K_1.

Now, to obtain an Eulerian cycle, we start at any vertex of K_1 and travel around K_1 until we reach a vertex that is part of a non-empty component of G_1. When we reach such a vertex, we follow the Eulerian cycle around that component. Then we resume traveling around K_1 until we reach the next such vertex. We continue that process until we arrive back at the vertex of K_1 that we chose to start with. Thus we have constructed an Eulerian cycle in G, showing that G is Eulerian for $n+1$ edges.

Then by induction, for any connected graph G, if the degree of every vertex of G is even, then G is Eulerian. $\square$

We already know that we are able to recover the degree sequence of the graph G. Then it follows from Theorem 9 that we are able to recover whether or not a graph is Eulerian.

CHAPTER IV

RECONSTRUCTIBLE GRAPHS

There are many classes of graphs that we already know are reconstructible. We are going to look at a few of these graphs and the proofs that they are reconstructible. We will assume that our graphs are not multigraphs or digraphs, since there is a proof that the Reconstruction Conjecture is false for both of these types of graphs [7].

We should also make a point before we look at these graphs. Proving that these graphs are reconstructible and coming up with a method for reconstructing these graphs are two very different problems. The following proofs will simply show that if we reconstruct these types of graph, then the reconstruction is unique up to isomorphism.

Theorem 10. *[1] Regular graphs are reconstructible.*

Proof. Let G be a regular graph of order $n \geq 3$, and let $G - v_i$ be the vertex-deleted subgraphs. From the definitions of these subgraphs, we know that each of them is missing one of the vertices of G. Because G is regular then we also know the degree of each of the vertices in G. Since this is a regular graph, each of the vertices have the same degree, r. Looking at any of the $G - v_i$ we can insert one more vertex, replacing the one that has been deleted. Now we replace edges from our inserted vertex to any of the v_i which have degree $r - 1$ until all the vertices have degree r. So G is reconstructed. □

Theorem 11. *[1] Complete graphs are reconstructible.*

Proof. Let G be a complete graph of order $n \geq 3$. By the definition of a complete graph, G is a $n-1$ regular graph. Since we know that regular graphs are reconstructible by Theorem 10, G is reconstructible as well. □

Theorem 12. *[1] If G is a disconnected graph which has at least one isolated vertex, then G is reconstructible.*

Proof. Let G be a graph with at least one isolated vertex, call it v. It follows that $G - v$ is one of the vertex-deleted subgraphs that we have, let it be H. Clearly the reconstruction of G is H adjoined with an isolated vertex v. □

Theorem 13. *[1] Disconnected graphs are reconstructible.*

Proof. We know from Theorem 5, that we can determine from the vertex-deleted subgraphs whether the graph G is disconnected of not. Assume that G is disconnected. Then, at most one of its vertex-deleted subgraphs are connected, by Theorem 6.

Let the order of G be $n \geq 3$. Put $V(G) = \{v_1, \cdots, v_n\}$. Also let q_i be the number of edges in each of the $G - v_i$ and q the number of edges in G. We know from previous work that we can compute the degree of each of the $v_i \in G$.

If for some $0 \leq j \leq n$, $q_j = q$, then it is obvious that the vertex v_j is an isolated vertex in the graph G. So then from Theorem 12, we know that we can uniquely reconstruct G.

Now, assume that G has no isolated vertices. If there are no isolated vertices, then it follows from Corollary 2, and the fact that for any n, $\frac{n+1}{2} \geq \frac{n}{2}$, that $q \geq \frac{n+1}{2}$. Therefore, G cannot be trivial if the order of G is at least 3 and there are no isolated vertices. Then by Corollary 1, G contains at least two vertices that are not cut vertices.

Then for some $G - v_i \simeq G - u$ such that there is a component F which contains the smallest number of vertices in any of the components of any of the $G - v_i$, let m be the number of vertices in this component. Then it is obvious that the deleted vertex u must be in this component F in G. If it was not in F, it would have to be in some other component F'. There would be a vertex-deleted subgraph G_j with a component that would have $m - 1$ vertices it in where v_j is in the component F'. This contradicts the assumption that m is the least number of vertices in any of the components of the subgraphs.

Now, because we are able to isolate the components that contain the deleted vertex in the way described above, then we will only consider the aforementioned subgraph $G - u$ which has the original component F. In this graph, label the components which have more than m vertices as $F_2, \cdots, F_k$. It is clear that these components are contained in the graph G. So we only have the graph F_1 left to identify somewhere in the other $G - v_i$ so that we can uniquely reconstruct G.
We must consider 3 cases:

1. Some component F_i, $2 \leq i \leq k$ has order at least $m + 3$.

Let b denote the number of these components of G which have order $m+1$, keeping in mind that it is very possible that $b = 0$. Then we select one subgraph G_j with k components such that $b + 1$ of these components have order $m + 1$. This means that v_j belongs in a component that has order greater than $m + 2$. If v_j was not in a component of order greater than $m+2$, then it would be in a component of order $m + 1$ meaning that there are $b + 2$ components of order $m + 1$ which contradicts our selection of G_j. The $b+1$ components that have order $m+1$ are all components of our original G, and one of which is F_1. G consists of these $b + 1$ components along with all of the previously stated components F_i, $i = 2, \cdots, k$ which have order greater than $m + 1$. Therefore G is reconstructed.

2. All components of F_i, $2 \leq i \leq k$ have order $m+2$.

Look at all the remaining $G-v_i$ which have k components, two of which have order $m+1$. Obviously, one of those two components will be F_1. If there is only one graph that appears in each pair, in other words, the two graphs are isomorphic, then F_1 is isomorphic to both of these graphs. If this is not the case, then every pair of components will be two non-isomorphic components. Call these two F' and F''. One of either F' or F'' is F_1, the other was obtained by deleting a non-cut-vertex from one of the F_i, $i = 2, \cdots, k$. So look at the F_i. Pick one and remove a non-cut-vertex from the component. You will then obtain a graph that is either isomorphic to F' or F''. This is computable because there are only $k-1$ components to check, each of which has only $m+2$ vertices. Which ever one you do not get by deleting the vertex, is your original F_1. Then this F_1 along with all of the F_i, $i = 2, \cdots, k$ form the original G and G i reconstructed uniquely.

3. At least one component among the F_i, $2 \leq i \leq k$ has order $m+1$, all the others have order $m+2$.

Considering all subgraphs, $G - v_i$ which have k components, one of which is a component of order m, then all components that have order greater than m will be components of G. A component H is a component of G if and only if it has order greater than m and it is a component of $G - v_i$ for some i where $G - v_i$ has at least one component of order m. If each $G - v_i$ has all but one of its components isomorphic to H, then every component of G is isomorphic to H.

If there is some component of G which is not isomorphic to H, then one should notice that the number of components in G which are isomorphic to H will be the same as the maximum number of components in one of the G_j which we are looking at that are isomorphic to H. Denote this maximum as c' in one of the $G - v_i$ with k components where one of the components has order m. If this is the case, then looking at the $G - v_i$ which gave us this maximum, we get that all the components of this $G - v_i$ which are isomorphic to H are components in G. Look at a G_j which has less than c' components isomorphic to H; specifically it will have $c' - 1$ components isomorphic to H. So v_j must have been deleted from one of the components that is isomorphic to H. The other $k - c'$ components which are not isomorphic to H are also components of G. Now G has been reconstructed.

In this case, there is one special case. Suppose G has components of order $m + 2$, H is a component of order $m + 1$ and every component of order $m + 1$ in each of the $G - v_i$ is isomorphic to H. With the aforementioned conditions, the number of components of G which are isomorphic to H is one more than the c' which was previously defined. However, this will not affect the reconstruction of G by the previously described process.

Therefore, since it has been shown for all possible cases, it follows that disconnected graphs are reconstructible. □

Theorem 14. *If G is a graph such that $\bar{G}$ is a disconnected graph, then we can reconstruct G.*

Proof. Assume that G is a connected graph such that the complement of G is disconnected. Then from the $G - v_i$ it is clear that $\bar{G}_i$ is the same as taking the complement of each of the $G - v_i$, i.e. $(\bar{G})_i = G \bar{-} v_i$. We know that since $\bar{G}$ is disconnected then we can reconstruct it by Theorem 13. From there, the definition of complement allows us to reconstruct G. □

One significant breakthrough in the effort to show the Reconstruction Conjecture either true or false was the proof that blocks are reconstructible. However, the work in block reconstruction only extends as far as reconstructing the distinct blocks of a graph G which has more than one block. We are not able to reconstruct the block of a graph with only one block [1].

A *block* B of a graph G is a maximal subgraph of G such that B does not contain any cut vertices. For our purposes we will assume that the block B has at least one edge [12].

There are preliminary theorems that we need to establish before we are able to tackle the major theorem of this section, which reconstructs a large number of graphs, though not all graphs.

Theorem 15. *[1] Let G and H be graphs such that $V(G) = \{v_1, \cdots, v_p\}$ and $V(H) = \{u_1, \cdots, u_p\}$ where $p \geq 3$ and $G - v_i \simeq H - u_i$. Then if G contains k subgraphs which are isomorphic to a graph F, such that $2 \leq v(F) < p$, then H also has k subgraphs which are isomorphic to F. Also, for $i = 1, \cdots, p$, the vertices u_i and v_i belong to the same number of subgraphs which are isomorphic to F.*

Proof. Let F be a graph such that $2 \leq v(F) < p$. Assume that G has k subgraphs that are isomorphic to the given graph F and that H has m subgraphs which are isomorphic to the given graph F. Put k_i as the number of subgraphs of G which contain v_i and are isomorphic to F. Also, put m_i as the number of subgraphs of H which contain u_i and are isomorphic to F.

Then it follows that because $|V(F)|\, k = \sum_{1}^{p} k_i$ and $|V(F)|\, m = \sum_{1}^{p} m_i$, that $k = \dfrac{\sum_1^p k_i}{|V(F)|}$ and $m = \dfrac{\sum_1^p m_i}{|V(F)|}$.

Likewise, $G - v_i \simeq H - u_i$ then it follows that the number of subgraphs of G which do not contain v_i and are isomorphic to F is the same as the number of subgraphs of H that do not contain u_i but are still isomorphic to F. Clearly $k - k_i = m - m_i$ for $i = 1, \cdots, p$. Since k, k_i, m, m_i are integers we can apply properties of integers and get that $k - m = k_i - m_i$. We see that

$$\sum_{i}^{p}(k_i - m_i) = |V(G)|(k-m) = |V(H)]|(k-m) = p(k-m)$$

So it follows that

$$\sum_{1}^{p}(k_i - m_i) = \sum_{1}^{p} k_i - \sum_{1}^{p} m_i = |V(F)|(k-m)$$

This implies that $k = m$ and therefore $k_i = m_i$ for $i = 1, \cdots, p$. □

Theorem 16. *[1] Let G and H be connected graphs with cut-vertices with $V(G) = \{v_1, \cdots, v_p\}$, $V(H) = \{u_1, \cdots, u_p\}$. Then let $B_1, \cdots, B_m$ for $m \geq 2$ be the blocks of G and $B'_1, \cdots, B'_n$ be the blocks of H. Then $m = n$ and $B_i \simeq B'_i$ for $i = 1, \cdots, m$ after a possible relabeling.*

Proof. Without loss of generality we can assume that the graph G is connected because we have already shown that if G were disconnected then we can reconstruct G which would include the reconstruction of the blocks of G. Because G is connected and $G - v_i \simeq H - u_i$ then by Theorem 6 we can see that H is also connected.

Due to the fact that H has cut vertices, then $n \geq 2$. We can order the blocks of G and the blocks of H so that

$$|B_1| \geq |B_2| \geq \ldots \geq |B_m|$$

$$|B'_1| \geq |B'_2| \geq \ldots \geq |B'_n|$$

Now we shall apply Theorem 15 and let $F \simeq B_1$. The graph H then has a subgraph $H_1 \simeq B_1$. Since B_1 is a block then clearly H_1 has no cut vertices. Then it follows that H_1 is a subgraph of B'_j for some j. Yet, $|H_1| = |B_1| \geq |B'_1| \geq \left|B'_j\right|$

so then we can conclude that $H_1 \simeq B'_j$. Therefore, with some relabeling (if needed) we can conclude that $B_1 \simeq B'_1$.

By induction we can assume that $B_i \simeq B'_i$ for $1 \leq i \leq k$ with $1 \leq k < m$. Next we want to look at the block B_{k+1}. By again using an application of Theorem 15 we know that G and H have the same number of subgraphs isomorphic to B_{k+1}. Even stronger, from here we can conclude that $\bigcup_{i=1}^{k} B_i$ and $\bigcup_{i=1}^{k} B'_i$ also have the same number of graphs isomorphic to B_{k+1}. By a similar argument as in the 1 case, we can see that H has a subgraph $H_{k+1} \simeq B_{k+1}$ which must be a subgraph of B'_j for some $j > k$. So then we know that the block B'_{k+1} exists. Consider two cases.

Case 1

Assume that $|B_{k+1}| \geq \left|B'_{k+1}\right|$. Then $|H_{k+1}| = |B_{k+1}| \geq \left|B'_{k+1}\right| \geq \left|B'_j\right|$ for $j < k$. This implies that $H_{k+1} \simeq B'_j$. Then after relabeling if necessary, we can conclude that $B_{k+1} \simeq B'_{k+1}$. By induction we get that $B_i \simeq B'_i$ for all $i = 1, \cdots, m$ and $i = 1, \cdots, n$ which implies that $m = n$.

Case 2

Assume that $\left|B'_{k+1}\right| \geq |B_{k+1}|$. Now if we apply Theorem 15, G and H have the same number of subgraphs which are isomorphic to B'_{k+1}. Also by the generalization from above we get that $\bigcup_{i=1}^{k} B_i$ and $\bigcup_{i=1}^{k} B'_i$ also have the same number of subgraphs isomorphic to B'_{k+1}. So G has a subgraph $G_{k+1} \simeq B'_{k+1}$, which must be a subgraph of B_j for some $j < k$. Which give us

$$|G_{k+1}| = \left|B'_{k+1}\right| \geq |B_{k+1}| \geq |B_j|$$

for $j > k$. This implies that $G_{k+1} \simeq B_j$. After any needed relabeling, we can conclude that $B_{k+1} \simeq B'_{k+1}$. So by induction we conclude that $B_i \simeq B'_i$ for all

$i = 1, \cdots, m$ and $i = 1, \cdots, n$ and therefore $m = n$. □

We should, at this point, note that we have shown that the individual blocks of a graph G can be reconstructed. This does not imply that we are able to reconstruct these graphs in their entirety. There are many graphs which have the same collection of blocks where the graphs themselves are not isomorphic [1].

Theorem 17. *[1] Let G and H be connected graphs having cut-vertices but no end-vertices. Let $V(G) = \{v_1, \cdots, v_p\}$, $V(H) = \{u_1, \cdots, u_p\}$, and $G - v_i \simeq H - u_i$ for $i = 1, \cdots, p$. Then $G \simeq H$.*

Proof. Let B_1 be an end block of G so then $\exists\, v_i$ such that $1 \leq i \leq p$ and $v_i \in V(B_1)$. Let v be the cut-vertex of G contained in B_1. Also let G_1 be the subgraph of G obtained by deleting all the vertices of B_1 except for v.

For $s \in \mathbb{Z}$, $s > 0$, let $G_{1,s}$ be the graph obtained by adding in s new vertices to the graph G_1 and then joining an edge from v to each of the new vertices. Because our graph G has no end vertices, there exists a cycle C_1 such that for every vertex $v_i \in B_1$, $v_i \in C_1$. Also, $G_{1,1}$ is a proper subgraph of G. We are guaranteed that it is not all of G because that would imply that the vertex we added in is an end-vertex. So by Theorem 15, we can say that there is a graph H such that it has a subgraph $H_{1,1}$ where $H_{1,1}$ is isomorphic to $G_{1,1}$. Let ϕ be an isomorphism such that $\phi : G_{1,1} \rightarrow H_{1,1}$.

Now, let w be the end-vertex of $H_{1,1}$. Then we will denote H_1 as $H_{1,1} - w$ and it follows directly that $H_1 \simeq G_1$. Let ϕ_1 be the restriction of ϕ to $V(G_1)$. Clearly, $\phi_1 : G_1 \rightarrow H_1$ and ϕ_1 is an isomorphism because ϕ was an isomorphism. Now we wish to show that we can extend the isomorphism ϕ_1 to σ where $\sigma : G \rightarrow H$ is an isomorphism of G onto H.

We will let $b(G)$ denote the number of blocks in a graph G. Then, from

Theorem 16 we can easily see that $b(H_1) = b(H) - 1$ and from this we also know that the blocks of G are the same as the blocks of H. However, from $G_1 \simeq H_1$ we cannot deduce that B_1 is isomorphic to the block of H which is missing from H_1. We will denote the missing block of H_1 as B_1'. From here B would need to be the block B_1' that we are missing, and we must show that B actually is isomorphic to B_1'.

First, it is clear that H_1 and B_1' have only one vertex in common, the cut-vertex that v maps to under the isomorphism ϕ. Then since we know that $w\phi(v)$ is an edge in H that is not in H_1, it must be that it is an edge in the block B_1'. It is also true that since $\phi_1(v)$ is a vertex in B_1' and in H_1, that $\phi(v)$ is the vertex in both B_1' and H_1. This tells us that $\phi(v) = \phi_1(v)$ is the cut-vertex in both B_1' and H_1. Now, it is enough to show that there exists an isomorphism from B_1 to B_1' which maps v to $\phi(v)$.

Now let $B_{1,1}$ be the graph obtained from B_1 by adding in one vertex and an edge joining it to v. We will define $B_{1,1}'$ similarly. Then since $\forall i = 1, \cdots, p$, $G - v_i \simeq H - u_i$, it follows that v_i is a cut-vertex of G if and only if u_i is a cut-vertex of H. Furthermore, we get that $\deg v_i = \deg u_i$ for $i = 1, \cdots, p$. Now, since $G_1 \simeq H_1$, we get that $\deg_G v = \deg_H \phi(v) = r + s$, where r edges of G_1 and s edges of B_1 are incident to v. Then $\phi(v)$ must be incident to exactly r edges of H_1 and s edges of B_1'. Then given that there are α subgraphs of $G_{1,s}$ which are isomorphic to $B_{1,1}$, it follows that G has $\alpha + r$ subgraphs isomorphic to $B_{1,1}$. But, since $H_{1,s} \simeq G_{1,s}$, the graph $H_{1,s}$ also contains α subgraphs isomorphic to $B_{1,1}$. By Theorem 15 we get that there must be $\alpha + r$ subgraphs of H which are isomorphic to $B_{1,1}$. So we can conclude that $B_{1,1}'$ contains at least one subgraph that has to be isomorphic to $B_{1,1}$. Because we already knew that $B_1 \simeq B_1'$ then we get that $B_{1,1} \simeq B_{1,1}'$ So we have found an isomorphism, ϕ, from $B_{1,1}$ to $B_{1,1}'$ which maps v

to $\phi(v)$. □

As previously mentioned, Theorem 17 does give us a very large set of graphs for which the Reconstruction Conjecture is true. However, there are still many graphs which do not satisfy the conditions laid forth in the theorem. Theorem 16 proves that we are able to reconstruct the individual blocks of a graph, but does not address the problem of correctly reconnecting the blocks in the graph. Therefore, though we are able to construct the blocks of the graph, this does not ensure the reconstructability of the graph itself [1].

CHAPTER V
TREES

First we will note that the primary sources for this chapter are [3] and [5] which have both used the original dissertation of Paul Kelly as a resource. For our purposes we will always assume that the tree that we are working with is a finite tree.

When needed, we are able to select a vertex in a tree T and call it the *root* of T. When we do this, we refer to the graph T as a *rooted tree*. We often will redraw the tree so that the root r is at the top of the graph and the remaining vertices are below the root. For convenience we will denote a rooted tree T with root r as (T, r). If the tree, T, is a path, and we select an r such that $\deg r = 1$, then we refer to (T, r) as a *rooted path*. Also, in a tree, any vertex v that has three or more edges incident to it is referred to as a *junction vertex*. If T is a tree, there are some vertices which are end vertices. If v is a vertex on any path of maximal length among all the paths of T which is also an end-vertex, then we call v a *peripheral vertex* of T. We will denote the set of peripheral vertices by $\prod(T)$. Any $T - v_i$ which is itself a tree will also be referred to as a *vertex-deleted subtree*. It should also be noted that if $T - v_i$ is a vertex-deleted subtree then the vertex v_i would have been an end-vertex of T. [3]

If we take a tree T and remove all the end vertices, then we will obtain a subtree. Continuing this process, eventually the subtree will either be a single vertex or a pair of vertices joined by a single edge. If the result is a single vertex, then we will refer to the tree T as a *central tree* and the remaining vertex as the *center of* T, c. If the result is a pair of vertices then we will refer to the tree T as a *bicentral tree* and the remaining vertices would be the *bicenter* of T. If T is a central tree, then a *branch of* T, denoted (B, c), is a rooted tree such that the center of (T, c) is the root, only one edge incident to c is included, and the collection of vertices, u, where there is a path from c to u that uses the edge are included. If T is a bicentral tree, then a *branch of* T is one of the components of the subgraph obtained by deleting the single edge which connects the bicenter of T. It follows from this that if T is a bicentral tree, then T has exactly two branches and if T is a central tree, then T has $deg\,(c)$ branches.

A branch of T is a *peripheral branch* if it contains a peripheral vertex of T. It also follows that every tree will have at least two peripheral branches, in a bicentral tree, both branches are peripheral. However, it is possible that a tree T will have more than two peripheral branches. This happens when there are more than two vertices in $\prod(G)$. Note, the number of peripheral branches cannot exceed the number of vertices in $\prod(G)$.

We will refer to a *v-reconstruction* of a graph G as a graph H where $V(G) = V(H)$, $G - v = H - v$ and G is hypomorphic to H. Since in a v-reconstruction of a graph G, any $v \in V(G)$ is also a $v \in V(H)$ then for a vertex that is in one of these sets while looking at a v-reconstruction of H, clearly $\deg_G(v) = \deg_H(v)$. We will also use the convention that the set of vertices in G which are adjacent to v, the *neighborhood of* v, will be denoted $N_G(v)$. Let Y_α denote a tree with has exactly one junction vertex, exactly three end vertices, and where the junction vertex is adjacent to two and only two of the end vertices. Also given any positive integer α the distance between the junction vertex and the non-adjacent end-vertex is α.

Looking at the set of vertices of $G, V(G)$, we will refer to $V_1(G)$ as the set of the end vertices of G. Evidently $V_1(G)$ is a subset of $V(G)$. Also, a vertex of a graph G is a *bad vertex*, v, if G has a vertex of degree $n = \deg(v) - 1$. Assuming that G and Q are graphs, we will let $s_Q(G, v)$ be the number of subgraphs of (G) which include the vertex v that are isomorphic to Q. We will also let $d_T(u, v)$ denote the distance in tree T between vertices u and v. Assuming that U is a non-empty subset of $V(T)$, let $d_T(v, U)$ denote the minimum $d_T(u, v)\ \forall u \in U$. If T is a tree with three or more vertices and (R, r) is a rooted tree, let $b_{(R,r)}(T)$ denote the number of branches of T that are isomorphic to (R, r). We say that a graph G is *z-reconstructible* if it has a vertex z such that every z-reconstruction of G is isomorphic to G.

Theorem 18. *1. If G and H are hypomorphic, then $|V(G)| = |V(H)|$.*

2. If G and H are hypomorphic and both have three or more vertices, then $|E(G)| = |E(H)|$.

Proof. 1. Since G and H are hypomorphic, then there exists an hypomorphism, σ, from G onto H. Then it follows that any hypomorphism of G onto H is a one to one, onto function from $V(G)$ to $V(H)$. Then it follows from the properties of bijections that $|V(G)| = |V(H)|$.

2. Let σ be a hypomorphism from G onto H. Then because each edge of G is in $|V(G)| - 2$ of the $G - v_i$ and each edge of H is in $|V(H)| - 2$ of the H_i and also $G - v \simeq H - \sigma(v)\ \forall v \in V(G)$, it follows that

$$\begin{aligned}
|E(G)|(|V(G)| - 2) &= \sum |E(G - v)|, \forall v \in V(G) \\
&= \sum |E(H - \sigma(v))|, \forall v \in V(G) \\
&= \sum |E(H - w)|, \forall w \in V(H) \\
&= |E(H)|(|V(H)| - 2)
\end{aligned}$$

Then it follows that $|E(H)| = |E(G)|$ because $|V(G)| \geq 3$ and $|V(H)| \geq 3$. □

Theorem 19. *If two graphs of order ≥ 3 are hypomorphic, then they are both connected or both disconnected.*

Proof. Let G and H be two graphs of order $n \geq 3$. Assume that σ is a hypomorphism from G onto H. We know from Corollary 1 that if G and H are connected then they both contain at least two vertices that are not cut vertices and that they both have at least two connected vertex-deleted subgraphs. If G and H are disconnected then they must have at most one connected vertex-deleted subgraph. Then because G and H have the same number of connected and disconnected vertex-deleted subgraphs, subsequently if G is connected, so is H and if G is disconnected then H must be also. □

Lemma 1. *Any graph hypomorphic to a tree with at least three vertices, is also a tree with at least three vertices.*

Proof. Assume that G is a graph which is hypomorphic to a tree T which has at least 3 vertices. By Theorem 18, the order of G is the same as the order of T. Let the order of G be n. Since T is a tree, T has $n-1$ edges. Additionally, because there is a hypomorphism between G and T, Theorem 18 also says that the number of edges in G and T is the same. Therefore, since T has $n-1$ edges it follows that G has $n-1$ edges as well. It also follows from Theorem 19 that G is connected. Thus by the definition of a tree, G is a tree. □

Lemma 2. *Every tree T with at least 3 vertices and at most one junction vertex is reconstructible.*

Proof. Let T be a tree with at least 3 vertices and at most one junction vertex. Let

G be a graph which is hypomorphic to T. By the definition of hypomorphic, there exists a hypomorphism σ of T onto G. So it follows from Lemma 1 that G is a tree.

Case 1

Assume T has no junction vertices. It follows that G also has no junction vertices. We can see that, by the definition of a junction vertex, it follows that T and G are both paths. Therefore, T is isomorphic to G and it then follows that G is the reconstruction of T.

Case 2

Assume T has one junction vertex. Assume that v is the junction vertex of T and by the definition of a hypomorphism, $\sigma(v)$ is the only junction vertex of G. Now, it follows that G is isomorphic to T and so G is the reconstruction of T. □

Lemma 3. *If G and H are hypomorphic graphs, $v \in V(G)$, and Q is a graph such that $|V(Q)| < |V(G)|$, then $s_Q(G, v) = s_Q(H, v)$.*

Proof. Since G and H are hypomorphic, then there exists a hypomorphism, σ, of G onto H. Since each subgraph of G which is isomorphic to Q is contained in $|V(G)| - |V(Q)|$ of our $G - v_i$ and each subgraph of H which is isomorphic to Q is contained in $|V(H)| - |V(Q)|$ of H_i and also $G - v \simeq H - \sigma(v)\ \forall v \in V(G)$, then it follows that

$$\begin{aligned} s_Q(G, v)(|V(G)| - |V(Q)|) &= \sum s_Q(G - v), \forall v \in V(G) \\ &= \sum s_Q(H - \sigma(v)), \forall v \in V(G) \\ &= \sum s_Q(H - w), \forall w \in V(H) \\ &= s_Q(H, v)(|V(H)| - |V(Q)|) \end{aligned}$$

So we can conclued that $s_Q(G, v) = s_Q(H, v)$ since $|V(Q)| - |V(H)| = |V(Q)| -$

$|V(G)|$ by Theorem 18. □

Theorem 20. *If σ is a hypomorphism of a graph G onto a graph H, where G and H both have at least three vertices, then it follows that $deg_G(v) = deg_H(\sigma(v))$ $\forall$ $v \in V(G)$.*

Proof. By Theorem 18 (ii) and the definition of hypomorphism, we can see that $|E(G)| = |E(H)|$ and $G - v \cong H - \sigma(v)$ $\forall$ $v \in V(G)$. Therefore

$$deg_G(v) = |E(G)| - |E(G-v)| = |E(H)| - |E(H - \sigma(v))| = deg_H(\sigma(v))$$

$$\forall v \in V(G).$$

□

Corollary 3. *If G and H are hypomorphic graphs with at least three vertices, then G and H have the same degree sequence.*

Proof. From Theorem 20 we can see that $deg_G(v) = deg_H(\sigma(v))$ for every $v \in V(G)$. Then from Theorem 18 we get $|V(G)| = |V(H)|$. So then it follows that G and H have the same degree sequence. □

Lemma 4. *If v is a vertex of a graph G and if H is a v-reconstruction of G and if Q is a graph such that $|V(Q)| < |V(G)|$, then $s_Q(G, v) = s_Q(H, v)$.*

Proof. Since $G - v \simeq H - v$, then by Kelly's Lemma, Lemma 3, we get that

$$\begin{aligned} s_Q(G,v) &= s_Q(G,v) - s_Q(G-v) \\ &= s_Q(H) - s_Q(H-v) \\ &= s_Q(H,v) \end{aligned}$$

□

Lemma 5. *Let z be a vertex of a graph G with at least three vertices. Let H be a z-reconstruction of G. Then for some non-negative integer m, there exists m*

distinct bad neighbors, $v_1, \cdots, v_m$*, of* z *in* G *and* m *distinct vertices,* $u_1, \cdots, u_m$ *of* $V(G) - N_G(z)$ *such that* $deg_G(u_i) = deg_G(v_i) - 1$*,* $\forall i = 1, \cdots, m$ *and* $N_H(z) \simeq (N_G(z) - \{v_1, \cdots, v_m\}) \bigcup \{u_1, \cdots, u_m\}$.

Proof. Since H is hypomorphic to G and $H - z \simeq G - z$, then it follows that $deg_G(z) = deg_H(z)$. For all nonnegative integers k the number of vertices in $N_H(z) - N_G(z)$ which have degree k in H must be equal to the number of vertices in $N_G(z) - N_H(z)$ which have degree k. So we write

$N_G(z) - N_H(z) = \{v_1, \cdots, v_m\}$

$N_H(z) - N_G(z) = \{u_1, \cdots, u_m\}$ with

$deg_G(v_i) = deg_H(w_i) = deg_G(w_i) + 1,\ i = 1, \cdots, m.$

We can see from this that $v_1, \cdots, v_m$ are bad in G and $u_1, \cdots, u_m$ are the vertices of G such that $deg_G(u_i) = deg_H(v_i) - 1$.

Then $N_H(z) = (N_G(z) - \{v_1, \cdots, v_m\}) \bigcup \{u_1, \cdots, u_m\}$. □

We should remember now that from the definition of a branch, we are guaranteed to have an assigned root for each branch no matter whether or not the tree itself is rooted.

Lemma 6. *[3] Let* T *be a tree with the following conditions,*

1. T *has exactly two branches and at least one of them is a rooted path*

   or

2. T *is a central tree and all of its peripheral branches are rooted paths.*

Then T *is reconstructible.*

Proof. From Lemma 2 we can assume that G has more than one junction vertex. Assume that the diameter of T is a. By the assumptions of the aforementioned lemma, T has a peripheral branch that is a rooted path of length ≥ 2. Let v be the peripheral vertex on that branch. Let also, w be the neighbor of v in T. Assume that S is a v-reconstruction of T. Then it remains be to proven that $T \simeq S$.

Since T has at least two junction vertices, clearly $deg\,(w) = 2$. Then by Lemma 5, either $N_S\,(v) = \{w\}$ or $N_S\,(v) = \{z\}$ for some $z \in V_1\,(T) - \{v\}$.

If $N_S\,(v) = \{w\}$ it is obvious that $S \simeq T$. Assume that $N_S\,(v) = \{z\}$ for some $z \in V_1\,(T) - \{v\}$. In this case, it is obvious that the set of junction vertices of T, $J\,(T)$ is the same as the set of junction vertices of S, $J\,(S)$. Thus S is a tree. Put the set J equal to $J\,(T)$. Then by Lemma 4, the smallest positive integer α such that v is in a subtree of T that is isomorphic to Y_α is the same as the smallest α such that v is in a subtree of S that is isomorphic to Y_α. In other words, $d_T\,(v, J) = d_S\,(v, J) = d_T\,(z, J) + 1$.

It also follows from the way that we defined v and the fact that the diameter of T is larger than both $D_T\,(v, z)$ and $d_T\,(v, J) \geq \frac{1}{2}a$, what follows is that the vz-path in T includes exactly one element in J, call it u. Then $d_T\,(u, v) = d_T\,(z, u) + 1$. Therefore, $S \simeq T$. □

As was previously stated, the proof that trees are reconstructible was the subject of Kelly's doctoral dissertation. For our purposes, we will simply be giving an outline of the proof that he presented at that time [3].

Theorem 21. *[3] Trees are reconstructible.*

Proof. Let T and S be hypomorphic trees of order greater than or equal to three. Then by Lemma 1, we need to prove that $S \simeq T$.

If Lemma 6 applies to either S or T, then we know that because they are hypomorphic and one of them is reconstructible, then $S \simeq T$. Assume that both S and T fail at least one of the hypotheses of Lemma 6.

Then by Lemma 3, $s_Q\,(S) = s_Q\,(T)$. Let l be a path of maximal length in

T. It follows that S must contain a path of length l and that it must be the longest path in S. This means that S and T have the same diameter, let $d = diam\,(S) = diam\,(T)$. Define a set $D\,(T)$ as the set of all the components which are created from T by deleting v for all $v \in T$ which have diameter d. Clearly, all the components that are in $D\,(T)$ are trees as well. We will let $D\,(S)$ represents the same thing for S. Since S and T are hypomorphic, every tree in $D\,(T)$ is isomorphic to a tree in $D\,(S)$ and the same for trees in $D\,(S)$ being isomorphic to trees in $D\,(T)$. After that if $b_{(R,r)}$ is the number of vertices in the largest branch in T and $b_{(R,r)}$ is also the number of vertices in the largest branch in S. Pick (R, r) a rooted tree of order $b_{(R,r)}$ where at least one subtree in $D\,(S)$ has a branch isomorphic to (R, r).

Clearly (R, r) is not a rooted path. Then we can choose a vertex, $v \in v_1\,(R) - \{r\}$ so that $d_R\,(r, w) = \frac{1}{2}a$ for some vertex $w \in V_1\,(M) - \{v\}$. Put $L = M - v$. Now pick an element $S - z$ of $D\,(S)$ such that

1. $b_{(R,r)}\,(S - z) \leq b_{(R,r)}\,(S - u)$, $\forall S - u \in D\,(S)$
2. $b_{(L,r)}\,(S - z) \leq b_{(L,r)}\,(S - u)$, $\forall S - u \in D\,(S)$ such that $b_{(R,r)}\,(S - z) = b_{(R,r)}\,(S - u)$.

Let $T - x \in D\,(T)$ such that $T - x \simeq S - u$. Since S does not satisfy Lemma 6, then we are assured that each branch of S is a member for $D\,(S)$. Also, because T does the same, we know that each branch of T is a member of $D\,(T)$. Then following the same idea that we used in the proof that disconnected graphs are reconstructible, we see that S is isomorphic to a tree obtained from the graph $S - u$ by replacing one of the branches that is isomorphic to (L, r) with one that is isomorphic to (R, r). If we do a similar style replacement with T, then combining that with the fact that $S - u \simeq T - x$ we can conclude that $S \simeq T$. Therefore, any two reconstructions are isomorphic and so trees are reconstructible. $\square$

CHAPTER VI
TOURNAMENTS

Since the Reconstruction Conjecture is not true for all graphs, the interest lies in discovering for each family of graphs whether it is true or false. In this section we will discuss the family of tournament graphs for which the Reconstruction conjecture is false.

Palmer and Harary were among the first to consider the Reconstruction Conjecture for tournaments. A tournament is a directed graph but is constructed from an undirected graph. We say that a *tournament* is a complete undirected graph which has had a direction assigned to all of its edges. Harary and Palmer quickly found counterexamples for tournaments that had 3 vertices and 4 vertices, which aroused interest in the question of for which graphs the Reconstruction Conjecture is true [11].

Shortly after they found these counterexamples, Harary and Palmer were able to prove that the reconstruction conjecture is true for tournaments that have order 5 or more, where the tournaments are not strongly connected. We consider a tournament T to be *strongly connected* if for all u and v in the tournament T, there is a $u - v$ path and a $v - u$ path in $E(T)$. Then, considering only tournaments that are not strongly connected, they were able to find counterexamples for the 5, 6, and 8 vertex cases. The conjecture was also proven true for all tournaments which have 7 vertices [11].

In a directed graph, each vertex has a certian number of edges which are directed into the vertex and a certian number of edges which are directed out of the vertex. We will let the number of edges that are directed into that vertex v be the *in-degree* of the vertex, denoted $in\,(v)$, and the number of edges that are directed out of that vertex be the *out-degree* of the vertex, denoted $out\,(v)$. The *score sequence* of a directed graph is the sequence of all of the out-degrees of the vertices of the graph. Harary and Palmer then proved that in a tournament of order greater than or equal to 5, the score sequence is completely determined by the score sequences of its subtournaments [11].

Before we can discuss why the failure of the Reconstruction Conjecture for tournaments, we need to introduce some definitions and theorems from Tournament Theory. A *tournament of order* p is a set of p vertices with exactly one directed edge joining each pair of distinct vertices. Additionally a vertex p_i *dominates* a vertex p_j if the edge that is joining p_i to p_j is directed from vertex p_i to vertex p_j, we denote this $p_i \rightarrow p_j$. Let $A = (V, E)$ denote a tournament with vertices from the set V and edges from the ordered set E. Then the *complement of* A, denoted A^c, is defined as $A^c = (V, E^c)$ where we say that $E^c = \{(u, v)\,|u, v \in V and\,(u, v) \notin E\}$. The *score* of a vertex p_i in the tournament A is the number of vertices $p_j \in A$ which p_i dominates. From here, it is convenient to define the *dominance matrix*, which is the $p \times p$ matrix $M_{p\times p}$, where the m_{ij} entry is 1 if p_i dominates p_j, and 0 otherwise. Clearly from this the diagonal of $M_{p\times p}$ is all zeros [11].

There are also some functions that are useful in our discussion of tournaments. For any nonzero integer k, we define the function $pow\,(k)$ as the largest integer i such that $2^i|k$ and $odd\,(k)$ is the quotient when k is divided by $2^{pow(k)}$. For example,

$$pow\,(192) = 6, odd\,(192) = 3.$$

Note that for any integer $k \neq 0$, $pow\,(k) \geq 0$ and $odd\,(k) \geq 1$ [3].

Now we want to define a specific family of tournaments. For all $n \in \mathbb{Z}^+$, we will let A_n denote the tournament that has $p = 2^n$; $V(A_n) = \{v_1, \cdots, v_p\}$, where $v_i \rightarrow v_j$ if and only if $odd\,(j-i) \equiv 1 \pmod 4$, for $i \neq j$.

Example. [11]

Using A_3, the dominance matrix is

$$\begin{bmatrix} 0 & 1 & 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \end{bmatrix}$$

□

There are a few facts that we need to recognize about the tournaments A_n.

Fact 1. *[11] The tournament A_n is its own complement.*

Proof. If we define a mapping of A_n onto A_n by $\phi(v_i) = v_{p+1-i}$, then ϕ is an isomorphism that reverses direction on all the edges of A_n. □

Fact 2. *[11] For a tournament A_n, and the vertices of $A_n = \{v_1, \cdots, v_p\}$ then the first 2^{n-1} vertices will have a score of 2^{n-1} and then the remaining vertices will have a score of $2^{n-1} - 1$.*

Proof. Fix i, $i = 1, \cdots, p$. We can pair up all of the vertices of A_n except for the vertices v_i and either $v_{i+\frac{p}{2}}$ if $i \leq 2^{n-1}$ or $v_{i-\frac{p}{2}}$ if $i \geq 2^{n-1}$. We pair them by creating a pair v_j, v_k where $k \equiv 2i - j \pmod p$. Now because v_i dominates v_l if and only if $odd\,(l-i) \equiv 1 \pmod 4$ then clearly v_i dominates exactly one of either v_j or v_k. It follows that the score of v_i is exactly $2^{n-1} - 1$. Also, for $i \leq 2^{n-1}$, we can see that the vertex v_i dominates $v_{i+\frac{p}{2}}$. □

Fact 3. *[11] The tournament A_n has only the identity automorphism.*

Proof by Induction. Start with $n = 1$. We can see that we are dealing with the graph A_1 which has exactly $2^1 = 2$ vertices. So by the definition of tournament, clearly there is only one edge. Now, either $v_1 \to v_2$ or $v_2 \to v_1$. By the definition of an automorphism, clearly we can only map v_1 to v_1 and likewise with v_2. So the only automorphism is the identity automorphism.

Now let $n > 1$. Assume that for A_n the only automorphism is the identity automorphism. We need to show that for A_{n+1}, the only automorphism is the identity automorphism. Now, clearly A_{n+1} has $2^n + 2^n$ vertices. Also, from Fact 2 we know that the first 2^n vertices have the same scores 2^{n-1} and the second 2^n vertices have the same scores $2^{n-1} - 1$. Since an automorphism must preserve score, we know that the first 2^n elements must permute within themselves. Because there are 2^n elements, these first elements produce a tournament T_1 that is isomorphic to A_n, so they must have only the identity automorphism. Similarly, the second 2^n elements produce a tournament T_2 that is also isomorphic to A_n so they also have only the identity automorphism. So $T_1 \bigcup T_2 \simeq A_{n+1}$ has only the identity automorphism.

So, by induction, for $n \in \mathbb{Z}$ we get that A_n has only the identity automorphism. □

Theorem 22. *[11] For all $k \in \mathbb{Z}$, such that $1 \leq k \leq 2^n$, the vertex-deleted subtournaments $A_n - v_k$ and $A_n - v_{p+1-k}$ are isomorphic and preserve direction.*

Proof. For each $i \in \mathbb{Z}$, $i \neq k$, we will let $p_i = pow\,(k - i)$ and we will let r_i be the remainder of i mod 2^{p_i+1}. We will define i' so that $i' \equiv i + 2^{p_i+1} + 1 - 2r_i \pmod{2^n}$. We claim that the mapping ϕ that sends v_i onto $v_{i'}$ is an isomorphism from $A_n - v_k$ to $A_n - v_{p+1-k}$.

Let i and j be $\in \mathbb{Z}$ such that $i \neq j \neq k$. Now we must consider two cases

Case 1

Assume that $pow\,(k-i) = pow\,(k-j)$. Clearly $r_i = r_j$. So we can conclude from this that $j' - i' = j - i$. We can see that v_i dominates v_j in $A_n - v_k$ if and only if $v_{i'}$ dominates $v_{j'}$ in $A_n - v_{p+1-k}$.

Case 2

Assume that $pow\,(k-i) \neq pow\,(k-j)$. Without loss of generality we can assume that $pow\,(k-i) > pow\,(k-j)$. Next it follows that $p_i > p_j$. We have that $pow\,(j-i) = pow\,(r_j - r_i) = p_j$. We can write $r_j - r_i = 2^{p_j}(1+2m)$ for $m \in \mathbb{Z}$. We obtain

$$\begin{aligned} j' - i' &= j - i + 2^{p_j+1} - 2^{p_i+1} - 2\,(r_j - r_i) \\ &= j - i - 2^{p_j+1}\left(2^{p_i-p_j} - 1\right) - 2^{p_j+1}(1+2m) \\ &= j - i - 2^{p_j+2}\left(2^{p_i-p_j-1} + m\right) \end{aligned}$$

So, when we combine this with the fact that $pow\,(j-i) = p_j$, we can conclude that $pow\,(j'-i) = p_j$ also. So we can divide through by 2^{p_j} giving the result

$$odd\,(j'-i') = odd\,(j-i) - 4\left(2^{p_i-p_j-1} + m\right)$$

which is the same as saying that

$$odd\,(j'-i') \equiv odd\,(j-i) \pmod{4},$$

So we can conclude that $v_i \rightarrow v_j$ if and only if $v_{i'} \rightarrow v_{j'}$.

Then the mapping ϕ that sends v_i onto $v_{i'}$ is an isomorphism of $A_n - v_k$ onto $A_n - v_{p+1-k}$ because a vertex v_i in A_n dominates another vertex v_j if and only if its mapping $v_{i'}$ dominates the mapping $v_{j'}$ of v_j in $A_n - v_{p+1-k}$. $\square$

Corollary 4. *[11] Each vertex-deleted subtournament of A_n is self complementary.*

Proof. Now, assuming the necessary conditions and then applying the above theorem, we know that $A_n - v_i \simeq A_n - v_{p+1-i}$ for all $1 \leq i \leq p$. From Fact 1 we can also get that the complement of $A_n - v_i$ is also isomorphic to $A_n - v_{p+1-i}$. So then putting the two together, we can conclude that $A_n - v_i$ is isomorphic to its complement. □

Initially, only finitely many counterexamples were known, and so the question remained open as to whether or not there was a certain number, call it m, such that for any tournament that has more than m vertices the Reconstruction Conjecture would be true. Then there was a proof found that an infinite family of tournaments was not reconstructible.

The first family of tournaments for which we will show the Reconstruction Conjecture is false are the tournaments of order $2^n + 1$. To show this family of tournaments is non-reconstructible, we need to define a few more terms. First, for each $n \in \mathbb{Z}^+$, we will define B_n as the tournament which is obtained from A_n by adding a vertex v_0, where v_0 dominates $v_2, v_4, \cdots, v_p$ and is dominated by $v_1, v_3, \cdots, v_{p-1}$. We also define C_n as the tournament obtained from A_n by adding in a vertex v_0, where v_0 dominates by $v_2, v_4, \cdots, v_p$ and dominates $v_1, v_3, \cdots, v_{p-1}$. We define a *transitive tournament* to be a tournament whose vertices can be indexed from $1, \cdots, n$ such that $v_i \rightarrow v_j$ if and only if $i > j$.[11]

Theorem 23. *[11] The tournaments B_n and C_n are not isomorphic.*

Proof by Induction. Considering the $n = 1$ case, we can see that B_1 is a transitive tournament with 3 vertices and C_1 is a cyclic tournament with three vertices. So clearly $B_1 \not\simeq C_1$.

Then in the $n = 2$ case, the unique vertex v_1 which has score 3 in B_2 is dominated by the unique vertex v_4 which has score 1, while in C_2 the unique vertex v_1 which has score 3 dominates the unique vertex v_3 of score 1. So since an isomorphism has to preserve the score sequence, we cannot have $B_2 \simeq C_2$.

Now let us look at $n \geq 3$. From Fact 2 we can see that the vertices of B_n which have score $2^{n-1} + 1$ are $v_1, v_3, v_{\frac{p}{2}-1}$. These vertices induce a subtournament of B_n which we will call T_1. Similarly we can see the vertices of C_n which have score $2^{n-1} + 1$ are $v_2, v_4, \cdots, v_{\frac{p}{2}}$ which generate a subtournament of C_n which we will call T_2. Then we can see that any isomorphism from B_n to C_n must be an extension of an isomorphism from T_1 to T_2. Now let $\phi\colon T_1 \rightarrow A_{n-2}$ be such that $\phi(v_i) = v_{i+1}$. Clearly, ϕ is an isomorphism of T_1 onto A_{n-2}. Similarly we will let $\sigma\colon T_2 \rightarrow A_{n-2}$ be such that $\sigma(v_i) = v_{\frac{i}{2}}$. Then just as before we can see that σ is an isomorphism of T_2 onto A_{n-2}.

From the properties of isomorphisms we can define a function $\phi^{-1}\sigma$ which is an isomorphism from T_1 onto T_2 such that $\phi^{-1}\sigma(v_i) = v_{i+1}$. By Fact 3, we can see that this is the only isomorphism that sends v_i to v_{i+1}. Hence we can see that any isomorphism that goes from B_n to C_n must take v_1 to v_2.

By a similar inspection of the vertices of B_n and C_n which have score $2^{n-1}-1$, we can see that any isomorphism from B_n to C_n must map v_p onto v_{p-1}. Then considering $v_p \rightarrow v_1$ in B_n and $v_2 \rightarrow v_{p-1}$ in C_n, we conclude that there can be no such isomorphism from B_n to C_n. Therefore, $B_n \not\simeq C_n$. $\square$

Theorem 24. *[11] The tournaments $B_n - v_0$ and $C_n - v_0$ are isomorphic and for $1 \leq k \leq p$, the tournaments $B_n - v_k$ and $C_n - v_{p+1-k}$ are isomorphic.*

Proof. It is clear that $B_n - v_0 \simeq C_n - v_0$ since they are both created by taking A_n and adding in a vertex and then some incident edges.

Then if we consider an isomorphism, ϕ, that goes from $A_n - v_k$ to $A_n - v_{p+1-k}$

we can obtain the second result of the theorem. First take the graph $A_n - v_k$ and add in another vertex v_0 in the same manner that was used to obtain the graph B_n. Similarly, take the graph $A_n - v_{p+1-k}$ and do the same with the method that was used to obtain the graph C_n. Then we can easily see that ϕ is an isomorphism from $B_n - v_k$ to $C_n - v_{p+1-k}$. □

So now because there is a hypomorphism between B_n and C_n, and we have proven that B_n is not isomorphic to C_n, then clearly the graphs B_n and C_n are tournaments which contradict the Reconstruction Conjecture.

Now we must look at another family of graphs which is not reconstructible. We shall start by defining some new tournaments which are needed. The tournament D_n shall be a tournament with $2^n + 2$ vertices formed by taking the tournament B_n and adding in a vertex v_{p+1} which dominates $v_1, v_3, \cdots, v_{p-1}$ and is dominated by $v_2, v_4, \cdots, v_p$ and v_0. Similarly we will define E_n as a tournament with $2^n + 2$ vertices; we acquire E_n from the tournament C_n by adding in a point v_{p+1} which dominates $v_2, v_4, \cdots, v_p$ and is dominated by $v_0, v_1, v_3, v_5 \cdots, v_{p-1}$.

Theorem 25. *[11] For $n > 1$, the tournaments D_n and E_n are not isomorphic.*

Proof. We use Fact 2 and consider the vertices that have score $2^{n-1} + 1$ in both D_n and E_n, specifically $v_0, v_2, \cdots, v_{\frac{p}{2}}$. These vertices in D_n generate a subtournament which is isomorphic to B_{n-1} and in E_n generate a subtournament which is isomorphic to C_{n-1}. By the previous logic we can see that any isomorphism between D_n and E_n must be an extension of an isomorphism between B_{n-1} and C_{n-1}. But by Theorem 23, there is no such isomorphism. From here we conclude that $D_n \not\cong E_n$. □

Theorem 26. *[11] For all $0 \leq k \leq p+1$, the tournaments $D_n - v_k$ and $E_n - v_{p+1-k}$ are isomorphic.*

Proof. Clearly, when $k = 0$ or $k = p + 1$ this theorem holds. So assume that $1 \leq k \leq p$. Then if we apply Theorem 22, we can easily see that the extension of this isomorphism which will map v_0 onto v_0 and v_{p+1} onto v_{p+1}, gives an isomorphism from $D_n - v_k$ to $E_n - v_{p+1-k}$. As a result, for all k such that $0 \leq k \leq p + 1$, we get that $D_n - v_k \simeq E_n - v_{p+1-k}$. □

So now we have shown two infinite families of tournaments for which the Reconstruction Conjecture is false.

CHAPTER VII

EDGE RECONSTRUCTION

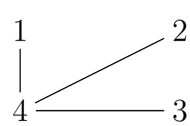
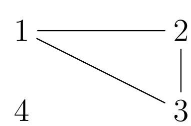
The primary resources for this chapter are [3] and [5] unless otherwise noted.

Edge reconstruction was a natural problem to consider in connection with the Reconstruction Conjecture. Many of the people who were working on the conjecture tried to prove many of the same things about edge-deleted graphs that they already had about vertex-deleted subgraph. As it turned out there were many types of edge-deleted subgraphs which could be proven reconstructible. We will discuss a few of the major results in this area.

In the process of dealing with edge reconstruction, we have defined some terminology that we need to be able to work with the edges. Let $E(G)$ be the set of all the edges which are in the original graph G. If $e_{i,j} \in E(G)$ then we will define the graph $G - e_{i,j}$ as an *edge-deleted subgraph of* G. Now define an *edge-hypomorphism* as a one-to-one function σ of a graph G onto a graph H such that $G - e_{i,j} \simeq H - \sigma(e_{i,j})$ for all $e_{i,j} \in E(G)$. The two graphs G and H are called *edge-hypomorphic* if such a σ exists. We also will say that a graph G is *edge-reconstructible* if for a each edge $(u, v) = e_{i,j}$, every edge reconstruction of $G - e_{i,j}$ is isomorphic to G. Note that in edge reconstruction, we only deal with a graph G which has $|E(G)| \geq 4$.

Example

Let H and G be the graphs illustrated below.

$G =$  $H =$ 

Then clearly if $E(G) = \{e_1, e_2, e_3\}$ and $E(H) = \{f_1, f_2., f_3\}$, $G - e_i$ is isomorphic with $H - f_i$ for $i = 1, 2, 3$. However, G and H are not isomorphic to each other.

Theorem 27 (Edge Reconstruction Conjecture)**.** *All simple graphs with more than four edges are edge reconstructible.*

In general, for most graphs G there are more edge-deleted subgraphs than there are vertex-deleted subgraphs. Therefore, we realize that having the edge-deleted subgraphs of a graph G for all edges $e \in E(G)$ has the possibility of retaining more information about the original graph G than the vertex-deleted subgraphs. Thus it seems possible that the edge reconstruction conjecture might be true.

For our purposes, we will let $Aut(G)$ represent the group of automorphisms of G and likewise, $Aut(H)$ be the group of automorphisms of H. This leads us to a very important preliminary theorem in edge reconstruction.

Theorem 28. *[3] Suppose that G is a spanning subgraph of the complete graph K_p that is not edge reconstructible. Then for every subset A of $E(G)$ such that $|A| \equiv |E(G)| \bmod 2$, there exists an automorphism ϕ of K_p such that $E(G \cap \phi(G)) = A$.*

Proof. If $E(G) = \emptyset$ then clearly $E(G \cap \phi(G)) = A$ for $|A| \equiv |E(G)| \bmod 2$. So we will assume that $E(G) \neq \emptyset$.

Since we are assuming that G is not edge reconstructible, then there exists a graph H such that G is edge hypomorphic to H but $G \not\simeq H$. However, since G and H are edge hypomorphic, we know that $|V(G)| = |V(H)| = p$. Then because we know that every graph of order p is isomorphic to a spanning graph of K_p, we can select such a spanning graph H' where $H \simeq H'$. So from here it follows that since $H \simeq H'$ and $H \not\simeq G$ then $H' \not\simeq G$.

Let σ be an edge-hypomorphism of G onto H, and let $|E(G)| = q$. Then, by the definition of an edge-hypomorphism and because σ is a bijection; we get that

$|E(H)| = q$. So, let P be a spanning subgraph of K_p such that $|E(P)| = q$ and let D be a subset of $E(P)$ where $|D| \equiv q \mod 2$.

Now, let $\mathcal{T}_G$ be the set of all ordered pairs (ϕ, S) such that ϕ is an automorphism of K_p, where S is a set of edges and $D \subseteq S \subseteq E(P \cap \phi(G))$. The fix S_0 where $D \subseteq S_0 \subseteq E(P)$. Then we will let $\mathcal{T}_G(-, S_0)$ be the set of all ordered pairs in $\mathcal{T}_G$ such that the second element is S_0. Additionally, let $\mathcal{T}_G(-, \equiv q)$ be the set of all ordered pairs in $\mathcal{T}_G$ where $|S| \equiv q \mod 2$, and let $\mathcal{T}_G(-, \not\equiv q)$ be the set of all ordered pairs in $\mathcal{T}_G$ where $|S| \not\equiv q \mod 2$.

For any fixed automorphism ϕ_o of K_p, we will let $\mathcal{T}_G(\phi_0, -)$ be the set of all ordered pairs belonging to $\mathcal{T}_G$ whose first component is ϕ_0. We will also let $\mathcal{T}(\phi_0, \equiv q)$ be the set of all the elements (ϕ_0, S) of $\mathcal{T}_G(\phi_0, -)$ such that $|S| \equiv q \mod 2$, let $\mathcal{T}_G(\phi_0, \not\equiv q)$ be the set of all the elements (ϕ_0, S) of $\mathcal{T}_G(\phi_0, -)$ such that $|S| \not\equiv q \mod 2$.

In a similar fashion we will let $\mathcal{T}_H$ be the set of all ordered pairs (ϕ, S) such that ϕ is an automorphism of K_p, where S is a set of edges and $D \subseteq S \subseteq E(P \cap \phi(H))$. Now, for our fixed S_0 where $D \subseteq S_0 \subseteq E(P)$. Then we will let $\mathcal{T}_H(-, S_0)$ be the set of all ordered pairs in $\mathcal{T}_H$ such that the second element is S_0, let $\mathcal{T}_H(-, \equiv q)$ be the set of all ordered pairs in $\mathcal{T}_H$ where $|S| \equiv q \mod 2$, and let $\mathcal{T}_H(-, \not\equiv q)$ be the set of all ordered pairs in $\mathcal{T}_H$ where $|S| \not\equiv q \mod 2$.

For any fixed automorphism ϕ_o of K_p, we will let $\mathcal{T}_H(\phi_0, -)$ be the set of all ordered pairs belonging to $\mathcal{T}_H$ whose first component is ϕ_0. We will also let $\mathcal{T}(\phi_0, \equiv q)$ be the set of all the elements (ϕ_0, S) of $\mathcal{T}_H(\phi_0, -)$ such that $|S| \equiv q \mod 2$, let $\mathcal{T}_H(\phi_0, \not\equiv q)$ be the set of all the elements (ϕ_0, S) of $\mathcal{T}_H(\phi_0, -)$ such that $|S| \not\equiv q \mod 2$.

Now, let $\mu(G, P)$ denote the number of automorphisms ϕ of K_p such that $\phi(G) = P$ and also let $v(G, P, D)$ denote the number of automorphisms ϕ of K_p

such that $E(P \cap \phi(G)) = D$. Similarly we will let $\mu(H, P)$ denote the number of automorphisms ϕ of K_p such that $\phi(H) = P$ and also let $v(H, P, D)$ denote the number of automorphisms ϕ of K_p such that $E(P \cap \phi(H)) = D$.
Consider a fixed set S_0 where

$$D \subseteq S_0 \subset E(P)$$

For a spanning subgraph J of K_p, we will define $\gamma(J)$ as the number of automorphisms ψ of K_p such that $\psi(S_0) \subseteq E(J)$. Given any automorphism ϕ of K_p, $(\phi, S_0) \in \mathcal{T}_G$ if and only if $D \subseteq S_0 \subseteq E(P \cap \phi(G))$, which, by the previous consideration on S_0 happens if and only if $S_0 \subseteq E(\phi(G))$. Another way of writing this is that $D \subseteq S_0 \subseteq E(P \cap \phi(G))$ if and only if $\phi^{-1}(S_0) \subseteq E(G)$. So from here it follows that $|\mathcal{T}_G(-, S_0)|$ is the same as the number of automorphisms ϕ of K_p where $\phi^{-1}(S_0) \subseteq E(G)$. So from here we can conclude that

$$|\mathcal{T}_G(-, S_0)| = \gamma(G).$$

After following the similar computations for $\mathcal{T}_H$ we can reach the conclusion that

$$|\mathcal{T}_H(-, S_0)| = \gamma(H).$$

Given that $\psi \in AutK_p$, and $\psi(S_0) \subseteq E(G)$ then we can conclude that $\psi(S_0) \subseteq E(G - e)$ for precisely $q - |S_0|$ edges, $e_{i,j}$, of G. From here we can obtain the equation

$$\gamma(G) = \frac{1}{q - |S_0|} \sum_{e \in E(G)} \gamma(G - e).$$

Similarly, given that $\psi \in AutK_p$, and $\psi(S_0) \subseteq E(H)$ then we can conclude that $\psi(S_0) \subseteq E(H - f)$ for precisely $q - |S_0|$ edges, f, of H. From here we can obtain the equation

$$\gamma(H) = \frac{1}{q - |S_0|} \sum_{f \in E(H)} \gamma(H - f).$$

Now, assuming that J and J' are isomorphic spanning subgraphs of K_p then we see that if $|J| = q_J$ and $|J'| = q_{J'}$, then clearly $q_J = q_{J'}$, let this number be denoted by q'. Now we can see that for all $e' \in E(J)$ then there exists an $f' \in E(J')$ such that $(J - e') \simeq (J' - f')$. Now from the definition of $\gamma(J)$ and $\gamma(J')$ we can easily see that

$$\sum_{e' \in E(J)} \gamma(J - e') = \sum_{f' \in E(J')} \gamma(J' - f').$$

So now one can clearly see that

$$\begin{aligned} \gamma(J) &= \frac{1}{q' - |S_0|} \sum_{e' \in E(J)} \gamma(J - e') \\ &= \frac{1}{q - |S_0|} \sum_{f' \in E(J')} \gamma(J' - f') \\ &= \gamma(J') \end{aligned}$$

Therefore, $\gamma(G - e) \simeq \gamma(H - \sigma(e))$ for all $e \in E(G)$, then

$$\sum_{e \in E(G)} \gamma(G - e) = \sum_{e \in E(G)} \gamma(H - \sigma(e)) = \sum_{f \in E(H)} \gamma(H - f).$$

Then by combining all of the above we get

$$|\mathcal{T}_G(-, S_0)| = |\mathcal{T}_H(-, S_0)|.$$

It should be noted that the above equation is no longer obtainable if $S_0 = E(P)$ since in the proof we divided by $q - |S_0| = |E(P)| - |S_0|$. Now, since we know that $|E(P)| = q$, then we can obtain

$$\sum_{S_i} |\mathcal{T}_G(-, S_i)| = \sum_{S_i} |\mathcal{T}_H(-, S_i)|, \forall S_i$$

such that

$$D \subseteq S_i \subseteq E(P)$$

and

$$|S_i| \not\equiv q \pmod 2$$

Which results in

$$|\mathcal{T}_G(-, \not\equiv q)| = |\mathcal{T}_H(-, \not\equiv q)|.$$

We can also get

$$\sum_{S_i} |\mathcal{T}_G(-, S_i)| = \sum_{S_i} |\mathcal{T}_H(-, S_i)|, \forall S_i$$

such that

$$D \subseteq S_i \subseteq E(P)$$

and

$$|S_i| \equiv q \pmod 2$$

Then the result is

$$|\mathcal{T}_G(-, \equiv q)| - |\mathcal{T}_G(-, E(P))| = |\mathcal{T}_H(-, \equiv q)| - |\mathcal{T}_H(-, E(P))|.$$

Then for any automorphism ϕ of K_p, we get that

$(\phi, E(P)) \in \mathcal{T}$ if and only if $D \subseteq E(P) \subseteq E(P \cap \phi(G))$.

and we also know that

$E(P) \subseteq E(P \cap \phi(G))$ if and only if $P = \phi(G)$.

Now since we know that $D \subseteq E(P)$ and $|E(P)| = |E(G)| = q$, then we can see that

$|\mathcal{T}_G(-, E(P))| = \mu(G, P)$ and $|\mathcal{T}_H(-, E(P))| = \mu(H, P)$.

So then we can rewrite

$$|\mathcal{T}_G(-, \equiv q)| - |\mathcal{T}_G(-, E(P))| = |\mathcal{T}_H(-, \equiv q)| - |\mathcal{T}_H(-, E(P))|$$

as

$$|\mathcal{T}_G(-, \equiv q)| - \mu(G, P) = |\mathcal{T}_H(-, \equiv q)| - \mu(H, P).$$

Now, if we look at a fixed automorphism ϕ_0 of K_p such that $E(P \cap \phi_0(G)) = D$, then one can see that for all S such that $D \subseteq S \subseteq E(P \cap \phi_0(G))$, if we let $||S||_O$ denote the number of sets S such that $|S|$ is odd and similarly let $||S||_E$ denote the number of sets S such that $|S|$ is even, then

$$||S||_O = ||S||_E.$$

It follows that the number of sets $||S||_E$ such that $(\phi_0, S) \in \mathcal{T}_G$ is the same as the number of sets $||S||_O$ such that $(\phi_0, S) \in \mathcal{T}_G$. Then we can easily conclude that

$$|\mathcal{T}_G(\phi_0, \equiv q)| = |\mathcal{T}_G(\phi_0, \not\equiv q)|.$$

For the same fixed automorphism ϕ_0 of K_p such that $E(P \cap \phi_0(H)) = D$, then one can see that for all S such that $D \subseteq S \subseteq E(H \cap \phi_0(H))$ gives the result that $||S||_O = ||S||_E$.

We can see that the number of sets $||S||_E$ such that $(\phi_0, S) \in \mathcal{T}_H$ is the same as the number of sets $||S||_O$ such that $(\phi_0, S) \in \mathcal{T}_H$. Then we can conclude that $|\mathcal{T}_H(\phi_0, \equiv q)| = |\mathcal{T}_H(\phi_0, \not\equiv q)|$.

Assume that $E\left(P\cap\phi_0\left(G\right)\right)=D$. Then it follows that D is the only set S where $D\subseteq S\subseteq E\left(P\cap\phi_0\left(G\right)\right)$, and then $\left(\phi_0,D\right)$ is the only element in $\mathcal{T}_G\left(\phi_0,-\right)$. Now because $|D|\equiv q\,(\text{mod }2)$ we see that

$\left|\mathcal{T}_G\left(\phi_0,\equiv q\right)\right|=1$ and $\left|\mathcal{T}_G\left(\phi_0,\not\equiv q\right)\right|=0$.

Using the same technique, when we assume that $E\left(P\cap\phi_0\left(H\right)\right)=D$, then it follows that D is the only set S where $D\subseteq S\subseteq E\left(P\cap\phi_0\left(H\right)\right)$, and then $\left(\phi_0,D\right)$ is the only element in $\mathcal{T}_H\left(\phi_0,-\right)$. Now because $|D|\equiv q\,(\text{mod }2)$ we see that

$\left|\mathcal{T}_H\left(\phi_0,\equiv q\right)\right|=1$ and $\left|\mathcal{T}_H\left(\phi_0,\not\equiv q\right)\right|=0$.

Then for the $v\left(G,P,D\right)$ automorphisms we see that

$\left|\mathcal{T}_G\left(\phi_0,\equiv q\right)\right|=1$ and $\left|\mathcal{T}_G\left(\phi_0,\not\equiv q\right)\right|=0$

holds from the previous definition of $v\left(G,P,D\right)$. Also for the other automorphisms ϕ_0 of K_p, $\left|\mathcal{T}_G\left(\phi_0,\equiv q\right)\right|=\left|\mathcal{T}_G\left(\phi_0,\not\equiv q\right)\right|$ holds. Then it follows that

$$\sum_{\phi_0\in AutK_p}\left|\mathcal{T}_G\left(\phi_0,\equiv q\right)\right|=v\left(G,P,D\right)+\sum_{\phi_0\in AutK_p}\left|\mathcal{T}_G\left(\phi_0,\not\equiv q\right)\right|.$$

Similarly for the $v\left(H,P,D\right)$ automorphisms we see that $\left|\mathcal{T}_H\left(\phi_0,\equiv q\right)\right|=1$ and $\left|\mathcal{T}_H\left(\phi_0,\not\equiv q\right)\right|=0$ holds from the previous definition of $v\left(H,P,D\right)$. Also for the other automorphisms ϕ_0 of K_p, $\left|\mathcal{T}_H\left(\phi_0,\equiv q\right)\right|=\left|\mathcal{T}_H\left(\phi_0,\not\equiv q\right)\right|$ holds. Then it follows that

$$\sum_{\phi_0\in AutK_p}\left|\mathcal{T}_H\left(\phi_0,\equiv q\right)\right|=v\left(H,P,D\right)+\sum_{\phi_0\in AutK_p}\left|\mathcal{T}_H\left(\phi_0,\not\equiv q\right)\right|.$$

So then because

$\sum_{\phi_0\in AutK_p}\left|\mathcal{T}_G\left(\phi_0,\equiv q\right)\right|=\left|\mathcal{T}_G\left(-,\equiv q\right)\right|$ and

$\sum_{\phi_0\in AutK_p}\left|\mathcal{T}_H\left(\phi_0,\equiv q\right)\right|=\left|\mathcal{T}_H\left(-,\equiv q\right)\right|$

We are able to obtain the two equations

$$|\mathcal{T}_G(-, \equiv q)| = v(G, P, D) + \sum_{\phi_0 \in Aut K_p} |\mathcal{T}_G(\phi_0, \not\equiv q)|$$

and

$$|\mathcal{T}_H(-, \equiv q)| = v(H, P, D) + \sum_{\phi_0 \in Aut K_p} |\mathcal{T}_H(\phi_0, \not\equiv q)|.$$

By combining the above equations with the equation

$|\mathcal{T}_G(-, \equiv q)| - \mu(G, P) = |\mathcal{T}_H(-, \equiv q)| - \mu(H, P)$, we are able to deduce

$$v(G, P, D) - v(H, P, D) = \mu(G, P) - \mu(H, P).$$

Thus we can conclude that for any spanning subgraph P of K_p with q edges, any subset D of $E(P)$ where $D \equiv q \pmod 2$ if A is any subset of $E(G)$ such that $|A| \equiv q \pmod 2$, then when we take $P = G$ and $D = A$ we get that

$$v(G, G, A) - v(H, G, A) = \mu(G, G) - \mu(H, G).$$

Now because $\mu(G, G) > 0$ and $\phi(G) \simeq G$ for ϕ the identity automorphism of K_p. Also, since $G \not\cong H$ we get that $\mu(G, H) = 0$. As a result, we are able to simplify the above equation to $v(G, G, A) - v(H, G, A) = \mu(G, G) > 0$.

In conclusion we get that

$$v(G, G, A) > 0.$$

So then we can clearly see that there is at least one automorphism ϕ of K_p such that $E(G \cap \phi(G)) = A$. □

Now that we have this result, we are able to show that a large quantity of graphs are edge reconstructible.

Theorem 29. *A graph G is edge reconstructible if* $2^{|E(G)|-1} > |V(G)|!$.

Proof. Let $|V(G)| = p$ and $|E(G)| = q$. Also, let K_p be the complete graph such that G is a spanning subgraph of K_p. Assume that $2^{q-1} > p!$. So $q > 0$ and there are 2^q subsets of $E(G)$. Since each of these subsets must be of either even or odd order, we can conclude that there are 2^{q-1} subsets of $E(G)$ whose order is congruent to q mod 2. Due to the nature of K_p, we see that there are $p!$ automorphisms of K_p. These automorphisms are in one-to-one correspondence with the permutations of $V(K_p)$. Since $p! < 2^{q-1}$, it cannot possibly be that for every subset of $E(G)$, with the order of the subset congruent to q mod 2, there exists an automorphism ϕ of K_p such that $E(G \cap \phi(G)) = A$. By Theorem 28, G is edge reconstructible. □

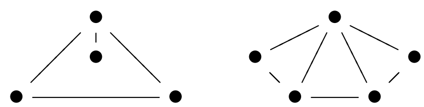
Theorem 30. *[3] A graph G with p vertices and q edges is edge reconstructible if* $q > \dfrac{(p \log p)}{(\log 2)}$.

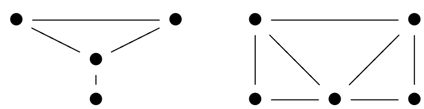
Proof. Assume that $q > \dfrac{(p \log p)}{(\log 2)} \geq 0$, so the graph G has at least one edge. So it follows that $p \geq 2$. Since $q > \dfrac{(p \log p)}{(\log 2)} \Rightarrow q \log 2 > p \log p$, it is evident that $2^q > p^p \geq 2(p!)$. So we are able to conclude that $2^{q-1} > (p!)$. Then by Theorem 29 it follows that G is edge reconstructible. □

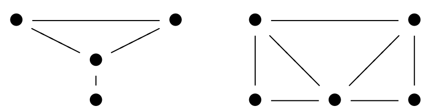
Since clearly for p large enough, all graphs with p vertices will have more than $\dfrac{(p \log p)}{(\log 2)}$ edges since the maximum number edges of a graph with p vertices is $\frac{1}{2}p(p-1)$. So by Theorem 30 we have proven that the Edge Reconstruction Conjecture is true for almost all simple graphs.

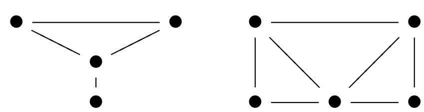
CHAPTER VIII

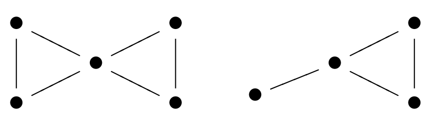
SAMPLE RECONSTRUCTION

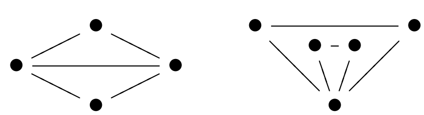
$G_1 =$ 

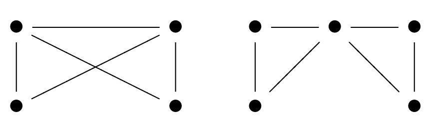
$G_2 =$ 

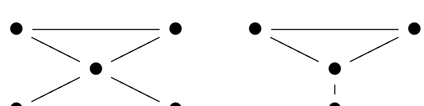
$G_3 =$ 

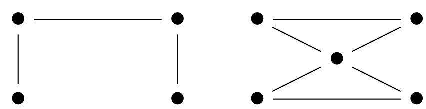
$G_4 =$ 

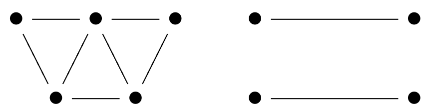
$G_5 =$ 

$G_6 =$ 

$G_7 =$ 

$G_8 =$ 

$G_9 =$ 

$G_{10} =$ 

First we can easily see that n, the number of vertices, for our original G is 10.

Using the formulas that we have previously introduced, we can compute q. We know that

$$q = \sum \frac{\mathrm{q_i}}{\mathrm{n-2}}.$$

Thus, we can easily see that $q_1 = 11, q_2 = 11, q_3 = 11, q_4 = 11, q_5 = 10, q_6 = 11, q_7 = 11, q_8 = 10, q_9 = 9, q_{10} = 9$.

So from that we get

$$q = \frac{11 + 11 + 11 + 11 + 10 + 11 + 11 + 10 + 9 + 9}{10 - 2}.$$

$$q = \frac{104}{8} = 13.$$

Another formula that we have previously introduced will give us the degree of each v_i, $deg v_i = q - q_i$.

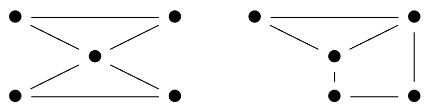
So, $deg\ v_1 = q - q_1$, $deg\ v_2 = q - q_2$, $deg\ v_3 = q - q_3$, $deg\ v_4 = q - q_4$, $deg\ v_5 = q - q_5$, $deg\ v_6 = q - q_6$, $deg\ v_7 = q - q_7$, $deg\ v_8 = q - q_8$, $deg\ v_9 = q - q_9$, $deg\ v_{10} = q - q_{10}$.

And, $deg\ v_1 = 13 - 11$, $deg\ v_2 = 13 - 11$, $deg\ v_3 = 13 - 11$, $deg\ v_4 = 13 - 11$, $deg\ v_5 = 13 - 10$, $deg\ v_6 = 13 - 11$, $deg\ v_7 = 13 - 11$, $deg\ v_8 = 13 - 10$, $deg\ v_9 = 13 - 9$, $deg\ v_{10} = 13 - 9$.

deg $v_1 = 2$, *deg* $v_2 = 2$, *deg* $v_3 = 2$, *deg* $v_4 = 2$, *deg* $v_5 = 3$, *deg* $v_6 = 2$, *deg* $v_7 = 2$, *deg* $v_8 = 3$, *deg* $v_9 = 4$, *deg* $v_{10} = 4$.

We know that the original graph, G, is disconnected because each of the $G - v_i$ subgraphs are disconnected.

Then because G is disconnected, the unique reconstruction of G is below. This reconstruction is unique up to isomorphism.

$G =$ 

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