CHODOUNSKY, DAVID, M.A. Relative Topological Properties. (2006)
Directed by Dr. Jerry Vaughan. 48pp.

In this thesis we study the concepts of relative topological properties and give some basic facts and relations among them. Our main focus is on various versions of relative normality, relative regularity and relative compactness. We give examples which answer some open questions and contract some conjectures in the literature. The theory of relative topological properties was introduced by A. V. Arhangel'skii and H. M. M Genedi in 1989.

Our three main results are (1) an example which presents a way to modify any Dowker space and get a normal space X such that $X \times [0, 1]$ is not κ-normal (Example 5.2.14). (2) A theorem which implies the existence of a non-Tychonoff space that is internally compact is a larger regular space (Theorem 6.2.6), and (3) a theorem that characterizes those subspaces of the Niemytzki plane that are normal subspaces.

RELATIVE TOPOLOGICAL PROPERTIES

by

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A Thesis Submitted to
the Faculty of The Graduate School at
The University of North Carolina at Greensboro
in Partial Fulfillment
of the Requirements for the Degree
Master of Arts

Greensboro
2006

Approved by

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APPROVAL PAGE

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ACKNOWLEDGMENTS

I would like to thank Jerry Vaughan for his guidance during the preparation of this thesis and Jan Rychtář for the invitation to UNCG. And I would like to thank Eva Murtinová, who introduced me to the topic of relative topological properties, gave me suggestions and advice for solving some open questions presented in this thesis, and helped me considerably with the formal writing of some texts investigating this topic.

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CHAPTER I
INTRODUCTION

In general topology we often encounter the question of how a certain space is located in a larger space. Examples of this are a Tychonoff space in its Čech-Stone extension, a Hausdorff space in its Katětov extension and a T_1 space in its hyperspace. This is the origin of the idea and motivation to investigate relative topological properties of a space Y in a superspace X.

The systematic study of relative topological properties was begun by A. V. Arhangel'skii and H. M. M. Genedi in a paper published in Russian in 1989 [4]. In 1996 Arhangel'skii wrote a survey article on this topic [2]. Parts of this thesis are based on that article.

Relative topological properties often generalize a global property in the sense that if the smaller space Y coincides with the larger space X, then the relative topological property should be the same as the global one. For example of some global properties we mention Hausdorfness, regularity, normality, compactness, Lindelöfness, countable compactness, pseudocompactness, paracompactness and metrizability. In this text we will mainly study various version of relative separation axioms and relative compactness. We will also see that some global properties can be generalized in several ways yielding several relative versions of the global property.

In Chapter 3 we give a short survey of relative topological properties obtained from regularity, and we also show how various relative versions of regularity arise. In Chapter 4 we review relative normality.

In Chapter 5 we discuss another version of relative normality, which has a close relation with κ-normality. The notion of κ-normality was introduced in 1972 by E. V. Schepin in [16]. In Chapter 5 we give an answer to two questions from Arhangel'skii's article [3] by constructing a normal space X, such that its product with the closed unit interval $X \times I$ is not κ-normal. This Example is a joint work with Eva Murtinová. In the last part of Chapter 5 we study relative normality of subspaces of the Niemytzki plane and we will derive a general condition for such a subspace to be relatively normal. From this condition we easily obtain a negative answer to a question of M. G. Tkačenko et al. in [17] by showing that the Niemytzki plane is not normal on many of its dense countable subspaces.

Finally in Chapter 6 we consider relative compactness, and answer two questions of Arhangel'skii from [3] by proving that there exists a non-Tychonoff space that is internally compact in a larger regular space.

We assume that the the reader is familiar with basic topological and set theoretic notions and principles. Some basic theorems and constructions, which will be used later, are reviewed in Chapter 2. We use the standard notation: the set of all natural numbers is denoted by ω, the set of all nonzero natural numbers $\mathbb{N}$, the set of real numbers $\mathbb{R}$, the set of all rational numbers $\mathbb{Q}$ and the set of irrational numbers $\mathbb{P}$.

For a subset A of a topological space (X, τ) the closure of A in (X, τ) is denoted by $\overline{A}$. If we want to emphasize the space or the topology we use the notation $\overline{A}^X$ or $\overline{A}^\tau$. The interior of the set A in the space X is denoted $int_X A$ or just $intA$.

CHAPTER II
PREREQUISITES

Our basic topological reference is [9]. For Set Theory e.g. [12] can be used.

2.1 Basic Topological Notions

Definition 2.1.1. A topological space X is T_1 if all singletons in X are closed in X.

Definition 2.1.2. A topological space X is *Hausdorff* (or T_2) if for each two different points x, y in X there are two disjoint open sets U and V such that $x \in U$ and $y \in V$.

Definition 2.1.3. A topological space X is *regular* (or T_3) if for each nonempty closed subset A of X and each point $x \in X \setminus A$ there are two disjoint open sets U and V such that $A \subset U$ and $x \in V$.

Definition 2.1.4. A topological space X is *Tychonoff* (or $T_{3\frac{1}{2}}$) if for each nonempty closed subset A of X and each point $x \in X \setminus A$ there is a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f[A] = \{1\}$.

Definition 2.1.5. A topological space X is *normal* (or T_4) if for each two disjoint nonempty closed subsets A and B of X there are two disjoint open sets U and V such that $A \subset U$ and $B \subset V$.

Definition 2.1.6. A topological space X is *compact* if X is Hausdorff and for each open cover $\mathcal{U}$ of X there exists a finite set $\mathcal{U}' \subset \mathcal{U}$ which covers X.

Definition 2.1.7. A topological space X is *Lindelöf* if X is Hausdorff and for each open cover $\mathcal{U}$ of X there exists a countable set $\mathcal{U}' \subset \mathcal{U}$ which covers X.

Definition 2.1.8. A mapping $f : X \to Y$ is *closed* if f is a continuous mapping and for each closed subset A of X the image $f[A]$ is a closed subset of Y.

Theorem 2.1.9. *Let $f : X \to Y$ be a closed mapping onto Y. Then:*
1) If X is T_1, then Y is T_1.
2) If X is normal, then Y is normal.

Definition 2.1.10. A mapping $f : X \to Y$ is a *perfect mapping* if X is a Hausdorff space, f is a continuous closed mapping and for each $y \in Y$ the preimage $f^{-1}[\{y\}]$ is a compact subset of X.

Theorem 2.1.11. *Let $f : X \to Y$ be a perfect mapping onto Y. Then:*
1) If X is Hausdorff, then Y is Hausdorff.
2) X is regular if and only if Y is regular.
3) If X is normal, then Y is normal.
4) X is compact if and only if Y is compact.
5) X is locally compact if and only if Y is locally compact.

In this thesis, we will also use relative versions of some of the above-mentioned topological properties. So if we need to emphasize that we are dealing with a general version of some property, we will e.g. use the notation "X is normal in itself" instead of just "X is normal".

Definition 2.1.12. Let Y be a subspace of a topological space X. The space Y is C^0 *embedded in* X if each continuous function $f : Y \to [0, 1]$ can be extended to a continuous function $F : X \to \mathbb{R}$.

Example 2.1.13. Let X be any topological space. Let us recall a construction of a space $X^\star$ which contains X and is called its *Alexandroff double*. Put $X^\star = X \times 2$ (where $2 = \{0, 1\}$) and topologize $X^\star$ as follows. All points of $X \times \{1\}$ are isolated and a basic open neighborhood of a point $x \in X \times \{0\}$ is the set $O \times 2 \setminus \{(x, 1)\}$ where O is an open subset of X containing x.

Example 2.1.14. Let X be a T_1 regular non-normal topological space. We will sketch the construction of a canonical T_1 regular non-Tychonoff space $J(X)$. The construction of the space $J(X)$ uses a method called *Jones machine*.

Pick two closed disjoint subsets A_0 and A_1 of X such that A_0 and A_1 cannot be separated by disjoint open neighborhoods. Add one new point z to the product $X \times \omega$. The base of topology at z consist of sets of form

$$\{z\} \cup \big(X \times (\omega \setminus 2n + i)\big) \cup \big((X \setminus A_i) \times \{2n - 1 + i\}\big)$$

for $n \in \omega \setminus 1$ and $i \in 2$. The resulting space $X \times \omega \cup \{z\}$ will be denoted $P(X)$. Finally identify each point $(a, 2n)$ in the set $A_0 \times \{2n\}$ with the corresponding point $(a, 2n + 1)$ in $A_0 \times \{2n + 1\}$ and each point $(a, 2n + 1) \in A_1 \times \{2n + 1\}$ with $(a, 2n + 2) \in A_1 \times \{2n + 2\}$ for every $n \in \omega$. This quotient space is the Jones space $J(X)$ and the quotient mapping will be denoted $q : P(X) \to J(X)$. Note that q is a perfect mapping.

It follows from the construction that $J(X)$ is a T_1 regular space and the closed set $A_1 \times \{0\}$ and the point z cannot be separated by a continuous real valued function, hence $J(X)$ is not Tychonoff. The space $J(X)$ inherits many properties from the original space X. For details see the original paper of F. B. Jones [13].

Definition 2.1.15. A subset A of a topological space X is called *nowhere dense* (in X) if $X \setminus \overline{A}$ is a dense set in X.

Theorem 2.1.16 (Baire Category Theorem for $\mathbb{R}$)**.** *The real line* $\mathbb{R}$ *is not a union of countably many nowhere dense sets in* $\mathbb{R}$*.*

All other topological notions which are used in this thesis, as well as basic facts from general topology can be found in [9].

2.2 H-closed Spaces

Definition 2.2.1. A Hausdorff space X is called *H-closed* if X is a closed subspace of each of its Hausdorff superspaces.

Theorem 2.2.2. *If X is a Hausdorff space, the following conditions are equivalent:*
1) X is H-closed
2) For every centered family $\mathcal{V}$ of open subsets of X the intersection $\bigcap\{\overline{V} : v \in \mathcal{V}\}$ is non-empty
3) Every open cover $\mathcal{U}$ of X has a finite subset $\mathcal{U}'$ such that $\bigcup\{\overline{U} : U \in \mathcal{U}'\} = X$

Proposition 2.2.3. *A regular space X is H-closed if and only if X is compact.*

For each Hausdorff space X there exists a unique H-closed space denoted τX and called the *Katetov extension of X*. Among other properties of τX let us mention these: X is an open dense subspace of τX and the set $\tau X \setminus X$ is discrete. For details see [9].

For more information about H-closed spaces see [14].

We will also use the notion of R-closed space.

Definition 2.2.4. A regular space X is called *R-closed* if X is a closed subspace of each of its regular superspaces.

2.3 Set Theory

Definition 2.3.1. Let κ be an infinite regular cardinal. The set A is a *closed unbounded* set (or *club*) if A is an unbounded subset of κ and contains all its limit points.

Definition 2.3.2. Let κ be an infinite regular cardinal. A set S is a *stationary* subset of κ if for each closed unbounded set $A \subset \kappa$, $A \cap S$ is nonempty.

Theorem 2.3.3 (Fodor's Lemma)**.** *Let κ be an uncountable regular cardinal and S a stationary subset of κ. Then each function $f : S \to \kappa$, such that $f(\alpha) < \alpha$ for each $\alpha \in S \setminus 1$, is constant on some stationary subset of κ.*

Theorem 2.3.4 (Solovay)**.** *Let κ be an uncountable regular cardinal and S a stationary subset of κ. Then κ is a union of κ many pairwise disjoint stationary subsets of κ.*

CHAPTER III
LOWER SEPARATION AXIOMS

3.1 Relative Regularity

Hereafter all spaces are considered to be T_1.

Definition 3.1.1. A topological space Y *is regular in* X, if for each $y \in Y$ and for each subset A of X which is closed in X and such that $y \notin A$, there are two disjoint sets U and V, open in X, such that $y \in U$ and $A \cap Y \subset V$.

If the larger space X is regular, then clearly each subspace of X is regular in X.

Proposition 3.1.2. *If the space Y is regular in X, then Y is a regular (in itself) subspace of X.*

Proof. Take a point $y \in Y$ and a closed set A in Y such that $y \notin A$. We will separate them by disjoint open neighborhoods to prove regularity. Since Y is regular in X there exist two disjoint open sets U and V in X separating y and $Y \cap \overline{A}^X$. Hence $U \cap Y$ and $V \cap Y$ are the desired disjoint neighborhoods. □

The next example is a version of a classical one (see, e.g. [9]).

Example 3.1.3. We will construct a regular space Y and a larger Hausdorff space X in which Y is not regular.

Let $P = \{1/(n+1) : n \in \omega\}$ be a subset of the real line $\mathbb{R}$. Add one new element $\mathbb{R} \setminus P$ to the usual topology of the real line $\mathcal{R}$ and denote the resulting topology τ. Let X be the space $(\mathbb{R}, \tau)$ and $Y = P \cup \{0\}$. Y is a discrete subset of X, thus it is regular, but Y is not regular in X.

Definition 3.1.4. A topological space is Y *internally regular in* X, if for every $y \in Y$ and every subset A of Y which is closed in X and such that $y \notin A$, there are two disjoint sets U and V open in X such that $y \in U$ and $A \subset V$.

It is easy to see that if Y is regular in X then Y is internally regular in X, but the converse is not true.

Example 3.1.5. Consider the space X from Example 3.1.3 and add one new point $\{z\}$ to get a larger space $Z = X \cup \{z\}$. The base of the topology at z consists of the sets $O \cup \{z\}$, where O is an open set in $\mathbb{R}$ such that $(P \setminus K) \subset O$ and K is a finite set. Now X is not regular in Z but X is internally regular in Z since no infinite subset of P is closed in Z. Note that Z is not Hausdorff.

A Hausdorff space X with a non-regular subspace Y and such that Y is internally regular in X, is constructed in Example 6.2.5.

Proposition 3.1.6. *If Y is a dense subspace of a space X, then Y is regular in X if and only if Y is regular.*

Proof. Let Y be regular. Pick any y in Y and A closed in X not containing y. Since Y is regular, there exist two disjoint open subsets U and V in Y separating y and $A \cap Y$. Take open sets U' and V' in X such that $U = U' \cap Y$ and $V = V' \cap Y$. Now $U' \cap V' \subset X \setminus Y$ since U and V are disjoint and so $U' \cap V' = \emptyset$ since Y is dense and $U' \cap V'$ is open. So U' and V' are disjoint open subsets of X separating y and $A \cap Y$ and Y is regular in X.

The other implication follows from Proposition 3.1.2. $\square$

Theorem 3.1.7 ([5])**.** *For a Hausdorff space Y the following conditions are equivalent:*

1) Y is regular in every larger Hausdorff space X

2) Y is internally regular in every larger Hausdorff space

3) Y is compact

Proof. 3⇒1: Let Y be a compact space and X a larger Hausdorff space. Let y be any point in Y and A a closed subset of X such that $y \notin A$. Put $A' = A \cap Y$. The set A' is a closed subset of Y and so A' is compact. Since X is Hausdorff, for each $x \in A'$ there are two open disjoint subsets of X, say U_x and V_x, such that $x \in U_x$ and $y \in V_x$. $\{U_x : x \in A'\}$ is an open covering of A' so we can fix a finite set $F \subset A'$ such that $A' \subset \bigcup\{U_x : x \in F\}$. This means that $\bigcap\{V_x : x \in F\}$ and $\bigcap\{U_x : x \in F\}$ are two disjoint open subsets of X separating y and $A \cap Y$, so Y is regular in X.

Since regularity of Y in any Hausdorff space X implies internal regularity of Y in X, it is sufficient to prove 2⇒3. Let Y be a non-compact Hausdorff space. We will construct a larger Hausdorff space X in which Y is not internally regular.

Fix a centered family $\mathcal{C}$ of closed subsets of Y, which has an empty intersection. We may pick a $C_0 \in \mathcal{C}$ which is a proper subset of Y and assume that all the sets in $\mathcal{C}$ are subsets of C_0. Choose $y_0 \in Y \setminus C_0$. We aim to extend Y to a Hausdorff space X, so that C_0 will be closed in X but y_0 and C_0 cannot be separated by disjoint open sets in X.

Let $X = Y \cup (C_0 \times \omega)$ and topologize X as follows. Let $Y \setminus (C_0 \cup y_0)$ be an open subspace and let all points of $C_0 \times \omega$ be isolated. A basic open neighborhood of y_0 has the form $U \cup (C \times \omega)$ for U an open neighborhood of y_0 in $Y \setminus C_0$ and $C \in \mathcal{C}$. A basic open neighborhood of $x \in C_0$ has the form $V \cup (V \cap C_0) \times (\omega \setminus n)$ where V is some open neighborhood of x in Y such that $y_0 \notin V$ and $n \in \omega$.

To show that this is a correctly defined base of topology in X we need to check that finite intersections of basic sets are open sets. Let $W = U \cup (C \times \omega)$ be a basic neighborhood of y_0 and let $Z = V \cup (V \cap C_0) \times (\omega \setminus n)$ be a basic open neighborhood of $x \in C_0$. Now $W \cap Z = (U \cap V) \cup (V \cap C) \times (\omega \setminus n)$ where $U \cap V \subset Y \setminus (C_0 \cup \{y_0\})$ and so $W \cap Z$ is an open set.

The other cases are trivial to check so the definition of topology works. Moreover, the topology of X coincides with the topology of Y on Y and C_0 is closed in X.

We will prove that Y is not internally regular in X. Let $W = U \cup (C \times \omega)$ be a basic open neighborhood of y_0. Take arbitrary $x \in C$ and let $Z = V \cup (V \cap C_0) \times (\omega \setminus n)$ be any basic open neighborhood of x. Now $\{x\} \times (\omega \setminus n) \subset W \cap Z$ and thus $W \cap Z$ is nonempty and $x \in \overline{W} \cap C_0$.

Now it only remains to show that X is Hausdorff. The only nontrivial case is again y_0 and $x \in C_0$. The space Y is Hausdorff, hence there exist disjoint open subsets U, V of Y such that $y_0 \in U$ and $x \in V$. Since the intersection of $\mathcal{C}$ is empty, there exists a $C \in \mathcal{C}$ such that $x \notin C$. Now $Z = V \cap (Y \setminus C)$ is an open neighborhood of x in Y. Then $x \in A = Z \cup (Z \cap C_0) \times (\omega \setminus n)$ for some $n \in \omega$ and $y_0 \in B = U \cup (C \times \omega)$, so A and B are disjoint open sets in X separating y_0 and x. Separation of other types of points in X is straightforward and therefore X is Hausdorff.

□

The first proof of the equivalence 1⇔3 in Theorem 3.1.7 was given in [5]. The authors used a different construction, which for each non-compact space Y gives a larger space X in which Y is not regular. The proof of the equivalence 2⇔3 in Theorem 3.1.7 uses a simplified construction from [10].

CHAPTER IV
RELATIVE NORMALITY

4.1 Relative Normality

Definition 4.1.1. Let Y be a subspace of a topological space X. Y is said to be *normal in* X if for every A and B which are disjoint closed subsets of X, there are two disjoint open sets U and V in X such that $A \cup Y \subset U$ and $B \cup Y \subset V$.

Proposition 4.1.2. *If a space Y is normal in some larger space X, then Y is a regular space.*

Proof. If Y is normal in X then Y is obviously regular in X and due to Proposition 3.1.2 Y is a regular space. □

Definition 4.1.3. A topological space Y is *strongly normal in* X if for every two disjoint subsets A and B of Y, which are closed in Y, there are two disjoint open subsets U and V of X, such that $A \subset U$ and $B \subset V$.

If the space Y is strongly normal in X, then clearly Y is normal in X.

Proposition 4.1.4. *If Y is strongly normal in X, then Y is a normal space.* □

Definition 4.1.5. A function $f : X \to \mathbb{R}$ is called a Y-*continuous* function for $Y \subset X$, if it is continuous at each point y of Y.

Definition 4.1.6. A topological space Y is *weakly C-embedded in* X, if each continuous function $f : Y \to \mathbb{R}$ can be extended into a Y-continuous function $F : X \to \mathbb{R}$.

Proposition 4.1.7. *If the space Y is dense in X and Y is normal, then Y is strongly normal in X.*

Proof. Take A and B to be two disjoint closed subsets of Y. Y is normal so there exist two disjoint open subsets U and V of Y separating A and B. Take U' and V' to be some open subsets of X such that $U' \cap Y = U$ and $V' \cap Y = V$. Note that $U' \cap V'$ is an open set in X which does not intersect Y so $U' \cap V'$ must by empty. Thus U' and V' are the desired sets separating A and B in X. □

Proposition 4.1.8. *If the space Y is closed in X and Y is normal in X, then Y is strongly normal in X.*

Proof. This Proposition holds true since each closed set in Y is also closed in X. □

The next Lemma is a version of a classical well known argument (see e.g. [9]).

Lemma 4.1.9. *Let $\mathcal{U}$ and $\mathcal{V}$ be two countable families of open subsets of a topological space X. Then there are two disjoint open sets U and V such that $\bigcup \mathcal{U} \setminus \bigcup\{\overline{V'} : V' \in \mathcal{V}\} \subset U$ and $\bigcup \mathcal{V} \setminus \bigcup\{\overline{U'} : U' \in \mathcal{U}\} \subset V$.*

Proof. Without loss of generality we may assume that $\mathcal{U} = \{U_i : i \in \omega\}$, $U_i \subset U_{i+1}$ for $i \in \omega$ and that $\mathcal{V}$ has an analogous property. Put $G_i = U_i \setminus \overline{V}_i$ and $H_i = V_i \setminus \overline{U}_i$ for each $i \in \omega$. Now for any j and $k \in \omega$, H_j and G_k are disjoint open sets in X hence also $U = \bigcup\{G_i : i \in \omega\}$ and $V = \bigcup\{H_i : i \in \omega\}$ are open disjoint subsets of X. And since $\bigcup \mathcal{U} \setminus \bigcup\{\overline{V}_i : i \in \omega\} \subset U$ and $\bigcup \mathcal{V} \setminus \bigcup\{\overline{U}_i : i \in \omega\} \subset V$, these sets U and V fulfill the condition required in the Lemma. □

Theorem 4.1.10. *If Y is regular in X and the space Y is Lindelöf, then Y is strongly normal in X.*

Proof. Let A and B be any two disjoint closed subsets of Y. For each $a \in A$ and $b \in B$ fix some open neighborhoods O_a and O_b of a and b in X such that

$\overline{O}_a \cap B = \emptyset$ and $\overline{O}_a \cap B = \emptyset$. The existence of such neighborhoods is guaranteed by regularity of Y in X. Since Y is Lindelöf there are countable sets A' and B' such that $A \subset \bigcup\{O_a : a \in A'\}$ and $B \subset \bigcup\{O_b : b \in B'\}$. We can now use Lemma 4.1.9 to get open sets U and V which separate A and B. □

Corollary 4.1.11. *If X is a regular space, then every Lindelöf subspace of X is strongly normal in X.*

Proof. Since X is regular, each subspace of X is regular in X. The result follows from Theorem 4.1.10. □

Theorem 4.1.12. *A subspace Y is strongly normal in X if and only if Y is normal (in itself) and weakly C-embedded in X.*

Proof. $\Rightarrow$ Let Y be strongly normal in X; then Y is normal by Proposition 4.1.4. We will slightly change the topology on X in the following way: The open base at all points $y \in Y$ remains the same as in X and all points in $X \setminus Y$ are isolated. Denote this new space X_Y. Clearly Y is a closed subspace of X_Y and X_Y is a normal space. For details see [9, Chapter 5.1] Let $f : Y \to \mathbb{R}$ be any continuous function. By the Tietze Lemma f can be extended to a continuous function $F : X_Y \to \mathbb{R}$. This function considered as a function $F : X \to \mathbb{R}$ is Y-continuous since the open base at all points of Y is the same in both spaces X and X_Y.

$\Leftarrow$ Take two disjoint closed nonempty subsets A and B of Y. Since Y is normal, there is a continuous function $f : Y \to \mathbb{R}$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$. This function can be extended to a Y-continuous function $F : X \to \mathbb{R}$ and $intF^{-1}[(-1, 1/2)]$ and $intF^{-1}[(1/2, 2)]$ are disjoint open sets in X separating A and B.

□

4.2 Relative Realnormality

Definition 4.2.1. The space Y *is realnormal in* X, if for every two nonempty disjoint closed sets A and B in X there is a Y-continuous function $f : X \to \mathbb{R}$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$.

Proposition 4.2.2. *Y is realnormal in X if and only if for every two nonempty disjoint closed sets A and B in X there is a Y-continuous function $f : X \to \mathbb{R}$ such that $f[A \cap Y] \subset \{0\}$ and $f[B \cap Y] \subset \{1\}$.*

Proof. Let Y be a subspace of some space X which fulfills the condition of the Proposition. We will show that Y is realnormal in X. Pick two nonempty disjoint closed subsets A and B of X. There is a Y-continuous function $f' : X \to \mathbb{R}$ such that $f'[A \cap Y] \subset \{0\}$ and $f'[B \cap Y] \subset \{1\}$. Put $f = f'$ on $X \setminus (A \cup B)$, $f = 0$ on A and $f = 1$ on B. We need to show that f is Y-continuous. For $y \in Y \setminus (A \cup B)$ there is a neighborhood O of y in X, such that O is disjoint from $A \cup B$, and $f = f'$ on O, so f is continuous at y. For $y \in Y \cap A$ and for each neighborhood O of y, we have $f[O] \subset f'[O]$ since $f'(y) = 0$. The case $y \in B$ is similar. □

Definition 4.2.3. Y is *weakly realnormal* in X, if for every two nonempty disjoint closed sets A and B in X there is a continuous function $f : Y \to \mathbb{R}$ such that $f[A \cap Y] \subset \{0\}$ and $f[B \cap Y] \subset \{1\}$.

Proposition 4.2.4. *If Y is weakly realnormal in X, then Y is a Tychonoff space.*

Proof. If A is a closed subset of Y and $x \in Y \setminus A$ then $x \notin \overline{A}^X$ and due to weak realnormality of Y in X there exists a continuous function $f : Y \to \mathbb{R}$ separating x and A. □

Theorem 4.2.5 ([8])**.** *Let X be any topological space (not necessarily T_1) and S a locally compact space. If Y is dense in X and $f : Y \to S$ is a continuous function,* *then there exists a function $F : X \to S$ which extends f and is continuous at each point $y \in Y$.*

Proof. For each $x \in X$ let $\{U_i^x : i \in I_x\}$ be a base of open neighborhoods at x in X. Denote $C_i^x = \overline{f[U_i^x \cap Y]}$. Observe that $C_i^x \neq \emptyset$ for each $x \in X$ and $i \in I_x$ since Y is dense in X. Let B be the set of all points x in X, for which there exists some $i \in I_x$ such that C_i^x is compact.

Claim 1. *Y is a subset of B.*

Take any $x \in Y$. For this x there exists an open neighborhood O of $f(x)$ such that $\overline{O}$ is compact since S is locally compact. Thus there is also some $i \in I_x$ such that $f[U_i^x \cap Y] \subset O$ because f is continuous. For this i the set $\overline{f[U_i^x \cap Y]}$ is compact and $x \in B$. □

Claim 2. *B is an open subset of X.*

Take any $x \in B$. There is some $i \in I_x$ such that $\overline{f[U_i^x \cap Y]}$ is compact and thus $U_i^x \subset B$. □

Now $\{C_i^x : i \in I_x\}$ may be viewed as a centered family of compact sets for each $x \in B$ and so the intersection $\bigcap\{C_i^x : i \in I_x\}$ is nonempty for such x. Define $F : B \to S$ such that $F(x) \in \bigcap\{C_i^x : i \in I_x\}$ for each $x \in B$.

Claim 3. $F \restriction Y = f$

Take any $x \in Y$ and pick any $s \neq f(x)$, $s \in S$. S is locally compact thus there is an open neighborhood U of $f(x)$ such that $s \notin \overline{U}$. Since f is continuous there is some $i \in I_x$ such that $f[U_i^x \cap Y] \subset U$, and then $s \notin C_i^x$ and $F(x) \neq s$. □

Claim 4. *F is Y-continuous.*

We will prove that F is continuous at each $y \in Y$: Pick any open neighborhood V of $F(y)$ in S. S is locally compact so S is regular and there exists an open neighborhood W of $F(y)$ such that $\overline{W}$ is a compact subset of V. f is continuous hence there exists an open neighborhood U of y such that $f[U \cap Y] \subset W$. Fix such a U and note that for each $x \in U \cap B$ there is an $i \in I_x$ such that $U_i^x \subset U$ with C_i^x compact. This implies that $F(x) \in C_i^x = \overline{f[U_i^x \cap Y]} \subset \overline{f[U \cap Y]} \subset \overline{W} \subset V$, hence $F[U \cap B] \subset V$ and F is continuous at y. □

Finally for $x \in X \setminus B$ let $F(x) \in S$ be arbitrary. The function F is still Y-continuous since Y is a subset of the open set B. □

Corollary 4.2.6. *Every dense subspace of a space X is weakly C-embedded in X.*

Corollary 4.2.7. *If Y is a Lindelöf subspace of a regular space X, then Y is weakly C-embedded in X.*

Proof. This is a corollary of Theorem 4.1.11 and Theorem 4.1.12. □

Corollary 4.2.8. *If Y is dense in X and weakly realnormal in X, then Y is realnormal in X.*

Proof. Let A and B be two nonempty disjoint closed subsets of X. Since Y is weakly realnormal in X, there exists a continuous function $f : Y \to \mathbb{R}$ separating $A \cap Y$ and $B \cap Y$. Due to Theorem 4.2.5 we can extend f into a Y-continuous function over X and Y is realnormal in X according to Proposition 4.2.2. □

Example 4.2.9 ([10, Example 6])**.** In [10], P.M. Gartside and A. Glyn gave an example of a Tychonoff space X with a dense subspace Y, such that Y is normal in X but not realnormal in X. This space was constructed under additional set-theoretic assumptions (MA + $\aleph_2 < 2^{\aleph_0}$).

Proposition 4.2.10. *If every closed subspace of a space X is weakly C-embedded in X, then X is normal.*

Proof. Let A and B be two nonempty disjoint closed subsets of X. Let us define a continuous function $f : A \cup B \to \{0,1\}$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$. Since $A \cup B$ is weakly C-embedded in X, we can extend f into a $A \cup B$-continuous function $F : X \to \mathbb{R}$. The sets $intF^{-1}[(-1, 1/2)]$ and $intF^{-1}[(1/2, 2)]$ are disjoint open subsets of X separating A and B. □

Theorem 4.2.11. *If X is a hereditarily normal space, then every subspace Y of X is strongly normal in X and weakly C-embedded in X.*

Proof. Let A and B be two disjoint closed sets in some subspace Y of X. Y is normal so there are two disjoint open sets U' and V' in Y such that $A \subset U'$ and $B \subset V'$. Let U and V be open sets in X such that $U \cap Y = U'$ and $V \cap Y = V'$. The space $U \cup V$ is an open normal subspace of X so there exist two disjoint open subsets O and P of X such that $U \setminus V \subset O$ and $V \setminus U \subset P$. Now the sets O and P also separate A and B in X, and Y is strongly normal in X. The fact that Y is weakly C-embedded in X follows from Theorem 4.1.12. □

CHAPTER V

MORE VERSIONS OF RELATIVE NORMALITY

5.1 Normality on a Subspace

Definition 5.1.1. A topological space Y is *internally normal in* X, if for every two disjoint subsets A and B of Y which are closed in X, there are disjoint sets U and V, open in X, such that $B \subset U$ and $A \subset V$.

Proposition 5.1.2. *Let Y be a dense subspace of a space X and Z be internally normal in Y. Then Z is internally normal in X.*

Proof. Let A and B be two disjoint subsets of Z which are closed in X. Then A and B are also closed in Y and thus there exist disjoint open subsets U and V of Y separating A and B. Take U' and V' open in X such that $U' \cap Y = U$ and $V' \cap Y = V$. Note that $U' \cap V'$ is an open set in X which does not intersect Y so $U' \cap V'$ must be empty. Thus U' and V' are the desired sets separating A and B in X. □

Corollary 5.1.3. *Every normal subspace Y of X which is dense in X, is internally normal in X.*

Proof. Y is internally normal in Y since Y is normal in itself so we can use Proposition 5.1.2 for $Z = Y$. □

The next Proposition follows directly from the definition.

Proposition 5.1.4. *If Y is normal in X, then Y is internally normal in X.*

But the next example shows that internal normality does not coincide with relative normality.

Example 5.1.5 ([3])**.** There is a Tychonoff space X with a dense subspace Y such that Y is internally normal in X and not normal in X.

Let L be the set of all limit ordinals in ω_1 and S, T two disjoint subsets of L stationary in ω_1. Put $M = (\omega_1 + 1) \setminus S$, $X' = \{(\alpha, \beta) : \beta \leq \alpha \leq \omega_1\}$, $X = X' \setminus \{(\omega_1, \omega_1)\}$ and $Y = (M \times M) \cap X$ and let π be the projection from X' to the second coordinate. The topology on X, Y and X' is inherited from $(\omega_1 + 1) \times (\omega_1 + 1)$. It is easy to see that X' is compact, X is locally compact and Y is dense in X since S contains only limit ordinals in ω_1.

Put $A = \{(\alpha, \alpha) : \alpha \in T\}$ and $B = \{(\omega_1, \alpha) : \alpha \in T\}$. Obviously, A and B are subsets of Y with disjoint closures in X. We will show that A and B cannot be separated by disjoint open sets in X so Y is not normal in X.

Let U be an open neighborhood of A in X. For each $\alpha \in T$ fix some $\delta(\alpha) < \alpha$ such that $V_\alpha = (\delta(\alpha), \alpha]^2 \cap X$ is a subset of U. By Fodor's Lemma (2.3.3) there exist $\beta < \omega_1$ and a stationary subset E of T such that $\delta(\alpha) = \beta$ for each $\alpha \in E$. This implies that $(\omega_1, \alpha) \in \overline{\bigcup\{V_\alpha : \alpha \in E\}} \subset \overline{U}$ for each $\alpha \in E$. Since E is a subset of T, $\overline{U}$ intersects B and A and B cannot be separated in X. As Y is dense in X, Y is not even weakly normal in X.

On the other hand, we will show that Y is internally normal in X. We will prove that each subset of Y, which is closed in X, is compact. This property will be defined in Chapter 6 as internal compactness and Lemma 6.2.2 then implies internal normality of Y in X.

Let P be a non-compact closed subset of X. X' is compact so P cannot be closed in X' and $P' = \overline{P}^{X'} = P \cup \{(\omega_1, \omega_1)\}$. Now $\pi[P']$ is a closed subset of $\omega_1 + 1$ since P' is compact, and $\omega_1 \in \pi[P']$. Thus $\pi[P] = \pi[P'] \setminus \{\omega_1\}$ is a closed unbounded set in ω_1 and $\pi[P]$ has nonempty intersection with the stationary set S. But this shows that P is not a subset of Y.

Example 5.1.6. There exists a Tychonoff space X such that for any dense subspace Y of X and any dense subspace Z of Y, Z is not internally normal in Y (and X does not contain a dense normal subspace). For details see [3, Theorem 1.4].

Proposition 4.1.2 states, that each space normal in some larger space is regular. Generally we can ask the question, whether a relative property implies any general property for the smaller space. Arhangel'skii stated this question in the following way [3, Question 10]: Let Y be a subspace of a regular space X such that Y is internally normal in X. Is then Y Tychonoff? We will give a negative answer to this question in Corollary 6.2.7.

Definition 5.1.7. A subset A of X *is concentrated on* Y, if $A \subset \overline{A \cap Y}^X$

It is easy to see that closed subsets of X concentrated on Y are closures of subsets of Y.

Definition 5.1.8. A space X is *normal on* its subspace Y, if for every two disjoint closed subsets A and B of X concentrated on Y there are disjoint open sets U and V in X such that $A \subset U$ and $B \subset V$.

If X is normal, then X is normal on every subspace Y of X.

Proposition 5.1.9. *If X is normal on Y, then Y is normal in X.*

Proof. Pick two disjoint closed subsets A and B of X. There are disjoint open subsets U and V of X such that $\overline{A \cap Y}^X \subset U$ and $\overline{B \cap Y}^X \subset V$. In particular, $A \cap Y \subset U$ and $B \cap Y \subset V$. □

"Normality on" is stronger that "normality in". There are examples showing that it is strictly stronger.

Example 5.1.10. There exists a countable dense subspace Y of the space $X = \mathbb{R}^{\mathfrak{c}}$ such that X is not normal on Y. But Y is Lindelöf and so Y is normal in X. For details see [1].

The notion of normality on a subspace does not coincide with strong normality either.

Example 5.1.11. Take any normal space X such that X contains a non-normal subspace Y. Then X is normal on Y. On the other hand, Y cannot be strongly normal in X since Y is not normal.

Proposition 5.1.12. *Let Y be closed in X, then the following are equivalent:*

1) X is normal on Y

2) Y is normal in X

3) Y is strongly normal in X

Proof. 1⇒2 holds due to Proposition 5.1.9 and 2⇒3 due to Proposition 4.1.8. If a set A is closed in X and concentrated on Y, then A is a closed subset of Y. Thus if Y is strongly normal in X, then X is normal on Y. □

5.2 On κ-normality

Definition 5.2.1. A space X is *densely normal* if there exists a dense subspace Y of X such that X is normal on Y.

Definition 5.2.2. A set A is a *regular closed* set if A is a closure of an open set.

Definition 5.2.3. A space X is *κ-normal* if every two disjoint regular closed sets in X can be separated by disjoint open neighborhoods.

The notion of κ-normality was introduced by E.V. Schepin in [16]. As we will see, κ-normality is an absolute property, but it has interesting relations with some versions of relative normality.

Lemma 5.2.4. *If Y is dense in X then each regular closed subset of X is concentrated on Y.*

Proof. Let O be an open subset of X. Suppose that $O \setminus \overline{O \cap Y}$ is nonempty. Then $(O \setminus \overline{O \cap Y}) \cap Y$ is also nonempty since Y is dense in X and that is a contradiction. Thus $O \subset \overline{O \cap Y}$ and $\overline{O} \subset \overline{O \cap Y}$. □

Theorem 5.2.5. *Every densely normal space is κ-normal.*

Proof. This follows immediately from Lemma 5.2.4. □

The converse is not true; there exists a κ-normal Tychonoff space which is not densely normal. For details see [1].

An important property of κ-normal spaces was proved in [16]. We will state this result without proof.

Theorem 5.2.6 ([16])**.** *If X is a κ-normal space, then every two disjoint regular closed sets A and B in X are functionally separated (there exists a continuous function $f : X \to \mathbb{R}$ such that $f[A] \subset \{0\}$ and $f[B] \subset \{1\}$).*

Theorem 5.2.7. *If a regular space X is normal on Y, then the space Y is Tychonoff.*

Proof. Pick any nonempty closed subset A of Y and a point $b \in Y \setminus A$. We can take two disjoint regular closed sets G and H in $\overline{Y}$ such that $A \subset G$ and $b \in H$. This is possible since the space $\overline{Y}$ is regular. Note that $\overline{Y}$ is normal on Y, thus densely normal and κ-normal. The result follows from Theorem 5.2.6. □

Theorem 5.2.8. *If X is normal on Y and Y is dense in X, then the space Y is realnormal in X.*

Proof. The space X is densely normal and therefore, by Theorem 5.2.5, X is κ-normal. Let A and B be two nonempty disjoint subsets of Y such that $\overline{A}^X \cap \overline{B}^X = \emptyset$. There are two disjoint regular closed subsets G and H of X such that $A \subset G$ and $B \subset H$. Now we can apply Theorem 5.2.6 to get a real valued continuous function on X separating G and H. □

Proposition 5.2.9. *If Y is a normal subspace of X and Y is C^0-embedded in X then X is normal on Y.*

Proof. Let A and B be two nonempty closed disjoint subsets of Y such that $\overline{A}^X \cap \overline{B}^X = \emptyset$. There is a continuous function $f : Y \to [0,1]$ such that $f[A] = \{0\}$ and $f[B] = \{1\}$ since Y is normal. Y is C^0-embedded in X so we can extend f to a continuous function F over X. Note that $F[\overline{A}^X] = 0$ and $F[\overline{B}^X] = 1$. The sets $F^{-1}[[0,1/2)]$ and $F^{-1}[(1/2,1]]$ are disjoint open sets in X separating $\overline{A}^X$ and $\overline{B}^X$ so X is normal on Y. □

Corollary 5.2.10. *If X has a dense normal subspace Y and Y is C^0-embedded in X, then the space X is densely normal and κ-normal.*

Example 5.2.11. Let Y be any normal space and X any space such that

$$Y \subset X \subset \beta Y$$

Then X is normal on Y by Corollary 5.2.10.

Example 5.2.12. Let X be a non-normal topological space which is dense in itself and let A and B be two closed subsets of X each of which si dense in itself and which cannot be separated by open neighborhoods. We can get such a space by setting $X = X' \times \mathbb{R}$ for any non-normal space X'. Let $X^\star = X \times 2$ be the Alexandroff double of X (see 2.1.13). $A \times \{1\}$ and $B \times \{1\}$ are open sets and thus $A \times 2$ and

$B \times 2$ are two disjoint regular closed sets in $X^\star$. These two sets cannot be separated by disjoint open sets in $X^\star$ since A and B cannot be separated in X. This shows that $X^\star$ is not κ-normal and not normal on $Y = X \times \{1\}$. Y is discrete, hence this example shows that no separation property (even discreteness) of the smaller space Y can be strong enough to guarantee that X has to be normal on Y.

Proposition 5.2.13. *Let Y be the set of all isolated points of a space X. If Y is dense in X (in particular, if X is scattered), then the following conditions are equivalent:*
1) X is κ-normal
2) X is normal on Y
3) X is densely normal

Proof. 1⇒2 All subsets of Y are open in X so the closure of each subset of Y is a regular closed set in X. The other implications are trivial or already proved. □

One of the famous questions in General Topology was the existence of a normal space X whose product with the closed unit interval I is not normal. Such spaces X are called Dowker spaces, and Dowker and Katětov proved that X is a Dowker space if and only if X is normal and not countably paracompact (see, eg. [9, Chapter 5.2]). The existence of such spaces X was proved in [15]. Two related questions, which follow, were given in [3].
1) Is the product of a normal space X and the closed interval I always κ-normal?
2) Let X be a normal space and B a compact Hausdorff space. Is the space $X \times B$ κ-normal?

The next example gives a negative answer to both of Arhangel'skii's questions. It is a modification of any Dowker space.

Example 5.2.14. Let Y be any Dowker space. Put $X' = (\omega+1) \times Y$ and change the product topology by declaring all points in $\omega \times Y$ to be isolated. The resulting space will be denoted X and the top level $\{\omega\} \times Y$ will be identified with Y. The space X is a normal space. Let A and B be two disjoint closed subsets of X. Then $A \cap Y$ and $B \cap Y$ are two disjoint closed subsets of Y and there exist disjoint open subsets U and V of Y separating $A \cap Y$ and $B \cap Y$, since Y is normal. The sets $(A \setminus Y) \cup ((\omega+1) \times U) \setminus B$ and $(B \setminus Y) \cup ((\omega+1) \times V) \setminus A$ are disjoint open sets in X separating A and B.

The construction of canonically closed subsets of $X \times I$ is analogous to the classical one (see, e.g., [9, Chapter 5.2]). Since Y is not countably paracompact, there exists a sequence $\{F_n : n \in \omega\}$ of closed subsets of Y such that $F_{n+1} \subset F_n$, $\bigcap\{F_n : n \in \omega\} = \emptyset$ and for each sequence $\{G_n : n \in \omega\}$ of open sets in Y, such that $F_n \subset G_n$, $\bigcap\{G_n : n \in \omega\}$ is nonempty.

For each $n \in \omega$, put

$$B_n = (\omega \setminus n) \times F_n \times \left(\frac{1}{2(n+1)}, \min\left(\frac{3}{2(n+1)}, 1\right)\right)$$

and

$$S_n = n \times Y \times \left[0, \frac{1}{2(n+2)}\right).$$

Note that B_n and S_n are open subsets of $(n+1) \times Y \times I$ and thus open sets in $X \times I$ and $B_n \cap S_m = \emptyset$ for each $n, m \in \omega$.

We will define regular closed subsets of $X \times I$:

$$F = \overline{\bigcup\{B_n : n \in \omega\}}$$

and

$$E = \overline{\bigcup\{S_n : n \in \omega\}}.$$

To prove that E and F are disjoint it is only necessary to show that $(Y \times \{0\}) \cap F = \emptyset$. Pick any $x \in Y \times \{0\}$, fix $n \in \omega$ such that $x \notin F_n$ and let O be an open neighborhood of x, where

$$O = (\omega + 1) \times (Y \setminus F_n) \times \left[0, \frac{1}{2(n+1)}\right).$$

We will show that O is disjoint from B_m for each $m \in \omega$ and thus disjoint from $\bigcup\{B_m : m \in \omega\}$. If $m \leq n$, then

$$O \subset (\omega + 1) \times Y \times \left[0, \frac{1}{2(n+1)}\right)$$

and

$$B_m \subset (\omega + 1) \times Y \times \left(\frac{1}{2(n+1)}, 1\right]$$

so O and B_m are disjoint. If $n < m$, then $F_m \subset F_n$ so $B_m \subset (\omega + 1) \times F_n \times I$ and this set is disjoint from O.

Now it is clear that

$$E = (Y \times \{0\}) \cup \bigcup\{S_n : n \in \omega\}$$

and

$$F = \bigcup\{B_n \cup F_n \times \left[\frac{1}{2(n+1)}, \min\left(\frac{3}{2(n+1)}, 1\right)\right] : n \in \omega\}.$$

The sets E and F cannot be separated by disjoint open neighborhoods. If $F \subset U$ and U is open then $F_n \times \{1/(n+1)\} \subset U$ for each n and thus $\{G_n : n \in \omega\}$, where $G_n = \pi_Y[U \cap (Y \times \{1/(n+1)\})]$, is a sequence of open sets in Y such that $F_n \subset G_n$ (π_Y is the projection from $Y \times I$ onto Y). This implies that there exists some $x \in \bigcap\{G_n : n \in \omega\}$. For this x we have $(x, 0) \in \overline{U} \cap E$ and therefore E and F cannot be separated. This shows that $X \times I$ is not κ-normal.

5.3 Some Examples

Let us now recall the definition of the Niemytzki plane $\mathbf{N}$ and establish some notation. Let $\mathbf{L} = \{(t,0) : t \in \mathbb{R}\}$, $\mathbf{E} = \{(r,s) : r \in \mathbb{R}, s \in \mathbb{R}^+\}$, $\mathbf{N} = \mathbf{L} \cup \mathbf{E}$. For $x = (r,s) \in \mathbf{E}$ and $0 < \varepsilon < s$ let $B_\varepsilon(x) = \{(r_1,s_1) \in \mathbf{E} : (r_1 - r)^2 + (s_1 - s)^2 < \varepsilon^2\}$ and for $x = (t,0) \in \mathbf{L}$ and $\varepsilon \in \mathbb{R}^+$ let $B_\varepsilon(x) = B_\varepsilon(t,\varepsilon) \cup \{x\}$. The Niemytzki plane is the set $\mathbf{N}$ with the topology generated by the sets $B_\varepsilon(x)$ for $x \in \mathbf{N}$ and $\varepsilon \in \mathbb{R}^+$. On the set $\mathbf{L}$ we will also use the topology of the real line denoted by $\mathcal{R}$. The set of all rational numbers will be denoted by $\mathbb{Q}$ and irrationals will be denoted by $\mathbb{P}$.

Lemma 5.3.1. *Let $Q = \{(t,0) : t \in \mathbb{Q}\}$ and $P = \{(t,0) : t \in \mathbb{P}\}$ be two disjoint closed subsets of $\mathbf{N}$. The sets P and Q cannot be separated by disjoint open neighborhoods in $\mathbf{N}$.*

Proof. Take an open set V in $\mathbf{N}$ such that $P \subset V$. For each $x \in \mathbb{P}$ we can fix some $n_x \in \mathbb{N}$ such that $B_{1/n_x}((x,0)) \subset V$ and put $P_n = \{x \in \mathbb{P} : n_x = n\}$ for each $n \in \mathbb{N}$. Since $\mathbb{Q} \cup \bigcup\{P_n : n \in \mathbb{N}\} = \mathbb{R}$, Theorem 2.1.16 implies that there exists a $q \in \mathbb{Q}$ such that $q \in \overline{P_m}^{\mathcal{R}}$ for some $m \in \mathbb{N}$. Thus $(q,0) \in \overline{\bigcup\{B_{1/m}((x,0)) : x \in P_m\}} \subset \overline{V}$ and hence $Q \cap \overline{V} \neq \emptyset$ so P and Q cannot be separated by disjoint open sets. $\square$

Since each countable regular space is strongly normal in every larger regular space (Theorem 4.1.10) it is natural to study normality of X on its countable subspaces. This topic is investigated in the article of Tkačenko, Tkachuk, Wilson and Yaschenko [17].

Example 5.3.2 ([17])**.** In this example a countable dense subset C of $\mathbf{N}$, such that $\mathbf{N}$ is not normal on C, is constructed. Let $A = \{(x,y) \in \mathbf{E} : x, y \in \mathbb{Q}\}$ and $Q = \{(x,0) : x \in \mathbb{Q}\}$. We will show that $C = A \cup Q$ works.

Put $Q = \{t_n : n \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ fix some $W_n = B_{\varepsilon(n)}(t_n)$ such that $\varepsilon(n) < 1/n$ and $\overline{W}_n \cap \overline{W}_m = \emptyset$ for $n \neq m$. Put $W = \bigcup\{W_n : n \in \mathbb{N}\}$ and $P = A \backslash W$.

The sets $F = \overline{P}$ and $Q = \overline{Q}$ are closed subsets of $\mathbf{N}$ concentrated on C and we will show that they are disjoint and cannot be separated by open neighborhoods. By Lemma 5.3.1 it is sufficient to show that $F \cap \mathbf{L} = \mathbf{L} \setminus Q$.

The intersection $F \cap Q$ is empty since W is an open neighborhood of Q. It remains to verify that each $z \in \mathbf{L} \setminus Q$ is in F. Suppose that there is $z \in \mathbf{L} \setminus Q$ such that $z \notin \overline{P}$. That implies that there is some $B_\varepsilon(z)$ such that $B_\varepsilon(z) \cap A \subset W$. For $\xi \in \mathbb{R}^+$ put

$$O_\xi = B_\varepsilon(z) \cap \{(x, y) \in \mathbf{E} : \xi < y\}.$$

There must be a finite set $X \subset \mathbb{N}$ such that

$$O_\varepsilon \cap A \subset \bigcup \{W_n : n \in X\}.$$

Therefore

$$O_\varepsilon \subset \bigcup \{\overline{W}_n : n \in X\}$$

and since O_ε is connected, there must be one $n \in X$ such that $O_\varepsilon \subset \overline{W}_n$. There is some $\delta \in \mathbb{R}^+$ such that $\delta < \varepsilon$ and $O_\delta \not\subset \overline{W}_n$, and we can use the same arguments to show that there is some $m \in \mathbb{N}$ such that $O_\delta \subset \overline{W}_m$. Clearly $n \neq m$ but $\emptyset \neq O_\varepsilon \subset \overline{W}_n \cap \overline{W}_m$, a contradiction. Thus $z \in \overline{P}$ and, consequently, $\mathbf{N}$ is not normal on C.

Example 5.3.3 ([17])**.** We will construct a separable Tychonoff space which is not normal on any countable dense subspace.

Put $L = \{(t, 0) : t \in \mathbb{R}\}$, $Y = \{(r, s) : r \in \mathbb{Q}, s \in \mathbb{Q}^+\}$ and $X = L \cup Y$. For $n \in \mathbb{N}$ and $x = (t, 0) \in L$ put $T_n(x) = \{(t, s) \in \mathbb{R}^2 : 0 < s < 1/n\}$. All points in Y are isolated and for $x \in L$ let $\mathcal{B}_x = \{\{x\} \cup (U \cap Y) : U \text{ open in } \mathbb{R}^2, T_n(x) \subset U, n \in \mathbb{N}\}$ be an open base at x. It is clear that X is zero-dimensional and $X = \overline{Y}$, so X is separable. Since each dense subspace must contain Y, it is sufficient to prove that X is not normal on Y.

Put $\{(t,0) : t \in \mathbb{Q}\} = \{t_n : n \in \mathbb{N}\}$ and for each $n \in \mathbb{N}$ fix W_n in the same way as in Example 5.3.2 and put

$$W = \bigcup\{W_n : n \in \mathbb{N}\}.$$

For each $n \in \mathbb{N}$ fix a sequence $P_n = \{p_n^m : m \in \mathbb{N}\}$ such that

$$p_n^m \in T_m(t_n) \cap W_n \cap Y.$$

Let $A = \bigcup\{P_n : n \in \mathbb{N}\}$ and $B = Y \setminus W$. We will prove that $\overline{A} = A \cup \{(t,0) : t \in \mathbb{Q}\}$ and $\overline{B} = B \cup \{(t,0) : t \in \mathbb{P}\}$.

$(W \cap Y) \cup \{(t,0) : t \in \mathbb{Q}\}$ is an open set disjoint from B so

$$\overline{B} \cap \{(t,0) : t \in \mathbb{Q}\} = \emptyset.$$

It is also clear that $\{(t,0) : t \in \mathbb{Q}\} \subset \overline{A}$.

Fix any point $z = (t,0)$ such that $t \in \mathbb{I}$, we need to prove that $z \notin \overline{A}$ and that $z \in \overline{B}$.

Claim. *For each two real numbers $b > a > 0$ there exists a positive real number $\varepsilon(a,b)$ such that $((t - \varepsilon(a,b), t + \varepsilon(a,b)) \times (a,b)) \cap A = \emptyset$*

Proof. The set $F = \{t_n : n \in \mathbb{N}, W_n \cap \{(r,s) \in Y : a < s\} \neq \emptyset\}$ is finite, therefore there is some $\varepsilon(a,b)$ such that $(t-\varepsilon(a,b), t+\varepsilon(a,b)) \cap F$ is empty. Now $(t-\varepsilon(a,b), t+\varepsilon(a,b)) \times (a,b)$ does not intersect A. □

Put $\varepsilon_n = \varepsilon(1/(n+2), 1/n)$ for each $n \in \mathbb{N}$. The set

$$U = \{z\} \cup \bigcup\{(t - \varepsilon_n, t + \varepsilon_n) \times (1/(n+2), 1/n) : n \in \mathbb{N}\}$$

is an open neighborhood of z disjoint from A, so $z \notin \overline{A}$.

Suppose that $z \notin \overline{B}$, so there exists an open neighborhood V' of z disjoint from B. Suppose that $V' = \{z\} \cup (V \cap Y)$, where V is open in $\mathbb{R}^2, T_n(z) \subset V$. There exists a sequence $\{\varepsilon_m : m \in \mathbb{N}\}$ of positive real numbers such that the set

$$H = \bigcup \{(t - \varepsilon_m, t + \varepsilon_m) \times (1/(n+m+2), 1/(n+m)) : m \in \mathbb{N}\}$$

is contained in V (apply compactness of the interval $[1/(n+m+2), 1/(n+m)]$). Now we can use the same arguments as in Example 5.3.2 to prove that H is not a subset of W, and it follows that $z \in \overline{B}$.

Finally, let us prove that the sets $Q = \{(t, 0) : t \in \mathbb{Q}\}$ and $I = \{(t, 0) : t \in \mathbb{P}\}$ cannot be separated by disjoint open sets in X. Take some open set V in X such that $I \subset V$. For each $x \in \mathbb{P}$ we can fix some $n_x \in \mathbb{N}$ such that $T_{n_x}((x, 0)) \subset V$ and put

$$P_n = \bigcup \{x \in \mathbb{P} : n_x = n\}$$

for each $n \in \mathbb{N}$. Since

$$\mathbb{Q} \cup \bigcup \{I_n : n \in \mathbb{N}\} = \mathbb{R},$$

Theorem 2.1.16 implies that there exists a $q \in \mathbb{Q}$ such that $q \in \overline{P_m}^{\mathcal{R}}$ for some $m \in \mathbb{N}$. Thus

$$(q, 0) \in \overline{\bigcup \{T_m((x, 0)) : x \in P_m\}} \subset \overline{V},$$

since each open neighborhood of $(q, 0)$ has to intersect $T_m((i, 0))$ for some $i \in P_m$. Thus $(q, 0) \in Q \cap \overline{V}$, so P and Q cannot be separated by disjoint open sets.

Notice that the space X is not first countable.

In the light of the previous examples, the authors of [17] raised the following problem ([17, Problem 3.4]): Is it true that the Niemytzki plane is not normal on each of its countable dense subspaces?

We answer this question in the negative by describing certain type of countable dense subspaces of $\mathbf{N}$ on which $\mathbf{N}$ is is normal.

Theorem 5.3.4. *Let G, H be disjoint closed subsets of $\mathbf{N}$. Then G and H can be separated by disjoint open sets if and only if there exist sets G_i and H_i for $i \in \mathbb{N}$ such that $G \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} G_i$, $H \cap \mathbf{L} = \bigcup_{i\in\mathbb{N}} H_i$ and $\overline{G_i}^{\mathcal{R}} \cap H = \emptyset = \overline{H_i}^{\mathcal{R}} \cap G$ for every $i \in \mathbb{N}$.*

We will use the following technical Lemma in the proof of Theorem 5.3.4.

Lemma 5.3.5. *For each $x \in \mathbf{E}$ there exists some $\iota \in \mathbb{R}^+$ such that $x \notin B_\varepsilon(y)$ implies $B_{\varepsilon/2}(y) \cap B_\iota(x) = \emptyset$ for each $y \in \mathbf{L}$ and each $\varepsilon \in \mathbb{R}^+$, $\varepsilon \leq 1$.*

Proof of Lemma 5.3.5. Without loss of generality we may assume $x = (0, a)$. Take any ι such that $\iota + \iota^2 \leq a^2/2$ and $\iota \leq a/2$. We will prove that this ι works. Let $y = (b, 0) \in \mathbf{L}$ and $\varepsilon \in \mathbb{R}^+$, $\varepsilon \leq 1$ be such that $x \notin B_\varepsilon(y)$ (and thus $\varepsilon^2 \leq b^2+(a-\varepsilon)^2$). We have to prove that $B_{\varepsilon/2}(y) \cap B_\iota(x) = \emptyset$. This fact can be reformulated as $(\iota + \varepsilon/2)^2 \leq b^2 + (a - \varepsilon/2)^2$.

Case 1: $a/2 \leq \varepsilon \leq 1$

$$(\iota + \varepsilon/2)^2 = \varepsilon^2/4 + \varepsilon\iota + \iota^2 \leq \varepsilon^2/4 + \iota + \iota^2 \leq a^2/2 + \varepsilon^2/4$$
$$\leq a\varepsilon + \varepsilon^2/4 \leq b^2 + (a - \varepsilon)^2 + a\varepsilon + \varepsilon^2/4 - \varepsilon^2 = b^2 + (a - \varepsilon/2)^2$$

Case 2: $0 < \varepsilon < a/2$

$$(\iota + \varepsilon/2)^2 = \iota^2 + \varepsilon\iota + \varepsilon^2/4 \leq \iota^2 + \iota + \varepsilon^2/4$$
$$\leq a^2/2 + \varepsilon^2/4 \leq a^2 - a\varepsilon + \varepsilon^2/4 \leq b^2 + (a - \varepsilon/2)^2.$$

□

Proof of Theorem 5.3.4. We will denote $G' = G \cap \mathbf{L}$, $H' = H \cap \mathbf{L}$.

First, let us show that if the condition is not fulfilled, then the sets G and H cannot be separated. Suppose U and V are open sets, such that $G \subset U$ and $H \subset V$. To each $x \in G'$ ($x \in H'$) assign $\varepsilon(x) \in \mathbb{R}^+$, for which $B_{\varepsilon(x)}(x) \subset U$ ($B_{\varepsilon(x)}(x) \subset V$, respectively). Now if $G_i = \{x \in G' : \varepsilon(x) > \frac{1}{i}\}$ and $H_i = \{x \in H' : \varepsilon(x) > \frac{1}{i}\}$ for $i \in \mathbb{N}$, then without lost of generality $(\exists j \in \mathbb{N})(\exists h \in \overline{G_j}^{\mathcal{R}})(h \in H')$. Otherwise G_i, H_i satisfy the given condition. This implies

$$\emptyset \neq \bigcup_{y \in G_j} B_{\varepsilon(y)}(y) \cap B_{\varepsilon(h)}(h) \subset U \cap V$$

and U and V are not disjoint.

Now let us fix sets G and H, which satisfy the condition given in the theorem, and construct the disjoint sets U and V. In the first (and crucial) step we will separate G' and H'. For $x = (t, 0) \in \mathbf{L}$ put

$$P_\varepsilon(x) = \{(r, s) \in \mathbf{E} : \varepsilon > s > (t - r)^2\} \cup \{x\}.$$

Now for $x \in G_1$ fix any $\varepsilon(x) \in (0, 1)$. For each $x = (t, 0) \in H_1$ fix an $\varepsilon(x) \in (0, 1)$ such that $\{(t', 0) \in \mathbf{L} : |t' - t| < 2\sqrt{\varepsilon(x)}\} \cap G_1 = \emptyset$. That is possible since $\overline{G_1}^{\mathcal{R}} \cap H_1 = \emptyset$. Thus

$$P_{\varepsilon(x)}(x) \cap \bigcup_{y \in G_1} P_{\varepsilon(y)}(y) = \emptyset$$

for every $x \in H_1$.

Further, we may assume that the sets G_i (H_i, respectively) are pairwise disjoint and we will continue inductively: to $x \in G_n$ (H_n, respectively) we assign $\varepsilon(x)$ in the same way. For $x = (t, 0) \in G_n$ let $\varepsilon(x) \in (0, 1)$ be such that

$$\{(t', 0) \in \mathbf{L} : |t - t'| < 2\sqrt{\varepsilon(x)}\} \cap \bigcup_{i<n} H_i = \emptyset$$

Such $\varepsilon(x)$ exists since $\overline{\bigcup_{i<n} H_i}^{\mathcal{R}} \cap G_n = \emptyset$. For x and $\varepsilon(x)$ chosen in this way

$$P_{\varepsilon(x)}(x) \cap \bigcup_{i<n} \bigcup_{y \in H_i} P_{\varepsilon(y)}(y) = \emptyset.$$

For $x \in H_n$ the construction (and also the resulting property) is similar. From the construction it follows that

$$\bigcup_{y \in G'} P_{\varepsilon(y)}(y) \cap \bigcup_{y \in H'} P_{\varepsilon(y)}(y) = \emptyset.$$

Since $B_{\varepsilon/2}(x) \subset P_{\varepsilon}(x)$ for $x \in \mathbf{L}$ and $\varepsilon \in (0,1)$,

$$U_1 = \bigcup_{x \in G'} B_{\varepsilon(x)/2}(x)$$

and

$$V_1 = \bigcup_{x \in H'} B_{\varepsilon(x)/2}(x)$$

are disjoint open sets in $\mathbf{N}$ and $G' \subset U_1$, $H' \subset V_1$.

In the second step we will separate G' from H: for each $x \in G'$ fix $\delta'(x) \in \mathbb{R}^+$ such that $B_{\delta'(x)}(x) \cap H = \emptyset$. For $x \in G'$ let $\delta(x) = \min\{\delta'(x)/2, \varepsilon(x)/2\}$. The set

$$U_2 = \bigcup_{x \in G'} B_{\delta(x)}(x)$$

is open and contains G'. We will prove that $\overline{U}_2 \cap H = \emptyset$. Let us show that $h \in H \Rightarrow h \notin \overline{U}_2$.

If $h \in H'$, then $U_1 \cap V_1 = \emptyset$ and $U_2 \subset U_1$, V_1 is open and $H' \subset V_1$. Thus $h \notin \overline{U}_2$. If $h \in H \cap \mathbf{E}$, then $h \notin B_{\delta'(x)}(x)$ for each $x \in G'$. From this and Lemma 5.3.5 it follows that there exists $\iota \in \mathbb{R}$ such that $B_\iota(h) \cap B_{\delta(x)}(x) = \emptyset$ for all $x \in G'$, so $B_\iota(h) \cap U_2 = \emptyset$ and $h \notin \overline{U}_2$. Similarly we can construct an open set V_2 such that $H' \subset V_2$, $\overline{V}_2 \cap G = \emptyset$ and $V_2 \subset V_1$, which implies $U_2 \cap V_2 = \emptyset$.

Finally, let us separate the whole sets. $\mathbf{E}$ is an open normal subspace of $\mathbf{N}$, $G \cap \mathbf{E}$ and $H \cap \mathbf{E}$ are disjoint closed subsets of $\mathbf{E}$, so there exist disjoint open subsets U_3,V_3 of $\mathbf{E}$ (and thus open in $\mathbf{N}$) such that $G \cap \mathbf{E} \subset U_3$, $H \cap \mathbf{E} \subset V_3$. Hence $U = (U_2 \cup U_3) \setminus \overline{V}_2$ and $V = (V_2 \cup V_3) \setminus \overline{U}_2$ are the desired disjoint open sets separating G and H. □

Lemma 5.3.6. $\mathbf{N}$ *is normal on* $\mathbf{E}$*.*

Proof. Consider G, H subsets of $\mathbf{E}$, $\overline{G} \cap \overline{H} = \emptyset$. We will show, that $\overline{G}$ and $\overline{H}$ fulfill the condition of Theorem 5.3.4 and thus they can be separated. Put

$$G_i = \{x \in \overline{G} \cap \mathbf{L} : B_{1/i}(x) \cap \overline{H} = \emptyset\}$$

and

$$H_i = \{x \in \overline{H} \cap \mathbf{L} : B_{1/i}(x) \cap \overline{G} = \emptyset\}$$

for $i \in \mathbb{N}$. It is obvious that $\overline{G} \cap \mathbf{L} = \bigcup_{i \in \mathbb{N}} G_i$ and $\overline{H} \cap \mathbf{L} = \bigcup_{i \in \mathbb{N}} H_i$, so it remains to show that $\overline{G_i}^{\mathcal{R}} \cap \overline{H} = \emptyset$ ($\overline{H_i}^{\mathcal{R}} \cap \overline{G} = \emptyset$, respectively).

For a contradiction assume that there is some $n \in \mathbb{N}$ and $h \in \overline{G_n}^{\mathcal{R}}$ such that $h \in \overline{H}$. Since $h \in \overline{H}$, there exists $h' \in H \cap B_{1/n}(h)$. Now $h \in \overline{G_n}^{\mathcal{R}}$,

$$B_{1/n}(h) \subset \bigcup_{x \in G_n} B_{1/n}(x)$$

and this implies that $h' \in B_{1/n}(g)$ for some $g \in G_n$. This is a contradiction. The case $(\exists n \in \mathbb{N})(\exists g \in \overline{H_n}^{\mathcal{R}})(h \in \overline{G})$ is similar. □

Corollary 5.3.7. $\mathbf{N}$ *is normal on each subset of* $\mathbf{E}$*.* □

So each countable dense subset of $\mathbf{E}$ (and such clearly exists) gives us an example of a countable dense subspace of $\mathbf{N}$ on which $\mathbf{N}$ is normal.

CHAPTER VI
RELATIVE COMPACTNESS

6.1 Relative Compactness

Definition 6.1.1. A topological space Y is *compact in* its superspace X, if every open cover of X has a finite subsystem which covers Y.

Observe that if Y is compact in X and Z is any subset of Y, then Z is compact in X. It is also easy to see that Y is compact in X if and only if Y is compact in $\overline{Y}^X$. These two facts together give the following Proposition.

Proposition 6.1.2. *If Y is compact in X and $Z \subset Y$ is closed in X, then Z is compact.*

Lemma 6.1.3. *Y is compact in X if and only if for each centered family $\mathcal{C}$ of subsets of Y the intersection $\bigcap\{\overline{P}^X : P \in \mathcal{C}\}$ is nonempty.*

Proof. Let Y be compact in X and $\mathcal{C}$ a family of subsets of Y such that the intersection $\bigcap\{\overline{P}^X : P \in \mathcal{C}\}$ is empty. Then the family $\mathcal{U} = \{X \setminus \overline{P}^X : P \in \mathcal{C}\}$ is an open cover of X. There exists a finite $\mathcal{U}' \subset \mathcal{U}$ cover of Y. Thus the family $\mathcal{C}$ is not centered.

For the other direction, let $\mathcal{U}$ be an open cover of X. Put

$$\mathcal{C} = \{Y \cap (X \setminus U) : U \in \mathcal{U}\}$$

The set $\{\overline{P}^X : P \in \mathcal{C}\}$ has empty intersection (note that $\overline{P}^X \cap Y = P$ for $P \in \mathcal{C}$), so there exists a finite set $\mathcal{C}' \subset \mathcal{C}$ such that $\bigcap \mathcal{C}' = \emptyset$. This implies that $\{U \in \mathcal{U} : Y \cap (X \setminus U) \in \mathcal{C}'\}$ is a finite cover of Y. $\square$

Theorem 6.1.4. *If X is a regular space then Y is compact in X if and only if $\overline{Y}^X$ is compact.*

Proof. If $\overline{Y}^X$ is compact then Y is compact in $\overline{Y}^X$ and thus in X.

Let Y be compact and dense in a regular space $X = \overline{Y}$. We will prove that X is compact. Let $\mathcal{U}$ be an open cover of X. Since X is regular, there exists an open cover $\mathcal{O}$ of X such that for each $O \in \mathcal{O}$ there is some $U \in \mathcal{U}$ such that $\overline{O} \subset U$. Y is compact in X so there is a finite $\mathcal{O}' \subset \mathcal{O}$ which covers Y. Thus there is a finite $\mathcal{U}' \subset \mathcal{U}$ such that

$$\bigcup\{\overline{O} : O \in \mathcal{O}'\} \subset \bigcup \mathcal{U}'.$$

Now

$$X = \overline{Y} \subset \overline{\bigcup \mathcal{O}'} = \bigcup\{\overline{O} : O \in \mathcal{O}'\} \subset \bigcup \mathcal{U}'$$

and X is compact. □

Proposition 6.1.5. *If X is a Hausdorff space, and Y is compact in X and dense in X, then X is an H-closed space.*

Proof. Let $\mathcal{C}$ be a centered family of open subsets of X. The family $\{C \cap Y : C \in \mathcal{C}\}$ is a centered family of subsets of Y since Y is dense in X. Lemma 6.1.3 and Theorem 2.2.2 now imply that X is H-closed. □

Theorem 6.1.6. *If X is Hausdorff and Y is compact in X, then Y is normal in X.*

Proof. Let us start with proving that Y is regular in X. Pick a closed subset A of X and $x \in Y \setminus A$. For each $a \in A$ there are two disjoint open subsets U_a and V_a in X such that $a \in U_a$ and $x \in V_a$. Y is compact in X so there is a finite set $A' \subset A$ such that $A \cap Y \subset \bigcup\{U_a : a \in A'\}$. The sets $\bigcup\{U_a : a \in A'\}$ and $\bigcap\{V_a : a \in A'\}$ are open disjoint sets which separate x and $A \cap Y$.

Now we will prove that Y is normal in X in the same way. Pick two nonempty disjoint closed subsets A and B of X. For each $a \in A$ there are two disjoint open subsets U_a and V_a in X such that $a \in U_a$ and $B \cap Y \subset V_a$. Y is compact in X, so there is a finite set $A' \subset A$ such that $A \cap Y \subset \bigcup\{U_a : a \in A'\}$. The sets $\bigcup\{U_a : a \in A'\}$ and $\bigcap\{V_a : a \in A'\}$ are open disjoint sets, and separate $A \cap Y$ and $B \cap Y$. □

Corollary 6.1.7. *If X is Hausdorff and Y is compact in X, then Y is a regular space.*

Proof. This follows from Theorem 6.1.6 and Proposition 4.1.2. □

Definition 6.1.8. A topological space Y is *potentially compact* if there is a Hausdorff space X such that Y is compact in X.

Proposition 6.1.9. *Every potentially compact space is regular and every Tychonoff space is potentially compact.*

Proof. If Y is a Tychonoff space then Y is compact in βY. The rest of the Proposition is Corollary 6.1.7. □

Theorem 6.1.10. *If a Hausdorff space A is a preimage of a potentially compact space under a perfect mapping, then A is a potentially compact space.*

Proof. Let Y be compact in X and let $f : A \to Y$ be a perfect mapping onto Y. We need to construct a Hausdorff space Z in which A is compact. Put $S = X \setminus Y$ and $Z = A \cup S$. A base $\mathcal{B}$ of the topology on Z will be defined as follows

$$\mathcal{B} = \{U : U \text{ open in } A\} \cup \{O(s, U) : U \text{ open in } X, s \in S \cap U\}$$

where $O(s, U) = \{s\} \cup f^{-1}[U \cap Y]$.

Claim 1. *Z is a Hausdorff space.*

Proof. Let a and b be two distinct points in Z. If $a, b \in A$ then there are two disjoint open sets separating a and b since A is Hausdorff. If $a, b \in S$, then there are disjoint open subsets U and V of X separating a and b in X and $O(a, U)$, $O(b, V)$ separate a and b in Z. So let $a \in A$ and $b \in S$. Put $y = f(a)$. Now y and b are two distinct points in X thus there are disjoint open sets U, V in X such that $y \in U$ and $b \in V$. Then $f^{-1}[U \cap Y]$ and $O(b, V)$ are neighborhoods separating a and b in Z. □

Claim 2. *For $E \subset A$ and $s \in S$; $s \in \overline{E}^Z$ if and only if $s \in \overline{f[E]}^X$.* □

Let $\mathcal{C}$ be a centered family of closed subsets of A which is closed under finite intersections. Put $\mathcal{C}_X = \{f[C] : C \in \mathcal{C}\}$ and $\overline{\mathcal{C}}_X = \{\overline{f[C]}^X : C \in \mathcal{C}\}$. The system $\mathcal{C}_X$ is a centered family of closed subsets of Y, thus the set $M = \bigcap \overline{\mathcal{C}}_X$ is nonempty. If there is some $c \in S \cap M$ then Claim 2 implies $\bigcap \mathcal{C} \neq \emptyset$.

Assume $M \subset Y$ and pick some $c \in M$. Then $B = f^{-1}[\{c\}]$ is a compact subset of A, and $\mathcal{C} \restriction B = \{C \cap B : C \in \mathcal{C}\}$ is a centered family of closed subsets of B which is closed under finite intersections, and $\mathcal{C} \restriction B$ does not contain the empty set (because $c \in \overline{f[C]}^X \cap Y = f[C]$ for each $C \in \mathcal{C}$). Hence $\emptyset \neq \bigcap \mathcal{C} \restriction B \subset \bigcap \mathcal{C}$ and A is compact in Z. □

The next example was constructed in [11] and it shows that being potentially compact is strictly weaker than being Tychonoff.

Example 6.1.11 ([11])**.** Let $f : X \to Y$ be a perfect mapping of a non-Tychonoff space X onto a Tychonoff space Y. Such an example of f, X and Y was constructed in [7]. Theorem 6.1.10 now implies that X is a non-Tychonoff potentially compact space.

There also exists an infinite potentially compact space on which every continuous real valued function is constant. Such a space was constructed in [6].

Proposition 6.1.12. *Let Y be an R-closed space. Then Y is potentially compact if and only if Y is compact.*

Proof. Let Y be a R-closed space compact in some space X. Choose any $x \in X \setminus Y$ and put $Y' = Y \cup \{x\}$. Y' is also compact in X and so Y' is regular (Corollary 6.1.7) and thus Y is closed in Y'. That means that for each $x \in X \setminus Y$ is $x \notin \overline{Y}^X$, i. e. Y is closed in X. Proposition 6.1.2 now implies that Y is compact. □

The last proposition offers an easy way to construct a regular space that is not potentially compact. That shows that the property of being potentially compact is strictly stronger than regularity.

Example 6.1.13. The Jones space (the space obtained by the Jones machine) over $(\omega_1 + 1) \times (\omega_1 + 1) \setminus \{(\omega_1, \omega_1)\}$ is a non-Tychonoff regular R-closed space, so Proposition 6.1.12 implies that it is an example of a regular non-potentially compact space. For details see [6].

6.2 Internal Compactness

Definition 6.2.1. A topological space Y is *internally compact in X* if every subspace of Y which is closed in X, is compact.

Theorem 6.2.2. *If the space Y is internally compact in a Hausdorff space X, then Y is internally normal in X.*

Proof. the proof of this Theorem is similar to the proof of Theorem 6.1.6. □

Theorem 6.2.3 states that the Jones machine introduced in Example 2.1.14 preserves internal compactness in the following sense. If a non-normal space Y is a subspace of X, then $J(Y)$ can be considered as a subspace of $J(X)$ in the natural way; the new point (in Example 2.1.14 denoted by z) is considered to be the same for both $J(Y)$ and $J(X)$. For A_0 and A_1 in $J(Y)$ we can use a pair of closed subsets of Y such that A_0 and A_1 cannot be separated in Y and $\overline{A_0}^X \cap \overline{A_1}^X = \emptyset$. For A_0 and A_1 in $J(X)$ we use $\overline{A_0}^X$ and $\overline{A_1}^X$.

Theorem 6.2.3. *If a non-normal space Y is internally compact in a regular space X, then $J(Y)$ is internally compact in $J(X)$.*

Proof. We will use the notation established in Example 2.1.14. Pick any centered system $\mathcal{C}$ of subsets of $J(Y)$ such that all sets in $\mathcal{C}$ are closed in $J(X)$. We have to prove that the intersection $\bigcap \mathcal{C}$ is nonempty.

Assume that $z \notin Z$ for some $Z \in \mathcal{C}$. Then $q^{-1}[Z] \subset Y \times n$ for some $n \in \omega$. Since $q^{-1}[Z] \cap (X \times \{j\})$ is a subset of j-th copy of Y and it is closed in j-th copy of X for each $j \in n$ and since Y is internally compact in X, the set $q^{-1}[Z]$ is a finite sum of compact sets and thus compact. Hence $\emptyset \neq \bigcap\{q^{-1}[C] : C \in \mathcal{C}\} = q^{-1}[\bigcap \mathcal{C}]$ and $J(Y)$ is internally compact in $J(X)$. □

Closely related to Theorem 6.1.4, Theorem 6.1.6 and Corollary 6.1.7, is the question, whether a certain version of relative compactness does imply any absolute version of some separation axiom for the smaller space Y. Arhangel'skii formulated one version of this problem in [3] as Question 9: Let Y be a subspace of a Hausdorff space X such that Y is internally compact in X. Is then true that Y is Tychonoff? What if we assume X to be regular?

A closely related Question 10 was also given in article [3]: Let Y be a subspace of a regular space X such that Y is internally normal in X. Is Y Tychonoff? We will construct examples that provide negative answers to these questions.

Example 6.2.5 was constructed by Eva Murtinová and gives a negative answer to the first part of Question 9 from [3].

Lemma 6.2.4. *For each ultrafilter $\mathcal{U}$ on ω there exists a maximal almost disjoint (MAD) system $\mathcal{A}$ on ω such that $\mathcal{A} \cap \mathcal{U} = \emptyset$.*

Proof. Fix any ultrafilter $\mathcal{U}$ and consider the system of all almost disjoint (AD) systems on ω satisfying the condition given in the Lemma, ordered by inclusion. This system is nonempty since the empty set is such a AD system. Since this system is closed under the union of increasing chains, Zorn's Lemma implies that there is a maximal such AD system $\mathcal{A}$. We will show that $\mathcal{A}$ is a MAD system. If not, there is an infinite set $A \in \mathcal{P}(\omega) \setminus \mathcal{A}$ such that $\mathcal{A} \cup \{A\}$ is an AD system. Split A into two infinite sets A_0, A_1 such that $A_0 \cup A_1 = A$. Since $\mathcal{U}$ is an ultrafilter, at least one of sets A_0 and A_1 does not belong to $\mathcal{U}$. Denote this set by A_i. Now $\mathcal{A} \cup \{A_i\}$ is an AD system contradicting the maximality of $\mathcal{A}$. □

Example 6.2.5. We will construct a non-regular space internally compact in a Hausdorff space. The idea is to construct a space $X = Y \cup Z$ with Y non-regular such that all "nontrivial" infinite subsets of Y have cluster points in Z. Then there are only few closed subsets of X contained in Y and these are arranged to be compact.

Fix a free ultrafilter $\mathcal{U}$ on ω and let $\mathcal{A}$ be a MAD system on ω constructed as in Lemma 6.2.4. Put $Y = \{y\} \cup ((\omega + 1) \times \omega)$, $F = \{\omega\} \times \omega \subset Y$. Let us endow the set $X = Y \cup \mathcal{A}$ with a topology by declaring each point of $\omega \times \omega$ isolated,

$$\{((\omega + 1) \setminus n_0) \times \{n\} : n_0 \in \omega\}$$

an open base in $(\omega, n) \in F$,

$$\{\{y\} \cup (\omega \times U) : U \in \mathcal{U}\}$$

an open base at y and

$$\{\{A\} \cup ((\omega + 1) \times (A \setminus n_0)) : n_0 \in \omega\}$$

an open base in $A \in \mathcal{A}$. This obviously defines a Hausdorff topology on X, while the closed subset F of Y cannot be separated from y, hence Y is not regular.

It remains to show that Y is internally compact in X. Consider a closed subset C of X, $C \subset Y$ and an infinite $B \subset C$ whose cluster point is to be found in C. Since C is closed, the set

$$\{n \in A : C \cap ((\omega + 1) \times \{n\}) \neq \emptyset\}$$

is finite for every $A \in \mathcal{A}$. Thus

$$N = \{n \in \omega : C \cap ((\omega + 1) \times \{n\}) \neq \emptyset\}$$

is almost disjoint from $\mathcal{A}$. It follows that N is finite. As B is infinite, there is an n_0 such that $B \cap (\omega \times \{n_0\})$ is infinite. Now (ω, n_0) is a cluster point of B.

Theorem 6.2.6. *There exists a non-normal space Y which is internally compact in a zero-dimensional space X.*

Proof. Through this proof, all points in the Čech-Stone compactification βD of any discrete space D will be identified with ultrafilters on D. For any discrete space D we will also define a subspace γD of βD as

$$\gamma D = \{p \in \beta D : (\exists P \in p)\, |P| \leq \omega\}.$$

Let A and B be two disjoint sets of size ω_2, put $C = A \times B$ and π_A, π_B will denote the natural projections of C onto A and B. The underlying sets for X and Y are

$$Y = A \cup B \cup C$$

and

$$X = \gamma A \cup \gamma B \cup \gamma C$$

and the topology is defined as follows: γC is an open subspace of X, other basic open sets of X are

$$O \cup \overline{\pi_A^{-1}[O \cap A] \setminus K}^{\gamma C}$$

for $|K| \leq \omega$, O open subset of γA and

$$O \cup \overline{\pi_B^{-1}[O \cap B] \setminus K}^{\gamma C}$$

for $|K| \leq \omega$, O open subset of γB. It is a routine to check, that we have defined a base for a topology on X correctly.

Claim 1. *X is a Hausdorff space.*

Proof. We need to show that each two distinct points a and b in X can be separated by disjoint open neighborhoods. If $a, b \in \gamma C$, then $\gamma C \subset \beta C$ implies that these two points can be separated. If $a, b \in \gamma A$, then there are disjoint open sets U and V separating a and b in γA thus

$$U \cup \overline{\pi_A^{-1}[U \cap A]}^{\gamma C}$$

and

$$V \cup \overline{\pi_A^{-1}[V \cap A]}^{\gamma C}$$

separate a and b in X. Case $a, b \in \gamma B$ is similar. If $a \in \gamma A$ and $b \in \gamma B$, then fix countable sets $U \subset A$ and $V \subset B$ such that $a \in \overline{U}^{\gamma A}$ and $b \in \overline{V}^{\gamma B}$. The sets

$$U \cup \overline{\pi_A^{-1}[U] \setminus (U \times V)}^{\gamma C}$$

and

$$V \cup \overline{\pi_B^{-1}[V] \setminus (U \times V)}^{\gamma C}$$

separate a and b in X. And if $a \in \gamma A$, $b \in \gamma C$, then fix countable sets $U \subset A$ and $V \subset C$ such that $a \in \overline{U}^{\gamma A}$ and $b \in \overline{V}^{\gamma C}$. The sets

$$U \cup \overline{\pi_A^{-1}[U] \setminus V}^{\gamma C} \text{ and } \overline{V}^{\gamma C}$$

separate a and b in X. □

Claim 2. *X is a zero-dimensional space.*

Proof. For each $x \in \gamma C$ there is an open base at x which consists of sets of form γK where $K \subset C$ such that $|K| \leq \omega$, and for such K is $\gamma K = \overline{K}^X$. For $x \in \gamma A$ there is an open base at x which consists of sets of form

$$B = \gamma O \cup \overline{\pi_A^{-1}[O \cap A] \setminus K}^{\gamma C}$$

where $K \subset C$, $|K| \leq \omega$ and $O \subset A$ such that $|O| \leq \omega$. For such O and K is B closed in X. The case $x \in \gamma B$ is similar. □

Claim 3. *A and B are closed subsets of Y which cannot be separated by disjoint open sets in Y. Moreover, $\overline{A}^X \cap \overline{B}^X = \emptyset$.*

Proof. Let U be open in Y and let $A' \subset U \cap A$ be some set of size ω_1. We will show that $\overline{U} \cap B$ is nonempty. For each $a \in A'$ fix a $K_a \in [C]^\omega$ such that

$$\pi_A^{-1}[\{a\}] \setminus K_a \subset U.$$

Hence

$$\pi_A^{-1}[A'] \setminus K \subset U$$

where

$$K = \bigcup\{K_a : a \in A'\}$$

and notice that $|K| \leq \omega_1$. Each

$$b \in B \setminus \pi_B[K]$$

(and such clearly exists) is an element of $\overline{U}$ because

$$\pi_B^{-1}[\{b\}] \cap U \supset A' \times \{b\}$$

and the product $A' \times \{b\}$ has cardinality ω_1.

$\overline{A}^X \cap \overline{B}^X = \emptyset$ is a consequence of $\overline{A}^X = \gamma A$ and $\overline{B}^X = \gamma B$. □

Claim 4. *If $G \subset Y$ is closed in X then $|G| < \omega$.*

Proof. Suppose $G \subset Y$, $\omega \leq |G|$. Then at least one of the sets $G \cap A$, $G \cap B$ and $G \cap C$ must be infinite. Assume that $\omega \leq |G \cap C|$. Then $\overline{G \cap C}^{\gamma C} \setminus (G \cap C) \subset \overline{G} \setminus Y$ is nonempty. Thus G is not closed. Cases $\omega \leq |G \cap A|$ and $\omega \leq |G \cap B|$ work similarly. □

The last claim implies that Y is internally compact in X and the Theorem is proved. □

Corollary 6.2.7. *There exists a non-Tychonoff space Y which is internally compact in a regular T_1 space X.*

Proof. Use Theorem 6.2.6 and Theorem 6.2.3. □

Corollary 6.2.7 provides an answer to the second part of Question 9. From Proposition 6.2.2 we now also get that there exists a non-Tychonoff space Y which is internally normal in a larger space X and that gives a negative answer to Question 10.

BIBLIOGRAPHY

[1] A. V. Arhangel'skii, *Normality and dense subspaces*, Proceedings AMS (2001), no. 130, 283–291.

[2] ———, *From Classic Topological Invariants to Relative Topological Properties*, Scientiae Mathematicae Japonicae (2002), no. 1, 153–201.

[3] ———, *Relative normality and dense subspaces*, Topology and its Applications (2002), no. 123, 27–36.

[4] A. V. Arhangel'skii and H. M. M. Genedi, *Beginnings of the Theory of Relative Topological Properties*, General Topology, Spaces and Mappings, MGU, Moscow (1989), 3–48.

[5] A. V. Arhangel'skii and J. Tartir, *A characterization of compactness by a relative separation property*, Questions and Answers in General Topology (1996), no. 14, 49–52.

[6] A. V. Arhangel'skii and I. V. Yaschenko, *Relatively compact spaces and separation properties*, Comment. Math. Univ. Carolinae (1996), no. 37:2, 343–348.

[7] J. Chaber, *Remarks on open-closed mappings*, Fundamenta Mathematicae (1972), no. 74, 197–208.

[8] C. Costantini and A. Marcone, *Extensions of functions which preserve the continuity on the original domain*, Topology and its Applications (2000), no. 103, 131–153.

[9] R. Engelking, *General Topology*, Heldermann Verlag Berlin, 1989.

[10] P. M. Gartside and A. Glyn, *Relative Separation Properties*, Preprint.

[11] K. P. Hart and J. Vermeer, *Non-Tychonoff e-compactifiable spaces*, Proceedings AMS (1983), no. 89:4, 725–729.

[12] T. Jech, *Set Theory*, Springer-VerLag, Heidelberg, 2002.

[13] F. B. Jones, *Hereditary separable, non-completely regular spaces*, Topology Conf., Virginia Polytechnic Inst. and State U. 1973, Springer Lecture Notes in Mathematics (1974), no. 375, 149–152.

[14] M. Katětov, *Über H-abgeschlossene und bikompakte Räume*, Časopis pěst. Mat. Fys. (1940), no. 69, 36–49.

[15] M. E. Rudin, *A normal space X for which $X \times I$ is not normal*, Fundamenta Mathematicae (1971), no. 73, 179–186.

[16] E. V. Schepin, *Real-valued functions, and spaces close to normal*, Sib. Matem. Journ. (1972), no. 13:2, 1182–1196.

[17] M. G. Tkačenko, V. V. Tkachuk, R. G. Wilson, and I. V. Yaschenko, *Normality on dense countable subspaces*, Scientiae Mathematicae Japonicae (2001), no. 4, 1–8.