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The author has defined weakly equivalent topologies and introduced the concepts of the weakly and somewhat separation axioms. These two types of separation axioms are investigated individually, and the relationships of each to the usual separation axioms are studied. The concepts of the strongly separation axioms and the strongly topological properties are also defined.

UNUSUAL SEPARATION AXIOMS AND

TOPOLOGICAL SPACES

by

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INTRODUCTION

In this paper various types of separation axioms are studied. The ideas behind these results originated in [2], [4], and a seminar held by Dr. K. R. Gentry in the spring of 1969.

In Chapters I and II, the separation axioms related to weakly equivalent topologies are studied. Also in Chapter II, somewhat homeomorphisms are defined and a major theorem is proved showing that weak topological properties are preserved under somewhat homeomorphisms.

In Chapter III, strongly topological properties and strongly separation axioms are introduced.

In Chapter IV, the somewhat separation axioms are defined and several theorems showing their relationship to the usual separation axioms are proved.

CHAPTER I

WEAKLY SEPARATION AXIOMS--PRELIMINARY RESULTS

DEFINITION 1. Let X be a set. A topology S for X is said to be weakly equivalent to a topology T for X provided, if $U \in S$ and $U \neq \emptyset$, then there exists a $V \in T$ such that $V \neq \emptyset$ and $V \subset U$; and if $W \in T$ and $W \neq \emptyset$, then there exists an $R \in S$ such that $R \neq \emptyset$ and $R \subset W$.

DEFINITION 2. A topological space (X,T) is weakly T_i $(i=0,1,2,3,4)$ provided there exists a topology S for X such that (X,S) is T_i and T is weakly equivalent to S.

THEOREM 1. If (X,T) is a topological space and (X,T) is weakly $T_{i+1}(i=0,1,2,3)$, then (X,T) is weakly T_i.

Proof: Since (X,T) is weakly T_{i+1}, there is a topology S for X such that S is weakly equivalent to T and (X,S) is T_{i+1}. Since (X,S) is T_{i+1}, (X,S) is T_i and thus (X,T) is weakly T_i.

EXAMPLE 1. Let $X = \{0,1,2\}$ and let $S = \{\emptyset,X,\{0\}\}$. Then (X,S) is weakly T_0 but not weakly T_1.

Proof: Let $T = \{\emptyset,X,\{0\},\{0,1\}\}$. Then T is weakly equivalent to S and (X,T) is T_0. Thus (X,S) is weakly T_0. The subsets $\{1\}$, $\{2\}$, and $\{1,2\}$ cannot be in a topology for X weakly equivalent to S, since X is the only member of S containing 1 or 2 and it is not contained in $\{1\}$, $\{2\}$, or $\{1,2\}$. Also, $\{0\}$ must be in any topology weakly equivalent to S, since

CHAPTER I

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DEFINITION 1. Let X be a set. A topology S for X is said to be weakly equivalent to a topology T for X provided, if $U \in S$ and $U \neq \emptyset$, then there exists a $V \in T$ such that $V \neq \emptyset$ and $V \subset U$; and if $W \in T$ and $W \neq \emptyset$, then there exists an $R \in S$ such that $R \neq \emptyset$ and $R \subset W$.

DEFINITION 2. A topological space (X,T) is weakly T_i $(i=0,1,2,3,4)$ provided there exists a topology S for X such that (X,S) is T_i and T is weakly equivalent to S.

THEOREM 1. If (X,T) is a topological space and (X,T) is weakly T_{i+1} $(i=0,1,2,3)$, then (X,T) is weakly T_i.

Proof: Since (X,T) is weakly T_{i+1}, there is a topology S for X such that S is weakly equivalent to T and (X,S) is T_{i+1}. Since (X,S) is T_{i+1}, (X,S) is T_i and thus (X,T) is weakly T_i.

EXAMPLE 1. Let $X = \{0,1,2\}$ and let $S = \{\emptyset,X,\{0\}\}$. Then (X,S) is weakly T_0 but not weakly T_1.

Proof: Let $T = \{\emptyset,X,\{0\},\{0,1\}\}$. Then T is weakly equivalent to S and (X,T) is T_0. Thus (X,S) is weakly T_0. The subsets $\{1\}$, $\{2\}$, and $\{1,2\}$ cannot be in a topology for X weakly equivalent to S, since X is the only member of S containing 1 or 2 and it is not contained in $\{1\}$, $\{2\}$, or $\{1,2\}$. Also, $\{0\}$ must be in any topology weakly equivalent to S, since

$\{0\}$ is the only non-empty set contained in the element $\{0\}$ of S. Since $\emptyset$ and X are in every topology, the topologies for X weakly equivalent to S must be exactly $S_1 = S$, $S_2 = \{\emptyset, X, \{0\}, \{0,1\}\}$, $S_3 = \{\emptyset, X, \{0\}, \{0,2\}\}$, and $S_4 = \{\emptyset, X, \{0\}, \{0,1\}, \{0,2\}\}$. Since none of these topologies is T_1, (X,S) is not weakly T_1.

EXAMPLE 2. Let X be an infinite set and let T be the cofinite topology for X (that is, T consists of $\emptyset$ and all compliments of finite sets). Then (X,T) is weakly T_1 but not weakly T_2.

Proof: Since (X,T) is T_1 [1, p. 138], (X,T) is weakly T_1.

Suppose there is a topology S for X such that S is weakly equivalent to T and (X,S) is T_2. Since X is infinite, there exist points p and q in X such that $p \neq q$. Since (X,S) is T_2, there exist elements V and U in S such that $p \in V$, $q \in U$, and $V \cap U = \emptyset$. Since $V \neq \emptyset$, $U \neq \emptyset$, and S is weakly equivalent to T, there exist sets V' and U' in T such that $V' \neq \emptyset$, $U' \neq \emptyset$, $V' \subset V$, and $U' \subset U$. Thus $U' \cap V' = \emptyset$. Now $U' \cap V' = X - [(X-U') \cup (X-V')]$ and since $U' \neq \emptyset$, $X - U'$ is finite. Similarly, $X - V'$ is finite and hence $(X-U') \cup (X-V')$ is finite. Since X is infinite, $U' \cap V' = X - [(X-U') \cup (X-V')]$ is infinite and hence cannot be empty. But this is impossible. Therefore, (X,T) is not weakly T_2.

THEOREM 2. If (X,T) is $T_i (i=0,1,2,3,4)$, then (X,T) is weakly T_i.

Proof: Since every topology is weakly equivalent to itself, the proof is trivial.

<u>EXAMPLE 3</u>. Let $X = \{0,1,2\}$ and let $T = \{\emptyset,X,\{0\}\}$. Then (X,T) is weakly T_0 but not T_0.

Proof: Let $T' = \{\emptyset,X,\{0\},\{0,1\}\}$. Since T is weakly equivalent to T' and (X,T') is T_0, (X,T) is weakly T_0. Clearly T is not T_0.

<u>EXAMPLE 4</u>. Let $X =$ an infinite set containing points a and b. Let $T = \{\emptyset\}\cup\{X\}\cup\{U|$ the compliment of U is a finite set containing a and $b\}$. Then (X,T) is weakly T_1 but not T_1.

Proof: Let $T' = \{\emptyset\}\cup\{V|$ the compliment of V is finite$\}$. If V is in T, then V is in T' and $V \subset V$. Let $U \in T'$ such that $U \neq \emptyset$. Let V^* be the compliment of U union $\{a,b\}$ and let U^* be the compliment of V^*. Then $U^* \neq \emptyset$ and $U^* \subset U$. Therefore T is weakly equivalent to T'. So T' is T_1 [1, p. 138], but T is not T_1 since neither a nor b is in an open set which does not contain the other.

In fact, the above example is not only a weakly T_1 space that is not T_1, but is a weakly T_1 space that is not T_0. This is easily seen since the only element of T containing a or b is X and it contains both.

<u>EXAMPLE 5</u>. Let $X = [0,1]$ and let U equal the usual topology. Let T be the topology for X gotten by excluding all the sets in U containing $1/2$ except X. Then (X,T) is weakly T_2 but not T_2.

Proof: Obviously, any set in T is also in U. Let $V \in U$. If V does not contain $1/2$, then $V \in T$. If $1/2$ is in V, then there exists an open interval $((1/2)-r,(1/2)+r)) = V^*$ such that $r > 0$ and $V^* \subset V$. Let $W = ((1/2)+(1/4)r,(1/2)+(3/4)r)$. Then $W \varepsilon T$ and $W \subset V^* \subset V$. Therefore T is weakly equivalent to U. Of course (X,U) is T_2. Thus (X,T) is weakly T_2. But (X,T) is not T_2 since given the two points $1/8$ and $1/2$, $1/2$ has only one set in T containing it, X, and X intersects any set containing $1/8$.

The above example is not only a weakly T_2 space that is not T_2, but is a weakly T_2 space that is not T_1.

In fact, (X,T) is weakly T_3 since (X,U) is T_3 (since it is metrizable), but (X,T) is not T_3 since (X,T) is not T_2.

<u>EXAMPLE 6</u>. Let $X = [0,1]$ and let U be the usual topology on $[0,1]$. Let T be the topology for X gotten by excluding all the sets in U containing $1/4$ or $3/4$ except for X. Then (X,T) is weakly T_2 but not even T_0.

Proof: Obviously, any set in T is also in U. Let $V \varepsilon U$. If V does not contain $1/4$ or $3/4$, then $V \varepsilon T$. Now suppose $1/4$ or $3/4$ or both are in V. (Assume without a loss of generality that at least $1/4$ is in V.) Then there exists an open interval $((1/4)-r,(1/4)+r) = M$ where $r > 0$ and $M \subset V$. Set $r^* =$ minimum of r and $1/8$. Then letting $W = ((1/4)+(1/4)r^*,(1/4)+(3/4)r^*)$, $W \subset M \subset V$, and W contains neither $1/4$ nor $3/4$, so $W \varepsilon T$. Thus T is weakly equivalent to U and (X,T) is weakly T_2. But choosing points $1/4$ and $3/4$, neither has a set in T containing it except X, and X contains both points. Therefore (X,T) is not T_0.

CHAPTER II

WEAKLY SEPARATION AXIOMS

DEFINITION 3. A property P of a topological space is called a topological property if, when true for a space (X,S), then it is true for all spaces homeomorphic to (X,S).

DEFINITION 4. Let P be a topological property. A topological space (X,T) is weakly P provided there exists a topology S for X such that S is weakly equivalent to T and (X,S) is P.

DEFINITION 5. A function $f : (X,S) \to (Y,T)$ is somewhat continuous provided, if $U \in T$ and $f^{-1}(U) \neq \emptyset$, then there exists a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U)$.

DEFINITION 6. A function $f : (X,S) \to (Y,T)$ is somewhat open provided, if $V \in S$ and $f(V) \neq \emptyset$, then there exists a $U \in T$ such that $U \neq \emptyset$ and $U \subset f(V)$.

DEFINITION 7. A map $f : (X,S) \to (Y,T)$ such that f is somewhat open, somewhat continuous, one-to-one, and onto is a somewhat homeomorphism.

THEOREM 3. If $f : (X,S) \to (Y,T)$ is a somewhat homeomorphism and (X,S) is weakly P, then (Y,T) is weakly P.

Proof: Let $f : (X,S) \to (Y,T)$ be a somewhat homeomorphism and let (X,S) be weakly P. Let S' be a topology for X such that S' is weakly equivalent to S and (X,S') has property P (we know such a topology exists by the definition of weakly P).

Define $T' = \{f(U) \mid U \in S'\}$. Then since $\emptyset$ and X are in S', $Y = f(X)$ and $\emptyset = f(\emptyset)$ are in T'. Let A and B be in T'. Then there are a U and a V in S' such that $f(U) = A$ and $f(V) = B$. Thus $A \cap B = f(U) \cap f(V)$ and since f is one-to-one, then $f(U) \cap f(V) = f(U \cap V)$. But $U \cap V \in S'$, thus $A \cap B \in T'$.

Also, given $A_\alpha \in T'$ for all $\alpha \in A$ (where A is an indexing set), then there is a $V_\alpha \in S'$ for each α such that $f(V_\alpha) = A_\alpha$ by the definition of T'. Since $V_\alpha \in S'$, then $f(V_\alpha) \in T'$. So $\bigcup_\alpha A_\alpha = \bigcup_\alpha f(V_\alpha)$ and since f is onto, $\bigcup_\alpha f(V_\alpha) = f[\bigcup_\alpha (V_\alpha)]$. But $\bigcup_\alpha V_\alpha \in S'$, thus $\bigcup_\alpha A_\alpha \in T'$. Therefore T' is a topology for Y.

Given $U \in S'$, $f(U) \in T'$ by the definition of T'. Therefore $f : (X,S') \to (Y,T')$ is an open mapping. Now, given $V \in T'$, then $V = f(U)$ for some $U \in S'$. Thus $f^{-1}(V) = f^{-1}(f(U)) = U \in S'$. Therefore f is a continuous map. The function $f : (X,S') \to (Y,T')$ is obviously one-to-one and onto since $f : (X,S) \to (Y,T)$ is, thus $f : (X,S') \to (Y,T')$ is a homeomorphism, and (Y,T') has property P.

Let $V^* \in T'$ where $V^* \neq \emptyset$. Then since f is a homeomorphism, $f^{-1}(V^*) \in S'$ and $f^{-1}(V^*) \neq \emptyset$. Since S' is weakly equivalent to S, there is a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(V^*)$. Since $f : (X,S) \to (Y,T)$ is a somewhat homeomorphism, there exists a $U \in T$ such that $U \neq \emptyset$ and $U \subset f(V)$. So, $U \subset f(V) \subset f(f^{-1}(V^*)) = V^*$ where $U \in T$.

Let $U^* \in T$ where $U^* \neq \emptyset$. Then since $f : (X,S) \to (Y,T)$ is a somewhat homeomorphism (and therefore somewhat continuous), $f^{-1}(U^*) \neq \emptyset$ and there is a $V \in S$ such that $V \neq \emptyset$ and $V \subset f^{-1}(U^*)$. Since S

is weakly equivalent to S', then there is a $W \in S'$ such that $W \neq \emptyset$ and $W \subset V$. So by the definition of T', $f(W) \in T'$ and $f(W) \neq \emptyset$ since $W \in S'$. Thus $W \subset V \subset f^{-1}(U^*)$ so $f(W) \subset f(V) \subset U^*$ where $f(W) \in T'$. Therefore T' is weakly equivalent to T.

Thus since (Y,T) is weakly equivalent to (Y,T') and (Y,T') has property P, then (Y,T) is weakly P.

<u>THEOREM 4</u>. Every non-discrete T_1 space is weakly compact.

Proof: Let (X,T) be a non-discrete T_1 space. Then there is a $p \in X$ such that $\{p\}$ is not open. Define $T' = [T - \{\text{all elements of } T \text{ containing } p\}] \cup \{X\}$. Then T' is obviously a topology for X. Clearly (X,T') is compact, since any open covering by T' of X must contain the set X in order to cover p.

Let $U' \in T'$ such that $U' \neq \emptyset$. Then $U' \in T$ and $U' \subset U'$. Let $U \in T$ such that $U \neq \emptyset$. If $p \notin U$, then $U \in T'$ and $U \subset U$. If $p \in U$, then since $\{p\}$ is not open, there exists a $q \in U$ such that $q \neq p$. Since (X,T) is T_1, there exists a $V \in T$ such that $q \in V$ and $p \notin V$. Then $U \cap V \in T$, $U \cap V \neq \emptyset$, and $p \notin U \cap V$. Thus $U \cap V \in T'$ and $U \cap V \subset U$. Therefore T is weakly equivalent to T' and (X,T) is weakly compact.

<u>THEOREM 5</u>. Every finite weakly T_1 space is discrete.

Proof: Let (X,S) be a finite weakly T_1 space. Then there is a topology T for X such that (X,T) is a T_1 space and T is weakly equivalent to S. Since (X,T) is a finite, T_1 space,

T is the discrete topology. Let $p \in X$. Then $\{p\} \in T$. Since S is weakly equivalent to T, $\{p\} \in S$. Thus S is the discrete topology.

CHAPTER III

THE STRONGLY THEOREMS

DEFINITION 8. Let P be a topological property. A topological space (X,T) is strongly P provided every topology S weakly equivalent to T has the property that (X,S) is P.

THEOREM 6. Let (X,T) be weakly T_2 and strongly compact. Then (X,T) is weakly T_3.

Proof: There exists a topological space (X,S) such that (X,T) is weakly equivalent to (X,S) and (X,S) is T_2. Therefore since (X,T) is strongly compact, then (X,S) is compact. Since every T_2, compact space is T_3 [1, p. 223], then (X,T) is weakly T_3.

THEOREM 7. Let (X,T) be weakly T_3 and strongly compact. Then (X,T) is weakly T_4.

Proof: The proof follows directly as in Theorem 6 since every T_3, compact space is T_4 [1, p. 223].

THEOREM 8. Every strongly T_2 space is discrete.

Proof: Let (X,S) be a strongly T_2 space. Suppose S is not discrete. Then there is a point p in X such that $\{p\} \notin S$. Let $T = [S - \{U \mid U \in S \text{ and } p \in U\}] \cup \{X\}$. Choose point p and any other point q in X. The only T-open set containing p is X, and it contains q. Therefore (X,T) is not T_2.

If $U \in T$, then $U \in S$ and $U \subset U$. Let $U \in S$ such that $U \neq \emptyset$. Then there is an $x \in U$ such that $x \neq p$ since $\{p\} \notin S$. Since (X,S) is T_2, there is a $V \in S$ such that $x \in V$ and $p \notin V$. Then $U \cap V \in S$ and $p \notin U \cap V$. Thus $U \cap V \in T$, $U \cap V \neq \emptyset$, and $U \cap V \subset U$. Therefore S is weakly equivalent to T. But this is impossible since (X,S) is strongly T_2. Therefore S must be discrete.

CHAPTER IV

THE SOMEWHAT SEPARATION AXIOMS

DEFINITION 9. A topological space (X,S) is said to be somewhat T_0 provided if p and q are in X, then either, if U is any open set containing p there is a non-empty open set V such that $V \subset U$ and $q \notin V$, or, if U' is any open set containing q there is a non-empty open set V' such that $V' \subset U'$ and $p \notin V'$.

DEFINITION 10. A topological space (X,S) is said to be somewhat T_1 provided if p and q are in X, then if U is any open set containing p there is a non-empty open set V such that $V \subset U$ and $q \notin V$, and if U' is any open set containing q there is a non-empty open set V' such that $V' \subset U'$ and $p \notin V'$.

DEFINITION 11. A space (X,S) is somewhat T_2 if given p and q in X and U and V in S with $p \in U$ and $q \in V$, then there exist open sets $U' \subset U$ and $V' \subset V$ such that $U' \neq \emptyset$, $V' \neq \emptyset$, and $U' \cap V' = \emptyset$.

DEFINITION 12. A space (X,T) is somewhat regular if, given a point p in X and a set A such that $A \neq \emptyset$ and A is closed (that is, A is the compliment of a T-open set with respect to X), with $p \notin A$, then if U and V are in T with $p \in U$ and $A \subset V$, there exist sets U' and V' in T such that $U' \subset U$ and $V' \subset V$ and such that $U' \neq \emptyset$, $V' \neq \emptyset$, and $U' \cap V' = \emptyset$.

DEFINITION 13. A space (X,T) is somewhat T_3 if (X,T) is somewhat regular and is T_1.

THEOREM 9. Let (X,T) be a T_0 space. Then (X,T) is somewhat T_0.

Proof: Since (X,T) is T_0, given p and q, $p \neq q$, there exists a T-open U about one of these points, say p, such that $q \notin U$. So given any T-open set U' about p, let $\overline{U} = U \cap U'$. Then $\overline{U}$ is non-empty, $\overline{U} \subset U'$ and $q \notin \overline{U}$. Therefore (X,T) is somewhat T_0.

THEOREM 10. Let (X,T) be a T_1 space. Then (X,T) is somewhat T_1.

Proof: Since (X,T) is T_1, given points p and q, $p \neq q$, there exist T-open sets U and V such that $p \in U$, $q \notin U$, $q \in V$, and $p \notin V$. Then, given any T-open sets U' and V' such that $p \in U'$ and $q \in V'$, there exist T-open sets $\overline{U} = U \cap U'$ and $\overline{V} = V \cap V'$ such that $\overline{U}$ and $\overline{V}$ are non-empty, $\overline{U} \subset U'$, $\overline{V} \subset V'$, and $q \notin \overline{U}$ and $p \notin \overline{V}$. Therefore, (X,T) is somewhat T_1.

THEOREM 11. Let (X,T) be a T_2 space. Then (X,T) is somewhat T_2.

Proof: Since (X,T) is T_2, given p and q are two distinct points in X, there exist T-open sets U and V such that $p \in U$, $q \in V$, and $U \cap V = \emptyset$. So, given U' and V' are T-open sets in X such that $p \in U'$ and $q \in V'$ there exist T-open sets $\overline{U} = U \cap U'$ and $\overline{V} = V \cap V'$. Thus, $\overline{U}$ and $\overline{V}$ are non-empty, $\overline{U} \subset U'$, $\overline{V} \subset V'$, and $\overline{U} \cap \overline{V} = \emptyset$. Therefore (X,T) is somewhat T_2.

<u>THEOREM 12</u>. Let (X,T) be a T_3 space. Then (X,T) is somewhat T_3.

Proof: Since (X,T) is T_3, given a point $p \notin A$ and a closed set A (the compliment of a T-open set) such that $A \neq \emptyset$, there exist T-open sets U and V, such that $p \in U$, $A \subset V$, and $U \cap V = \emptyset$. Thus given T-open sets U' and V' such that $p \in U'$ and $A \subset V'$, there exist T-open sets $\overline{U} = U \cap U'$ and $\overline{V} = V \cap V'$. Thus $\overline{U}$ and $\overline{V}$ are non-empty, $\overline{U} \subset U'$, $\overline{V} \subset V'$, and $\overline{U} \cap \overline{V} = \emptyset$. Therefore (X,T) is somewhat T_3.

<u>EXAMPLE 7</u>. Let $X = \{a,b,c\}$ and let $T = \{\emptyset,\{a\},X\}$. Then (X,T) is a somewhat T_0 space that is not T_0.

Proof: The space (X,T) is not T_0 since given points b and c, neither point has an open set containing it except X, and X contains both points. Given the two points b and c, the only set containing either is X, so $\{a\} \subset X$ and $\{a\}$ contains neither b nor c. If the two points are other than b and c, then a must be one of the points, and any set that a is in must contain $\{a\}$. And as before, $\{a\}$ contains neither b nor c. Therefore (X,T) is somewhat T_0.

<u>EXAMPLE 8</u>. Let $X = [0,1]$ and let $\overline{U}$ be the usual topology on $[0,1]$. Let T be the topology for X gotten by

excluding all the sets in $\overline{U}$ containing $1/4$ or $3/4$ except X. Then (X,T) is somewhat T_2 but not T_1.

Proof: The space (X,T) is not T_1 since, given points $1/4$ and $3/4$, the only open set containing either is X and it contains both.

Let us prove that (X,T) is somewhat T_2 by dividing the proof into four cases. First, let p and q be any points in X except $1/4$ or $3/4$, $p \neq q$, and let U_1 be any set in T containing p and let V_1 be any set in T containing q. Since the usual topology is T_1, there exist sets U_2 and U_3 in $\overline{U}$ such that $p \in U_2$ and U_3 and $1/4 \notin U_2$ and $3/4 \notin U_3$. For the same reason, there exist sets V_2 and V_3 in $\overline{U}$ such that $q \in V_2$ and V_3 and $1/4 \notin V_2$ and $3/4 \notin V_3$. Also, since the usual topology is T_2, there exist sets U_4 and V_4 in the usual topology such that $p \in U_4$, $q \in V_4$, and $U_4 \cap V_4 = \emptyset$.

Then letting $V = V_1 \cap V_2 \cap V_3 \cap V_4$ and $U = U_1 \cap U_2 \cap U_3 \cap U_4$, $V \subset V_1$, $V \neq \emptyset$ since $q \in V$, $U \subset U_1$, $U \neq \emptyset$ since $p \in U$, and $U \cap V = \emptyset$. Both U and V are in (X,T) since both are in the usual topology yet neither contain $1/4$ or $3/4$.

In the second case, let $p = 1/4$, $q \neq 3/4$, and $q \neq p$, and let U_1 be any set in T containing p and V_1 be any set in T containing q. Since neither $\{p\}$ nor $\{p,3/4\}$ is open in the usual topology, U_1 contains a point p^* such that $p^* \neq p$ and $p^* \neq 3/4$. Since the usual topology is T_1, there exist sets V_2 and $V_3 \in \overline{U}$ such that $q \in V_2$ and V_3, and $p \notin V_2$ and $3/4 \notin V_3$. In the same way there exist sets U_2 and $U_3 \in \overline{U}$ such that $p^* \in U_2$ and U_3, and $3/4 \notin U_2$ and $p = 1/4 \notin U_3$. Also, since the usual topology is T_2, there exist sets U_4 and $V_4 \in \overline{U}$ such that $p^* \in U_4$, $q \in V_4$, and $U_4 \cap V_4 = \emptyset$. Then $U = U_1 \cap U_2 \cap U_3 \cap U_4$ and $V = V_1 \cap V_2 \cap V_3 \cap V_4$, where $V \subset V_1$, $V \neq \emptyset$ since $q \in V$, $U \subset U_1$, $U \neq \emptyset$ since $p^* \in U$, and $U \cap V = \emptyset$. So U and V are in (X,T) since both are in the usual topology yet neither contain $1/4$ nor $3/4$.

The third case where $p \neq 1/4$, $q = 3/4$, and $q \neq p$ with $p \in U_1 \in T$ and $q \in V_1 \in T$ is proved in exactly the same manner as the second case.

In the fourth case, let $p = 1/4$, and $q = 3/4$ with $U_1 \in T$ such that $p \in U_1$ and $V_1 \in T$ such that $q \in V_1$. Then, as in the second case, there exists a point $p^* \in U_1$ such that $p^* \neq p$ and $p^* \neq q$ and there exists a point $q^* \in V_1$ such that $q^* \neq p$ and $q^* \neq q$ and $p^* \neq q^*$.

Then there exist open sets in the usual topology, U_2 and U_3, such that $p^* \in U_2$ and U_3 and $p \notin U_2$ and $q \notin U_3$. In the same way, there exist sets V_2 and $V_3 \in \overline{U}$ such that $q^* \in V_2$ and V_3, and $p \notin V_2$ and $q \notin V_3$. Also, there exist sets V_4 and U_4 in the usual topology such that $p \in U_4$ and $q \in V_4$, and $U_4 \cap V_4 = \emptyset$.

Let $U = U_1 \cap U_2 \cap U_3 \cap U_4$ and $V = V_1 \cap V_2 \cap V_3 \cap V_4$. Then $U \subset U_1$, $U \neq \emptyset$ since $p^* \in U$, $V \subset V_1$, $V \neq \emptyset$ since $q^* \in V$, and $U \cap V = \emptyset$. Sets U and V are in (X,T) since both are in the usual topology yet neither contain $1/4$ or $3/4$. Thus in all four cases, (X,T) is somewhat T_1.

<u>THEOREM 13</u>. A somewhat T_1 space is somewhat T_0.

Proof: The theorem follows directly from the definitions.

<u>THEOREM 14</u>. Let (X,T) be a somewhat T_2 space. Then (X,T) is somewhat T_1.

Proof: Let p and q be distinct points in X. Let U be an open set containing p. If $q \in U$, then there are non-empty open sets U' and V' such that $U' \subset U$, $V' \subset U$ and $U' \cap V' = \emptyset$. Since $U' \cap V' = \emptyset$, either $q \notin U'$ or $q \notin V'$. Say $q \notin U'$. Then $U' \subset U$ and $q \notin U'$.

If $q \notin U$, then there are non-empty open sets U' and V' such that $U' \subset U$, $V' \subset X$, and $U' \cap V' = \emptyset$. Then $U' \subset U$ and $q \notin U'$.

In the same way, if V is an open set containing q, then there is a $V' \in T$ such that $V' \neq \emptyset$ and $V' \subset V$ with $p \notin V'$.

Thus (X,T) is a somewhat T_1 space.

<u>THEOREM 15</u>. Let (X,T) be a somewhat T_3 space. Then (X,T) is somewhat T_2.

Proof: Let (X,T) be somewhat T_3. Let p and q be points in X such that $p \neq q$. Since somewhat T_3 implies T_1, $\{q\}$ is a closed set in X. Therefore $p \notin \{q\}$ and given $p \in U$ and $\{q\} \subset V$ (that is, $q \in V$) where U and V are in T, there exist sets U' and V' in T such that $U' \subset U$ and $V' \subset V$ such that $U' \neq \emptyset$, $V' \neq \emptyset$, and $U' \cap V' = \emptyset$. Therefore (X,T) is somewhat T_2.

EXAMPLE 9. Let $X = \{a,b,c\}$ and let $T = \{\{a\},\emptyset,X\}$. Then (X,T) is somewhat T_0 but not somewhat T_1.

Proof: Choose point a and any other point of X. The only two sets in T containing a are X and $\{a\}$. If you choose $a \in \{a\}$, then $\{a\} \subset \{a\}$ and no other point is in $\{a\}$. If $a \in X$, then $\{a\} \subset X$, and no other point is in $\{a\}$.

Choose points b and c in X. The only set in T containing either is X and $\{a\} \subset X$. And $\{a\}$ contains neither b nor c. Therefore (X,T) is somewhat T_0.

Choose a and b in X. The only set in T containing b is X and the only non-empty sets in T contained in X are $\{a\}$ and X. But a is in both of these sets. Therefore (X,T) cannot be somewhat T_1.

EXAMPLE 10. Let R = the real numbers. Let $T = \{\text{interval } (a,\infty) \mid a \in R\} \cup \{\emptyset\} \cup \{R\}$. Then (R,T) is somewhat T_1 but not somewhat T_2.

Proof: Choose points p and q $(p \neq q)$ in R. Let U be any set in T containing p. Then let $V = (p+1,\infty) \cap (q+1,\infty)$. Then V is a non-empty set in T such that $V \subset U$ and $q \notin V$. Let U' be a set in T containing q. Then $V \subset U'$ and $p \notin V$. Therefore (R,T) is somewhat T_1.

But choose points p and q in R with $p \in U$ and $q \in U'$ where U and U' are in T. Then, since all T-open sets in U or U' are rays of the form (x,∞), with x a real number, all will intersect in some point. Therefore (R,T) is not somewhat T_2.

<u>EXAMPLE 11</u>. Let R be the real numbers. Let $X = R \cup \{i\}$ where i is the point $(0,1)$ in the plane. Let U be the usual topology for the reals and let $T = U \cup \{X\}$. Then (X,T) is a topological space that is somewhat T_2 but not T_1.

Proof: The space (X,T) is not T_1 since the only open set containing i is X.

Let p and q be in X with $p \neq q$. Let U' and V' be in T such that $p \in U'$ and $q \in V'$. If neither p nor q is i, then since (R,U) is T_2, there exist sets U^* and V^* of U such that $p \in U^*$, $q \in V^*$ and $U^* \cap V^* = \emptyset$. Then $U' \cap U^*$ and $V' \cap V^*$ are the required T-open sets.

Thus we need only to consider the case when either p or q is i. For the sake of convenience, we will assume that p is i. Then $U' = X$. Since $q \neq i$, $q \in R$. Since R is non-degenerate, there exists an element $r \in R$ such that $r \neq q$. Since (R,U) is T_2, there exist sets U^* and V^* of U such that $r \in U^*$, $q \in V^*$, and $U^* \cap V^* = \emptyset$. Sets U^* and $V' \cap V^*$ are the required T-open sets. Thus (X,T) is somewhat T_2.

<u>EXAMPLE 12</u>. Let (X,T) be as it is in Example 11. Then (X,T) is somewhat T_2 but is not somewhat T_3.

Proof: The space (X,T) is somewhat T_2 as shown in Example 11. But given points 0 and i, the only T-open set containing i is X and therefore contains 0. But $\{0\}$ is a T-closed set not containing i. Thus (X,T) is not somewhat T_3.

In Chapter III, we saw that every finite weakly T_1 space was discrete. But here we have the following result for finite somewhat T_1 spaces.

EXAMPLE 13. Let $X = \{a,b,c,d\}$ and let $T = \{\emptyset,\{b\},\{c\},\{b,c\},X\}$. Then (X,T) is a finite, somewhat T_1 space that is not discrete.

Proof: The only open set containing a or d is X. Thus, given points a and b, $\{c\} \subset X$ and $\{b\} \subset \{b\}$, $\{b,c\}$, or X; given points a and c, $\{b\} \subset X$ and $\{c\} \subset \{c\}$, $\{b,c\}$, or X; given points a and d, $\{b\} \subset X$; given points b and c, $\{b\} \subset \{b\}$, $\{b,c\}$, or X and $\{c\} \subset \{c\}$, $\{b,c\}$, or X; given points b and d, $\{c\} \subset X$ and $\{b\} \subset \{b\}$, $\{b,c\}$, or X; given points c and d, $\{b\} \subset X$ and $\{c\} \subset \{c\}$, $\{b,c\}$, or X. So, in any case, (X,T) is somewhat T_1. Obviously (X,T) is finite and is not discrete.

EXAMPLE 14. Let $X = \{a,b,c,d\}$ and $T = \{\emptyset,\{b\},\{c\},\{b,c\},X\}$. Then (X,T) is somewhat T_1 but not weakly T_1.

Proof: This is easily seen by using Example 13 and Theorem 5.

SUMMARY

The weakly separation axioms behave more like the usual separation axioms than did the somewhat axioms. The weakly topological properties were preserved under somewhat homeomorphisms. In Chapters III and IV, a comparison was made between finite weakly T_1 spaces and finite somewhat T_1 spaces. The finite weakly T_1 spaces were discrete. However, the finite somewhat T_1 spaces were not necessarily discrete. The relationship between the strongly separation axioms and the weakly and somewhat separation axioms remains as a possible area of investigation.

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