VERNER, JONATHAN, M.A. The 17^{th} Topological Type of ω^*. (2006)
Directed by Dr. Jerry Vaughan. 35pp.

It has long been known, that $\omega^* = \beta\omega \setminus \omega$ is not homogeneous, that is, that there are at least two topologically different points. In fact, it was proved by Frolík in [Fro67a] that the Čech-Stone growth of any non-pseudocompact space X is not homogeneous. Frolík's result, which was "non-constructive" in nature, prompted interest in finding specific, topological reasons for the nonhomogeneity of the Čech-Stone growths of non-pseudocompact spaces in general and ω^* in particular.

To this day, there have been discovered a total of 16 mutually exclusive topologically described classes of points, called topological types, in ω^*, such as Kunen's weak P-points. We investigate along these lines, defining another topological type called a uniquely ω-accessible point. Such a point is known to exist in ω^* under MA and in this work we investigate a method possibly leading to a proof in ZFC. The main result of this thesis is the construction of two irresolvable spaces one with a remote point and the other with a weak P-point. We also present a construction of irresolvable spaces, which is similar to [Hew43] but is used in the context of Boolean algebras.

THE 17^{TH} TOPOLOGICAL TYPE OF ω^*

by

Jonathan Verner

A Thesis Submitted to
the Faculty of The Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
2006

Approved by

____

Dr. Jerry Vaughan

APPROVAL PAGE

This thesis has been approved by the following committee of the Faculty of The Graduate School at The University of North Carolina at Greensboro.

Committee Chair ________________________________

Committee Members ________________________________

____

____

Date of Acceptance by Committee

____

Date of Final Oral Examination

ACKNOWLEDGMENTS

I would like to thank Jan Rychtář for the invitation to UNCG and Jerry Vaughan for his guidance during the preparation of this thesis.

I would also like to thank Petr Simon of Charles University, Prague for introducing me into set-theoretic topology as well as for his advice, help and guidance throughout my study of ω^*. The committee members who have expended significant efforts to review this work also deserve special mention. The comments of Jan Rychtář had helped me to particularly improve the manuscript.

TABLE OF CONTENTS

INTRODUCTION

The starting point for the study of the nonhomegeneity of ω^* was W. Rudin's proof ([Rud56]), that, under CH, there are P-points in ω^*. Since there are obviously points in ω^* which are not P-points, this shows that, supposing CH, ω^* is not homogeneous. The continuum hypothesis in his proof can be weakened to Martin's axiom but, by a deep and hard result of Shelah ([Wim82]), it is consistent with ZFC that there are no P-points in ω^*. In 1967 Frolík gave a surprising answer to the question of whether ω^* is or is not homogeneous ([Fro67a], [Fro67b]). It is not; in fact, there are $2^{\mathfrak{c}}$ pairwise "topologically different" points (i.e. there is no homeomorphism taking one to another) in ω^*. (In his paper he shows that X^* is not homogeneous for any non-pseudocompact space X.) The problem with his proof was that it was based on cardinality arguments and did not yield a "topological" description of even two different points. A next step forward was Kunen's proof of the existence of weak P-points in ZFC ([Kun78]). He proved that there are points in ω^*, which are not limit points of any countable set. Obviously not every point of ω^* is a weak P-point, so this also gives a proof of the nonhomegeneity of ω^*. And it actually shows two concrete topologically distinct points (a weak P-point and a non-weak P-point) attesting to the nonhomegeneity, so it is an "effective" proof in the sense of van Douwen [vD81], because it provides a topological property which one class of points has and another does not. The next result and a huge step forward was van Mill's description of sixteen distinct topological properties of points in ω^* ([vM82]). We continue this line of developmentent by looking for a seventeenth property — topological type.

There are, a priori, two approaches to finding a specific point in ω^*. One can use transfinite induction to construct the point, at each step ensuring the necessary properties. This process is usually aided by some independent matrix. The other way is to find a space with the needed point and embed it in an appropriate way into ω^*. We have adopted the second approach, which seems to uncover the nature of the problem while not going into all the details of finding the right matrix. We will still need some matrix but, presumably, a much simpler one. The following definition states what type of points we are looking for.

Definition 0.1. A point is *uniquely ω-accessible* in a space X if it is in the closure of a countable set, not in the closure of a discrete set, and any two countable sets, whose limit point it is, intersect.

The definition says, that there is, really, only a "single" countable set, whose closure contains the point (i.e. the countable sets form a filter base). See [VD93] for a similar notion of accessibility, but unlike in that paper we include the requirement that the point is not in the closure of a discrete set, since the existence of such a point was proven by Van Mill ([vM82]). When trying the first approach of constructing the point by induction, it is soon seen, that one needs to evade certain "parts" of ω^*. For example one certainly wants to evade a subspace, which has two countable disjoint dense sets. Hence the following definition is relevant:

Definition 0.2 ([Hew43]). A space is *irresolvable* if any two dense sets intersect.

If we have a uniquely ω-accessible point and if we take a countable set, whose limit point it is, with the subspace topology, we indeed get an irresolvable space. For our purposes we will need a slightly weaker condition which will only talk about countable dense sets. For the respective definitions see chapter 5.

The plan is to find a suitable irresolvable space, whose compactification contains a uniquely ω-accessible point and which can be embedded into ω^* without losing this point. The second part can be guaranteed under certain additional conditions on the space (extremal disconnectedness, π-weight $\leq \mathfrak{c}$) and we will deal with it in the second chapter. One way to get a uniquely ω-accessible point in an irresolvable space is to find in it a remote, weak P-point. Then this point cannot be accessed by a countable set from the growth (since it is a weak P-point) and cannot be in the closure of a nowhere dense set in X. Thus if we can show, that any two sets, dense in a given open set, must intersect, we are almost done. The following definition gives the additional conditions:

Definition 0.3 ([Hew43]). A space is *open hereditarily irresolvable* (OHI for short) if any two sets, which are dense in the same open set, intersect.

In the third chapter, we will look more closely at weak P-points, giving first some existence theorems and then constructing a suitable space with a weak P-point. The fourth chapter will concentrate on remote points and will be structured in a roughly similar way to the third chapter. In the fifth chapter we will investigate irresolvable spaces and maximal topologies in the setting of Boolean algebras. It will provide us theorems which, starting from a certain space, give us a finer OHI topology on the space which nevertheless preserves the properties we are interested in. In the final chapter we give a summary of the results and also a plan of how the construction of a uniquely ω-accessible point would go through. The main results of this thesis are Theorem 4.5 and Theorem 3.11.

CHAPTER I
BASIC DEFINITIONS

We assume the reader is familiar with basic topological and set-theoretic concepts. In this chapter we list for reference some of the ones we will be using. We also give some basic facts about the Čech-Stone compactification and introduce some elementary definitions and theorems from Boolean algebras. In the last part we give several simple topological lemmas and also relate the properties of Boolean algebras to those of topological spaces. For further details, see [Eng] for unexplained topological terms, [Jech] for set-theoretic notions and [HBA] for Boolean algebraic ones. Also we do not include most proofs since they are elementary and may be found in, e.g., [Eng] or [HBA].

1.1 Set Theory & Topology

First we introduce notation. The Greek letters κ, λ, θ will denote infinite cardinal numbers, α, β ordinal numbers, k, n, m, i, j natural numbers. The first infinite cardinal will be denoted by ω and $\mathfrak{c}$ will be the cardinality of the powerset of ω. For two sets X, Y their *symmetric difference* is denoted by $X \Delta Y = (X \setminus Y) \cup (Y \setminus X)$. The symbol $\mathcal{FR}(X)$ will stand for the generalized Fréchet filter on X, that is $\mathcal{FR}(X) = \{F \subseteq X : |X \setminus F| < |X|\}$. A *filter base* for a filter $\mathcal{F}$ is a system of sets from the filter such that any set in the filter contains a set from the filter base. The character of a filter (denoted by $\chi(\mathcal{F})$) is the minimal cardinality of a filter base for $\mathcal{F}$. If X is a set let its powerset be denoted by $\mathcal{P}(X)$. The symbols $[X]^\kappa, [X]^{<\kappa}$ shall denote the set of all subsets of X of cardinality κ and less than κ

respectively. ${}^{X}Y$ shall denote the set of all functions from X to Y. The cardinality of a set X shall be denoted by $|X|$ and 2^X shall be the cardinality of ${}^{X}2$. We shall say that a system of sets is *centered* (or, equivalently, has the *finite intersection property*), if any finite subsystem has nonempty intersection.

Turning to topology, we note, that all topological spaces we will consider will be (at least) Hausdorff (i.e. T_2). Other separation properties we will use are T_0, T_1, regularity (T_3) and complete regularity ($T_{3\frac{1}{2}}$). Note that $T_{3\frac{1}{2}} \Rightarrow T_3 \Rightarrow T_2 \Rightarrow T_1 \Rightarrow T_0$. We will also need the notion of a *regular open* set, i.e. a set, which is equal to the interior of its closure. The following fact will be useful:

Fact 1.1. *In an infinite space without isolated points a regular open set is either infinite or empty, as is its complement.*

Among the separation properties we may also count *total disconnectedness* (i.e. two points can be separated by closed-and-open (*clopen* for short) sets), *zero-dimensionality* (that is, the space has a base consisting of clopen sets) and *extremal disconnectedness* (i.e., the closure of any open set is open). Note that any discrete space has all the listed separation properties. A further notion, which proves to be useful, is *extremal disconnectedness at a point.* We say that a space X is *extremally disconnected* at $p \in X$ if p is not in the closure of two disjoint open sets. Let us note a simple lemma:

Lemma 1.2. *A T_0 zero-dimensional space satisfies $T_{3\frac{1}{2}}$ and hence has a base consisting of regular open sets.*

For a topological space X, denote by $\tau(X)$ the topology of X. Let $\overline{A}^X$ be the closure of A in X and, if X is clear from the context, we will drop it. $Clopen(X)$ is the set of closed and open sets of X. A subset of a topological space is *dense* if its closure is the whole space or, equivalently, if it meets any nonempty open set.

It is called *nowhere dense* (n.w.d. for short) if its closure has empty interior (or, equivalently, if the complement of its closure is dense), and it is called *somewhere dense* otherwise.

Lemma 1.3. *The nowhere dense sets in a space form an ideal.*

The *weight* of a space (denoted by $w(X)$) is the minimal cardinality of a *base* for the space (i.e., a system of open sets of X such that any open set is a union of sets from the system). A *π-base* for a space is a family of nonempty open sets such that any nonempty open set of the space contains a set from the π-base. The π-character of a space (denoted $\pi\chi(X)$) is the minimal cardinality of a π-base. A *local base* at a point x is a system of open sets containing x such that any open set containing x contains a set from the local base. A *local π-base* at x is a system of nonempty open sets such that any open neighborhood of x contains a set from the π-base. Define the character and π-character of a point $x \in X$ as the minimal cardinality of a base and π-base respectively at x.

A space is *compact* if any cover of the space by open sets contains a finite subcover or, equivalently, if any centered system of closed sets has nonempty intersection. It is *locally compact* if any point has an (open) neighborhood with compact closure and it is *nowhere locally compact* if the closure of any nonempty open set is noncompact. Note that a subset of a compact, T_2 space is compact iff it is closed and any compact subset of a T_2 space is closed in this space.

A *homeomorphism* between two topological spaces is a continuous bijection which has a continuous inverse. A continuous map (function) is *open*, if the images of open sets are open. It is *quasiopen*, if the images of nonempty open sets have nonempty interiors. It *closed* if the images of closed sets are closed and it is *irreducible*, if the image of a proper closed subspace of the domain is never onto. A closed map is *perfect* if the preimages of points are compact. For a space X we say that EX is its *projective cover* iff it is extremally disconnected and admits an irreducible perfect map onto X. EX (sometimes called the *absolute* of X) can be shown to exist for any completely regular space X.

A topological space is *homogeneous* if, for any two points x, y, there is a homeomorphism $f_{x,y}$ from the space onto itself such that $f(x) = y$. A *topological type* is a ("topologically defined") class of points of a topological space, such that no point outside of this class can be mapped to a point inside it via a homeomorphism.

1.2 Čech-Stone compactification

For any completely regular space X there is a compact space βX, such that X embeds densely into βX and any continuous function from X into a compact space can be continuously extended to βX. The space βX is called the Čech-Stone compactification of X. The book [Wal74] is a standard reference for Čech-Stone compactifications. We refer the reader to this book for the proofs in this section which we omit.

Dealing with Čech-Stone compactifications, it is customary that X^* stands for the (Čech-Stone) *growth* of X, i.e. $X^* = \beta X \setminus X$. Let us now give a definition of four concrete topological types relevant to (Čech-Stone) growths:

Definition 1.4. A point $p \in X^*$ is a *remote point* of X iff it is not in the closure (in βX) of a n.w.d. subset of X. A slightly weaker requirement on $p \in X^*$ is that it is not a limit point of a countable discrete subset of X. We call such points *ω-far*. A point $p \in X$ is a *κ-O.K. point* of X iff for any countable sequence $\langle U_n : n \in \omega \rangle$ of neighborhoods of p there is a system $\{V_\alpha : \alpha < \kappa\}$ of neighborhoods of p such that for any finite $K \in [\kappa]^{<\omega}$, the following is true:

$$\bigcap_{\alpha \in K} V_\alpha \subseteq U_{|K|}$$

Note that, if $\kappa < \lambda$, then any λ-O.K. point is also a κ-O.K. point and if $\mathcal{B}$ is a base for the topology of X, then the definition is equivalent if we only consider sequences of neighborhoods from the base. A point $p \in X$ is a *weak P-point* of X if it is not a limit point of any countable set. A closed subset Y of a space X is a *weak P-set* (*κ-O.K. set*) if Y is a weak P-point (κ-O.K. point) of the quotient space X/Y. Note that a weak P-set does not contain a limit point of a countable set disjoint from it.

Proposition 1.5. *If X is a T_1 space and p is an ω_1-O.K. point of X, then p is a weak p-point of X.*

Proof. If $\{x_n : n \in \omega\} \subseteq X \setminus \{p\}$, then because X is T_1 we can choose a descending sequence of neighborhoods U_n of p such that U_n misses x_n. Then, because p is ω_1-O.K., choose $\{V_\alpha : \alpha < \omega_1\}$ neighborhoods of p, so that the intersection of any n of them is contained in U_n. Then each x_n is contained in only finitely many of them, so there is $\alpha < \omega_1$ which misses all of them, so p is not in the closure of $\{x_n : n \in \omega\}$. $\square$

A similar argument can be used to show the following proposition:

Proposition 1.6. *If X is regular and Y is a closed subset of X which is an ω_1-O.K. subset of X, then Y is a weak P-set of X.*

The following facts will be useful.

Fact 1.7 ([Wal74],1.59)**.** *X^* is compact iff X is locally compact.*

Fact 1.8 ([Wal74],2J.3)**.** *A space X is extremally disconnected iff βX is.*

Fact 1.9 ([vD81],5.2)**.** *βX is extremally disconnected at each remote point of X, and if X is nowhere locally compact, X^* is also extremally disconnected at each remote point of X.*

Proposition 1.10. *Let X be extremally disconnected. If $p \in X^*$ is a remote point and $p \in \overline{D_0}^{\beta X} \cap \overline{D_1}^{\beta X}$ for two sets $D_0, D_1 \subseteq X$, then there is an open $G \subseteq X$ such that $G \subseteq \overline{D_0}^{X} \cap \overline{D_1}^{X}$.*

Proof. First observe, that a point in an extremally disconnected space cannot be in the closure of two disjoint open sets. Let $G_i = int(\overline{D_i}^X)$. The set $N_i = G_i \setminus D_i$ is n.w.d. Then, since p is remote and cannot be in the closure of N_i, p is in the closure of both G_0, G_1, hence by our observation $G = G_0 \cap G_1$ is nonempty. □

1.3 Boolean algebras

Definition 1.11. A set B, together with operations $\vee, \wedge, -$ and constants $\mathbf{0}, \mathbf{1} \in B$ is a *Boolean algebra*, if the following is satisfied

(i) $(\forall a \in B)(a \wedge a = a \vee a = a)$

(ii) $(\forall a \in B)(a \wedge -a = \mathbf{0} \ \& \ a \vee -a = \mathbf{1})$

(iii) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$

(iv) $a \wedge b = b \wedge a$, $a \vee b = b \vee a$

(v) $\mathbf{0} \neq \mathbf{1}$.

We shall write $\mathbb{B} = \langle B, \wedge, \vee, \mathbf{0}, \mathbf{1} \rangle$, and also $\mathbb{B}$ instead of B (e.g. $|\mathbb{B}|$ denotes $|B|$ etc.). The canonical order $\leq$ on any Boolean algebra is defined in the following way:

(vi) $a \leq b$ iff $a \wedge b = a$

Also for $a, b \in \mathbb{B}$ we define

(vii) $a - b := a \wedge -b$

and for $A \subseteq \mathbb{B}$, $\bigvee A := \sup_{\leq} A$, $\bigwedge A := \inf_{\leq} A$. The operation $\vee$ is called a *join*, $\wedge$ is called a *meet*.

Definition 1.12. A Boolean algebra $\mathbb{B}$ is κ*-complete*, if for every $A \in [\mathbb{B}]^{<\kappa}$ the following exist: $\bigwedge A$, $\bigvee A$. $\mathbb{B}$ is *complete*, if for any subset $A \subseteq \mathbb{B}$ the following exist: $\bigwedge A$, $\bigvee A$.

Definition 1.13. If $\mathbb{B}, \mathbb{C}$ are two Boolean algebras, and $h : \mathbb{B} \to \mathbb{C}$ is a function we say that h is a *homomorphism*, if it preserves the operations (it does not need to preserve infinite suprema and infima). An injective homomorphism is called an *embedding*. An onto embedding is called an *isomorphism*. If a homomorphism h respects all (existing) suprema and infima, we call it *regular*. A regular homomorphism of complete Boolean algebras will be called a *complete homomorphism*.

Definition 1.14. If $A \subseteq \mathcal{P}(X)$ is closed under intersections, unions and complements (into X) then $\mathbb{A} = \langle \mathcal{P}(X), \cap, \cup, \emptyset, X \rangle$ is an *algebra of sets*. If the $\bigwedge, \bigvee$ agree with $\bigcap, \bigcup$, we say that it is a *complete algebra of sets*.

Definition 1.15. If $\mathbb{A} \subseteq \mathbb{B}$ and the identity mapping from $\mathbb{A}$ into $\mathbb{B}$ is a homomorphism, we say that $\mathbb{A}$ is a *subalgebra* of $\mathbb{B}$ (and write $\mathbb{A} \leq \mathbb{B}$. If the identity is a regular homomorphism, we say that $\mathbb{A}$ is a *regular subalgebra* of $\mathbb{B}$ and if $\mathbb{A}, \mathbb{B}$ are complete, we call $\mathbb{A}$ a *complete subalgebra* of $\mathbb{B}$.

Definition 1.16. If $\mathbb{B}$ is a Boolean algebra, two elements $a, b \in \mathbb{B}$ are *disjoint*, (denoted by $a \perp b$) if $a \wedge b = \mathbf{0}$. A set $X \subseteq \mathbb{B}$ is called *disjoint*, if every two distinct members of the set are disjoint. We define $\mathbb{B}^+ := \mathbb{B} \setminus \{\mathbf{0}\}$. An element $b \in \mathbb{B}^+$ is an *atom*, if $(\forall a \in \mathbb{B}^+)(a \leq b \to a = b)$, or, equivalently, if there are no two disjoint b_0, b_1 below a (we say, that a cannot be split). We further define $At(\mathbb{B}) := \{b \in \mathbb{B}^+ : \ b \text{ is an atom } \}$. We say, that $\mathbb{B}$ is *atomary*, if $\bigvee At(\mathbb{B}) = \mathbf{1}$. If an algebra has no atoms, we say that it is *atomless*.

Definition 1.17. For each $b \in \mathbb{B}^+$ the set $\mathbb{B} \restriction b := \{a \wedge b : \ a \in \mathbb{B}\}$ together with operations $\vee, \wedge, -$ (complements taken in b, i.e. $-a = b - a$) and constants $\mathbf{0}, b$ forms a Boolean algebra We call it the *factor algebra of* $\mathbb{B}$ *with respect to* b.

Lemma 1.18. *Suppose* $\mathbb{B}$ *is a Boolean algebra,* $M, N \subseteq \mathbb{B}$*,* $\langle M_\alpha : \alpha < \kappa \rangle$ *is a sequence of subsets of* $\mathbb{B}$ *and* $\bigvee M$*,* $\bigvee N$*,* $\bigvee_{\alpha<\kappa} M_\alpha$ *exist. Then the following holds:*

(De Morgan laws) $-\bigvee M = \bigwedge\{-m : m \in M\}$

(distributivity) $a \wedge \bigvee M = \bigvee\{a \wedge m : m \in M\}$

(distributivity) $\bigvee M \wedge \bigvee N = \bigvee\{m \wedge n : m \in M, \ n \in N\}$

(associativity) $\bigvee_{\alpha<\kappa}(\bigvee M_\alpha) = \bigvee\{m : (\exists \alpha < \kappa)(m \in M_\alpha)\}$

Definition 1.19. For a Boolean algebra $\mathbb{B}$ and a set $A \subseteq \mathbb{B}$ we define an *elementary meet* over the set A to be

$$\bigwedge_{i=0}^{n} \varepsilon(i) a_i,$$

for any $\{a_0, \dots, a_n\} \subseteq A$ and $\varepsilon : n \to \{-1, 1\}$ (where $-1a_i = -a_i$ and $1a_i = a_i$. A is said to be *independent*, if all elementary meets over A are nonzero. The minimal cardinality of a maximal (with respect to inclusion) independent subset of $\mathbb{B}$ is denoted by $\mathfrak{i}(\mathbb{B})$. We say that $\mathbb{B}$ has *hereditary independence* κ if $(\forall b \in \mathbb{B})(\mathfrak{i}(\mathbb{B} \restriction b) \geq \kappa)$.

Lemma 1.20 (Normal form Theorem)**.** *Let* $\mathbb{B}$ *be a Boolean algebra and* $A \subseteq \mathbb{B}$ *be a subset. Then every element in* $\langle A \rangle$*, the algebra generated by* A*, can be written in the form of a finite join of elementary meets over* A*.*

As a special case of this lemma, we will be using the following corollary

Corollary 1.21. *Let* $\mathbb{C}$ *be a Boolean algebra,* $\mathbb{B}$ *a subalgebra and* $c \in \mathbb{C}$*. Then every atom of* $\langle \{c\} \cup \mathbb{B} \rangle$ *can be written in the form* $c \wedge b$ *or* $-c \wedge b$ *for some* $b \in \mathbb{B}$*.*

Definition 1.22. A subset F of a Boolean algebra $\mathbb{B}$ is a *filter* if it contains the meet of any two of its members, does not contain $\mathbf{0}$ and is upwards closed with respect to the canonical ordering on the Boolean algebra. It is an *ultrafilter* if for each $b \in \mathbb{B}$ it contains either b or $-b$. An *ideal* and *prime ideal* are the "dual" notions to the notion of a filter (and ultrafilter respectively). That is $I \subseteq \mathbb{B}$ is an ideal (prime ideal) if $F = I^* := \{-i : i \in I\}$ is a filter (ultrafilter). In that case we say that I is dual to F and also write $I = F^*$. If I is an ideal in a Boolean albebra. $\mathbb{B}$, let $\mathbb{B}/I$ consist of the quotient classes modulo the I-equivalence relation (i.e. $a \simeq b \iff (a - b) \vee (b - a) \in I$) where the operations are defined using representatives. It is easy to check that this gives $\mathbb{B}/I$ the structure of a Boolean algebra and we call this algebra the *quotient* algebra of $\mathbb{B}$ mod I. If I is an ideal on $\mathcal{P}(\omega)$ and $A, B \in \mathcal{P}(\omega)$, we say that A is I-almost contained in B if $A \setminus B \in I$. Similarly we write $A =_I B$ if $(A \setminus B) \cup (B \setminus A) \in I$.

Definition 1.23. Any Boolean algebra $\mathbb{B}$ gives rise to a compact, zerodimensional space (a *Boolean space*), the *Stone space* of $\mathbb{B}$, denoted by $St(\mathbb{B})$. The points of $St(\mathbb{B})$ are ultrafilters on the Boolean algebra $\mathbb{B}$ and the base for the topology consists of sets $\hat{A}$ for $A \in \mathbb{B}$, where $\hat{A} = \{p \in St(\mathbb{B}) : A \in p\}$.

The following lemmas and definitions are a motivation for our endeavors in Chapter 5. They tie together Boolean-algebraic properties of Boolean algebras with the topological properties of certain topological spaces. If $\mathbb{B}$ is an algebra of sets, i.e. $\mathbb{B} \leq \mathcal{P}(X)$, then $\mathbb{B}$ is a base for some zero-dimensional topology on X. Call this topology $\tau_{\mathbb{B}}$. (Every zero-dimensional space arises in this way.) The next lemmas (and definitions) describe this topology. (Until the end of this section the space X will always have $\mathbb{B}$ as the basis for its topology.)

Definition 1.24. A set $C \subseteq \mathbb{B}$ is *independent* with respect to $D \subseteq \mathbb{B}$ if for all

$c \in C, d \in D^+$ we have $c \wedge d \neq \mathbf{0}$. For $D \subseteq \mathbb{B}$ define $idp(D)$ to be the set of all members of $\mathbb{B}$ independent with respect to D.

Definition 1.25. A subalgebra $\mathbb{B}$ of $\mathcal{P}(X)$ is said to have the T_2 property if for distinct $x, y \in X$, there are disjoint $b_x, b_y \in \mathbb{B}$, so that $x \in b_x, y \in b_y$.

Lemma 1.26. *$\mathbb{B}$ is atomless if $(X, \tau_{\mathbb{B}})$ is T_2 and* crowded *(i.e. has no isolated points). If X is T_2 then $\mathbb{B}$ is atomary iff $(X, \tau_{\mathbb{B}})$ is discrete.*

Lemma 1.27. *$\mathbb{B}$ has T_2 iff $(X, \tau_{\mathbb{B}})$ is T_2.*

Lemma 1.28. *$D \subseteq X$ is dense in $(X, \tau_{\mathbb{B}})$ iff it is independent with respect to $\mathbb{B}$.*

Lemma 1.29. *$\mathbb{B}$ is complete iff $(X, \tau_{\mathbb{B}})$ is extremally disconnected.*

Proof. Let $\mathbb{B}$ be complete and $U = \cup\mathcal{U}$ for some $\mathcal{U} \subseteq \mathbb{B}$. Then $\bigvee \mathcal{U}$ (which is, usually, *different* from $\cup\mathcal{U}$, even though for any $a, b \in \mathbb{B}$ $a \vee b = a \cup b$) is the closure of U, hence the closure of an open set is open. On the other hand if X is extremally disconnected and $\mathcal{U} \subseteq \mathbb{B}$, then $\overline{\cup\mathcal{U}} = \bigvee \mathcal{U}$, i.e. $\mathbb{B}$ is complete. □

Lemma 1.30. *The π-weight of $(X, \tau_{\mathbb{B}})$ is equal to the density of $\mathbb{B}$, and the weight of $(X, \tau_{\mathbb{B}})$ is less or equal to the cardinality of $\mathbb{B}$.*

Since we will be mostly concerned with extremally disconnected spaces, the following characterization, which ties together Boolean algebras with Čech-Stone compactifications, is useful:

Proposition 1.31. *If Y is extremally disconnected then βY is homeomorphic to $St(Clopen(Y))$, where $St(\mathbb{A})$ is the space of ultrafilters on $\mathbb{A}$.*

Corollary 1.32. *For an extremally disconnected Y the algebra $Clopen(Y)$ is isomorphic to $Clopen(\beta Y)$.*

CHAPTER II
EMBEDDING SPACES INTO THE GROWTH OF INTEGERS

In this chapter we give needed definitions and quote theorems, which will allow us to embed certain spaces into ω^* in such a way, that will preserve unique ω-accessibility of points. We will prove this fact after giving the theorems. All of the listed results are known. The following theorem is the main theorem of the chapter.

Theorem 2.1. *The Čech-Stone compactification of any extremally disconnected space X of weight $\leq \mathfrak{c}$ can be embedded as an ω_1-O.K. set into ω^*.*

This theorem is just a topological reformulation of a theorem of Kunen and Baker ([KB02], Theorem 5.6; but see also [Sim85]). Before we quote it, we need some definitions. They will not be needed in the other chapters and may be safely skipped.

Definition 2.2. A set function $\hat{}\,:[\theta]^{<\omega} \longrightarrow [\kappa]^{<\omega}$ is called a (θ, κ)-*hatfunction.* If for any two sets $A \subseteq B$ in the domain of $\hat{}$ we have, that $\hat{B} \subseteq \hat{A}$, we say that $\hat{}$ is *monotone.* A set $P \subseteq X$ in a topological space is a $\hat{}$-*set* iff for any sequence $\langle U_K : K \in [\kappa]^{<\omega}\rangle$ of neighborhoods of P there is a sequence $\langle V_\alpha : \alpha < \theta\rangle$ of neighborhoods of P such that for any A in the domain of $\hat{}$ the following is true:

$$\bigcap_{\alpha \in A} V_\alpha \subseteq U_{\hat{A}}$$

The function which assigns to each $A \in [\theta]^{<\omega}$ its cardinality shall be called the θ-O.K. function.

Definition 2.3. A sequence $\mathbb{B}$ $\langle M_\alpha : \alpha \in A\rangle$ of subsets of $\mathbb{B}$ is a *matrix* in $\mathbb{B}$ *independent with respect to a filter* $\mathcal{F}$ on $\mathbb{B}$, if for any $A_0 \in [A]^{<\omega}$, $c \in \mathcal{F}$ and $f : A_0 \longrightarrow \cup\{M_\alpha : \alpha \in A\}$ so that $f(\alpha) \in M_\alpha$ the following intersection is nonzero:

$$c \wedge \bigwedge_{\alpha \in A_0} f(\alpha)$$

Definition 2.4. If $\mathcal{G}$ is a filter on $\mathbb{B}$ then $M \subseteq \mathbb{B}$ is a $\hat{}$-step family on $(\mathbb{B}, \mathcal{G})$ iff it is of the form:

$$M = \{e_K : K \in [\kappa]^{<\omega}\} \cup \{a_\alpha : \alpha < \theta\} \cup \left\{ e_K \bigwedge_{\alpha \in A} a_\alpha : K \in [\kappa]^{<\omega}, A \in [\theta]^{<\omega}, \hat{A} \subseteq K \right\}$$

and satisfies:

(i) $\{e_K : K \in [\kappa]^{<\omega}\}$ is a partition of unity (i.e. it is a set of disjoint elements with supremum $\mathbf{1}$).

(ii) For each $A \in [\theta]^{<\omega}$

$$-\left(\bigwedge_{\alpha \in A} a_\alpha \wedge \bigvee \{e_K : \hat{A} \not\subseteq K\} \right) \in \mathcal{G}$$

(iii) For each $A \in [\theta]^{<\omega}$ and $K \in [\kappa]^{<\omega}$ if $\hat{A} \subseteq K$ then

$$e_K \wedge \bigwedge_{\alpha \in A} a_\alpha \notin \mathcal{G}^*,$$

where $\mathcal{G}^*$ is the dual ideal to the filter $\mathcal{G}$.

Theorem 2.5 (Kunen, Baker)**.** *Let $\mathbb{B}$ be a complete Boolean algebra of size 2^κ with $\mathcal{G} \subseteq \mathcal{F}$ two filters on $\mathbb{B}$. Let $\hat{}$ be any monotone (θ, κ)-hatfunction. Assume that $\mathbb{M} = \langle \mathcal{M}^i : i \in 2^\kappa\rangle$ is a matrix independent with respect to $\mathcal{F}$ so that each $\mathcal{M}^i$ is a $\hat{}$-step family on $(\mathbb{B}, \mathcal{G})$. Then for every complete Boolean algebra $\mathbb{A}$ of size $\leq 2^\kappa$,* *there is an* $h : \mathbb{B} \twoheadrightarrow \mathbb{A}$ *such that* $h''\mathcal{F} = \{1\}$ *and such that* $h^*(st(\mathbb{A})) \subseteq st(\mathbb{B}/\mathcal{F})$ *is a* $\hat{}$*-set in* $st(\mathbb{B}/\mathcal{G})$*.*

Proof of 2.1. Let $\mathbb{A}$ be a basis of X of size $\leq \mathfrak{c}$ consisting of clopen sets. Since X is extremally disconnected $\mathbb{A}$ is a complete Boolean algebra. Let $\mathbb{B}$ be $\mathcal{P}(\omega)$, $\mathcal{F}, \mathcal{G}$ the Fréchet filter on ω, $\hat{}$ the ω_1-O.K. hatfunction from definition 2.2 and $\mathbb{M}$ the matrix given by Theorem 3.9 of [KB02] (which is proved in [KB01]). Then the preceding theorem gives us an embedding of $st(\mathbb{A}) \approx \beta X$ as an ω_1-O.K. set into $st(\mathcal{P}(\omega)/fin) \approx \omega^*$. $\square$

We also mention a useful corollary of a theorem of van Mill ([vM82]):

Theorem 2.6 (van Mill)**.** *The projective cover of a continuous image of* ω^* *can be embedded as a* $\mathfrak{c}$*-O.K. set in* ω^**.*

Proposition 2.7. *If* $p \in Y \subseteq X$ *is an* ω*-uniquely accessible point of* Y *and if* Y *is a closed* ω_1*-O.K. in* X *which is* T_3*, then* p *is an* ω*-uniquely accessible point of* X*.*

Proof. Suppose $C, D \in [X]^\omega$ are two disjoint sets with $p \in \overline{C} \cap \overline{D}$. Then, since Y is a weak P-set of X by Proposition 1.6, $p \in \overline{C \cap Y} \cap \overline{D \cap Y}$ and, by ω-unique accessibility of p in Y we have, that $\emptyset \neq (C \cap Y) \cap (D \cap Y) \subseteq C \cap D$. $\square$

CHAPTER III

WEAK P-POINTS

Now we turn our attention to finding weak P-points (in fact ω_1-O.K. points) in general growths. We will give two versions of a theorem of van Mill since it is not clear, which is actually the most useful. In the second part we will construct a suitable space, to which, after some further modifications, Theorem 3.4 can be applied.

3.1 Two existence theorems

Definition 3.1. A system of closed subsets of a topological space X is called *precisely n-linked* if the intersection of n members of this system is noncompact but the intersection of any $n+1$ members of this system is compact. A system $\{A(\beta, n) : \beta \in B, n \in \omega\}$ is a *linked* system, if each $\{A(\beta, n) : \beta \in B\}$ is precisely n-linked and for each β, $A(\beta, n) \subseteq A(\beta, n+1)$. A system $\{A_\alpha(\beta, n) : \alpha \in A, \beta \in B, n \in \omega\}$ is an $|A|$ *by* $|B|$ *independent linked system* with respect to some closed (i.e. consisting of closed sets) filter $\mathcal{F}$ if each $\{A_\alpha(\beta, n) : \beta \in B, n \in \omega\}$ is a linked system and for any $A_0 \in [A]^{<\omega}$, $F \in \mathcal{F}$, $n \in {}^{A_0}\omega$, $\beta \in {}^{A_0}B$ the following intersection is noncompact:

$$F \cap \bigcap_{\alpha \in A_0} A_\alpha(\beta(\alpha), n(\alpha)).$$

A filter $\mathcal{F}$ on a topological sum $\sum X_n$ is called *nice*, provided for each $F \in \mathcal{F}$ the set $\{n \in \omega : F \cap X_n = \emptyset\}$ is finite.

Next we give a fact due to Kunen ([Kun78]). In 3.7 we actually describe such an independent linked system.

Fact 3.2. *There is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked system on the integers with respect to the Fréchet filter.*

The following theorem is Theorem 2.4 of van Mill [vM82].

Theorem 3.3. *If Z is a compact space of weight at most $\mathfrak{c}$, $\mathcal{F}$ is a nice filter on $X = \omega \times Z$ and Y is a continuous image of ω^* then the projective cover EY of Y can be embedded as a $\mathfrak{c}$-O.K. set into X^* such that $EY \subseteq \cap\{F^* : F \in \mathcal{F}\}$.*

A slight modification of the proof of this theorem yields the following theorem, which can also be useful:

Theorem 3.4. *If X is a space of weight $\leq \mathfrak{c}$ admitting a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked system with respect to some filter $\mathcal{C}$, then there is a $\mathfrak{c}$-O.K. point of $Y = \omega \times X$ in Y^*, which lies in the intersection $\bigcap\{F^* : F \in \mathcal{F}\}$, where*

$$\mathcal{F} = \left\{ \bigcup_{n \in A} \{n\} \times F(n) : F \in {}^{\omega}\mathcal{C}, \mathcal{A} \in \mathcal{FR}(\omega) \right\}$$

Proof. Let $\{A_\alpha(\beta, n) : \alpha, \beta < \mathfrak{c}, n < \omega\}$ be the independent linked system on ω from Theorem 3.2, $\{B_\alpha(\beta, n) : \alpha, \beta < \mathfrak{c}, n < \omega\}$ be the respective independent linked system on X. Note, that $\mathcal{F}$ is a nice filter on Y (and if $\mathcal{C}$ was remote, then so is $\mathcal{F}$), and that the following sets form an independent linked system mod $\mathcal{F}$ on Y:

$$C_\alpha(\beta, n) = \bigcup_{m \in A_\alpha(\beta, n)} \{m\} \times B_\alpha(\beta, n).$$

Let $\mathcal{B}$ be a base of Y of cardinality $\leq \mathfrak{c}$ and $\{\langle D_n^\alpha : n \in \omega \rangle : \alpha < \mathfrak{c}\}$ be an enumeration of all sequences of closures of sets from $\mathcal{B}$ satisfying $D_{n+1}^\alpha \subseteq \mathrm{int} D_n^\alpha \setminus$

$(n \times X)$. Without loss of generality let each such sequence be listed cofinally many times. By induction on $\alpha < \mathfrak{c}$ we construct $\mathcal{F}_\alpha \supseteq \mathcal{F}$ and $K_\alpha \subseteq \mathfrak{c}$ satisfying

(i) $\{C_\beta(\mu, n) : \beta \in K_\alpha,\ \mu < \mathfrak{c},\ n < \omega\}$ is an independent linked system mod $\mathcal{F}_\alpha$ for all $\alpha < \mathfrak{c}$.

(ii) $\mathcal{F}_\alpha \subseteq \mathcal{F}_\beta$ for all $\alpha < \beta$ are centered systems of closed sets

(iii) $K_\beta \subseteq K_\alpha$ for all $\alpha < \beta$ and $K_\beta \setminus K_{\beta+1}$ is finite.

(iv) If $D^\alpha_n \in \mathcal{F}_\alpha$ for all $n \in \omega$, then there are $\{E^\alpha_\gamma : \gamma < \mathfrak{c}\} \subseteq \mathcal{F}_{\alpha+1}$ witnesses to the O.K. property for D^α.

Let $K_0 = \mathfrak{c}$ and $\mathcal{F}_0 = \mathcal{F}$. If α is limit, then let $\mathcal{F}_\alpha = \bigcup\{\mathcal{F}_\beta : \beta < \alpha\}$ and $K_\alpha = \bigcap\{K_\beta : \beta < \alpha\}$. Now suppose we have constructed $K_\alpha, \mathcal{F}_\alpha$ and that D^α satisfies the condition in (iv). Choose $\beta \in K_\alpha$ and let $K_{\alpha+1} = K_\alpha \setminus \{\beta\}$. Define

$$E^\alpha_\gamma = \bigcup_{n<\omega} \underbrace{C_\beta(\gamma, n) \cap D^\alpha_n}_{\in \mathcal{F}}$$

and let $\mathcal{F}_{\alpha+1}$ be generated by F_α and $\{E^\alpha_\gamma : \gamma < \mathfrak{c}\}$. Note that, for any $A_0 \in [\mathfrak{c}]^n$

$$\left(\bigcap_{\gamma \in A_0} E^\alpha_\gamma\right) \setminus D^\alpha_n \subseteq \left(\bigcap_{\gamma \in A_0} C_\beta(\gamma, n)\right)$$

and the last term is compact, giving us (iv).

If we let $H = \cup\{\mathcal{F}_\alpha : \alpha < \mathfrak{c}\}$ then any $p \in Y^*$ containing H will (by (iv)) be a remote, $\mathfrak{c}$-O.K. point of Y^*. $\square$

Some further analysis of the previous proof shows, that requiring an independent linked system is, in fact, not needed. We can weaken the conditions to only require for each $n \in \omega$ a precisely n-linked system of closed sets, independent with respect to a remote filter. Finding such a system is presumably much easier.

Since the proof of the modified theorem is somewhat involved and we do not use it anywhere, we do not state it precisely or prove it.

If we drop the hypothesis that $\mathcal{C}$ is remote, it is easy to see that the previous proof will still give as an $\mathfrak{c} - O.K.$ point. We will need this for Theorem 5.23, so we state it here as a separate theorem:

Theorem 3.5. *If X is a space of weight $\leq \mathfrak{c}$ admitting a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked system with respect to some nice filter $\mathcal{C}$, then there is a $\mathfrak{c}$-O.K. point of $Y = \omega \times X$ in Y^*, which lies in the intersection $\bigcap\{F^* : F \in \mathcal{C}\}$.*

3.2 A crowded space with an independent linked system

In this section we proceed to construct a completely regular space containing a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family with respect to the filter of finite sets. Using the methods of Chapter 5, we will be able to modify this space to an irresolvable, extremally disconnected space while retaining the independent linked family.

The following definition and a theorem are due to Simon ([Sim85]).

Definition 3.6. Let $\mathbf{X} = \{\langle k, f\rangle : k \in \omega, f \in {}^{\mathcal{P}(k)}\mathcal{P}(\mathcal{P}(k))\}$ and for $X, Y \in \mathcal{P}(\omega)$, $n \in \omega$ let

$$F(Y, X, n) = \{\langle k, f\rangle : |f(Y \cap k)| \leq n \;\&\; X \cap k \in f(Y \cap k)\}.$$

Further let $O(Y, X, n) = \mathbf{X} \setminus F(Y, X, n)$.

Theorem 3.7. *The family $\{F(Y, X, n) : X, Y \in \mathcal{P}(\omega), n \in \omega\}$ is a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked family with respect to the Fréchet filter on $\mathbf{X}$, with the discrete topology.*

Let us prove the following lemma:

Lemma 3.8. *For any* $\{Y_0, \dots, Y_n\} \in \mathcal{P}(\omega)$ *and any* $X_0, \dots, X_n$, $Y'_0, \dots, Y'_m$, $X'_0, \dots, X'_m$ *subsets of* ω *and any* $i_0, \dots, i_n, j_0, \dots, j_m$ *natural numbers, the following intersection is either empty or infinite:*

$$F(Y_0, X_0, i_0) \cap \cdots \cap F(Y_n, X_n, i_n) \cap O(Y'_0, X'_0, j_0) \cap \cdots \cap O(Y'_m, X'_m, j_m)$$

Proof. Suppose, that $\langle k, f\rangle$ is in the intersection and suppose, without loss of generality, that there is $l \leq m$, that the following three groups of conditions are satisfied (and show that $\langle k, f\rangle$ is in the intersection):

$$|f(Y_0 \cap k)| \leq i_0 \quad \& \quad X_0 \cap k \in f(Y_0 \cap k) \tag{3.1}$$

$$\vdots$$

$$|f(Y_n \cap k)| \leq i_n \quad \& \quad X_n \cap k \in f(Y_n \cap k)$$

$$|f(Y'_0 \cap k)| > j_0 \tag{3.2}$$

$$\vdots$$

$$|f(Y'_l \cap k)| > j_l$$

$$X'_{l+1} \cap k \notin f(Y'_{l+1} \cap k) \tag{3.3}$$

$$\vdots$$

$$X'_m \cap k \notin f(Y'_m \cap k)$$

By Theorem 3.7, the condition (3.1) is satisfied by infinitely many $\langle k, f\rangle$s. Now if $\langle k', f'\rangle$ satisfies the conditions (3.1) for some $k' > k$, then the set

$$\{Y_0 \cap k', \dots, Y_n \cap k', Y'_0 \cap k', \dots, Y'_n \cap k',\}$$

has greater cardinality than the corresponding set with k in place of k'. So we can easily define a function $g \in \mathcal{P}(\mathcal{P}(k'))^{\mathcal{P}(k')}$, such that the conditions (3.1) and

(3.2) are satisfied. Notice now that satsifying condition 3.3 is also possible (the only reason why it would fail is that it would go against the second part of a condition in 3.1; but then already $\langle k, f\rangle$ would not have satisfied 3.1 and 3.3 since, if $X' \cap k' = X \cap k'$, then also $X' \cap k = X \cap k$). Thus for every $\langle k, f\rangle$ in the intersection, there is $\langle k', g\rangle$ with $k' > k$ in the intersection, so the intersection must be infinite. $\square$

Lemma 3.9. *For any $\langle k, f\rangle \neq \langle k', f'\rangle \in \mathbf{X}$, $\mathcal{Y} \in [\mathcal{P}(\omega)]^{<\omega}$, there are $n \in \omega$, $Y \in \mathcal{P}(\omega) \setminus \mathcal{Y}$ and $X \in \mathcal{P}(\omega)$ so that $F(Y, X, n)$ separates the two points.*

Proof. Consider three cases.

Case 1. If $k = k'$, then there are infinitely many Y's, which satisfy the following $f(Y \cap k) \neq f'(Y \cap k' = Y \cap k)$ (because, if not, then, necessarily, $f \neq f'$). Choose one such Y not in $\mathcal{Y}$, $n \geq |f(Y \cap k)|$ and $X \in f(Y \cap k)\Delta f'(Y \cap k')$. Then $F(Y, X, n)$ separates the points.

Case 2. If $k \neq k'$ and there is Y so that $f(Y \cap k) \neq f'(Y \cap k')$. Then there are infinitely many such Ys (modifying a Y above $\max\{k, k'\}$ preserves the inequality) so we can proceed similarly to the first case.

Case 3. If $k \neq k'$ and for all Y, $f(Y \cap k) = f'(Y \cap k')$, then choose any Y not in $\mathcal{Y}$ and X so that $|\{X \cap k, X \cap k'\} \cap f(Y \cap k)(= f'(Y \cap k'))| = 1$ (assume $k' < k$, then $f(Y \cap k) \subseteq \mathcal{P}(k') \subsetneq \mathcal{P}(k)$ so $\mathcal{P}(k) \setminus f(Y \cap k) \neq \emptyset$). $\square$

Proposition 3.10. *There is a topology τ on $\mathbf{X}$ so that all sets of the form $F(Y, X, n)$ are closed, the topology is totally disconnected, has no isolated points and the regular open sets form a basis for τ.*

Proof. Enumerate $[X]^2$ as $\{\{\langle k_n, f_n\rangle, \langle k'_n, f'_n\rangle\} : n \in \omega\}$ and by induction (using the previous lemma) find $Y_n \in \mathcal{P}(\omega) \setminus \{Y_0, \dots, Y_{n-1}\}$, $X_n \in \mathcal{P}(\omega)$ and $m_n \in \omega$ so that $F(Y_n, X_n, m_n)$ separates $\langle k_n, f_n\rangle$ from $\langle k'_n, f'_n\rangle$. Then let τ be the topology generated by $\{O(X, Y, n) : n \in \omega, X, Y \in \mathcal{P}(\omega)\} \cup \{F(Y_n, X_n, m_n) : n \in \omega\}$. By Lemma 3.8, this topology has no isolated points (by Lemma 1.2). For the last part, use a Theorem of Van Douwen ([vD81],1.5, 1.6) to refine this topology. $\square$

Corollary 3.11. *There is a countable, crowded, totally disconnected space with a base consisting of regular open sets and having a* $\mathfrak{c}$ *by* $\mathfrak{c}$ *independent linked system of closed sets.*

CHAPTER IV
REMOTE POINTS

As stated in the introduction and as will be seen in the last chapter, remote points are an essential tool for constructing ω-uniquely accessible points. We will first list some general conditions guaranteeing the existence of remote points in a large class of spaces and then give a concrete construction of a suitable space with a remote point. After modifying this space using the methods of Chapter 5 we will use it in the last chapter.

4.1 General theorems

This section will list some conditions under which we can have remote points. It will only be an overview; we do not include any proofs. The notion of a remote point was introduced by Fine and Gillman in [FG62] as a method for studying the nonhomogeneity of βX. The existence of remote points for spaces of countable π-weight was proved independently by van Douwen in [vD81] and Chae and Smith in [CS80]. The assumption of nonpseudocompactness in the theorems is due to the fact that any pseudocompact space of π-weight less than the first measurable cardinal has no remote points (see [Ter79]).

Theorem 4.1 ([Dow84]). *Any ccc nonpseudocompact space of π-weight ω_1 has a remote point. (A space is ccc if any system of disjoint open subsets of the space is at most countable.)*

Theorem 4.2 ([Dow82]). *Under MA any ccc nonpseudocompact space of π-weight at most $\mathfrak{c}$ has a remote point.*

Theorem 4.3 ([HP88]). *A nonpseudocompact space with a σ-locally finite π-base has a remote point.*

4.2 An irresolvable space with remote points

We will construct our space by constructing a topology on the integers. The following theorem is standard:

Theorem 4.4. *There is an ideal I on ω such that $\mathcal{P}(\omega)/I$ has hereditary independence $\mathfrak{c}$.*

Proof. The complete Boolean algebra $\mathbb{B} = Compl(Clopen(2^{\mathfrak{c}}))$ has hereditary independence $\mathfrak{c}$ and is σ-centered so there is an ideal I on ω such that $\mathbb{B}$ is isomorphic to $\mathcal{P}(\omega)/I$. □

Theorem 4.5. *There is a crowded, T_0, zero-dimensional, irresolvable topology on ω with a closed filter missing all sets with empty interior.*

Proof. The plan is to construct a subset M of $\mathcal{P}(\omega)$ independent in $\mathcal{P}(\omega)/I$ and then let the set of elementary meets over this subset be a base for the topology. The subset itself will then be the required filter. Now at each step we will look at a subset of ω and enlarge M so that either the subset will have nonempty interior in the resulting topology or our filter will miss it.

Let $\{P_\alpha : \alpha < \mathfrak{c} \ \& \ \alpha = 0 \bmod 2\}$ be an enumeration of $\mathcal{P}(\omega)$ and let $\{\langle Z_\alpha, n_\alpha\rangle : \alpha < \mathfrak{c} \ \& \ \alpha = 1 \bmod 2\}$ be an enumeration of $\{\langle Z, n\rangle : Z \in I^*, n \in Z\}$. First construct, by induction on ω, $M_0 \subseteq \mathcal{P}(\omega)$, which is independent in $\mathcal{P}(\omega)/I$ and for each pair of natural numbers i, j there is an $m \in M_0$ such that $i \in m$ and

$j \in \omega \setminus m = -m$. This is easy, since if a member of an independent family is finitely modified, the family remains independent and there are only countably many pairs of integers. Now construct an increasing chain $\langle M_\alpha : \alpha < \mathfrak{c}\rangle$ of subsets of $\mathcal{P}(\omega)$ which are independent in $\mathcal{P}(\omega)/I$, at a limit ordinal α, $M_\alpha = \bigcup\{M_\beta : \beta < \alpha\})$ and such, that the following condition is satisfied if $\alpha = 1 \bmod 2$

(i) There is an $m \in M_{\alpha+1}$ such that either $n_\alpha \in m \subseteq Z_\alpha$ or $n_\alpha \in -m \subseteq Z_\alpha$.

and at the same time either of the following is true if $\alpha = 0 \bmod 2$:

(ii) The set P_α I-almost contains an elementary meet over the set $M_{\alpha+1}$.

(iii) There is an $m \in M_{\alpha+1}$ such that $P_\alpha \cap m = \emptyset$.

Suppose we have constructed $\langle M_\beta : \beta \leq \alpha\rangle$.

Case 1 If $\alpha = 1 \bmod 2$ then choose $m \in \mathcal{P}(\omega)$ such that $M_\alpha \cup \{m'\}$ is independent in $\mathcal{P}(\omega)/I$ (we can do that since $|M_\alpha| \leq \alpha \cdot \omega < \mathfrak{i}(\mathcal{P}(\omega)/I) = \mathfrak{c}$) and find m so that $n_\alpha \in m \subseteq Z_\alpha$ and $m' \Delta m \in I$. Then $M_{\alpha+1} = M_\alpha \cup \{m\}$ is independent and (i) is satisfied.

Case 2 Suppose $\alpha = 1 \bmod 2$ and there is $m \in \mathcal{P}(\omega)$ so that P_α is I-almost contained in an elementary meet over the set $M_\alpha \cup \{m\}$ which is at the same time independent in $\mathcal{P}(\omega)/I$. Then let $M_{\alpha+1} = M_\alpha \cup \{m\}$ and (ii) is staisfied.

Case 3 Suppose $\alpha = 1 \bmod 2$ and that for any $m \in \mathcal{P}(\omega)$ for which $M_\alpha \cup \{m\}$ is independent in $\mathcal{P}(\omega)/I$, P_α is not almost contained in any elementary meet over $M_\alpha \cup \{m\}$. Then, necessarily, $Q = \omega \setminus P_\alpha \notin I$. Let $\mathbb{B} := \mathcal{P}(\omega)/I \restriction Q$. We claim that the following is true:

Claim 1 $M' = M_\alpha \upharpoonright \mathbb{B} = \{m \cap Q : m \in M\}$ is an independent subset of $\mathbb{B}$. If it were not, then for some elementary meet k over M, $k \cap Q \in I$. But, since M is independent, $k \notin Q$, so k is I-almost contained in $\omega \setminus Q = P_\alpha$ — a contradiction.

Now $\mathbb{B}$ has independence continuum (since $\mathcal{P}(\omega)/I$ has hereditary independence continuum), so M' is not maximal and there is an $m \subseteq Q$ such that $M' \cup \{m\}$ is independent in $\mathbb{B}$. Then also $M_{\alpha+1} = M_\alpha \cup \{m\}$ is independent in $\mathcal{P}(\omega)/I$ and $m \cap P_\alpha = \emptyset$.

So we have constructed the sequence $\langle M_\alpha : \alpha < \mathfrak{c}\rangle$ satisfying (i–iii). Now let Let $\mathcal{M}$ contain $\bigcup\{M_\alpha : \alpha < \mathfrak{c}\}$ and be maximal independent in $\mathcal{P}(\omega)/Z_0$. Then let τ be the topology generated by elementary meets over $\mathcal{M}$. This topology is zero-dimensional and T_0 ($M_0 \subseteq M \subseteq \tau$ and M_0 separates all points of ω). It is crowded, since the elementary meets of an independent system are in I^+ and I^+ contains no finite sets. Also notice that all $U \in I^*$ are τ-open. To see this, let A be the set of indices of pairs of the form $\langle U, n\rangle$ in our enumeration of $\{\langle Z, k\rangle : Z \in I^*, k \in Z\}$. Then, for each $\alpha \in A$ there is, by condition (i), $m_\alpha \in M_\alpha \subseteq M \subseteq \tau$ such that $n_\alpha \in m_\alpha \subseteq U$. Then $U = \bigcup\{m_\alpha : \alpha \in A\}$ so it is open. To see that (ω, τ) is irresolvable, first note that for any D dense and U open, $D \cap U \notin I$. Also, if both D and $\omega \setminus D$ are dense, then $\{D, \omega \setminus D\} \cap M = \emptyset$. Now, if D and $\omega \setminus D$ were dense, then both would intersect any elementary meet over M in a set from I^+, so $M \cup \{D, \omega \setminus D\}$ would be independent — a contradiction with the maximality of M. Now, let $\mathcal{F}$ be a filter of closed sets extending the centered system $\{\mathcal{M}\}$. It is a filter of closed sets, since all sets in M are clopen in τ. If $N \subseteq \omega$ has empty interior, then for some $\alpha < \mathfrak{c}$, $N = P_\alpha$. Since it has empty interior, it does not contain an elementary meet over M hence, a fortiori, over $M_{\alpha+1}$. So, by condition (ii) in the inductive construction, there is an $m \in M_{\alpha+1} \subseteq M \subseteq \mathcal{F}$ such that $m \cap N = \emptyset$. $\square$

CHAPTER V
BUILDING IRRESOLVABLE SPACES

In this chapter, methods for constructing special kinds of topologies will be introduced and the properties of the resulting topologies will be investigated. Our main result will be the existence of a countable, hereditarily open irresolvable, extremally disconnected space whose Čech-Stone compactification will have "special" properties. Our method is very similar to [Hew43], but we will use it in the context of extremally disconnected spaces and Boolean algebras. Also some of the results can be derived from [vD93]. First we give purely algebraic results. After that, we will look at the topological consequences.

5.1 Building complete atomless algebras

Lemma 5.1. *Let $\mathbb{B}$ be an atomless subalgebra of a complete algebra $\mathbb{C}$. If $D \subseteq \mathbb{B}$ has no supremum in $\mathbb{B}$, then there is a $c \in \mathbb{C} \setminus \mathbb{B}$ so that $\langle \mathbb{B} \cup \{c\} \rangle$ is atomless.*

Proof. Let c' be the supremum of D in $\mathbb{C}$. Define $U = \{u \in \mathbb{B} : u \geq c'\}$ and $D' = \{d \in \mathbb{B} : d \leq c'\} \supseteq D$. Now let c be the infimum of U in $\mathbb{C}$ ($c \notin \mathbb{B}$ otherwise it would be a supremum of D in$\mathbb{B}$). Suppose a were an atom of $\langle \mathbb{B} \cup \{c\} \rangle$, then, by Lemma 1.21, there are two cases:

Case 1. For some $b \in \mathbb{B}$, $a = b \wedge c$. But then all $d \in D'$ are less then $-b$ (otherwise for some $d \in D'$ we would have that $d \wedge b \neq \mathbf{0}$, but $d \leq c' \leq c$, so $\mathbf{0} \neq d \wedge b \leq c \wedge b = a$ which is a contradiction since $\mathbb{B}$ is atomless). So $-b \in U$ (since $D' \leq -b$ so $D \leq -b$ so $-b \geq c'$ because c' is the supremum of D) and $-b \geq c$ which contradicts $a \neq \mathbf{0}$.

Case 2. For some $b \in \mathbb{B}$, $a = b \wedge -c$. But then $b \leq u$ for all $u \in U$ (otherwise for some $u \in U$ we would have $-u \wedge b \neq \mathbf{0}$; then $u \geq c$ so $-u \leq -c$, and we would have $\mathbf{0} \neq -u \wedge b \leq -c \wedge b = a$, a contradiction with $\mathbb{B}$ being atomless) Suppose, aiming for a contradiction, that there is an atom a in $\langle \mathbb{B} \cup \{c\} \rangle$. We conclude that $b \leq c$, so $a = -c \wedge b = \mathbf{0}$, a contradiction. □

Proposition 5.2. *If $\mathbb{B}$ is an atomless subalgebra of a complete algebra $\mathbb{C}$, then there is a complete algebra $\mathbb{B}'$, which is a subalgebra of $\mathbb{C}$ (not necessarily a complete subalgebra of $\mathbb{C}$), which is atomless and contains $\mathbb{B}$. Denote this algebra $\mathbf{c}(\mathbb{B}, \mathbb{C})$ or just $\mathbf{c}(\mathbb{B})$ if $\mathbb{C}$ is clear from the context*[1].

Proof. Order the atomless subalgebras of $\mathbb{C}$ by inclusion. Since the union of a chain of atomless algebras is an atomless algebra we can use Zorn's lemma to get a maximal atomless subalgebra of $\mathbb{C}$ containing $\mathbb{B}$. This algebra is necessarily complete, since otherwise we could use Lemma 5.1 to extend the chain. □

Lemma 5.3. *Let $\mathbb{B}$ be an atomless subalgebra of $\mathbb{C}$, D a centered set of elements independent with respect to $\mathbb{B}$ and $c \in \mathbb{C}$ independent with respect to $\mathbb{B}$. Then either $D \cup \{c\}$ is centered, or for some $d \in D$ $\langle \{d\} \cup \mathbb{B} \rangle$ is atomless.*

Proof. Suppose the $D \cup \{c\}$ is not centered and choose $d \in D$ so that $d \wedge c = \mathbf{0}$. Suppose further that $a \in \langle \{d\} \cup \mathbb{B} \rangle$ is an atom. Then, using Lemma 1.21, we can write written as $a = -d \wedge b$ for some $b \in \mathbb{B}$ (it cannot be of the form $d \wedge b$ otherwise, since $\mathbb{B}$ is atomless, we could split b into b_1, b_2 and, since d is independent with respect to $\mathbb{B}$, $d \wedge b_1 \neq \mathbf{0} \neq d \wedge b_2$ would be a split of a). Now there is a $b' \in \mathbb{B}$, such that $-d \wedge b' = \mathbf{0}$ (split b and use the fact, that a is an atom). Then $b' \leq d$ and, since $d \wedge c = \mathbf{0}$, $b' \wedge c = \mathbf{0}$ contradicting that c is independent with respect to $\mathbb{B}$. □

[1]Note that there will actually be many such algebras. For most of our needs, we will not care which of these algebras is chosen to be $\mathbf{c}(\mathbb{B})$

Lemma 5.4. *Let $\mathbb{B}$ be an atomless subalgebra of $\mathbb{C}$, $b \in \mathbb{B}$, $c \in \mathbb{C}$. If $\langle(\mathbb{B} \restriction b) \cup \{c \wedge b\}\rangle$ is atomless, then so is $\langle \mathbb{B} \cup \{c \wedge b\}\rangle$.*

Proof. Denote $d = c \wedge b$. If $a \in \langle \mathbb{B} \cup \{d\}\rangle$ were an atom, it would have to be (using Lemma 1.21 and the fact that we could split $d \wedge b'$) of the form $a = -d \wedge b'$ for some $b' \in \mathbb{B}$. Now, $(a \geq) - d \wedge b' \wedge b = \mathbf{0}$ (otherwise we could again split a), hence wlog, $b' \wedge b = \mathbf{0}$ (i.e. $b' \leq -b$). But, since $-d \geq -b$, we have that $a \geq -b \wedge b' = b'$ and we can split a, which is a contradiction. □

Proposition 5.5. *Let $\mathbb{B}$ be a subalgebra of a complete Boolean algebra $\mathbb{C}$. Then for any $c \in \mathbf{c}(\mathbb{B})$ the set of elements of $\mathbb{C}$ independent with respect to $\mathbf{c}(\mathbb{B}) \restriction c$ is centered.*

Proof. Let $\mathbb{B}' = \mathbf{c}(\mathbb{B}) \restriction c$ and (using Zorn's lemma) let D be a maximal centered set of elements (of $\mathbb{C}$) independent with respect to $\mathbb{B}'$. Suppose that there is some $e \in \mathbb{C} \setminus \mathbb{D}$, independent with respect to $\mathbb{B}'$. Then, using Lemma 5.3, we get $d \in D$ (necessarily $d \wedge c \notin \mathbb{B}'$, since it is independent with respect to $\mathbb{B}'$), such that $\langle \{d \wedge c\} \cup \mathbb{B}'\rangle$ is atomless. But then, using Lemma 5.4, we get a contradiction with the fact that $\mathbb{B}$ is a maximal atomless subalgebra of $\mathbb{C}$. □

Proposition 5.6. *If $\mathcal{F}$ is a filter on $\mathcal{P}(\kappa)$ extending the generalized Fréchet filter on κ and $\mathbb{B}$ is a subalgebra of $\mathcal{P}(\kappa)$ such that $\mathcal{F}$ is independent with respect to $\mathbb{B}$, then there is an algebra $\mathbb{A}$ containing $\mathbb{B}$ which is maximal among algebras with respect to which $\mathcal{F}$ is independent. Denote this algebra by $\mathbf{i}(\mathbb{B}, \mathcal{F})$[2]. If $\chi(\mathcal{F}) \leq \kappa$ then this algebra is atomless.*

Proof. The existence of $\mathbb{A} = \mathbf{i}(\mathbb{B}, \mathcal{F})$ is a simple consequence of Zorn's lemma. Now any $a \in \mathbb{A}$ must have cardinality κ (otherwise $\kappa \setminus a \in \mathcal{F}$ contradicting the

[2]Note that, again, there will actually be many such algebras. For most of our needs, we will not care which of these algebras is chosen to be $\imath(\mathbb{B}, \mathcal{F})$

independence of $\mathcal{F}$ with respect to $\mathbb{A}$). Reasoning similarly for any $f \in \mathcal{F}$ and any $a \in \mathbb{A}$, $f \cap a$ has cardinality κ. Suppose, aiming towards a contradiction, that $a \in \mathbb{A}$ is an atom. Let $\{F_\alpha : \alpha < \kappa\}$ be a base for $\mathcal{F}$. By induction on $\alpha < \kappa$ choose two distinct

$$c_\alpha^0, c_\alpha^1 \in F_\alpha \cap a \setminus \{c_\beta^i : \beta < \alpha, i < 2\}$$

Then, if $c = \{c_\alpha^0 : \alpha < \kappa\}$, $\mathcal{F}$ is independent with respect to $\langle \mathbb{A} \cup \{c\} \rangle$, and that is a contradiction with the maximality of $\mathbb{A}$. $\square$

Lemma 5.7. *If $\mathcal{F}$ is the filter of cofinite subsets of κ, then $\mathbf{c}(\mathbf{i}(\mathbb{B}, \mathcal{F})) = \mathbf{i}(\mathbb{B}, \mathcal{F})$.*

Proof. This is evident, since if $\mathbb{B}$ is atomless, then $\mathcal{F} \subseteq idp(\mathbb{B})$, so the equality holds by the maximality of $\mathbf{i}(\mathbb{B}, \mathcal{F})$. $\square$

Proposition 5.8. *If $\mathbb{B}$ is an atomless subalgebra of a $\mathcal{P}(X)$ which has T_2, then there is a subalgebra $\mathbb{B}'$ of $\mathcal{P}(X)$ containing $\mathbb{B}$ with the following properties: $\mathbb{B}'$ is complete (not necessarily a complete subalgebra of $\mathcal{P}(X)$), atomless, has T_2 and for any $b \in \mathbb{B}$ the set of elements independent with respect to b is centered. We may also require $idp(\mathbb{B}')$ to contain a chosen filter of character $\leq |X|$.*

Proof. Just take $\mathbb{B}'$ to be $\mathbf{c}(\mathbb{B})$, use Proposition 5.5 and note, that any algebra larger than $\mathbb{B}$ will also have T_2. The additional requirement may be satisfied by taking $\mathbf{i}(\mathbb{B}, \mathcal{F})$ instead of $\mathbf{c}(\mathbb{B})$ and using Lemma 5.7. $\square$

Theorem 5.9. *Any crowded, totally disconnected topology τ with a base of regular open sets on a space X can be extended to a zerodimensional topology with no isolated points.*

Proof. Let $\mathbb{B}$ be the algebra of clopen sets of τ and let $\mathbb{B}'$ be a maximal atomless subalgebra of $\mathcal{P}(X)$ containing $\mathbb{B}$ and satisfying that for any $b \in \mathbb{B}'$ and $U \in RO(\tau)$ $b \cap U$ and $b \cap (X \setminus U)$ are empty or infinite. This algebra exists, because $\mathbb{B}$ is atomless (τ is totally disconnected and crowded). Now $\mathbb{B}'$ is a base for a topology containing τ and having no isolated points: it has no isolated points since it is atomless, so we only need to prove, that any set in $RO(\tau)$ can be written as a union of sets from $\mathbb{B}'$. Suppose not. Then for some $U \in RO(\tau)$ and $x \in U$ we have, that for any $b \in \mathbb{B}'$ $x \in b$ implies $b \cap X \setminus U$ is nonempty. By the total disconnectedness of τ there is b, so that $x \in b$. But then $\langle \mathbb{B}' \cup \{b \cap U\} \rangle$ is atomless and satisfies the other conditions, contradicting the maximality of $\mathbb{B}'$: $b' \wedge (b \cap U)$ cannot be finite (and hence an atom) because $\mathbb{B}'$ satisfies our conditions and $b' \wedge -(b \cap U) = b' - b \vee b' \cap (X \setminus U)$, which also can't be finite (and hence an atom). $\square$

5.2 Topological consequences

Let us now exploit the previous theorems to construct some interesting topologies.

Definition 5.10. We will call a space *κ-dense centered* iff any two dense subsets of cardinality at most κ intersect. A space is *open hereditarily κ-dense centered* iff every open subspace is κ-dense centered. Call a space *maximal* ([Hew43]), if any finer topology contains isolated points.

The following two theorems can be found in [Hew43] or [vD93].

Lemma 5.11 ([vD93],1.5, 1.6)**.** *Any crowded totally disconnected topology can be refined to a topology having a base consisting of regular open sets.*

Theorem 5.12 ([vD93],2.2)**.** *If X is maximal, then all n.w.d. subsets of X are closed.*

Corollary 5.13. *If X is maximal, then all n.w.d. subsets of X are discrete.*

Proposition 5.14. *The topology $\tau_{\mathbf{c}(\mathbb{B})}$ is maximal.*

Proof. The finer topology will be regular (since $\mathbf{c}(\mathbb{B})$ is extremally disconnected). So there must be a regular open set g in the finer topology, which is not open in $\mathbf{c}(\mathbb{B})$. Then $\mathbf{c}(\mathbb{B}) \cup \{g\}$ has an atom (that is, contains a finite set) which is of the form $g \wedge b$ or $-g \wedge b$. If the first is an atom, then it gives rise to an isolated point in the finer topology. If the second one is, then we get an isolated point if we notice, that $-g$ is regular closed, so if it has finite intersection with a clopen set, it also gives rise to an isolated point. $\square$

Proposition 5.15. *If the topology given by* $\mathbf{c}(\mathbb{B})$ *is nowhere locally compact then its compact sets are precisely the finite ones.*

Proof. If a set is finite, then it is compact. On the other hand, if it is infinite, then it is either discrete (and hence noncompact) or has nonempty interior, but then it cannot be compact, since the space is nowhere locally compact. $\square$

Definition 5.16. Call a space *κ-fine* if it is an extremally disconnected, T_2 hereditarily κ-dense centered space without isolated points. *Fine* is ω-fine.

Theorem 5.17. *On any zerodimensional, T_2 space X without isolated points, there is a finer topology $d\tau$, so that X with this topology is an $|X|$-fine space. We may further require that the dense sets in $(X, d\tau)$ extend a given filter of character $\leq |X|$.*

Proof. Let $\mathbb{B}$ be the clopen sets of our space X. And let $\mathbb{B}'$ (from Proposition 5.8) be the base of the finer topology. Since $\mathbb{B}'$ is a complete algebra, X is extremally disconnected. Since it is atomless, no finite subset of X is open, hence X does not have isolated points. The algebra has T_2 which implies that X is T_2. Now notice, that a set d dense in some clopen subset c of X, regarded as an element $d \in \mathcal{P}(X)$ is independent with respect to $\mathbb{B}' \restriction c$ (it must intersect all nonempty open sets). The additional requirement follows, if we have $\mathbb{B}'$ satisfy the additional requirement of Proposition 5.8. $\square$

Lemma 5.18. *A subset of a κ-fine space X is nowhere dense iff its complement is dense.*

Proof. The implication from left to right is trivial. Suppose then, that N is not nowhere dense and $X \setminus N = D$ is dense. Then, N and D are dense on some clopen subset of the nonempty open set $\mathrm{int}(\overline{N})$, hence they must intersect, and this gives us a contradiction. $\square$

Corollary 5.19. *The dense sets in a κ-fine space form a filter.*

Proof. Obvious, since the nowhere dense sets form an ideal. $\square$

Proposition 5.20. *A countable fine space X is nowhere locally compact.*

Proof. Fix a closed set $b \in \mathrm{Clopen}(X) = \mathbb{B}$. Enumerate all members of b as $\{x_n : n \in \omega\}$. Now construct a decreasing sequence of closed sets b_n such that for each $n \in \omega$, $x_n \notin b_{n+1}$: Let $b_0 = b$ and, if all b_is were constructed for $i \leq n$, split b_n into two parts (by atomlessness of $\mathbb{B}$). One of them must contain x_n, so let the other be b_{n+1}. This gives us a decreasing system of closed sets with empty intersection, i.e. b is not compact. $\square$

Proposition 5.21. *A disjoint sum of κ-fine spaces is a fine space.*

Proof. Obvious. $\square$

We have thus proved the following theorems, which are the main results of this chapter:

Theorem 5.22. *On any countable zerodimensional space without isolated points there is a finer topology giving a fine space. We can also assume that the remote points of X are precisely the ω-far points.*

Proof. The last condition can be satisfied using corollary 5.13 which says that in the topology of $\mathbf{c}(\mathbb{B})$ the nowhere dense sets are discrete. $\square$

Theorem 5.23. *There is a countable fine space Y whose Čech-Stone compactification contains a weak p-point. Thus βY is ω-dense centered.*

Proof. Let (X, τ') be the space from Corollary 3.11. Then, using Theorem 5.11 and, extend τ' to a zerodimensional topology and then, using Theorem 5.17, extend it to a finde topology τ. Now, since (X, τ) is nowherelocally compact we can use Proposition 5.15 to see, that the ideal of finite sets coincides with the ideal of compact sets. Let $Y = \omega \times X$. Then the $\mathfrak{c}$ by $\mathfrak{c}$ independent linked system in X satisfies the conditions of Theorem 3.5. If we take $\mathcal{C}$ to be the filter dual to the filter of finite sets it is nice and the Theorem 3.5 gives us p a $\mathfrak{c} - O.K.$ point in Y^*. By Proposition 1.5 p is a weak P-point. Then, if D_1, D_2 were disjoint countable dense in βY, then $U = \beta Y \setminus \overline{D_1 \cup D_2}$ is nonempty, since $p \in U$. So D_1 and D_2 are dense in $U \cap Y$ so they must intersect since Y is a fine space — a contradiction. $\square$

CHAPTER VI
CONCLUSION

In Chapter 3 we constructed a crowded totally disconnected space with a base of regular open sets with a $\mathfrak{c}$ by $\mathfrak{c}$ independent linked system. Using Theorem 5.9 and 5.22 we get an crowded, extremally disconnected OHI space of weight $\leq \mathfrak{c}$, which has a weak P-point (by Theorem 3.4) and can be embedded into ω^* as a weak P-set (Theorem 2.1).

Also, in Chapter 4, we constructed a crowded totally disconnected space with a base of regular open sets with a filter missing all sets with empty interior. Again, using Theorems 5.9 and 5.22 we get an crowded, extremally disconnected OHI space, this time with a remote point. This space can, by the same Theorem 2.1, be embedded as a weak P-set into ω^*.

The last step is to get an extremally disconnected OHI space X with both of the above properties, that is: a space which has a remote weak P-point p and has weight $\leq \mathfrak{c}$. Once we have such a space and embed its Čech-Stone compactification into ω^* as a weak P-set, this point will become a uniquely ω-accessible point of ω^*: Since βX is a P-set of ω^*, using Proposition 1.6, there is no countable $D \subseteq \omega^* \setminus \beta X$ such that $p \in \bar{D}$. Since p is a weak P-point of X^*, applying Proposition 1.5 there is no countable $D \subseteq X^*$, such that $p \in \bar{D}$. Joining the previous two observations with the fact that p is remote, any countable set D such that $p \in \bar{D}$ must be dense in some open set of X. But then, since X is extremally disconnected, any two such sets must be dense in a common open set (Use Proposition 1.10, and the fact that X is OHI to conclude that they have nonempty intersection).

BIBLIOGRAPHY

[CS80] S. B. Chae and J. H. Smith, *Remote points and G-spaces*, Topology Appl. **11** (1980), 243–246.

[Dow82] Alan Dow, *Weak p-points in compact ccc f-spaces*, Trans. Am. Math. Soc. **269** (1982), 557–565.

[Dow84] ______, *Remote points in spaces with π-weight ω_1*, Fundamenta Math. **124** (1984), 197–205.

[Eng] Ryszard Engelking, *General Topology*, Sigma series in pure mathematics, Heldermann Verlag Berlin, Berlin, 1989.

[FG62] N. J. Fine and L. Gillman, *Remote points in β*, Proc. Am. Math. Soc. **13** (1962), 29–36.

[Fro67b] Frolík, Zdeněk, *Nonhomogeneity of $\beta P - P$*, Comm. Math. Univ. Carolin. **8** (1967), 705–709.

[Fro67a] ______, *Sums of ultrafilters*, Bull. Am. Math. Soc. **73** (1967), 87–91.

[Hew43] Edmund Hewitt, *A problem of set-theoretic topology*, Duke Math. J. **10** (1943), 309–333.

[HP88] M. Henriksen and T. J. Peters, *Locally finite families, completely separated sets and remote points*, Proc. Am. Math. Soc. **103** (1988), 989–995.

[Jech] Thomas Jech, *Set Theory*, Springer-Verlag, 2002.

[KB01] Kenneth Kunen and Joni Baker, *Limits in the uniform ultrafilters*, Trans. Am. Math. Soc. **353** (2001), 4083–4093.

[KB02] ______, *Matrices and ultrafilters*, Recent progress in General Topology II, North-Holland, 2002, pp. 59–81.

[Kun78] Kenneth Kunen, *Weak P-points in N^**, vol. Proc. Bolyai János Soc. Colloq. on Topology, pp. 741–749, 1978.

[HBA] Monk and Bonnet (eds.), *Handbook of boolean algebras*, Elsevier Science, 1989.

[Rud56] Walter Rudin, *Homogeneity problems in the theory of Čech compactifications*, Duke Math. J. **23** (1956), 409–419.

[Sim85] Petr Simon, *Applications of independent linked families*, Colloq. Math. Soc. János Bolyai **41** (1985), 561–580.

[Ter79] T. Terada, *On remote points in $\nu X - X$*, Proc. Am. Math. Soc. **77** (1979), 264–266.

[vD81] Eric K. van Douwen, *Remote points*, Dissertationes Math. **188** (1981).

[vD93] ———, *Applications of maximal topologies*, Topology Appl. **51** (1993), 125–139.

[VD93] Jerry Vaughan and Alan Dow, *Accessible and biaccessible points in contrasequential spaces*, Annals of the New York Academy of Sciences **704** (1993), 92–102.

[vM82] Jan van Mill, *Sixteen types in $\beta\omega - \omega$*, Topology Appl. **13** (1982), 43–57.

[Wal74] Russell C. Walker, *The Stone-Čech Compactification*, Springer-Verlag, 1974.

[Wim82] E. Wimmers, *The Shelah P-point independence theorem*, Israel Journal of Mathematics **43** (1982), no. 1, 28–48.