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Vector spaces are free modules over a division ring. Steinitz proved that they have the following property: given a linearly independent subset S of the module, there is a basis of the module which contains S. In general, free modules over a ring R do not have the Steinitz property. A ring R having the property is called a Steinitz ring. The purpose of this study is to determine precisely when a ring R is a Steinitz ring. We shall show that R is a right Steinitz ring if and only if R is a local, right perfect ring.

STEINITZ RINGS

by

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CHAPTER I

PRELIMINARIES

The complete characterization of Steinitz rings requires a certain degree of sophistocation in fundamental concepts in algebra. In an effort to present this paper so that it is readable for anyone with a minimal background in algebra, certain concepts need to be defined and discussed. In order to preserve continuity later, we shall establish the necessary foundations initially and refer back to it as needed.

This chapter will include definitions and theorems that are essential background for the study of right Steinitz rings. Proofs of theorems will be omitted when they are readily accessible in Joachim Lambek's Lectures on Rings and Modules [4] or when the proofs are well-known, but involve material that is not directly related to this study.

1.1. Basic Results.

Although the concepts of product and coproduct are rather standard, there is some variability among texts. Therefore in order to avoid any ambiguity we shall define these concepts precisely.

1.1 Definition: Let $\{A_i \mid i \in I\}$ be a set of right R-modules. The set of mappings $a: I \to \bigcup_{i\in I} A_i$ where $a(i) \in A_i$ for each $i \in I$, denoted $\prod_{i\in I} A_i$, is called the product of the set $\{A_i \mid i \in I\}$, and is itself a right R-module with operations defined componentwise:

$$(a + b)(i) = a(i) + b(i)$$

$$(ar)(i) = a(i)r, \qquad a,b \in \prod_{i\in I} A_i,\ r \in R.$$

The subset of $\prod_{i\in I} A_i$ consisting of the mappings where $a(i) \neq 0$ for at most finitely many $i \in I$ is easily seen to be a submodule and is called the coproduct or direct sum of the set $\{A_i \mid i \in I\}$ of R-modules. (Note that the product and coproduct agree for finite collections of R-modules.) The coproduct is denoted $\coprod_{i\in I} A_i$.

Two standard mappings are frequently associated with the product and coproduct. The canonical epimorphisms, or projections, which are generally associated with the product, and the canonical monomorphisms, or injections, generally associated with the coproduct, are defined as follows.

Let $\{\pi_j \mid j \in I\}$ be the canonical epimorphisms (projections) associated with the product. Then for each $j \in I$ the mapping $\pi_j: \prod_{i\in I} A_i \to A_j$ is defined by $\pi_j(f) = f(j)$.

Let $\{\kappa_j \mid j \in I\}$ be the canonical monomorphisms (injections) associated with the coproduct. Then for each $j \in I$ the mapping $\kappa_j\colon A_j \to \coprod_{i \in I} A_i$ is defined by

$$\kappa_j(x)(i) = \begin{cases} 0 \text{ if } i \neq j \\ x \text{ if } i = j \end{cases} \quad \text{for } x \in A_j.$$

A standard result concerning the coproduct, whose proof is omitted, is that of associativity.

<u>1.2 Theorem</u>: Let $\{M_i \mid i \in I\}$ be a set of R-modules. Let I be the disjoint union of the sets $\{I_j \mid j \in J\}$. Then $\coprod_{i \in I} M_i \cong \coprod_{j \in J} (\coprod_{i \in I_j} M_i)$.

<u>1.3 Theorem</u>: If $M_i \cong B \times C$ for each $i = 1,2,\ldots$, then $\coprod_{i=1}^{\infty} M_i \cong B \times \coprod_{i=1}^{\infty} M_i$.

<u>1.4 Theorem</u>: If $A_i \subseteq B_i$, then $\coprod_{i \in I} B_i/A_i \cong \coprod_{i \in I} B_i / \coprod_{i \in I} A_i$.

Proof: To prove the theorem, let us define the map $\Psi\colon \coprod_{i \in I} B_i \to \coprod_{i \in I} B_i/A_i$, as follows. For $b \in \coprod_{i \in I} B_i$, $\Psi(b)(i) = b(i) + A_i$. It is easy to show that Ψ is a homomorphism. Ker $\Psi \cong \coprod_{i \in I} A_i$, so Ψ is a monomorphism, and it is clearly onto. So Ψ is an isomorphism. By the First Isomorphism Theorem we know that Im $\Psi \cong \coprod_{i \in I} B_i/\text{Ker } \Psi$. Conclude that $\coprod_{i \in I} B_i/A_i \cong \coprod_{i \in I} B_i / \coprod_{i \in I} A_i$. //

We shall say that an R-module A is <u>simple</u> if it has exactly two submodules, 0 and A. A useful property of simple modules is the following. For the proof refer to [4, p. 20].

<u>1.5 Theorem</u>: Let M be a right ideal of R, then R/M is a simple right R-module if and only if M is a maximal right ideal.

We shall say that an element r of the ring R is right invertible if there exists an element s of R so that rs = 1. And we say that r is a unit if it is both right invertible and left invertible. (It is not difficult to see that the two "inverses" are the same.) Since we know that a division ring is a ring in which every nonzero element is a unit, it is not difficult to see that any of the following equivalent conditions are necessary and sufficient to say that a ring R is a division ring.

1.6 Theorem: The following conditions concerning the ring R are equivalent:

(1) 0 is a maximal right ideal.

(2) R is simple as a right R-module.

(3) Every nonzero element is right invertible.

(4) Every nonzero element is a unit.

Proof: See [4, p. 51].

I.2. Exact Sequences, Projectives, and Frees.

This section will be devoted to a discussion of exact sequences, projective modules, and free modules. We begin by defining exact sequence and what it means to say an exact sequence splits.

1.7 Definition: Let $\{A_i\}$ be a nonempty sequence of R-modules with a corresponding sequence of mappings $f_i: A_i \to A_{i-1}$ so that $\operatorname{Ker} f_i = \operatorname{Im} f_{i+1}$. Then the sequence

$$\cdots \longrightarrow A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \longrightarrow \cdots$$

is called an exact sequence. An exact sequence of the form

$$0 \longrightarrow A \xrightarrow{f} B \xrightarrow{\pi} C \longrightarrow 0$$

is called a short exact sequence; f is a monomorphism and π is an epimorphism. Note that $A \cong \operatorname{Im} f$, a submodule of B, and $C \cong B/\operatorname{Im} f = B/\operatorname{Ker} \pi$.

1.8 Definition: An exact sequence $M \xrightarrow{\mu} S \longrightarrow 0$ is said to split if there is a map $S \xrightarrow{\rho} M$ so that $\mu\rho = 1_S$ (the identity map on S). An exact sequence $0 \longrightarrow S \xrightarrow{j} M$ splits if there is a map $M \xrightarrow{\kappa} S$ so that $\kappa j = 1_M$. The maps ρ and κ are called splitting homomorphisms. Given that the exact sequence $A \longrightarrow M \underset{\tau}{\overset{\mu}{\rightleftarrows}} S \longrightarrow 0$ splits where τ is the splitting homomorphism, we then know that $\mu\tau = 1_S$, so τ is a monomorphism. Furthermore, we may write $M = \operatorname{Ker} \mu \oplus \operatorname{Im} \tau$, and $\operatorname{Im} \tau \cong S$.

A definition of a free module would presuppose familiarity with the concepts of basis and independence, hence, to avoid possible ambiguity, we shall define these terms prior to defining free module.

1.9 Definition: Let M be a right R-module. Then the set $\{m_i \mid i \in I\}$ of $m_i \in M$ is *independent* provided $\sum_{i \in I} m_i r_i = 0$, where $r_i \in R$, only if $r_i = 0$ for all $i \in I$.

1.10 Definition: Let M be a right R-module. Then a *basis* for M is a set $B = \{b_i \mid i \in I\}$ so that B is independent and B generates M, that is, every element $m \in M$ can be expressed uniquely in the form $m = \sum_{i \in I} b_i r_i$, where $r_i \neq 0$ for at most finitely many $i \in I$.

Now we may define free module.

1.11 Definition: Let F be an R-module. Then F is *free* if it has a basis.

The relationship between free modules and projective modules is very important to this study, hence we shall now define projectivity.

1.12 Definition: An R-module P is called *projective* if given the diagram

$$\begin{array}{ccccc} & & P & & \\ & & \downarrow{\scriptstyle \mu} & & \\ A & \xrightarrow{\pi} & B & \longrightarrow & 0 \end{array}$$

there is a mapping $\bar{\mu}$ so that in the diagram

$$\begin{array}{ccccc} & & P & & \\ & {\scriptstyle \bar{\mu}}\swarrow & \downarrow{\scriptstyle \mu} & & \\ A & \longrightarrow & B & \longrightarrow & 0 \end{array}$$

$\pi\bar{\mu} = \mu$. (The latter diagram is said to be commutative.)

1.13 Theorem: A module P is projective if and only if every exact sequence $M \longrightarrow P \longrightarrow 0$ splits.

Proof: See [3, p. 8].

As in a vector space, a mapping whose domain is a free module is completely determined by specifying its action on base elements. Any set map of a base of a free module into a module can be extended to a linear map. So maps of free modules are often given by specifying set maps on base elements.

1.14 Theorem: Free modules are projective.

Proof: Suppose we have the diagram

$$\begin{array}{ccccc} & & F & & \\ & & \downarrow{\scriptstyle \mu} & & \\ A & \xrightarrow{\pi} & C & \longrightarrow & 0 \end{array} \qquad \text{(row exact)}.$$

Let us exhibit an R-homomorphism $\bar{\mu}: F \longrightarrow A$ so that the diagram commutes. Let B be a base for F. Consider the set $\prod_{b \varepsilon B} \pi^{-1}(\mu(b))$. The set is not empty since π is an epimorphism. Choose $f \varepsilon \prod_{b \varepsilon B} \pi^{-1}(\mu(b))$. Then we have a set map f from B into A so that $\pi f = \mu|_B$. Let $\bar{\mu}$ be the linear map of F to A induced by f. Then $\pi\bar{\mu} = \mu$, so F is projective. //

1.15 Theorem: An R-module F is free is and only if it is isomorphic to a coproduct of copies of R_R.

Proof: Let F be a free R-module. Then F has a basis $B = \{b \varepsilon B\}$. For each $x \varepsilon F$, $x = \sum_{b \varepsilon B} br_b$. Define $\alpha(x): F \longrightarrow \coprod_{b \varepsilon B} (R_R)_b$ by $\alpha(x)(b) = r_b$. The map α is easily seen to be a well defined isomorphism, hence $F \cong \coprod_{b \varepsilon B} (R_R)_b$.

Assume F is isomorphic to a coproduct of copies of R. A coproduct $\coprod_{i \varepsilon I} ({}_R R)_i$ of copies of R has a standard base, $\{\kappa_i(1) \mid i \varepsilon I\}$, hence is free. Conclude that F is free. //

<u>1.16 Theorem</u>: $\coprod_{i \varepsilon I} M_i$ is a projective module if and only if each M_i for $i \varepsilon I$ is a projective module.

Proof: See [4, p. 82].

<u>1.17 Theorem</u>: Every module is the image of a free module.

Proof: Let M be an R-module and G a generating set for M. Let Ψ be a set mapping from a set A onto G. There is a free module F with base A, and Ψ induces a map from F to M. Ψ must be onto M since its image contains a generating set for M. //

<u>1.18 Theorem</u>: Let P be an R-module. Then P is projective if and only if P is isomorphic to a direct summand of a free module.

Proof: Assume P is projective. Let F be a free module. Then there is a map from F onto P, $F \xrightarrow{\pi} P \longrightarrow 0$. Since P is projective, the exact sequence splits and P is isomorphic to a direct summand of F.

Assume P is isomorphic to a direct summand of a free module, F. Since free modules are projective and from 1.16 Theorem direct summands of projective modules are projective, then P is projective. //

An immediate corollary to the preceding theorem is that a module is projective if and only if it is isomorphic to a direct summand of a free module with an infinite base. For we can always add on as a direct summand another free module with an infinite base if needed.

Another result we need to include in this section is that nonzero free modules have zero annihilator, but this presupposes familiarity with the annihilator of a module.

1.19 Definition: Let M Be an R-module. Then the set $\{r \mid mr = 0 \text{ for all } m \in M\}$ is called the annihilator of M. Note that the annihilator of M is clearly a two-sided ideal of R.

1.20 Theorem: Every nonzero free module has zero annihilator.

Proof: Let F be a free R-module with base $B = \{b_i \mid i \in I\}$. For any base element b, if r annihilates F then $br = 0$ which means that $r = 0$. //

A final result we need to include in this section is the following.

1.21 Theorem: If every simple R-module is free, then R is a division ring.

Proof: Let S be a simple R-module. Since S is free, S is isomorphic to a coproduct of copies of R_R by 1.15 Theorem. Since S is simple, it cannot be isomorphic to more than one copy of R, so $S \cong R$ and R_R is simple, hence a division ring, by 1.6 Theorem. //

I.3. Functors.

<u>1.22 Definition</u>: If $\mathcal{C}$ is a class of modules and maps, and $\mathcal{D}$ is a class of modules and maps, we shall say that $F\colon \mathcal{C} \longrightarrow \mathcal{D}$ is a (covariant) <u>functor</u> if, for all A,B, modules in $\mathcal{C}$ and f, a map $A \xrightarrow{f} B$, there exist F(A), F(B), modules in $\mathcal{D}$ and F(f), a map in $\mathcal{D}$, such that $F(f)\colon F(A) \longrightarrow F(B)$. We also require that F satisfy the following rules:

(1) F applied to the commutative diagram

$$\begin{array}{ccc} & B & \\ {}^{f}\nearrow & & \searrow^{g} \\ A & \xrightarrow[gf]{} & C \end{array}$$

yields the commutative diagram

$$\begin{array}{ccc} & F(B) & \\ {}^{F(f)}\nearrow & & \searrow^{F(g)} \\ F(A) & \xrightarrow[F(gf)]{} & F(C) \end{array} \quad .$$

(2) If $i\colon A \longrightarrow A$ is the identity map on A, then F(i) is the identity map on F(A).

A functor H is called an <u>exact</u> <u>functor</u> if it preserves exact sequences. That is, if

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

is an exact sequence, then

$$0 \longrightarrow H(A) \longrightarrow H(B) \longrightarrow H(C) \longrightarrow 0$$

is an exact sequence also. If <u>only</u>

$$H(A) \longrightarrow H(B) \longrightarrow H(C) \longrightarrow 0$$

is exact, we say that H is right exact; and if only

$$0 \longrightarrow H(A) \longrightarrow H(B) \longrightarrow H(C)$$

is exact, we say that H is left exact.

1.23 Definition: If A and B are R-modules, the additive group of all R homomorphisms of A into B is denoted $\mathrm{Hom}_R(A,B)$ or Hom (A,B).

If H is a functor from R-modules to S-modules and if for all $\alpha, \beta \in \mathrm{Hom}_R(A,B)$ one has $H(\alpha + \beta) = H(\alpha) + H(\beta)$, then H is called an additive functor. It is well-known (see [2, p. 20]) that additive functors map split exact sequences to split exact sequences. We will need this fact in Chapter III.

The tensor product of modules is vital to our study, so we define it here. Let A_R, ${}_RB$ be given right and left R-modules, and G an abelian group. We shall define the tensor product in the following way:

1.24 Definition: A map from A x B into G is called balanced if

$$\phi(a_1 + a_2, b_1) = \phi(a_1, b_1) + \phi(a_2, b_1),$$
$$\phi(a_1, b_1 + b_2) = \phi(a_1, b_1) + \phi(a_1, b_2),$$
$$\phi(a_1 r, b_1) = \phi(a_1, rb_1),$$

where $a_1, a_2 \in A$; $b_1, b_2 \in B$; and $r \in R$.

1.25 Definition: A tensor product of A_R , ${}_RB$ is an abelian group X together with a balanced map $\theta: A \times B \longrightarrow X$ such that given any abelian group Y and a balanced map $\tau: A \times B \longrightarrow Y$, there exists a unique map $\bar{\tau}$ of abelian groups such that $\bar{\tau}\theta = \tau$. Elements of the tensor product are of the form $\sum_{i=1}^{n} a_i \otimes b_i$ and have the following properties:

$$(a_1 + a_2) \otimes b_1 = a_1 \otimes b_1 + a_2 \otimes b_1,$$
$$a_1 \otimes (b_1 + b_2) = a_1 \otimes b_1 + a_1 \otimes b_2,$$
$$a_1 r \otimes b_1 = a_1 \otimes rb_1,$$

for $a_1, a_2 \in A$; $b_1, b_2 \in B$; and $r \in R$.

Now we shall briefly describe how the tensor product is a functor.

Fix a left R-module, ${}_RC$ and show that $\otimes C$ is a functor from Mod R to Mod Z. Given $A \xrightarrow{\alpha} B$, define $A \otimes C \xrightarrow{\alpha\otimes 1} B \otimes C$ as follows. Take $\alpha \times 1: A \times C \longrightarrow B \times C$ where $\alpha \times 1(a,c) = (\alpha(a), c)$. Composing this with the canonical balanced map from $B \times C$ to $B \otimes C$ we have a balanced map from $A \times C$ to $B \otimes C$. Then by definition there is a unique map, denoted $\alpha \otimes 1$, from $A \otimes C$ to $B \otimes C$ such that the diagram

$$\begin{array}{ccc} A \otimes C & \xrightarrow{\alpha\otimes 1} & B \otimes C \\ \uparrow & & \uparrow \\ A \times C & \xrightarrow[\alpha \times 1]{} & B \times C \end{array}$$

commutes. It is easy now to check that the functor properties hold.

It is well-known (see [2, chapter 2]) that $\otimes_R C$ is a right exact, additive, covariant functor from R-modules to abelian groups.

A functor we shall frequently use is defined in the following way. Let F and G be R-modules and $\alpha: F \longrightarrow G$ a homomorphism. Then a functor U from R-modules to R/J-modules can be illustrated by the following diagram

$$\begin{array}{ccc} F & \xrightarrow{\alpha_F} & F/FJ = UF \\ \alpha \downarrow & & \downarrow U\alpha \\ G & \xrightarrow{\alpha G} & G/GJ = UG \end{array}$$

where α_F and α_G are canonical epimorphisms and $U\alpha(f + FJ) = g + GJ$. The above square always commutes. It is not difficult to see that $U\alpha$ is well-defined. To check commutativity, trace an element $f \varepsilon F$ through the maps.

$$\begin{array}{ccc} f & \longmapsto & f + FJ \\ \downarrow & & \downarrow \\ \alpha(f) & \longmapsto & \alpha(f) + GJ \end{array} \quad .$$

We shall use this functor repeatedly later referring to it as "the bar functor, U" and denoting $U\alpha$ and UF by $\bar{\alpha}$ and $\bar{F}$.

We need two results involving tensor products that we can now prove.

<u>1.26 Theorem</u>: If F is an R-module, and J an ideal of R, then $F/FJ \cong F \otimes_R R/J$.

Proof: To prove $F/FJ \cong F \otimes_R R/J$, it will suffice to exhibit an isomorphism. Define $\Psi_F\colon F/FJ \longrightarrow F \otimes_R R/J$ by $\Psi_F(f + FJ) = f \otimes (1 + J)$. One easily checks that Ψ_F is a well-defined isomorphism of R/J-modules. If we exhibit an inverse for Ψ_F, it will be clear that Ψ_F is indeed an isomorphism. As a consequence of a balanced map from $F \times R/J$ to F/FJ we may define $\Phi_F\colon F \otimes_R R/J \longrightarrow F/FJ$ by $\Phi_F(f \otimes (r + J)) = fr + FJ$. It is not difficult to see that Φ_F and Ψ_F are inverse isomorphisms. Conclude that $F/FJ \cong F \otimes_R R/J$. Furthermore, Ψ_F is a functor isomorphism. If $\alpha\colon F \longrightarrow G$, then consider

$$\begin{array}{ccc} F/FJ & \xrightarrow{\bar{\alpha}} & G/GJ \\ \downarrow \Psi_F & & \downarrow \Psi_G \\ F \otimes R/J & \xrightarrow{\alpha \otimes 1} & G \otimes R/J \end{array}$$

$\alpha \otimes 1 \colon F \otimes R/J \longrightarrow G \otimes R/J$ is so that $\alpha \otimes 1(f \otimes r + J) = \alpha(f) \otimes r + J$. $\bar{\alpha}\colon F/FJ \longrightarrow G/GJ$ is so that $\bar{\alpha}(f + FJ) = \alpha(f) + GJ$. The diagram commutes, and $\otimes R/J$ and U are isomorphic functors. //

1.27 Theorem: If A is an R-module, then $A_R \otimes R \cong A_R$.

Proof: To prove that $A_R \otimes R \cong A_R$, it will suffice to exhibit an isomorphism. Define $\Psi\colon A_R \otimes R \longrightarrow A_R$ by $\Psi(a \otimes r) = ar$. This arises as a consequence of a balanced map from $A_R \times R$ to A_R. If we also define $\Phi\colon A_R \longrightarrow A_R \otimes R$ by $\Phi(a) = a \otimes 1$, it is not difficult to see that Φ and Ψ are inverse isomorphisms. Hence conclude that $A_R \otimes R \cong A_R$. //

Another vital concept we need to discuss is that of flat modules.

1.28 Definition: A right R-module F is called flat if whenever $\kappa\colon {}_RA \longrightarrow {}_RB$ is a monomorphism, then $1 \otimes \kappa\colon F \otimes_R A \longrightarrow F \otimes_R B$ is also a monomorphism.

As noted above, the functor $F \otimes_R$ is always right exact. It is exact precisely when F is a flat R-module.

1.29 Theorem: Every free module and every projective module is flat.

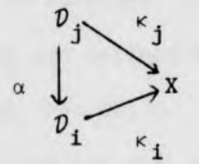
Proof: See [4, p. 133].

<u>1.30 Theorem</u>: Let A and F be right R-modules where F is flat. Then A is flat if and only if $FI \cap K = KI$ for every left ideal I.

Proof: See [4, p. 133].

I.4. Colimits and Direct Limits.

Suppose that we are given a set of modules $\{\mathcal{D}_i \mid i \in I\}$ and for each pair $\mathcal{D}_i$, $\mathcal{D}_j$ a set of homomorphisms H_{ij} from $\mathcal{D}_j$ to $\mathcal{D}_i$. Then we call the set of modules and maps a <u>diagram</u>. A set of maps $\{\mathcal{D}_i \xrightarrow{\kappa_i} X\}_{i \in I}$ is said to be <u>compatible</u> with the diagram if



commutes for every $\alpha \in H_{ij}$ and for every i and j.

We say that the set of maps $\{\mathcal{D}_i \xrightarrow{\kappa_i} X\}_{i \in I}$ is a <u>colimit</u> for the diagram if

(1) it is a compatible set for the diagram, and

(2) given any $\{\mathcal{D}_i \xrightarrow{\gamma_i} Y\}$ compatible with the diagram, there is a unique map $X \xrightarrow{\gamma} Y$ so that $\gamma\kappa_i = \gamma_i$ for each $i \in I$.

Let $(I, \leq)$ be a partially ordered set. I is <u>directed</u> if given $\alpha, \beta \in I$ there is $\gamma \in I$ such that $\alpha \leq \gamma$ and $\beta \leq \gamma$. Given a directed set $(I, \leq)$ a diagram called a <u>directed system of R-modules</u> is described by giving for each $\alpha \in I$ a module $\mathcal{D}_\alpha$ and for each pair $\alpha, \beta \in I$ with $\alpha \leq \beta$ a map $\phi_{\beta\alpha}: \mathcal{D}_\alpha \longrightarrow \mathcal{D}_\beta$ so that

(1) for each $\alpha \in I$, $\phi_{\alpha\alpha} = 1_{\mathcal{D}_\alpha}$,

(2) if $\alpha \leq \beta \leq \gamma$, then $\phi_{\gamma\alpha} = \phi_{\gamma\beta}\phi_{\beta\alpha}$.

We denote the directed system $(\mathcal{D}_\alpha, I)$. A colimit $(\mathcal{D}_\alpha \xrightarrow{\kappa_\alpha} X)_{\alpha \in I}$ for a directed system of modules is called a <u>direct limit</u> and X is denoted $\varinjlim_{\alpha\in I} \mathcal{D}_\alpha$. This is a functor and, in fact, an exact functor from directed systems over I to R-modules. $\varinjlim_{\alpha\in I} \mathcal{D}_\alpha$ is usually constructed as follows. Let X be $\dot{\bigcup}_{\alpha\in I} \mathcal{D}_\alpha$, that is the disjoint union of the $\mathcal{D}_\alpha$'s. Define an equivalence relation $\sim$ on X by $x_\alpha \sim x_\beta$ if for some $\gamma \geq \alpha$, $\gamma \geq \beta$ one has $\phi_{\gamma\alpha}(x_\alpha) = \phi_{\gamma\beta}(x_\beta)$. The set of equivalence classes $X/\sim$ is $\varinjlim_{\alpha\in I} \mathcal{D}_\alpha$. We will use $\bar{x}_\alpha$ to denote the $\sim$ - equivalence class of x_α. The κ_α maps are defined by $\kappa_\alpha(x_\alpha) = \bar{x}_\alpha$. For more of the basic properties of $\varinjlim$ see [6, p. 15].

Let $(I, \leq)$ be a directed set and $\{F_i \mid i \in I\}$ be a directed system of right R-modules with maps $\alpha_{ji}: F_i \longrightarrow F_j$ where $i \leq j$. Let D be the functor from the category of left R-modules $_R M$ to the category of directed systems of abelian groups $AbGp^I$ over I where $A \xrightarrow{\alpha} B$ in $_R M$ maps to the systems with groups $\{F_i \otimes A \mid i \in I\}$ and $\{F_i \otimes B \mid i \in I\}$ and the obvious maps $\alpha_{ji} \otimes 1_A$ and $\alpha_{ji} \otimes 1_B$ when $i \leq j$ and where $\mathcal{D}_\alpha$ is the map of these directed systems defined by the maps $F_i \otimes A \xrightarrow{1\otimes\alpha} F_i \otimes B$.

We have also the functor L from $AbGp^I$ to AbGp where $L = \varinjlim_I$. So $L \circ D: {_R M} \longrightarrow AbGp$. Another functor from $_R M$ to AbGp is given simply by $T = (\varinjlim F_i) \otimes_R$.

<u>1.31 Theorem</u>: $L \circ D \cong T$.

Proof: Given $A \in {_R M}$ we define an isomorphism of AbGp, $\phi_A: TA \longrightarrow LDA$. Let $\bar{x}_j \in \varinjlim F_i$ and $a \in A$. Then $\phi_1: \varinjlim F_i \times A \longrightarrow \varinjlim (F_i \otimes A)$ given by $\phi_1(\bar{x}_j, a) = \overline{x_j \otimes a}$ is a balanced, well-defined map and defines the map ϕ_A of AbGp, $\psi_A(\bar{x}_j \otimes a) = \overline{x_j \otimes a}$.

To see that ϕ_A is an isomorphism we define its inverse, Ψ_A. There is the set of natural maps $F_i \otimes A \xrightarrow{\alpha_i \otimes 1_A} (\varinjlim F_i) \otimes A$ where $F_i \xrightarrow{\alpha_i} \varinjlim F_i$ is the canonical map sending $x_i \in F_i$ to $\bar{x}_i$. Examining the diagram

$$\begin{array}{ccc} F_i \otimes A & \xrightarrow{\alpha_i \otimes 1_A} & \\ \downarrow{\scriptstyle \alpha_{ij} \otimes 1_A} & & (\varinjlim F_i) \otimes A \\ F_j \otimes A & \xrightarrow{\alpha_j \otimes 1_A} & \end{array}$$

which clearly commutes, there is a unique map $\Psi_A \colon \varinjlim (F_i \otimes A) \longrightarrow (\varinjlim F_i) \otimes A$ such that for each $i \in I$, Ψ_A composes with the canonical map $F_i \otimes A \longrightarrow \varinjlim (F_i \otimes A)$ and yields $\alpha_i \otimes 1_A$. That is, $\Psi_A\ (\overline{x_i \otimes a}) = \alpha_i \otimes 1_A\ (x_i \otimes a) = \alpha_i(x_i) \otimes a = \bar{x}_i \otimes a$. It is easy to see that ϕ_A and Ψ_A are inverse isomorphisms.

Now suppose we have a map $A \xrightarrow{\gamma} B$ of left R-modules. We want to show that

$$\begin{array}{ccc} TA & \xrightarrow{T\gamma} & TB \\ \downarrow{\scriptstyle \phi A} & & \downarrow{\scriptstyle \phi B} \\ LDA & \xrightarrow{LD\gamma} & LDB \end{array}$$

commutes. First if $\bar{x}_i \otimes a \in TA$, then $T\gamma$ maps it to $\bar{x}_i \otimes \gamma(a)$. If $\overline{y_j \otimes a_1} \in LDA$, then $LD\gamma$ maps it to $\overline{y_j \otimes \gamma(a_1)}$. Thus choosing $\bar{x}_i \otimes a \in TA$ we have $\phi_B \circ T\gamma(\bar{x}_i \otimes a) = \phi_B(\bar{x}_i \otimes \gamma(a)) = \overline{x_i \otimes \gamma(a)}$ and $LD\gamma \circ \phi_A(\bar{x}_i \otimes a) = LD\gamma(\overline{x_i \otimes a}) = \overline{x_i \otimes \gamma(a)}$ so that the diagram commutes. //

1.32 Corollary: If $\{F_i \mid i \in I\}$ is a directed system of flat R-modules, then $\varinjlim F_i$ is flat.

Proof: Let the monomorphism $0 \longrightarrow A \xrightarrow{\alpha} B$ be given. Then for $\{F_i \mid i \in I\}$ we know that we have $0 \longrightarrow F_i \otimes A \xrightarrow{1\otimes\alpha} F_i \otimes B$. Note that $F_i \otimes A$ and $F_i \otimes B$ are directed systems of abelian groups and the indicated maps are maps of directed systems of abelian groups. Take the direct limit,

$$0 \longrightarrow \varinjlim (F_i \otimes A) \longrightarrow \varinjlim (F_i \otimes B),$$

which is exact since $\varinjlim$ is an exact functor. By the previous theorem we know that we have

$$\begin{array}{ccc} 0 \longrightarrow \varinjlim (F_i \otimes A) & \longrightarrow & \varinjlim (F_i \otimes B) \\ \phi_A \Big\| \wr & & \Big\| \wr\ \phi_B \\ (\varinjlim F_i) \otimes A & \xrightarrow{1\otimes\alpha} & (\varinjlim F_i) \otimes B \end{array}.$$

commuting. Thus $1 \otimes \alpha$ is a monomorphism. Hence $\varinjlim F_i$ is flat. //

1.33 Lemma: $(\coprod_{i\in I} S_i) \otimes F \cong \coprod_{i\in I} (S_i \otimes F)$, where the S_i and F are R-modules.

Proof: See [4, p. 124].

I.5. Radicals.

Before we may define the prime radical of a ring, a preliminary definition is required.

1.34 Definition: An ideal P of a ring R is called a prime ideal if whenever I and J are ideals of R, then $IJ \subseteq P$ implies that either $I \subseteq P$ or $J \subseteq P$.

1.35 Definition: Let R be a ring. The prime radical of R, denoted rad R, is the intersection of all the prime ideals of R.

1.36 Definition: An element $x \in R$ is called nilpotent if $x^n = 0$ for some integer $n \geq 1$. An ideal is called nil if all of its elements are nilpotent. An ideal K of R is called nilpotent if $K^n = 0$ for some $n \geq 1$.

A definition we need for reference later is that of semiprime.

1.37 Definition: A ring R is called semiprime if rad $R = 0$.

It is well-known that this is equivalent to saying that R has no nonzero nilpotent ideals.

Preliminary to defining the Jacobson radical of a ring is defining the radical of an R-module.

1.38 Definition: Let A be an R-module. The radical of A, denoted Rad A, is the intersection of the maximal submodules of A. If A has no maximal submodules, then Rad $A = A$.

1.39 Definition: The intersection of all of the maximal right ideals of R is called the Jacobson radical of R, denoted Rad R.

The intersection of a collection of right ideals is a right ideal, so Rad R is a right ideal. Rad R is also the intersection of all maximal left ideals, so is itself a left ideal, hence a two-sided ideal.

It can be shown that one always has that rad R $\subseteq$ Rad R.

1.40 Theorem: The radical is the largest right ideal K so that, for all $r \in K$, $1 - r$ is a unit.

Proof: See [4, p. 57].

In order to obtain the next result concerning the radical, a preliminary definition is required.

1.41 Definition: Let A and B be R-modules where $B \subseteq A$. B is a small submodule of A provided that, given a submodule C of A, $B + C = A$ implies that $C = A$.

1.42 Theorem: Rad A contains every small submodule of A.

Proof: Let B be a small submodule of A. Assume that $B \not\subseteq$ Rad A. Then there exists a maximal submodule K such that $B \not\subseteq K$. $B + K = A$ since K is a maximal submodule of A. Thus $K = A$ since B is small. This contradicts $B \not\subseteq K$. //

We need another result concerning small modules.

1.43 Theorem: A submodule of a small submodule is small.

Proof: Let B be a small submodule of C, and let A be a submodule of B. Show that A is small in C. Assume $A + K = C$. Then $B + K = C$. B is small in C, so $K = C$. Hence A is small in C. //

1.44 Theorem: Rad A is the subset of A consisting of all the elements of A which lie in the kernel of any mapping from A into a simple R-module.

Proof: Let $x \in \text{Rad } A$ and $\alpha: A \longrightarrow S$ be a mapping to a simple module S. Then if $\alpha = 0$, $x \in \text{Ker } \alpha$. If $\alpha \neq 0$, then α is onto and then $\text{Ker } \alpha$ is a maximal submodule of A. Then $x \in \text{Ker } \alpha$.

Conversely, suppose $x \notin \text{Rad } A$. We may show that there is a map $A \xrightarrow{\alpha} S$ where S is simple and $\alpha(x) \neq 0$. For since $x \notin \text{Rad } A$ there is a maximal submodule B of A with $x \notin B$. Then take α to be the usual canonical epimorphism from A onto the simple module A/B. //

<u>1.45 Theorem</u>: If $\alpha \in \text{Hom}_R(A,B)$, then $\alpha(\text{Rad } A) \subseteq \text{Rad } B$.

Proof: Let $x \in \text{Rad } A$ and $\gamma: B \longrightarrow S$ be any map of B into a simple module S. Then considering $\gamma\alpha: A \longrightarrow S$ we have $\gamma\alpha(x) = 0$, that is, $\alpha(x) \in \text{Ker } \gamma$, by the previous theorem. //

We now state without proof the Dual Basis Lemma.

<u>1.46 Theorem</u> (Dual Basis Lemma): An R-module A is projective if and only if there is a set $\{a_i \mid i \in I\} \subseteq A$ and a set $\{f_i \mid i \in I\} \subseteq \text{Hom}_R(A_R, R_R)$ such that for every $a \in A$, $f_i(a) = 0$ except for finitely many $i \in I$ and for every $a \in A$, $a = \sum_{i \in I} a_i f_i(a)$.

<u>1.47 Theorem</u>: For any projective module P_R, if $J = \text{Rad } R$, then $PJ = \text{Rad } P$.

Proof: Let $\{P_i \mid i \in I\} \subseteq P$ and $\{f_i \mid i \in I\} \subseteq \text{Hom}_R(P_R, R_R)$ such that for $p \in P$, $f_i(p) = 0$ except for finitely many $i \in I$ and for each $p \in P$, $p = \sum_{i \in I} p_i f_i(p)$. By 1.44 Theorem, we know that $\alpha(\text{Rad } P) \subseteq \text{Rad } R$ if $\alpha \in \text{Hom}_R(P,R)$. It follows that if $p \in \text{Rad } P$ then $f_i(p) \in \text{Rad } R = J$. Thus $p = \sum_{i \in I} p_i f_i(p)$ implies $p \in PJ$. So we have $\text{Rad } P \subseteq PJ$.

We know that $PJ = \sum_{p \in P} pJ$. Define $\phi_p : R \longrightarrow P$ by $\phi_p(r) = pr$. Then $\phi_p(J) \subseteq \text{Rad } P$ and so $pJ \subseteq \text{Rad } P$. Thus $PJ \subseteq \text{Rad } P$ and $\text{Rad } P = PJ$. //

I.6. Semisimple Rings.

Preliminary to defining semisimple rings is defining the socle of a module and semisimple modules.

1.48 Definition: Let A be a right R-module. Then the socle of A_R, denoted Soc A_R, is the sum of all the simple submodules of A_R. If A_R has no simple submodules, then Soc $A_R = 0$. (Note that Soc A and Rad A are dual notions.)

1.49 Definition: A right R-module A is called a semisimple module if Soc A = A.

1.50 Definition: If R_R is a semisimple module we say R is a semisimple ring.

If M is either a submodule of a semisimple, a homomorphic image of a semisimple, or a sum of semisimple modules, then M is itself semisimple. Every submodule of a semisimple module is a direct summand.

1.51 Theorem: Every module over a semisimple ring is semisimple.

Proof: Suppose M is a module over R where R_R is semisimple. If $m \in M$ there is a map $\phi_m: R_R \longrightarrow M_R$ where $\phi_m(r) = mr$. The image of this mapping is clearly mR so mR is semisimple since it is a homomorphic image of R_R. Then since $M = \sum_{m \in M} mR$, M is also semisimple. //

1.52 Theorem: If R is semisimple, then every simple submodule is isomorphic to a minimal right ideal.

Proof: If S is simple, then $S \cong R/M$ where M is a maximal right ideal of R, by 1.5 Theorem. Since R is semisimple, and submodules of semisimple modules are direct summands, then

$$0 \longrightarrow M \longrightarrow R \longrightarrow R/M \longrightarrow 0$$

splits. Thus R/M is isomorphic to a right ideal of R. //

1.53 Theorem: Semisimple modules are annihilated by Rad R, that is, if S is a semisimple module, then $S \cdot \text{Rad } R = 0$.

Proof: It is sufficient to assume that S is simple. But then given $x \in S$, $x \neq 0$ one has $S = xR \cong R/x^r$ where x^r is the right annihilator of x. Since S is simple, then x^r is a maximal right ideal by 1.5 Theorem. Since Rad R is contained in any maximal right ideal, it follows that $x \cdot \text{Rad } R = 0$, whence $S \cdot \text{Rad } R = 0$. //

I.7. Chain Conditions.

In this section we shall define and very briefly discuss Artinian and Noetherian rings, hence chain conditions, to establish some level of understanding for Artinian rings, then conclude the section with the statement of the Wedderburn-Artin Theorem.

<u>1.54 Definition</u>: A right R-module A is called <u>Artinian</u> (<u>Noetherian</u>) if every non-empty set of submodules has a minimal (maximal) member.

An equivalent definition is: A_R is Artinian (Noetherian) if every descending (ascending) sequence of submodules is eventually stationary, that is, A_R has descending (ascending) chain condition. We say R is right Artinian (Noetherian) when the module R_R is Artinian (Noetherian).

Now we conclude the section with the statement of the Wedderburn-Artin Theorem:

<u>1.55 Theorem</u> (Wedderburn-Artin): The following statements are equivalent concerning the ring R:

(1) R is a semisimple ring.

(2) R is isomorphic to a finite product of rings, each of which is a total matrix ring over a division ring. That is, $R \cong S_1 \times S_2 \times \cdots \times S_k$ where S_i is the ring of $n_i \times n_i$ matrices over a division ring Δ_i.

(3) Rad R = 0 and R is right Artinian.

Since condition (2) is symmetric other characterizations of semisimple rings are obtained by replacing "right" by "left." Note that from (3) since rad R $\subseteq$ Rad R a semisimple ring is always semiprime.

I.8. Idempotents and Lifting.

In this section we shall define idempotents and discuss what it means to lift idempotents, and stipulate when it can be done.

1.56 Definition: An element e of R is called an idempotent if $e^2 = e$. Idempotents e and f are orthogonal if $ef = fe = 0$.

If e is an idempotent, then any $r \in R$ can be expressed uniquely in the form $r = er + (1 - e)r$ where $er \in eR$ and $(1 - e)r \in (1 - e)R$. Hence we see that R is the direct sum of the right ideals eR and $(1 - e)R$, and R is isomorphic to the product $eR \times (1 - e)R$.

1.57 Theorem: Every minimal right ideal of a semiprime ring R is generated by a nonzero idempotent.

Proof: See [4, p. 63].

Let I be any ideal of R. Then we say that idempotents may be lifted modulo I, if, for every element $x \in R$ such that $x^2 - x \in I$ there exists an element $e^2 = e \in R$ so that $e - x \in I$. That is, if every coset of R/I which is idempotent in that ring really contains a genuine idempotent of R.

1.58 Theorem: If N is any nil ideal of R, then idempotents may be lifted modulo N.

Proof: See [4, p. 72].

1.59 Theorem: Assume that idempotents may be lifted modulo N. Then any finite or countable orthogonal set of nonzero idempotents modulo N may be lifted to an orthogonal set of nonzero idempotents of R.

Proof: See [4, p. 73].

We shall say that an idempotent e is <u>minimal</u> if it generates a minimal ideal, and we shall say that e is a <u>semisimple</u> <u>idempotent</u> if e has the form $e = f_1 + f_2 + \cdots + f_n$, where the f_i are orthogonal minimal idempotents.

We shall say that a ring R has property (*) if it has the property that nonzero left modules have nonzero left socle.

<u>1.60 Theorem</u>: If R has property (*), then R/Rad R has property (*).

Proof: Let M be a nonzero left R/J-module where J = Rad R. Given an $m \in M$ and an $r \in R$ define rm to be $(r + J)m$. This makes M a nonzero left R-module. Let ${}_RS$ be a simple R-submodule of M. Then S is a simple R/J-submodule of M, with the obvious multiplication. If S is not simple as an R/J-module, let S_1 be and R/J-submodule of S. Then S_1 is an R-submodule of S which contradicts that S is simple as an R-module. So S must be simple as an R/J-submodule of M. //

<u>1.61 Theorem</u>: Assume R is semiprime and not semisimple, and that R has property (*). Given a semisimple idempotent e let T_e be the set of minimal idempotents orthogonal to e. Then T_e is not empty.

Proof: $R = Re + R(1 - e)$. Since R is not semisimple, then $R(1 - e) \neq 0$. By property (*) there is a minimal idempotent f with $f \in R(1 - e)$. Thus $fe = 0$. Then $(1 - e)f$ is an idempotent orthogonal to e. If $(1 - e)f = 0$, then $f = ef + (1 - e)f = ef$. Whence $f = ff = fef = 0$. Contradiction. Thus $(1 - e)f \neq 0$. Whence $Rf = R(1 - e)f$. Thus $(1 - e)f \in T_e$. //

<u>1.62 Theorem</u>: If $e^2 = e \in R$ and I is an ideal of R, then $(e + I)(R/I) \cong eR/eI$ as R-modules.

Proof: It will suffice to exhibit an isomorphism. Define $\Psi: (e + I)(R/I) \longrightarrow eR/eI$ by $\Psi:(e + I)(r + I) \longmapsto er + eI$. The isomorphism is obvious. //

I.9. König's Graph Theorem and Some Useful Lemmas.

In this section we shall prove several lemmas we need for later reference. One of the lemmas depends on König's Graph Theorem and three are from a paper by Hyman Bass, which will be discussed later. We begin with preliminary definitions.

1.63 Definition: Let K be an ideal in a ring R. We say that K is right T-nilpotent if, given any sequence $\{a_i\}$ of elements in K, there exists an n so that $a_n \ldots a_1 = 0$.

1.64 Definition: Let $\{A_n \mid n = 0,1,2,\ldots\}$ be a sequence of finite sets and $\{f_n \mid n = 0,1,2,\ldots\}$ a sequence of functions where $f_n\colon A_n \longrightarrow 2^{A_{n+1}}$. We see that f_n assigns to each element of A_n a finite subset of A_{n+1}. The pair $(\{A_n\}, \{f_n\})$ is called a graph. A path in this graph is a set of elements $\{a_m\}$ such that $a_o \in A_o$ and $a_m \in f_{m-1}(a_{m-1})$ for $m \geq 1$. The length of the path $\{a_m\}$ is either ∞ or one less than the cardinality of $\{a_m\}$ when this is finite.

1.65 Theorem (König's Graph Theorem): If the graph $(\{A_n\}, \{f_n\})$ has paths of arbitrary length, then it has a path of infinite length.

A simple proof of this theorem is found in B. L. Osofsky's paper [5]. We use it to prove the following result of Bass.

1.66 Lemma: Let B be a right R-module and K a right T-nilpotent ideal of R. If $BK = B$, then $B = 0$.

Proof: The proof is by contradiction. Suppose $BK = B$ and $B \neq 0$. Suppose $b \in B$. Since $B = BK$ there is a function $g \in \coprod_{b\in B} K_b$ such that $b = \sum_{b_1 \in B} b_1 g(b_1)$. Using the axiom of choice we select for each $b \in B$ an $f_b \in \coprod_{b_1 \in B} K_{b_1}$, such that $b = \sum_{b_1 \in B} b_1 f_b(b_1)$. Recall that the support Sg of $g \in \coprod_{b \in B} K_b$ is the set of $b \in B$ where $g(b) \neq 0$. Now choose $b \in B$, $b \neq 0$, $b = \sum_{b_1 \in Sf_b} b_1 f_b(b_1)$ and we can replace each b_1 by $\sum_{b_2 \in Sf_{b_1}} b_2 f_{b_1}(b_2)$ whence taking $A_1 = \dot{\bigcup}_{b_1 \in Sf_b} Sf_{b_1}$ (where $\dot{\bigcup}$ means disjoint union) then $b = \sum_{b_2 \in A_1} b_2 f_{b_1}(b_2) f_b(b_1)$. Extending this process of replacement indefinitely leads us to the following construction. Let $A_o = Sf_b$, and $A_n = \dot{\bigcup}_{b_n \in A_{n-1}} Sf_{b_n}$ for $n \geq 1$. Since the union is disjoint, given an element x_{n+1} of A_n there is a unique ancestor $x_n \in A_{n-1}$ such that $x_{n+1} \in Sf_{x_n}$. Thus we will label the successive ancestors of x_{n+1} with $x_n, x_{n-1}, \ldots, x_2, x_1$ where $x_j \in A_{j-1}$. Define $g_{n-1}: A_{n-1} \to 2^{A_n}$, $n \geq 1$ as follows. Given $b_n \in A_{n-1}$ let $g_{n-1}(b_n)$ be the subset of Sf_{b_n} consisting of those b_{n+1} such that

$f_{b_n}(b_{n+1}) \cdot f_{b_{n-1}}(b_n) \cdot \ldots \cdot f_{b_2}(b_3) \cdot f_{b_1}(b_2) \cdot f_b(b_1) \neq 0$. Then $(\{A_n\}, \{g_n\})$ is a graph.

Observe that one has always for each $n = 0,1,2,\ldots$

$$b = \sum_{b_{n+1} \in A_n} b_{n+1} \cdot f_{b_n}(b_{n+1}) \cdot f_{b_{n-1}}(b_n) \cdot \ldots \cdot f_{b_1}(b_2) \cdot f_b(b_1).$$

Now given any positive integer k since $b \neq 0$ we must have for some $b_{k+1} \in A_k$ that $f_{b_k}(b_{k+1}) \cdot f_{b_{k-1}}(b_k) \cdot \ldots \cdot f_{b_1}(b_2) \cdot f_b(b_1) \neq 0$. It is immediate then that $\{b_1, b_2, \ldots, b_{k+1}\}$ is a path in the graph of length k. Thus our graph has paths of arbitrary length and by König's Graph Theorem, there must be a path $\{b_1, b_2, \ldots\}$ of infinite length.

$b_1 \in A_o$ means $f_b(b_1) \neq 0$,

$b_2 \in g_o(b_1)$ means $f_{b_1}(b_2)\cdot f_b(b_1) \neq 0$,

. . .

$b_n \in g_{n-2}(b_{n-1})$ means $f_{b_{n-1}}(b_n)\cdot f_{b_{n-2}}(b_{n-1})\cdot\ldots\cdot f_{b_1}(b_2)\cdot f_b(b_1) \neq 0$,

and so on. It is then clear that examining the sequence $\{f_b(b_1), f_{b_1}(b_2), \ldots\}$ which lies in K we have a contradiction to the fact that K is right T-nilpotent. //

The following two lemmas are from Bass's paper. For the following two lemmas, let $\{a_n\}_{n=1}^{\infty}$ be any sequence of elements in a ring R. Let F be a free right R-module with basis $x_1, x_2, \ldots,$ and let G be the submodule of F generated by $\{x_n - x_{n+1}a_n\}_{n=1}^{\infty}$. The image of x_n in F/G will be denoted by z_n.

<u>1.67 Lemma</u>: If $J_k = \{r \in R \mid a_{k+n}\ldots a_k r = 0$ for some $n\}$ then J_k is the right annihilator ideal of z_k, z_k^r.

Proof: $z_k = z_{k+1}a_k = \ldots = z_{k+n+1}a_{k+n}\ldots a_k = \ldots$. So $J_k \subseteq z_k^r$.

Suppose $r \in z_k^r$ so that $z_k r = 0$, that is, $x_k r \in G$. Then $x_k r = \sum_i (x_i - x_{i+1}a_i)r_i$. $(x_1 - x_2a_1)r_1 + (x_2 - x_3a_2)r_2 + \ldots = x_k r$. Or equivalently, $x_1r_1 + x_2(r_2 - a_1r_1) + x_3(r_3 - a_2r_2) + \ldots +$ $x_{k-1}(r_{k-1} - a_{k-2}r_{k-2}) + x_k(r_k - a_{k-1}r_{k-1}) + \ldots = x_k r$. So $r_i = 0$ if $i < k$ and

$$r = r_k$$
$$0 = r_{k+1} - a_k r_k$$
$$0 = r_{k+2} - a_{k+1}r_{k+1}$$
$$\vdots$$

$$0 = r_{k+n} - a_{k+n-1}r_{k+n-1} .$$

Thus $r_{k+n} = a_{k+n-1}r_{k+n-1} = a_{k+n-1}a_{k+n-2}r_{k+n-2} = \dots =$ $a_{k+n-1}a_{k+n-2}\dots a_k r_k = a_{k+n-1}\dots a_k r$. Since $r_{k+n} = 0$ for n sufficiently large, we have $r \in J_k$. Thus $z_k^r \subseteq J_k$ and $J_k = z_k^r$. //

1.68 Lemma: Suppose G is a direct summand of F. Then the chain $\{Ra_n\dots a_1\}$ of principal left ideals terminates.

Proof: We identify F/G with the corresponding direct summand of F writing $F = F/G \oplus G$ and $x_n = z_n + g_n$ with $g_n \in G$. As an element of F we have

$$z_n = x_1c_{1n} + x_2c_{2n} + \dots + x_kc_{kn} + \dots .$$

As usual the column finite matrix (c_{ij}) represents the projection of F onto F/G. Since this endomorphism of F is idempotent then (c_{ij}) is an idempotent matrix. Let I be the left ideal generated by $\{c_{11}, c_{21}, c_{31}, \dots, c_{n1}, \dots\}$, the coordinates of z_1. Since $z_1 = z_2a_1 = z_3a_2a_1 = \dots = z_{n+1}a_na_{n-1}\dots a_1 = \dots$ we have $c_{k1} = c_{k2}a_1 = c_{k3}a_2a_1 = \dots = c_{k,n+1}a_na_{n-1}\dots a_1$ for any k. Then each generator of I and hence I itself is contained in $\bigcap_n (Ra_n\dots a_1)$.

We prove the lemma by showing that $a_ma_{m-1}\dots a_1 \in I$ for some m. Since (c_{ij}) is idempotent $c_{j1} = \sum_{k=1}^{\infty} c_{jk}c_{k1}$ for all j. Since $c_{k1} = 0$ for $k > n+1$ for some n then we have $c_{j1} = \sum_{k=1}^{n+1} c_{jk}c_{k1}$ for all j. Since for $k \leq n$ $z_k = z_{n+1}a_na_{n-1}\dots a_k$ then $c_{jk} = c_{j,n+1}a_na_{n-1}\dots a_k$ for all j. Then for all j $c_{j1} = \sum_{k=1}^{n+1} c_{j,n+1}a_na_{n-1}\dots a_kc_{k1} =$ $c_{j,n+1}(\sum_{k=1}^{n+1} a_na_{n-1}\dots a_kc_{k1})$. Let $\gamma = \sum_{k=1}^{n+1} a_na_{n-1}\dots a_kc_{k1}$. Then we have

$z_1 = z_{n+1}\gamma$. Since also $z_1 = z_{n+1}a_na_{n-1}\cdots a_1$ then $\gamma - a_na_{n-1}\cdots a_1 \in z^r_{n+1}$. By the previous lemma, for some $h \geq 1$, $a_{n+h}\cdots a_{n+1}(\gamma - a_n\cdots a_1) = 0$. Let $m = n+h$. Then $a_m\cdots a_1 = a_{n+h}\cdots a_{n+1}\gamma \in I$ since $\gamma \in I$. //

1.69 Lemma: If A is small in B and B is a submodule of C then A is small in C.

Proof: Suppose $A + C_1 = C$. Then
$$B = B \cap C = B \cap (A + C_1) = A + (B \cap C_1)$$
by the modular law. So $B \cap C_1 = B$ since A is small in B, that is $B \subseteq C_1$. Then $C_1 = B + C_1 \supseteq A + C_1 = C$. So $C_1 = C$. //

1.70 Lemma: If $\alpha \in \mathrm{Hom}_R(A,B)$ and A_1 is small in A then $\alpha(A_1)$ is small in B.

Proof: By the previous lemma we may assume α is onto. Then suppose $\alpha(A_1) + B_1 = B$. Let $A_2 = \alpha^{-1}(B_1)$. Since α is onto $\alpha(A_2) = B_1$. Then $\alpha(A_1 + A_2) = \alpha(A)$. So $A_1 + A_2 + \mathrm{Ker}\,\alpha = A$. Then $A_2 + \mathrm{Ker}\,\alpha = A$ since A_1 is small in A. Then $B_1 = \alpha(A_2) = \alpha(A_2 + \mathrm{Ker}\,\alpha) = \alpha(A) = B$. //

1.71 Lemma: If P is projective and $J = \mathrm{Rad}\,R$ then $P = PJ$ implies $P = 0$.

Proof: Let $F = P \oplus Q$ where F is free with base $\{x_i \mid i \in I\}$. Since $P = PJ$ then $P \subseteq FJ$. Let $x_i = y_i + z_i$ where $y_i \in P$ and $z_i \in Q$. Expanding $y_i = \sum_j x_ja_j$ we know $a_{ji} \in J$ for all i and j. Since $z_i = x_i - y_i$ then $z_i = \sum_j x_j(\delta_{ji} - a_{ji})$. We shall show the z_i's are independent. Let $z_1,\ldots,z_n$ be a finite set of z_i's.

Consider the projection of F onto the summand generated by $x_1, \ldots, x_n$. Then we let z_i' be the image of z_i under this projection. It suffices to show that $z_1', \ldots, z_n'$ are independent. We have $z_i' = \sum_{j=1}^{n} x_j(\delta_{ji} - a_{ji})$. Let I be the $n \times n$ identity matrix and $A = (a_{ji})_{j,i = 1,\ldots,n}$. A is in Rad R_n where R_n is the $n \times n$ matrix ring over R. It follows that $I - A$ is invertible. In particular its columns are independent and since its columns form the coordinates of the z_i' one sees that the z_i' are independent.

Now let $x_1a_1 + \ldots + x_na_n \in P$. Under the projection of F onto Q (with kernel P) we get $z_1a_1 + \ldots + z_na_n = 0$. Therefore $a_1 = \cdots = a_n = 0$. Hence $P = 0$. //

CHAPTER II

LOCAL RINGS AND PERFECT RINGS

This chapter is devoted to the study of local rings and right perfect rings. Our consideration of local rings will consist of studying a theorem which asserts three equivalent conditions which are necessary and sufficient for a ring to be local. We shall examine right perfect rings by studying a theorem proved by Hyman Bass in his paper Finitistic Dimension and a Homological Generalization of Semi-primary Rings [1], which he called Theorem P. A discussion of results that arise from studying Theorem P will conclude the chapter.

II.1. Local Rings.

A ring R is said to be a local ring if it satisfies one of the equivalent conditions of the following theorem.

<u>2.1 Theorem</u>: Let R be a ring in which $0 \neq 1$. The following conditions are equivalent:

(1) R/Rad R is a division ring.

(2) R has exactly one maximal right ideal.

(3) For every element r of R, either r or 1 - r is right invertible.

Proof: (1) $\Rightarrow$ (2). Assume R/Rad R is a division ring and show that R has exactly one maximal right ideal. Since R/Rad R is a division ring, it is simple as an R-module, by 1.6 Theorem. Therefore Rad R is a maximal right ideal by 1.5 Theorem. We know that Rad R is the intersection of all maximal right ideals, hence there must be only one. Therefore R has exactly one maximal right ideal, namely Rad R.

(2) $\Rightarrow$ (3). Assume that R has a unique maximal right ideal, and show that for every r of R either r or 1 - r is right invertible. Then rR is a proper right ideal, hence contained in the only maximal right ideal, which must be Rad R. If 1 - r is not right invertible, then (1 - r)R is also proper, hence contained in Rad R. But if $r \in$ Rad R and $1 - r \in$ Rad R, then $1 \in$ Rad R, which is impossible. Conclude that if r is not right invertible, 1 - r must be.

(3) $\Rightarrow$ (1). Assume for every element r of R, either r or $1 - r$ is right invertible, and show that $R/\text{Rad } R$ is a division ring. By 1.6 Theorem we see that to show that a ring is a division ring, it suffices to show that every nonzero element is right invertible. Hence, to show that $R/\text{Rad } R$ is a division ring it will suffice to show that for every $r \notin \text{Rad } R$, r is right invertible. Suppose $r \notin \text{Rad } R$ and r is not right invertible. Then rR is a proper right ideal. Also if $s \in rR$ then s is not right invertible. It follows that $1 - s$ is right invertible for all $s \in rR$. By 1.40 Theorem, $\text{Rad } R$ is the largest right ideal I so that for all $s \in I$, $1 - x$ is right invertible. Therefore $rR \subsetneqq \text{Rad } R$ which contradicts $r \notin \text{Rad } R$. Conclude that if $r \notin \text{Rad } R$, then r is right invertible.

Since the condition that $R/\text{Rad } R$ is a division ring is a symmetric one, other characterizations of local rings would result by replacing "left" by "right" in the above theorem.

II.2. Perfect Rings.

For our study of perfect rings we shall study a version of Bass's Theorem P. Preliminary to this, however, it is required that we define a few fundamental concepts prerequisite to understanding Theorem P.

2.1 Definition: A projective cover of a module M is an epimorphism $P \longrightarrow M$ with small kernel, where P is projective.

2.2 Lemma: Let $P(A) \xrightarrow{\pi} A$ be a projective cover of A. Suppose we have $0 \longrightarrow K \longrightarrow P_1 \xrightarrow{\alpha} A \longrightarrow 0$ exact with P_1 projective. Then $P_1 = P \oplus Q$ where $P \cong P(A)$, $Q \subseteq K$ and $K \cap P$ is small in P.

Proof: Since P_1 is projective there is a mapping β making

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & P_1 & \xrightarrow{\alpha} & A & \longrightarrow & 0 \\ & & & & \beta \downarrow & \nearrow \pi & & & \\ & & & & P(A) & & & & \end{array}$$

commute, that is $\pi\beta = \alpha$. It is easy to see that since α is onto then $P(A) = \text{Im}\ \beta + \text{Ker}\ \pi$. Thus since $\text{Ker}\ \pi$ is small, $P(A) = \text{Im}\ \beta$, that is, β is onto. Then since $P(A)$ is projective β splits. Let μ be the splitting map, $\beta\mu = 1_{P(A)}$. Let $P = \text{Im}\ \mu$ and $Q = \text{Ker}\ \beta$. Then $P_1 = P \oplus Q$ and $P \cong P(A)$. Clearly $Q = \text{Ker}\ \beta \subseteq \text{Ker}\ \pi\beta = \text{Ker}\ \alpha = K$.

Also since $\text{Ker}\ \pi$ is small in $P(A)$ then by 1.70 Lemma $\mu(\text{Ker}\ \pi)$ is small in P. But $\mu(\text{Ker}\ \pi) = \mu(\text{Ker}\ \alpha\mu) = K \cap \text{Im}\ \mu = K \cap P$. //

2.3 Definition: A ring R is called right perfect if every right R-module has a projective cover.

Now, our version of Theorem P.

2.4 Theorem P: Let R be a ring and J its Jacobson radical. Then the following statements are equivalent.

(1) J is right T-nilpotent and R/J is semisimple.

(2) R is right perfect.

(3) Flat right R-modules are projective.

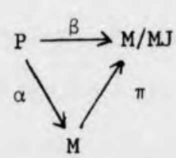
(4) R satisfies the descending chain condition on principal left ideals.

(5) R has no infinite set of orthogonal idempotents, and every nonzero left R-module has nonzero socle.

Proof: (1) ⇒ (2). Assume that J is right T-nilpotent and R/J is semisimple and show that R is right perfect. The proof proceeds as follows. Let M be a right R-module and consider the R/J-module M/MJ. Since R/J is semisimple, then M/MJ is isomorphic to a coproduct of minimal right ideals of R/J. Since minimal right ideals of R/J are generated by idempotents, and idempotents lift modulo J (since J is right T-nilpotent) then there is a set of idempotents of R $\{e_i \mid i \in I\}$ so that

$$M/MJ \cong \coprod_{i\in I} e_iR/e_iJ \cong \coprod_{i\in I} e_iR/\coprod_{i\in I} e_iJ = P/PJ.$$

Claim that $P = \coprod_{i\in I} e_iR$ is a projective cover of M. Consider the diagram



where β maps P canonically onto P/PJ and through the above isomorphisms onto M/MJ. π is the canonical epimorphism of M onto M/MJ, and α arises from the projectivity of P so that the diagram commutes. We desire to show that α is an epimorphism with small kernel.

<u>Assume</u> <u>MJ</u> <u>is</u> <u>small</u> <u>in</u> <u>M</u>. Since β is an epimorphism, $\pi\alpha$ is an epimorphism so that $\text{Im}\,\alpha + MJ = M$, therefore $\text{Im}\,\alpha = M$ and α is an epimorphism.

<u>Assume</u> <u>that</u> <u>PJ</u> <u>is</u> <u>small</u> <u>in</u> <u>P</u>. Then $\text{Ker}\,\alpha \subseteq \text{Ker}\,\beta = PJ$. Then $\text{Ker}\,\alpha$ is small in P since it is a submodule of a small submodule PJ. Thus we are finished provided the two underlined assumptions are shown to be valid.

Recall 1.66 Lemma, if K is a right T-nilpotent ideal of R, and M is a right R-module, then $M = MK$ only if $M = 0$. Now suppose $C = M$ or P and $S \subseteq C$ with $S + CJ = C$. Then $(C/S)J = C/S$, therefore by the lemma $C/S = 0$ which implies that $C = S$. So CJ is small in C.

(2) $\Rightarrow$ (3). Assume that R is right perfect and show that flat modules are projective. First, we show if R is right perfect then $J = \text{Rad}\,R$ is right T-nilpotent. Let $\{a_i\}_{i=1}^{\infty}$ be a sequence in J. Let F and G be the free R-modules constructed for 1.67 Lemma. Consider

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0.$$

Since R is right perfect then F/G has a projective cover and it follows from 2.2 Lemma that $F = P \oplus Q$ where $Q \subseteq G$ and $G \cap P$ is small in P, and P is a projective cover of F/G. Each generator x_i of F can be written $x_i = (x_i - x_{i+1}a_i) + x_{i+1}a_i \in G + FJ$.

So
$$\begin{aligned} F &= G + FJ \\ &= [(G \cap P) \oplus Q] + [PJ \oplus QJ] \\ &= [(G \cap P) + PJ] \oplus [Q + QJ] \\ &= [(G \cap P) + PJ] \oplus Q \end{aligned}$$

$$P = (G \cap P) + PJ$$

$$P = PJ$$

Then $P = 0$ by 1.71 Lemma. So $F/G = 0$. Then $z_1 \cdot 1 = x_1 + G = 0$, that is, $1 \in J_1 = z_1^r$. So $a_n a_{n-1} \cdots a_1 = 0$ for some n by 1.67 Lemma.

Now let F be a flat right R-module. Take a projective cover of F.

$$0 \longrightarrow K \longrightarrow P \longrightarrow F \longrightarrow 0 .$$

Take a left ideal I. Recall by 1.29 Theorem that $KI = PI \cap K$. Since J is a left ideal, then $KJ = PJ \cap K$. Since P is projective PJ is Rad P (1.47 Theorem), and Rad P contains all small submodules of P (1.42 Theorem). So $K \subseteq PJ$. Also $K \subseteq PJ \cap K$, so we have $K = PJ \cap K$. Therefore $KJ = K$, which we know implies that $K = 0$, (1.66 Lemma), hence $P \cong F$ and F is projective.

(3) $\Rightarrow$ (4). Assume that flat modules are projective, and show that R satisfies the descending chain condition on principal left ideals.

Let F be the free module whose basis is $x_1, x_2, \ldots$, and let G_n be the submodule of F generated by $x_1 - x_2a_1, x_2 - x_3a_2, x_3 - x_4a_3, \ldots, x_n - x_{n+1}a_n$. We claim that G_n is free, which requires only to show that the generating set is independent. Consider the linear combination $(x_1 - x_2a_1)c_1 + (x_2 - x_3a_2)c_2 + \cdots + (x_{n-1} - x_na_{n-1})c_{n-1} + (x_n - x_{n+1}a_n)c_n = 0$, which is equivalent to $x_1c_1 + (c_2 - a_1c_1)x_2 + (c_3 - a_2c_2)x_3 + \cdots + (c_n - a_{n-1}c_{n-1})x_n + (-c_na_n)x_{n+1} = 0$. We know that $\{x_1, x_2, x_3, \ldots, x_n, x_{n+1}\}$ is an independent set, hence we get $c_1 = 0$; $c_2 - a_1c_1 = 0$, therefore $c_2 = 0$; and so on, hence we have $c_1 = c_2 = \cdots = c_n = 0$. Conclude that the generating set for G_n is independent, hence a basis, and G_n is free. Now F and G_n are free. Is F/G_n free? To answer this, consider the epimorphism $\alpha: F \longrightarrow F/G_n$. Since we know that an epimorphism maps a generating set to a generating set, then the images of $x_1, x_2, \ldots, x_n, \ldots$ under α is a generating set for F/G_n. Let $\overline{x_i} = \alpha(x_i) = x_i + G_n$. Now consider the sequence $\overline{x_1}, \overline{x_2}, \overline{x_3}, \ldots, \overline{x_n}, \overline{x_{n+1}}, \ldots$. For any $x \in F$ we may write $x = \sum_{i=1}^{\infty} x_ir_i$, where $r_i \neq 0$ for only finitely many i. And for any $x \in F/G_n$ we may write $x = \sum_{i=1}^{\infty} \overline{x_i}r_i$, where $r_i \neq 0$ for only finitely many i. Since $x_1 - x_2a_1$, $x_2 - x_3a_2, \ldots, x_n - x_{n+1}a_n$ is a basis for G_n, we observe that $\overline{x_1} - \overline{x_2}a_1 = 0$, or $\overline{x_1} = \overline{x_2}a_1$, and $\overline{x_2} = \overline{x_3}a_2, \ldots, \overline{x_n} = \overline{x_{n+1}}a_n$. Hence it is clear that $\overline{x_{n+1}}, \overline{x_{n+2}}, \ldots$ is a generating set for F/G_n. Is the set independent? Assume the set is not independent. Then let

$\sum_{k=n+1}^{\infty} \overline{x_k}r_k = 0$ where at least one $r_k \neq 0$. If $\sum_{k=n+1}^{\infty} \overline{x_k}r_k = 0$, we know that $\sum_{k=n+1}^{\infty} x_kr_k \in G_n$. For an element to be in G_n, it must be a linear combination of $x_1 - x_2a_1, \ldots, x_n - x_{n+1}a_n$. Hence we have

$$\sum_{k=n+1}^{\infty} (x_kr_k) = \sum_{k=1}^{n} (x_k - x_{k+1}a_k)\, c_k = x_1c_1 + (c_2 - a_1c_1)x_2 + (c_3 - a_2c_2)x_3 + \cdots + (c_n - a_{n-1}c_{n-1})x_n + (-a_nc_n)x_{n+1} = 0.$$

Since the x_i are independent, then equating coefficients we see that $r_k = 0$ for all $k = n+1, n+2, \ldots$ which contradicts the assumption that at least one $r_k \neq 0$, hence $\overline{x_{n+1}}, \overline{x_{n+2}}, \ldots$ is independent, therefore a basis for F/G_n. Conclude that F/G_n is free. We know from a previous result that free modules are flat, so F/G_n is flat. Now $G = \bigcup_{i=1}^{\infty} G_i$. As in the proof that G_n is free, it is easy to see that G is free. It is now required that we show that $F/G = \varinjlim F/G_n$.

For all $i \leq j$ it is clear that G_i is a submodule of G_j. Hence we are guaranteed a well-defined epimorphism $\alpha_{ji}: F/G_i \longrightarrow F/G_j$ defined by $\alpha_{ji}(f + G_i) = f + G_j$. If we consider the set of positive integers, I, with the usual relation "$\leq$" then $(I, \leq)$ is a directed set. Then for each $i \in I$, we have a module, F/G_i, and for each pair $i,j \in I$ with $i \leq j$, we have a map $\alpha_{ji}: F/G_i \longrightarrow F/G_j$ where

(1) for each $i \in I$, $\alpha_{ii}: F/G_i \longrightarrow F/G_i$ is the identity map, and

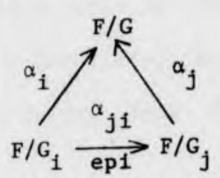
(2) if $i \leq j \leq k$, then $\alpha_{kj}\alpha_{ji} = \alpha_{ki}$, which describes a directed system, denoted $(F/G_i, I)$. We know that $\{F/G_i \xrightarrow{\alpha_i} F/G\}$ is a colimit for the diagram of modules and maps previously mentioned (that is, the set of modules F/G_i for $i \in I$ and maps α_{ji}, $i \leq j$, $i,j \in I$) if:

(1) it is a compatible set for the diagram, and

(2) given any $\{F/G_i \xrightarrow{\gamma_i} B\}$ compatible with the diagram, there is a unique map $F/G \xrightarrow{\gamma} B$ so that $\gamma\alpha_i = \gamma_i$ for each $i \in I$.

(1) Show compatibility:

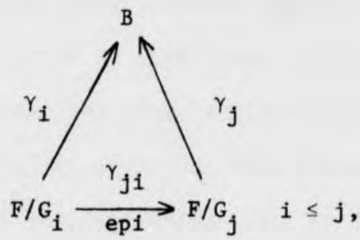
Consider the diagram



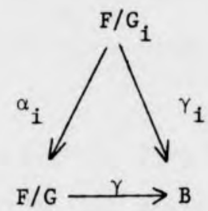
where $i \leq j$

$\alpha_i(f + G_i) = f + G$ and $\alpha_{ji}(f + G_i) = f + G_j$. The diagram clearly commutes, hence we have compatibility.

(2) Consider the diagram



and assume compatibility. Then we need to find a unique map $F/G \xrightarrow{\gamma} B$ so that $\gamma\alpha_i = \gamma_i$ for each $i \in I$.



Find γ. If we define γ: $F/G \longrightarrow B$ by $\gamma(f + G) = \gamma_i(f + G_i)$, the diagram commutes, if well-defined. It is not difficult to show that the map does not depend on the selection of $i \in I$ or the selection of $f \in f + G$. The map γ as defined above is so that $\gamma\alpha_i = \gamma_i$, and it is clear that for the diagram to commute, γ must be defined in this way. Conclude that $\{F/G_i \xrightarrow{\alpha_i} F/G\}$ is a colimit for the diagram.

Recall that $(F/G_i, I)$ is a directed system, hence the colimit $\{F/G_i \xrightarrow{\alpha_i} F/G\}$ is a direct limit of $(F/G_i, I)$, and $F/G = \varinjlim F/G_i$. By a previous result (1.32 Corollary), we know that the direct limit of flat modules is flat. Hence, since F/G_n is flat, F/G is flat.

By hypothesis, flat modules are projective, therefore F/G is projective. Consider the sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0 .$$

Since F/G is projective, the exact sequence splits; conclude that G is a direct summand of F, so by 1.68 Lemma, conclude that the chain $\{Ra_n a_{n-1} \cdots a_1\}$ of principal left ideals terminates, and R has descending chain condition on principal left ideals.

(4) $\Rightarrow$ (5). Assume R has descending chain condition on principal left ideals, and show that R has no infinite set of orthogonal idempotents, and every nonzero left R-module has nonzero socle. Assume R has an infinite set of orthogonal idempotents $\{e_i \mid i \in I\}$. Then the following chain of principal right ideals is always increasing:

$$e_1R \subset (e_1 + e_2)R \subset (e_1 + e_2 + e_3)R \subset \cdots$$

Consider the left annihilator of eR where $e^2 = e \in R$. Clearly $R(1 - e) \subseteq (eR)^{\ell} = \{x \in R \mid xer = 0 \text{ for all } r \in R\}$. Every $x \in R$ is of the form $x = xe + x(1 - e)$, and $xe = 0$ since $xer = 0$ for all $r \in R$. So $x = x(1 - e)$. So $(eR)^{\ell} \subseteq R(1 - e)$, and $(eR)^{\ell} = R(1 - e)$. Therefore $(e_1R)^{\ell} = R(1 - e_1)$ and $((e_1 + e_2)R)^{\ell} = R(1 - (e_1 + e_2))$. $(1 - e_1) \in (e_1R)^{\ell}$ but $(1 - e_1) \notin ((e_1 + e_3)R)^{\ell}$ since $(1 - e_1)$ does not annihilate $(1 - (e_1 + e_2))$. Clearly every $x \in ((e_1 + e_2)R)^{\ell}$ annihilates e_1R so we see that $R(1 - e_1) \supset R(1 - (e_1 + e_2))$. It is clear that we get a chain of principal left ideals

$$R \supset R(1 - e_1) \supset R(1 - (e_1 + e_2)) \supset R(1 - (e_1 + e_2 + e_3)) \supset \cdots .$$

This contradicts the hypotesis that R has no decreasing infinite chain of principal left ideals.

Now let A be a nonzero left R-module, and show that A has nonzero socle, that is, show that there exists a simple module S so that $S \subseteq A$. Let $a \in A$ be nonzero. If Ra is not simple, there must exist an $r_1 \in R$ so that $Rr_1a \neq 0$ and $Ra \supset Rr_1a$. If Rr_1a is not simple, there must exist an $r_2 \in R$ so that $Rr_2r_1a \neq 0$ and $Rr_1a \supset Rr_2r_1a$. If continuing in this way we never reach a simple submodule of A, then we produce a chain

$$Ra \supset Rr_1a \supset Rr_2r_1a \supset Rr_3r_2r_1a \supset \cdots$$

which obviously gives rise to the chain

$$R \supset Rr_1 \supset Rr_2r_1 \supset Rr_3r_2r_1 \supset \cdots$$

which is a decreasing infinite chain of principal left ideals, which contradicts the hypothesis. Conclude that there is a simple submodule of A, hence every nonzero left R-module has nonzero socle.

(5) $\Rightarrow$ (1). Assume that R has no infinite set of orthogonal idempotents and every nonzero left R-module has nonzero socle. Show that $J = \text{Rad } R$ is right T-nilpotent and R/J is semisimple. We shall define inductively $J_o = 0$, $J_{\alpha+1}$ is such that $J_{\alpha+1}/J$ is the socle of the left R-module J/J_α. If α is a limit ordinal, $J = \bigcup_{\beta<\alpha} J_\beta$. Claim that for some α_o, $J = J_{\alpha_o}$. Suppose this is not true. Then the sequence $\{J_\alpha\}$ contains sets of arbitrarily large cardinality. This is impossible since the cardinality of J is an upperbound for the cardinality of any J_α. Hence $J = J_{\alpha_o}$ for some

ordinal α_o. Thus if $a \in J$, we may define $h(a)$ to be the smallest α so that $a \in J_\alpha$. (Note that $h(a)$ can never be a limit ordinal since, if $a \in J_\alpha$ for a limit ordinal α, then $a \in \bigcup_{\beta<\alpha} J_\beta$, which implies that $a \in J_\beta$ for some $\beta < \alpha$.) So if $a \neq 0$, we may write $h(a) = \beta + 1$ for some β. By definition $J_{\beta+1}/J_\beta$ is the socle of J/J_β, hence is semisimple. Recall that semisimple modules are annihilated by Rad R (1.53 Theorem), so $J(J_{\beta+1}/J_\beta) = 0$, which means that $JJ_{\beta+1} \subseteq J_\beta$. Hence we see that for any $b \in J$, $h(ba) < h(a)$.

Let $\{a_n\}$ be a sequence of elements of J. Then if $a_n a_{n-1} \cdots a_1 \neq 0$ for all n, then $\{h(a_n a_{n-1} \cdots a_1)\}$ is a decreasing infinite chain of ordinals, which we know is impossible. Therefore, J is right T-nilpotent.

Claim that R/J contains no infinite set of orthogonal idempotents. If R/J does contain an infinite set of orthogonal idempotents, we know that countably many may be lifted to R, by 1.59 Theorem, but this contradicts the hypothesis that R contains no infinite set of orthogonal idempotents. We shall show that R/J is semisimple by showing that it is equal to its left socle. We know that $Rad(R/J) = 0$, so R/J is a semiprime ring, and every minimal left ideal is generated by an idempotent (1.56 Theorem). By 1.60 Theorem we see that since nonzero left R-modules have nonzero left socle, then nonzero left R/J-modules have nonzero left socle. Then we can assume that R is semiprime (by replacing R by R/J). Thus we know by 1.56 Theorem that minimal left ideals have the form Re for a minimal idempotent e. We wish to show that R is semisimple, which can be done by showing that R is equivalent to its left socle.

Assume R is not semisimple. From 1.61 Theorem we see that given a semisimple idempotent e, the set T_e of minimal idempotents orthogonal to e is nonempty. Let $g \in \prod_{\substack{e,\ \text{semisimple} \\ \text{idempotents}}} T_e$. Since ${}_R R \neq 0$, then ${}_R R$ has nonzero left socle, which means that R has a minimal left ideal Re. Define $e_1 = e$, $e_2 = g(e_1)$, ... , $e_n = g(e_1 + e_2 + \cdots + e_{n-1})$, ... Then $\{e_i\}_{i=1}^{\infty}$ is an infinite set of orthogonal idempotents. This contradicts the hypothesis. Conclude that R is semisimple. //

Another closely related topic to perfect rings is that of homological dimension. Bass discusses relationships between the two concepts and proves results that are useful in this paper, hence we shall define what the dimensions are and state the results Bass proves in his paper.

2.5 Definition: Let A be a right R-module. A projective resolution of A is an exact sequence

$$\cdots \longrightarrow P_n \xrightarrow{\alpha_n} P_{n-1} \xrightarrow{\alpha_{n-1}} \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_o \xrightarrow{\varepsilon} A \longrightarrow 0$$

where each P_i is projective. That every R-module has a projective resolution is a consequence of the fact that every R-module is the image of a free, hence projective, R-module (1.17 Theorem).

2.6 Definition: The projective dimension, denoted Pd(A), of an R-module A is the smallest positive integer n such that

$$0 \longrightarrow P_n \longrightarrow P_{n-1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_o \longrightarrow A \longrightarrow 0$$

is a projective resolution of A. If no such n exists, $Pd(A) = \infty$, and $Pd(A) = 0$ if and only if A is projective.

2.7 Definition: The right global dimension of a ring R (r.gl.dim.R) is $\sup \{ Pd(A) \mid A \text{ is a right R-module} \}$.

2.8 Definition: The right finitistic projective dimension of a ring R (rFPDR) is $\sup \{ Pd(A) \mid A \text{ is a right R-module, and } Pd(A) < \infty \}$.

We may observe from the definitions that if rFPDR = 0, then a projective submodule of a projective module is a direct summand.

For consider the monomorphism

$$0 \longrightarrow P_1 \longrightarrow P_2$$

where P_1 and P_2 are projective and $P_1 \subseteq P_2$. Then we may write

$$0 \longrightarrow P_1 \longrightarrow P_2 \longrightarrow P_2/P_1 \longrightarrow 0.$$

This is a projective resolution for P_2/P_1, and it is finite. Since rFPDR = 0, P_2/P_1 is projective, so the sequence splits. Conclude that P_1 is a direct summand of P_2.

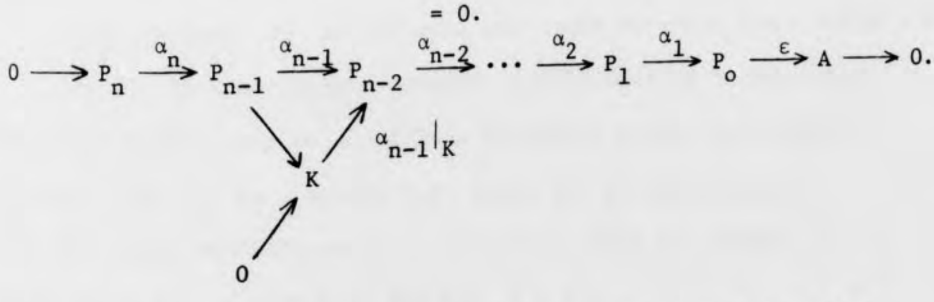
It is not difficult to show that the converse of the statement is also true. If given a projective submodule of a projective module we have that it is a direct summand, then rFPDR = 0. For consider a finite projective resolution

$$0 \longrightarrow P_n \xrightarrow{\alpha_n} P_{n-1} \xrightarrow{\alpha_{n-1}} P_{n-2} \xrightarrow{\alpha_{n-2}} \cdots \xrightarrow{\alpha_2} P_1 \xrightarrow{\alpha_1} P_o \xrightarrow{\varepsilon} A \longrightarrow 0.$$

By hypothesis, P_n is a direct summand of P_{n-1}, so $P_n = P_{n-1} \oplus K$.

We see that $\alpha_{n-1}|_K$ is a monomorphism, that is,

$$\begin{aligned} \text{Ker } \alpha_{n-1}|_K &= (\text{Ker } \alpha_{n-1}) \cap K \\ &= P_{n-1} \cap K \\ &= 0. \end{aligned}$$



This yields a shorter projective resolution. We may continue this process until we get a projective resolution

$$0 \longrightarrow \overline{P_o} \longrightarrow A \longrightarrow 0,$$

which means that rFPDR = 0.

Now, two theorems that Bass proves in his paper have useful results for our study hence will be stated here.

<u>2.9 Theorem</u>: The following are equivalent for any ring R:

(1) A finitely generated projective submodule of a projective right R-module is always a direct summand.

(2) The right annihilator of a finitely generated proper left ideal is always nonzero.

Proof: See [1, p. 478].

<u>2.10 Theorem</u>: Suppose that R is right perfect and that R satisfies the conditions of 2.9 Theorem. Then rFPDR = 0.

Proof: See [1, p. 479].

The following theorem will establish a desirable relationship between a local, right perfect ring and the right finitistic projective dimension of the ring.

<u>2.11 Theorem</u>: If R is local and right perfect, then rFPDR = 0.

Proof: By the previous theorem it will suffice to show that the right annihilator of a finitely generated proper left ideal is nonzero. Let L be a proper left ideal of R, and suppose that the right annihilator of L is zero. That is, assume $L^r = 0$ where $L^r = \{r \in R \mid zr = 0 \text{ for all } z \in L\}$.

Given an $x \in R$ let $L_x = \{z \in L \mid zx \neq 0\}$. Then if $x \neq 0$, $L_x \neq \phi$. By the axiom of choice we know that $\prod_{x \neq 0} L_x \neq \phi$. Let $f \in \prod_{x \neq 0} L_x$. Given $x \neq 0$, $f(x) \in L$, and $f(x) \cdot x \neq 0$. Since L is a proper left ideal, we know that L is contained in a maximal left ideal. Since R is local, there is only one maximal left ideal, Rad R. So $L \subseteq$ Rad R. Select $x_1 \neq 0$. Define $x_2 = f(x_1)$, $x_3 = f(x_2 x_1)$, ... , $x_n = f(x_{n-1} x_{n-2} \cdots x_2 x_1)$, ... , hence we get a sequence such that $x_2 x_1 \neq 0$, $x_3 x_2 x_1 \neq 0$, ... , $x_n x_{n-1} \cdots x_2 x_1 \neq 0$, ... But this contradicts right T-nilpotence of the radical (since R is right perfect). Therefore conclude that $L^r \neq 0$, and rFPDR = 0. //

This concludes our discussion of local rings and right perfect rings, which prepares us to discuss the rings of our interest, Steinitz rings.

CHAPTER III

STEINITZ RINGS

When studying vector spaces, which are modules over a division ring, one finds that the following property holds true: given any independent subset S in a vector space V, a basis for V can be found which contains S. This result is due to Steinitz. Since vector spaces are free modules over a division ring, one may suspect that the property Steinitz proved for vector spaces may hold for free modules over rings other than division rings. We hence state the Steinitz property more generally to be the following: given a free right R-module F and an independent subset T, a basis for F can be found that contains T. Considering the Steinitz theorem, it would behoove one to ask, precisely when would the theorem hold for a general free right module over a ring? We shall call a ring R a (right) Steinitz ring when the Steinitz property holds for free R-modules.

When considering Steinitz rings, some properties may immediately be seen. If F is any free module, then any independent set A generates a free submodule, G. Since a basis for F can be found that contains A, the set of elements of the basis that are not in A generates a free submodule H. Since $F = G \oplus H$ and $F/G \cong H$, we may conclude that the factor module of a free module by a free submodule is free when the ring is Steinitz. Similarly, we see from the above discussion that when G is a free submodule of a free module F that $F = G \oplus H$ (where H is as above). Hence we may

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Now we shall prove a theorem for a result that will be very helpful to our study.

3.1 Theorem: If R is right Steinitz, then R has descending chain condition on principal left ideals, or equivalently, R is right perfect.

Proof: Let

$$R \supseteq Ra_1 \supseteq Ra_2a_1 \supseteq Ra_3a_2a_1 \supseteq \cdots$$

be a descending chain of principal left ideals. Let F be the free module with a countable base generated by $\{x_i \mid i = 1,2,3,\ldots\}$, and as in 2.4 Theorem, Let G be the submodule of F generated by $\{x_n - a_n x_{n+1} \mid n = 1,2,\ldots\}$. We know from the proof of 2.4 Theorem that G is free. Since R is right Steinitz, G being a free submodule of the free module F is a direct summand of F. As we saw in 2.4 Theorem, sequences of the a_i are eventually stationary, which means that we cannot have a decreasing infinite chain of principal left ideals, which concludes the proof. //

Therefore, we see that if R is right Steinitz, then it is right perfect, by 2.4 Theorem (Theorem P of Bass).

Another result which is useful for our characterization of Steinitz rings is the following.

3.2 Theorem: If R is right Steinitz, then projective modules are free.

Proof: Let P be a projective module. Then by 1.18 Theorem, P is isomorphic to a direct summand of a free module F where F has an infinite base, so $F \cong P \oplus Q$. We know that for any free module B with an infinite base, $B \cong \sum_{i=1}^{\infty} (B)_i$, hence we have $F \cong \sum_{i=1}^{\infty} (F)_i$. Thus we may write $F \cong F \oplus F \oplus F \oplus \cdots$. Recall that $F \cong P \oplus Q$, so $F \cong (P \oplus Q) \oplus (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots$. By associativity of the coproduct, this yields

$$F \cong P \oplus (Q \oplus P) \oplus (Q \oplus P) \oplus \cdots$$

or

$$F \cong P \oplus (P \oplus Q) \oplus (P \oplus Q) \oplus \cdots$$

or

$$F \cong P \oplus F = F_1.$$

So $P \cong F_1/F$ which we know is free by our previous discussion. //

A well-known result due to Kaplansky is that over a local ring all projective modules are free. From the previous theorem we see that over Steinitz rings projective modules are free. These facts may prompt us to suspect that perhaps Steinitz rings are local. We shall now seek to answer the question: are Steinitz rings local?

We know that to show that a ring R is local, it is sufficient to show that R/J is a division ring where J is the Jacobson radical of R. A previous result (1.21 Theorem) tells us that if every R-module is free, then R is a division ring. Therefore, to show that Steinitz rings are local rings, it will suffice to show that all R/J-modules are free.

3.3 Theorem: If R is right Steinitz, then all R/J-modules are free.

Proof: Since we know that R is right perfect, we know that R/J is semisimple. Let M be an R/J-module. Then $M \cong \coprod_{i\varepsilon I} S_i$, where the S_i are simple modules (since every module over a semisimple ring is semisimple by 1.51 Theorem). Since semisimple rings are also semiprime, R/J is semiprime. Therefore any simple submodule of R/J is isomorphic to a minimal right ideal (1.52 Theorem). We recall that for a semiprime ring minimal right ideals are generated by idempotents (1.56 Theorem). Hence we see that if S_i is any simple submodule of M, then $S_i = \bar{e}_i R/J$ for each $i \varepsilon I$ where $\bar{e}_i = e_i + J$ is an idempotent of R/J. Now since R is right perfect, we know that we may lift idempotents, so we may assume that e_i is an idempotent of R. Hence we see from 1.62 Theorem that we may write $\bar{e}_i R/J \cong e_i R/e_i J$ for an idempotent e_i of R. So $M \cong \coprod_{i\varepsilon I} e_i R/e_i J$. From 1.4 Theorem we have that $\coprod_{i\varepsilon I} e_i R/e_i J \cong \coprod_{i\varepsilon I} e_i R/\coprod_{i\varepsilon I} e_i J$, hence $M \cong \coprod_{i\varepsilon I} e_i R/\coprod_{i\varepsilon I} e_i J$. It is clear that $\coprod_{i\varepsilon I} e_i R$ is a projective module P, so we may write $M \cong \coprod_{i\varepsilon I} e_i R/\coprod_{i\varepsilon I} e_i J \cong P/PJ$, for P, projective. Since R is right Steinitz, projective modules are free, hence $P \cong \coprod_{s\varepsilon S} (R)_s$ and $PJ \cong \coprod_{s\varepsilon S} (J)_s$. Hence $P/PJ \cong \coprod_{s\varepsilon S} (R)_s/\coprod_{s\varepsilon S} (J)_s \cong \coprod_{s\varepsilon S} (R/J)_s$. So $M \cong P/PJ$ is free as an R/J-module. //

We may now say that right Steinitz rings are local, right perfect rings. The obvious question we now investigate is: are the conditions local and right perfect sufficient for a ring to be a right Steinitz ring? If we can now show this, then we will have completed our characterization of Steinitz rings. The last part of this chapter will be devoted to proving that if a ring R is local and right perfect, then it is right Steinitz. Note that for a local ring R, we know that R/J is a division ring where J is the Jacobson radical, and division rings we know are Steinitz. We wish to employ the Steinitz properties of R/J to show that R is right Steinitz.

To prove that R is right Steinitz, we need to show that, given a free R-module, F, and an independent subset, S, we can find a basis for F containing S. Moreover, we shall show that given a base B of F, there is a set $B_1 \subseteq B$ so that $S \cup B_1$ is a base of F. Let F be a free R-module and S an independent subset of F. Consider the mapping $\alpha_F\colon F \longrightarrow F/FJ$. Let $\overline{F} = F/FJ$ and $\overline{S} = \{s + FJ \mid s \in S\}$. We know that the bar functor maps R-modules to R/J-modules. Using this functor and the fact that R/J is right Steinitz, in fact a division ring, we shall show that given a base B of F and an independent set $S \subseteq F$, there is a set $B_1 \subseteq B$ so that $B_1 \cup S$ is a base for F. First we need to prove four theorems which establish a relationship between R-modules and R/J-modules that we need. Essentially the first three theorems will allow us to transfer the hypotheses from R-modules to R/J-modules where we know that the Steinitz property holds true. The fourth theorem will allow us to "drag back" to R-modules the conclusion we desire, that is the base containing the independent set.

3.4 Theorem: If F is a free R-module, then $\overline{F} = F/FJ$ is free as an R/J-module.

Proof: From a previous theorem (1.26 Theorem), we know that $F/FJ \cong F \otimes_R R/J$. Recall that for any free module F we have that $F \cong \coprod_{i\in I} (R_R)_i$ (1.15 Theorem), hence we have

$$F/FJ \cong F \otimes_R R/J \cong \coprod_{i\in I} (R_R)_i \otimes_R R/J .$$

Recall that from 1.33 Lemma we have that $\coprod_{i\in I} S_i \otimes F \cong \coprod_{i\in I} (S \otimes F)_i$, therefore

$$\coprod_{i\in I} (R_R)_i \otimes_R R/J \cong \coprod_{i\in I} (R_R \otimes R/J)_i .$$

Recall finally that $A_R \otimes R \cong A$ for any R-module A by 1.27 Theorem. Hence we see that $F/FJ \cong \coprod_{i\in I} (R/J)_i$ which implies that F/FJ is free as an R/J-module. //

3.5 Theorem: If B is a base for a free R-module F, then $\overline{B} = \{b + FJ \mid b \in B\}$ is a base for $\overline{F} = F/FJ$.

Proof: We know from the previous theorem that if F is free as an R-module, then $\overline{F} = F/FJ$ is free as an R/J-module. We derived this result by manipulating with a series of isomorphisms. We did not, however, study the isomorphisms carefully to see exactly what the mappings were. It will be beneficial to do this now. We know from the previous theorem that $\overline{F} \cong \coprod_{b\in B} (R/J)_b$. Hence if we can determine the isomorphism, and find the image of a basis for $\coprod_{b\varepsilon B} (R/J)_b$ under the isomorphism, we will have a basis for $\overline{F}$. In particular, if we can show that $\overline{B}$ maps to a basis for $\coprod_{b\varepsilon B} (R/J)_b$ under the isomorphism, then we will have shown that $\overline{B}$ is a base for $\overline{F}$, which is what we desire. To this end, let us consider the isomorphisms that we know exist from the previous theorem, and determine the mappings explicitly. Recall:

$$\overline{F} = F/FJ \cong F \otimes_R R/J \cong \coprod_{b\varepsilon B} (R_R)_b \otimes_R R/J \cong \coprod_{b\varepsilon B} (R_R \otimes R/J)_b \cong \coprod_{b\varepsilon B} (R/J)_b.$$

Let: $\gamma_1: F/FJ \longrightarrow F \otimes_R R/J$ where $\gamma_1(x + FJ) = x \otimes \overline{1}$, $\overline{1} = 1 + J$;

$\gamma_2: F \otimes_R R/J \longrightarrow \coprod_{b\varepsilon B} (R_R)_b \otimes_R R/J$ where $\gamma_2(x \otimes \overline{1}) = \sum \kappa_b(r_b) \otimes \overline{1}$,

where $x = \sum_{b\varepsilon B} br_b$;

$\gamma_3: \coprod_{b\varepsilon B} (R_R)_b \otimes_R R/J \longrightarrow \coprod_{b\varepsilon B} (R_R \otimes R/J)_b$ where $\gamma_3(y \otimes \overline{1})(b) = y(b) \otimes \overline{1}$;

$\gamma_4: \coprod_{b\varepsilon B} (R_R \otimes R/J)_b \longrightarrow \coprod_{b\varepsilon B} (R/J)_b$ where $\gamma_4(z)(b) = \overline{r}$, where

$r \otimes \overline{1} = z(b)$, $z \varepsilon \coprod_{b\varepsilon B} (R_R \otimes R/J)_b$.

In order to find the image of $\overline{B}$ under the isomorphisms, select an element from $\overline{B}$ and determine its image under the composition of the mappings. Select $b \varepsilon B$, then $\overline{b} \varepsilon \overline{B}$ is of the form $\overline{b} = b + FJ$.

$$\gamma_1(b + FJ) = b \otimes \overline{1} \text{ where } \overline{1} = 1 + J,$$

$$\gamma_2(b \otimes \overline{1}) = \kappa_b(1) \otimes \overline{1},$$

$$\gamma_3(\kappa_b(1) \otimes \overline{1}) = \tilde{\kappa}_b(1 \otimes \overline{1}),$$

$$\gamma_4(\tilde{\kappa}_b(1 \otimes \overline{1})) = \approx{\kappa}_b(\overline{1}),$$

where κ_b, $\tilde{\kappa}_b$, $\tilde{\tilde{\kappa}}_b$ are the canonical monomorphisms associated with the various coproducts. If we observe that $\{\tilde{\tilde{\kappa}}_b(\overline{1}) \mid b \in B\}$ is the standard basis for $\coprod_{b \in B} (R/J)_b$, then we see that the image of $\overline{B}$ under the isomorphisms is a basis for $\coprod_{b \in B} (R/J)_b$, hence $\overline{B}$ must be a basis for $\overline{F}$. //

<u>3.6 Theorem</u>: Assume R is a local, right perfect ring. If S is an independent subset of F, then $\alpha_F(S) = \overline{S} = \{s + FJ \mid s \in S\}$ is independent in $\overline{F} = F/FJ$.

Proof: Since R is local and right perfect, we know by 2.11 Theorem that rFPDR = 0. Knowing this fact and the result of the following lemma, the conclusion of the theorem follows immediately.

<u>Lemma</u>: If rFPDR = 0, F is free, $S \subseteq F$, S is independent, and α_F: $F \longrightarrow F/FJ = \overline{F}$, then $\overline{S} = \alpha_F(S) = \{s + FJ \mid s \in S\}$ is independent in $\overline{F}$.

Proof: Let G be the free submodule of F generated by S. Then we have the exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0 \ .$$

Recall that if rFPDR = 0, then a projective submodule of a projective module is a direct summand. Hence our sequence is split exact. Since an additive functor preserves split exact sequences, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{j} & F & \longrightarrow & F/G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_G} & & \downarrow{\scriptstyle \alpha_F} & & \downarrow{\scriptstyle \alpha_{F/G}} & & \\ 0 & \longrightarrow & \overline{G} & \xrightarrow{\overline{j}} & \overline{F} & \longrightarrow & \overline{F/G} & \longrightarrow & 0 \end{array} .$$

Since S is a basis of G, $\widetilde{S} = \alpha_G(S) = \{s + GJ \mid s \in S\}$ is a basis of $\overline{G}$, by the previous theorem. Observe that $\overline{j}(\widetilde{S}) = \overline{S}$. Recalling that monomorphisms preserve independence, since $\overline{j}$ is a monomorphism, we have that $\overline{S}$ is independent in $\overline{F}$. //

<u>3.7 Theorem</u>: Assume that $J = \text{Rad } R$ is right T-nilpotent, F is projective, $B \subseteq F$. If $\overline{B}$ is a basis of $\overline{F}$, then B is a basis of F.

Proof: Show that B generates F. Let G be the submodule generated by B. Then we have the exact sequence

$$0 \longrightarrow G \longrightarrow F \longrightarrow F/G \longrightarrow 0 .$$

Consider the maps α_G, α_F. $\alpha_{F/G}$, then we have

$$\begin{array}{ccccccccc} 0 & \longrightarrow & G & \xrightarrow{j} & F & \longrightarrow & F/G & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_G} & & \downarrow{\scriptstyle \alpha_F} & & \downarrow{\scriptstyle \alpha_{F/G}} & & \\ & & \overline{G} & \xrightarrow{\overline{j}} & \overline{F} & \longrightarrow & \overline{F/G} & \longrightarrow & 0 \end{array} .$$

$\overline{G}$ is generated by $\overline{B}$. But $\overline{B}$ generates $\overline{F}$, so $\overline{j}$ is an epimorphism, hence $\overline{F/G} = 0$. Recall that if A is an R-module and J is right T-nilpotent, then $\overline{A} = A/AJ = 0$ only if $A = 0$ (1.66 Lemma). So $\overline{F/G} = 0$ implies $F/G = 0$ which means $F = G$. Therefore, B generates F.

Show that B is a basis for F. Let K be the free module with base B. Let

$$K \xrightarrow{\alpha} F \longrightarrow 0$$

be the epimorphism induced by the identity map on B, (since B generates F). Let $N = \text{Ker}\ \alpha$. Then we have the exact sequence

$$0 \longrightarrow N \longrightarrow K \xrightarrow{\alpha} F \longrightarrow 0 .$$

Since F is projective, the exact sequence splits, which gives rise to the following diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & N & \longrightarrow & K & \xrightarrow{\alpha} & F & \longrightarrow & 0 \\ & & \downarrow{\scriptstyle \alpha_N} & & \downarrow{\scriptstyle \alpha_K} & & \downarrow{\scriptstyle \alpha_F} & & \\ 0 & \longrightarrow & \overline{N} & \longrightarrow & \overline{K} & \xrightarrow{\overline{\alpha}} & \overline{F} & \longrightarrow & 0 \end{array} \quad .$$

We know that $\overline{\alpha}$ is an epimorphism. We need that $\overline{\alpha}$ is an isomorphism, so it will suffice to show that $\text{Ker}\ \overline{\alpha} = 0$. Observe that $\overline{K}$ is free with base $\{b + KJ\}$ and $\overline{F}$ is free with base $\{b + FJ\}$. Every element of $\overline{K}$ is uniquely expressible as $\sum_{b \varepsilon B} (b + KJ)r_b$, since $\{b + KJ\}$ generates $\overline{K}$. Find $\text{Ker}\ \overline{\alpha}$. If $\overline{\alpha}(\sum(b + KJ)r_b) = \sum(b + FJ)r_b = 0$ then $r_b = 0$ for all $b \varepsilon B$ and $\sum(b + KJ)r_b = 0$. Thus $\text{Ker}\ \overline{\alpha} = 0$, which is $\overline{N} = 0$. By right T-nilpotence of Rad R, we have that $N = 0$. So α is an isomorphism which implies that F is free with base B. //

In conclusion, we state the following theorem which has motivated this paper.

<u>3.8 Theorem</u>: The following statements are equivalent for the ring R.

(1) R is right Steinitz.

(2) R is local and right perfect.

(3) Given a free R-module F, a base B of F, and an independent subset S of F, there is a set $B_1 \subseteq B$ so that $S \cup B_1$ is a base for F.

Proof: (1) ⇒ (2). Assume that R is right Steinitz, and show that R is local and right perfect. If R is right Steinitz, we know by 3.1 Theorem that R has descending chain condition on principal left ideals, but this is, by 2.4 Theorem (Theorem P of Bass), equivalent to saying that R is right perfect. So if R is right Steinitz, then R is right perfect.

By 3.3 Theorem, we have that if R is right Steinitz, then all R/J-modules are free. Recall from 1.21 Theorem, that if every R-module is free, then R is a division ring. Therefore R/J is a division ring, which means that R is local by 2.1 Theorem.

(2) ⇒ (3). Assume that R is local and right perfect, and show that given a free R-module F, a base B of F, and an independent subset S of F, there is a set $B_1 \subseteq B$ so that $B_1 \cup S$ is a base for F. By 3.4 Theorem, we have that if F is a free R-module, then $F/FJ = \overline{F}$ is free as an R/J-module. Since R is local, R/J is a division ring, hence the Steinitz property holds for R/J-modules. By 3.5 Theorem, we know that if B is a base of F, then $\overline{B} = \{b + FJ \mid b \in B\}$ is a base for $\overline{F} = F/FJ$. By 3.6 Theorem, we know that if S is independent in F, then $\overline{S} = \{s + FJ \mid s \in S\}$

is independent in $\overline{F}$. By the Steinitz properties of R/J, we know that there is a subset of $\overline{B}$, $\overline{B_1}$, so that $\overline{B_1} \subseteq \overline{B}$ and $\overline{B_1} \cup \overline{S}$ is a base for $\overline{F}$. By 3.7 Theorem, since R is perfect, we know that if $\overline{B}$ is a base for $\overline{F}$, then B is a base for F. Choose a set $B_1 \subseteq B$ so that $\overline{B_1} = \{b_1 + FJ \mid b_1 \in B_1\}$. Then we see that $B_1 \cup S$ is a base for F which concludes this implication.

(3) $\Rightarrow$ (1) is obvious. //

APPENDIX

The following examples are offered to elucidate the concept of a right Steinitz ring.

I. A local ring which is not right Steinitz.

Lemma: A commutative integral domain D which is Steinitz is a field.

Proof: For clearly there could be no nonzero nilpotent element. Thus $\text{Rad } D$ being T-nilpotent must be zero, whence $D \cong D/\text{Rad } D$ is a field. //

Let R be the subring of $\mathcal{Q}$, the rational numbers, consisting of those elements a/b where b is an odd integer. It is easy to prove that R has the unique maximal ideal M where $M = \{a/b \in R \mid a \text{ is even}\}$. Since $\mathcal{Q}$ is a field then R is certainly an integral domain and since $\mathcal{Q}$ is a prime field R cannot be a field. Thus R is local but not Steinitz.

II. A right perfect ring which is not right Steinitz.

Lemma: Let R_n denote the ring of $n \times n$ matrices over R. R_n is right Steinitz if and only if R is right Steinitz and $n = 1$.

Proof: It is clear that if R is right Steinitz and $n = 1$, then R_1 is right Steinitz.

Let R_n be right Steinitz. Let A be the matrix with the first row identical to the first row of the identity matrix and all other rows consist entirely of zeroes. Clearly A is a zero-divisor hence a nonunit. Also $I - A$ is a nonunit if $n > 1$. Thus R_n is not local unless $n = 1$. Conclude $n = 1$ and R is right Steinitz. //

Then the ring F_2 of 2×2 matrices over a field F is right perfect (it is even semisimple) by the Wedderburn-Artin Theorem, but cannot be right Steinitz.

III. A ring which is right Steinitz but not left Steinitz.

The following example is a version of an example from Bass's paper.

Let F be any division ring. Let F_w be the ring of column-finite matrices over F. Thus F_w consists of matrices each of which has countably many rows and columns and where each column has only finitely many nonzero entries. The addition and multiplication of F_w is as usual for matrices. Let N be the subset of F_w consisting of the matrices which have zero entries on and below the diagonal and having finitely many nonzero entries above the diagonal. Let R be the set of matrices of the form $xI + n$ where $x \in F$, $n \in N$ and I is the identity matrix. R is easily seen to be a subring of F_w.

(1) N is an ideal in R.

It is easy to see that N is an additive subgroup of R closed under multiplication by elements of R on both sides.

(2) N is a nil ideal.

Let A be a matrix from N. Since A has only finitely many nonzero entries there is a least integer k such that for $\ell \geq k$ the ℓth column of A is all zero. A little study of matrix multiplication implies that if B is any matrix in R then BA has its ℓth column all zeroes for $\ell \geq k$. Let A have the form

$$A = \begin{bmatrix} 0 & * & * & \cdots & * & 0 & 0 & \cdots \\ 0 & 0 & * & \cdots & * & 0 & 0 & \cdots \\ 0 & 0 & 0 & \cdots & * & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdots & 0 & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdot & \cdots \end{bmatrix}$$

where the *'s indicate possible nonzero entries and there are $k - 2$ *'s in row 1. A look at A^2 will show it has zeroes where A has zeroes and has the above form with *'s replaced by zeroes on the first super-diagonal.

$$A^2 = \begin{bmatrix} 0 & 0 & * & * & \cdots & * & 0 & \cdots \\ 0 & 0 & 0 & * & \cdots & * & 0 & \cdots \\ 0 & 0 & 0 & 0 & \cdots & * & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & 0 & 0 & \cdots \\ \cdot & \cdot & \cdot & \cdot & \cdots & \cdot & \cdot & \cdots \end{bmatrix}$$

where now there are $k - 3$ *'s in row 1. The upward progression of the zero diagonal continues until by the time we reach A^{k-1} we must have a zero matrix.

(3) <u>$N = \text{Rad } R$ and R is local</u>. Since Rad R contains any nil ideal then $N \subseteq \text{Rad } R$. Consider R/N. Let $a + N$ be a coset. $a = xI + n$ for some $x \in F$, $n \in N$ so $a + N = xI + N$. Thus the mapping $x \longmapsto xI + N$ is from F onto R/N. It is clearly a ring mapping and

nonzero so it is an isomorphism since F is simple. Thus $R/N \cong F$ so N is a maximal ideal and N = Rad R and R is local.

(4) <u>N is right T-nilpotent</u>.

Let $A_1, A_2, A_3, \ldots$ be a sequence in N. We examine the products $A_1, A_2A_1, A_3A_2A_1, \ldots$ to see if we eventually get zero. The argument is nearly the same as in the argument that N is nil. If k is the least integer so that A_1 has zero columns from the kth column on then the same k works for $A_2A_1, A_3A_2A_1, \ldots$ As in the previous argement the zero diagonal progresses upward until one has (if not before) $A_{k-1}A_{k-2}\cdots A_3A_2A_1 = 0$.

(5) <u>N is not left T-nilpotent</u>.

Consider the following sequence in N.

$$A_1 = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

$$A_2 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 1 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . \\ 0 & 0 & 0 & 0 & 0 & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . \end{bmatrix}$$

$$A_3 = \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

and so on. In general A_n is the matrix with all rows zero except the nth and the nth row of A_n is the $n + 1$st row of the identity matrix. Computing $A_1, A_1A_2, A_1A_2A_3, \ldots$ one gets

$$\begin{bmatrix} 0 & 1 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ 0 & 0 & 0 & 0 & 0 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}$$

and so on where the 1 in the top row moves to the right one step each time. One never gets a zero matrix. Thus N is not left T-nilpotent. It follows that R is right Steinitz but not left Steinitz.

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