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This thesis presents some basic theory of analytic functions. Beginning with the definition of the derivative, necessary and sufficient conditions for a function to be analytic at a point are developed.

In order to prove that the derivative of an analytic function is itself analytic, the line integrals are defined. Then Cauchy's theorem and Cauchy's integral formula are stated. With the help of Cauchy's integral formula the theorem that an analytic function has derivatives of all orders is proved. Proofs for Morera's theorem and Liouville's theorem are also given.

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APPROVAL PAGE

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INTRODUCTION

This paper presents some basic theory of analytic function. The class of analytic functions is formed by the complex functions of a complex variable which possess a derivative wherever the function is defined. Beginning with the definition of the derivative, the necessary and sufficient conditions for a function to be analytic at a point are developed.

In order to prove that the derivative of an analytic function is itself analytic, the line integrals are defined. The exact differential and its relation to line integrals are discussed. Then Cauchy's theorem is stated. Since the proof of Cauchy's theorem is so complicated and can be found in most texts on complex variable, it is skipped.

Lastly Cauchy's integral formula is developed. The formula is an ideal tool for the study of local properties of analytic functions. Particularly it helps to show that an analytic function has derivatives of all orders, which are then analytic. With this result, the converse of Cauchy's theorem, known as Morera's theorem, can be proved. Last, the paper discusses Liouville's theorem concerning entire function.

CHAPTER I

ANALYTIC FUNCTIONS

1. PRELIMINARIES

Definition 1.1: A neighborhood of a point z_0 is the set of points z for which

$$|z - z_0| < \varepsilon$$

where ε is some positive constant.

Definition 1.2: A set of points is said to be open if every point z_0 of the set has a neighborhood lying wholly within the set.

Definition 1.3: An open set is called a connected open set or a region if besides being open, it has the property that any two points p, q of the set can be joined by a broken line lying wholly within the set.

Definition 1.4: Let z be any point of some δ-neighborhood, where that neighborhood is within the region of definition of a function f. Consider h as our complex variable, the derivative f' or df/dz of f at z is then defined by the equation

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}, \tag{1}$$

if the limit exists. That is, if the complex number f'(z), the derivative, exists, then to every positive number ε there corresponds a number δ such that

$$\left|\frac{f(z+h) - f(z)}{h} - f'(z)\right| < \varepsilon \tag{2}$$

whenever $0 < |h| < \delta$.

Note: It is an immediate consequence of equation (1) that f is necessarily continuous at any point z where its derivative exists. Indeed, from

$$f(z+h) - f(z) = h\cdot[f(z+h) - f(z)]/h,$$

we obtain

$$\lim_{h\to 0} [f(z+h) - f(z)] = 0\cdot f'(z) = 0.$$

If we write

$$f(z) = u(z) + iv(z),$$

it follows, moreover, that u(z) and v(z) are both continuous.

The usual rule for forming the derivative of a sum, a product, or a quotient for complex functions are all valid. The derivative of a composite function is determined by the chain rule.

Note: Observe that if a complex function is real-valued, either it has no derivative at all or it is a constant function. To illustrate our point, let f(z) be a real function of a complex variable whose derivative exists at $z = a$. Then $f'(z)$ is on one side real, for it is the limit of the quotients

$$\frac{f(a+h) - f(a)}{h}$$

as h tends to zero through real values. On the other side it is also the limit of the quotients

$$\frac{f(a+ih) - f(a)}{ih}$$

and as such purely imaginary. Therefore $f'(a)$ must be zero. Thus f is a constant function.

2. ANALYTIC FUNCTIONS

Definition 2.1: A function f of the complex variable is analytic at a point z_0 if its derivative exists not only at z_0 but at every point z in some neighborhood of z_0. It is analytic in a region of the z plane if it is analytic at every point in that region. The terms "holomophic" or "regular" are sometimes introduced to denote analyticity in regions of certain types.

Note: It is possible for a function to have a derivative at a point without being analytic at the point. For example, the function f defined by

$$f(z) = |z|^2$$

has derivative at 0 but no other point of z plane.

Definition 2.2: An entire function is one that is analytic at every point of the z plane; that is, throughout the entire plane.

The sum and the product of two analytic functions are again analytic. The same is true of the quotient $f(z)/g(z)$ of two analytic functions, provided that $g(z)$ does not vanish.

When we consider the derivative

$$f'(z) = \lim_{h \to 0} \frac{f(z+h) - f(z)}{h}$$

of a complex-valued function, it is understood that the limit of the difference quotient must be the same regardless of the way in which h approaches zero. If we choose real values for h, then the imaginary part y is kept constant, and the derivative becomes a partial derivative with respect to x. Writing

$$f(z) = u(z) + iv(z),$$

we have thus

$$f'(z) = \frac{\partial f}{\partial x} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}.$$

Similarly, if we substitute purely imaginary values ik for h, we obtain

$$f'(z) = \lim \frac{f(z+ik) - f(z)}{ik} = -i\frac{\partial f}{\partial y} = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}.$$

It follows that f(z) must satisfy the partial differential

$$\frac{\partial f}{\partial x} = -i\frac{\partial f}{\partial y} \tag{3}$$

which resolves into the real equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}. \tag{4}$$

These are the Cauchy-Riemann differential equation which must be satisfied by the real and imaginary part of any analytic function.

We remark that the existence of the four partial derivatives in (4) is implied by the existence of f'(z). Using (4) we can write down four formally different expressions for f'(z):

$$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial u}{\partial y} = \frac{\partial v}{\partial y} + i\frac{\partial v}{\partial x}. \quad (5)$$

The theorem which presents necessary conditions for the existence of $f'(z)$ is now established.

Theorem 2.1: If the derivative $f'(z)$ of a function

$$f = u + iv$$

exists at a point z, then the partial derivatives of the first order, with respect to x and y, of each of the components u and v must exist at that point and satisfy the Cauchy-Riemann conditions (4). Also, $f'(z)$ is given in terms of those partial derivatives by formula (5).

Conditions on u and v that ensure the existence of the derivative are given in the following theorem:

Theorem 2.2: Let u and v be real and single-valued functions of x and y which, together with their partial derivatives of the first order, are continuous at a point (x_0, y_0). If those partial derivatives satisfy the Cauchy-Riemann conditions at that point, then the derivative $f'(z_0)$ of the function $f = u + iv$ exists, where $z = x + iy$ and $z_0 = x_0 + iy_0$.

Proof: Since u and its partial derivatives of the first order are continuous at (x_0, y_0), those functions are defined throughout some neighborhood of that point. When $(x_0 + \Delta x, y_0 + \Delta y)$ is a point in the neighborhood, we can write

$$\begin{aligned}\Delta u &= u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0)\\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \epsilon_1 \Delta x + \epsilon_2 \Delta y,\end{aligned}$$

where $\partial u/\partial x$ and $\partial u/\partial y$ are the values of the partial derivatives at the point (x_0, y_0) and where ε_1 and ε_2 approach zero as both Δx and Δy approach zero. The above formula for Δu is established in advanced calculus in connection with the definition of the differential of the function u.

A similar formula may be written for Δv. Therefore,

$$\begin{aligned}\Delta f = f(z_0+\Delta z) - f(z_0) &= \Delta u + i\Delta v \\ &= \frac{\partial u}{\partial x}\Delta x + \frac{\partial u}{\partial y}\Delta y + \varepsilon_1\Delta x + \varepsilon_2\Delta y \\ &+ i(\frac{\partial v}{\partial x}\Delta x + \frac{\partial v}{\partial y}\Delta y + \varepsilon_3\Delta x + \varepsilon_4\Delta y).\end{aligned}$$

Assuming now that the Cauchy-Riemann conditions are satisfied at the point (x_0, y_0), we can replace $\partial u/\partial y$ by $-\partial v/\partial x$ and $\partial v/\partial y$ by $\partial u/\partial x$ and write the last equation in the form

$$\Delta f = \frac{\partial u}{\partial x}(\Delta x + i\Delta y) + i\frac{\partial v}{\partial x}(\Delta x + i\Delta y) + \delta_1\Delta x + \delta_2\Delta y,$$

where δ_1 and δ_2 approach zero as Δz approaches zero ($\Delta z = \Delta x + i\Delta y$). It follows that

$$\frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} + \delta_1\frac{\Delta x}{\Delta z} + \delta_2\frac{\Delta y}{\Delta z}. \tag{6}$$

Since $|\Delta x| \leq |\Delta z|$ and $|\Delta y| \leq |\Delta z|$, then

$$|\frac{\Delta x}{\Delta z}| \leq 1, \; |\frac{\Delta y}{\Delta z}| \leq 1,$$

so that the last two terms on the right of equation (6) tend to zero with Δz. Therefore, at the point z_0,

$$f'(z) = \lim_{\Delta z \to 0} \frac{\Delta f}{\Delta z} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x};$$

that is, the derivative $f'(z_0)$ exists, and the theorem is proved.

We shall see later that analyticity at a point z_0, puts a very severe restriction on a function. It implies the existence of all higher derivatives in a neighborhood of z_0. This is in marked contrast to the behavior of real-valued functions where it is possible to have existence and continuity of the first derivative without existence of the second derivative.

CHAPTER II

COMPLEX INTEGRATION

1. PRELIMINARIES

<u>Definition 1.1</u>: A <u>continuous arc</u> in the plane is defined as a set of points (x, y) such that

$$x = x(t), \quad y = y(t) \quad (t_1 \leq t \leq t_2) \qquad (1)$$

where x and y are continuous functions of the real parameter t. The definition establishes a continuous mapping of the points t from the interval $[t_1, t_2]$ to the arc and an ordering of the points (x, y) according to increasing values of t.

<u>Definition 1.2</u>: If no two distinct points of t correspond to the same point (x, y), the arc is called a <u>Jordan arc</u>.

<u>Definition 1.3</u>: If $x(t_1) = x(t_2)$ and $y(t_1) = y(t_2)$ and no other two values of t correspond to the same point (x, y), the continuous arc is called <u>simple closed curve</u> or a <u>Jordan curve</u>.

<u>Definition 1.4</u>: The <u>opposite arc</u> of equation (1) is the arc $x = x(-t)$, $y = y(-t)$, $-t_2 \leq t \leq -t_1$. Opposite arcs are denoted by c and -c.

<u>Definition 1.5</u>: If the functions in equation (1) have continuous derivatives x'(t) and y'(t) which do not vanish simultaneously for any value of t, the arc has a

continuously turning tangent. The arc or curve is then said to be smooth. Its length exists and is given by the formula

$$L = \int_{t_1}^{t_2} \sqrt{[x'(t)^2 + y'(t)^2]}\, dt$$

Definition 1.6: A contour is a continuous chain of a finite number of smooth arcs. If the contour is closed and does not intersect itself, it is a piecewise smooth Jordan curve, called a closed contour.

2. LINE INTEGRALS

Definition 2.1: Let $f(z)$ be continuous at all points of a curve c (Figure 1) which we shall assume has a finite length, i.e., c is a rectifiable curve.

Subdivide c into n parts by means of points $z_1, z_2, \ldots\ldots\ldots, z_{n-1}$, chosen arbitrarily, and call $a = z_0$, $b = z_n$. On each arc joining z_{k-1} to z_k (where $k = 1, \ldots\ldots, n$) choose a point ξ_k. Form the sum

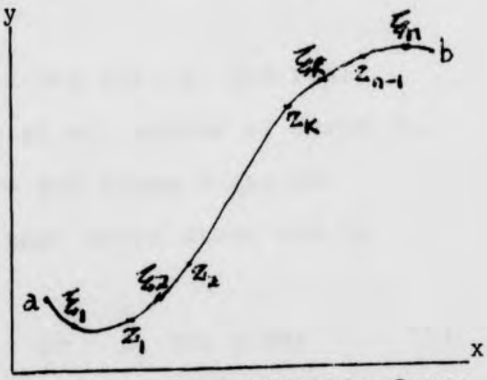

Figure 1. A Rectifiable Curve

$$S_n = f(\xi_1)(z_1 - a) + f(\xi_2)(z_2 - z_1) + \ldots + f(\xi_n)(b - z_{n-1}). \qquad (2)$$

On writing $z_k - z_{k-1} = \Delta z_k$, this becomes

$$S_n = \sum_{k=1}^{n} f(\xi_k)(z_k - z_{k-1}) = \sum_{k=1}^{n} f(\xi_k)\Delta z_k. \quad (3)$$

Let the number of subdivisions n increase in such a way that the largest of the chord lengths $|\Delta z_k|$ approaches zero. Then the sum S_n approaches a limit which does not depend on the mode of subdivision and we denote this limit by

$$\int_a^b f(z)dz \quad \text{or} \int_c f(z)dz \quad (4)$$

called the <u>complex line integral</u> or briefly <u>line integral</u> of f(z) along the curve c, or the <u>definite integral</u> of f(z) from a to b along the curve c. In this case f(z) is said to be integrable along c.

Note: If f(z) is analytic at all points of a region R and if c is a curve lying in R, then f(z) is certainly integrable along c.

<u>Definition 2.2</u>: If M(x, y) and N(x, y) are real functions of x and y continuous at all points of curve c, then <u>real line integral</u> of Mdx + Ndy along c can be defined in a manner similar to that given above and is denoted by

$$\int_c [M(x, y)dx + N(x, y)dy] \quad \text{or} \quad \int_c Mdx + Ndy \quad (5)$$

the second notation being used for brevity.

Note: If c is piecewise smooth and has parametric equations $x = x(t)$, $y = y(t)$ where $t_1 \le t \le t_2$, the value of (5) can be given by

$$\int_{t_1}^{t_2}\{M[x(t), y(t)]x'(t)dt + N[x(t), y(t)dt\}.$$

If $f(z) = u(x, y) + iv(x, y) = u + iv$ the complex line integral (4) can be expressed in terms of real line integral as

$$\int_c f(z)dz = \int_c (u + iv)(dx + idy)$$
$$= \int_c[udx - vdy] + i\int_c[vdx + udy]. \qquad (6)$$

For this reason (6) is sometimes taken as a definition of a complex line integral.

If $f(z)$ and $g(z)$ are integrable along c, then we have the following important properties of integrals:

(1) $\int_c[f(z) + g(z)]dz = \int_c f(z)dz + \int_c g(z)dz$

(2) $\int_c Af(z)dz = A\int_c f(z)dz$ where A = any constant

(3) $\int_a^b f(z)dz = -\int_b^a f(z)dz$

(4) $\int_a^b f(z)dz = \int_a^m f(z)dz + \int_m^b f(z)dz$ where points a, b, m are on c.

(5) $|\int_c f(z)dz| \le ML$

where $|f(z)| \le M$, i.e., M is an <u>upper</u> <u>bound</u> of $|f(z)|$ on c, and L is the length of c.

An important class of integrals is characterized by the property that the integral over an arc depends only on its end points. In other words if c_1 and c_2 have the same initial point and the same end point, we require that $\int_{c_1} Mdx + Ndy = \int_{c_2} Mdx + Ndy$. To say that an integral depends only on the end points is equivalent to saying that the integral over any closed curve is zero. Indeed, if c is a closed curve, then c and -c have the same end points, and if the integral depends only on the end points, we obtain

$$\int_c = \int_{-c} = -\int_c$$

and consequently $\int_c = 0$. Conversely, if c_1 and c_2 have the same end points, then $c_1 - c_2$ is a closed curve, and if the integral over any closed curve vanishes, it follows that

$$\int_{c_1} = \int_{c_2}.$$

The following theorem gives a necessary and sufficient condition under which a line integral depends only on the end points.

Theorem 2.2: The line integral $\int_c Mdx + Ndy$, defined in a region R, depends only on the end points of c if and only if there exists a function U(x, y) in R with the partial derivative $\partial U/\partial x = M, \partial U/\partial y = N$.

Proof: The sufficiency follows at once, for if the condition is fulfilled we can write, with the usual notations,

$$\begin{aligned}\int_c Mdx + Ndy &= \int_a^b [\frac{\partial U}{\partial x}x'(t) + \frac{\partial U}{\partial y}y'(t)]dt \\ &= \int_a^b \frac{dU}{dt}[x(t), y(t)]dt \\ &= U[x(b), y(b)] - U[x(a), y(a)],\end{aligned}$$

and the value of this difference depends only on the end points. To prove the necessity we choose a fixed point

$(x_0, y_0) \varepsilon R$, join it to (x, y) by a polygon c, contained in R, whose sides are parallel to the coordinate axes (Figure 2) and define a function by

$$U(x, y) = \int_c Mdx + Ndy.$$

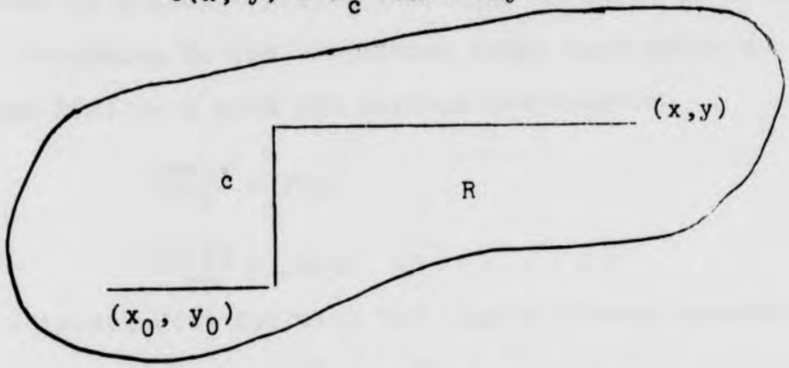

Figure 2. A Polygonal Curve

Since the integral depends only on the end points, the function is well-defined. Moreover, if we choose the last segment of c horizontal, we can keep y constant and let x vary without changing the other segments. On the last segment we can choose x for parameter and obtain

$$U(x, y) = \int^x M(x, y)dx + \text{const.},$$

the lower limit of the integral being irrelevant. From this expression it follows at once that $\partial U/\partial x = M$. In the same way, by choosing the last segment vertical, we can show that $\partial U/\partial y = N$.

It is customary to write $dU = (\partial U/\partial x)dx + (\partial U/\partial y)dy$ and to say that an expression $Mdx + Ndy$ which can be written in this form is an <u>exact differential</u>. Observe that M, N and U can be either real or complex. The function U, if it exists, is uniquely determined up to an additive constant, for if two functions have the same partial derivative their difference must be constant.

When is $f(z)dz = f(z)dx + if(z)dy$ an exact differential? According to the definition there must exist a function $F(z)$ in R with the partial derivatives

$$\frac{\partial F(z)}{dx} = f(z)$$

$$\frac{\partial F(z)}{dy} = if(z).$$

If this is so, $F(z)$ fulfills the Cauchy-Riemann equation

$$\frac{\partial F}{\partial x} = -i\frac{\partial F}{\partial y};$$

since $f(z)$ is by assumption continuous (otherwise $\int_c f(z)dz$ would not be defined) $F(z)$ is analytic with the derivative $f(z)$ (Chapter 1, Section 1). Therefore, the integral $\int_c fdz$, with continuous f, depends only on the end points of c if and only if f is the derivative of an analytic function in R.

Note: Under these circumstances we shall prove later that $f(z)$ is itself analytic.

As an immediate application of the above result we find that

$$\int_c (z - a)^n dz = 0 \tag{7}$$

for all closed curves c, provided that the integer n is ≥ 0. In fact, $(z - a)^n$ is the derivative of $(z - a)^{n+1}/(n + 1)$, a function which is analytic in the whole plane.

If n is negative, but $\neq -1$, the same result holds for all closed curves which do not pass through a, for in the complementary region of the point a the indefinite integral is still analytic and single-valued.

For $n = -1$, (7) does not hold. Consider a circle c with the center a, represented by the equation

$$z = a + re^{i\theta} \qquad 0 \leq \theta \leq 2\pi$$

we obtain

$$\int_c \frac{dz}{(z - a)} = \int_0^{2\pi} i\,d\theta = 2\pi i.$$

3. CAUCHY'S THEOREM--THE CAUCHY-GOURSAT THEOREM

Definition 3.1: A region is called simply-connected if it has the property that whenever a closed contour c lies within R, all points inside c are also in R. A region which is not simply-connected is called multiply-connected.

Theorem 3.1: If a function f is analytic at all points interior to and on a closed contour c then

$$\int_c f(z)dz = 0. \tag{1}$$

This fundamental theorem often called Cauchy's integral theorem or briefly Cauchy's theorem is valid for both simply- and multiply-connected regions. It was first proved by use of Green's theorem with the added restriction that $f'(z)$ be continuous in R. However, Goursat gave a proof which removed this restriction. For this reason the theorem is sometimes called the

Cauchy-Goursat theorem when one desires to emphasize the removal of this restriction. Goursat's detail proof for Cauchy's theorem can be found in most texts on complex variables and usually consists of three parts. The theorem is first proved for the case where c is a triangle and then where c is an arbitrary polygon and finally for the case where c is an arbitrary contour.

Now let f(z) be analytic in a simply-connected region R, Then the following theorems hold.

Theorem 3.2: If a and z are any two points in R, then

$$\int_a^z f(z)dz$$

is independent of the path in R joining a and z

Proof: By Cauchy's theorem,

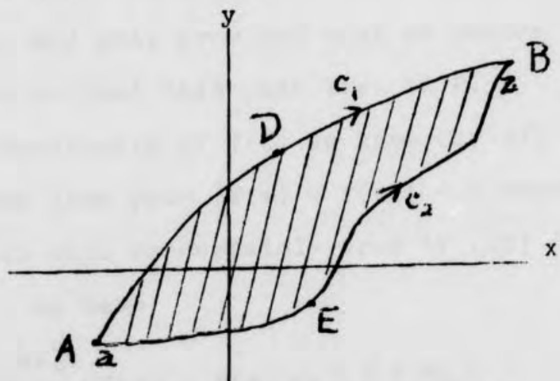

Figure 3. A Simply-Connected Region

$$\int_{ADBEA} f(z)dz = 0$$

or

$$\int_{ADB} f(z)dz + \int_{BEA} f(z)dz = 0.$$

Hence $\int_{ADB} f(z)dz = -\int_{BEA} = \int_{AEB} f(z)dz.$

Thus $\int_{c_1} f(z)dz = \int_{c_2} f(z)dz = \int_a^z f(z)dz$

which yields the required result.

Theorem 3.3: If a and z are any two points in R and

$$F(z) = \int_a^z f(w)dw \tag{2}$$

then F(z) is analytic in R and $F'(z) = f(z)$.

Proof: We have

$$\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z) = \frac{1}{\Delta z}\left[\int_a^{z+\Delta z} f(w)dw - \int_a^z f(w)dw\right] - f(z)$$

$$= \frac{1}{\Delta z}\int_z^{z+\Delta z} [f(w) - f(z)]dw. \tag{3}$$

By Cauchy's theorem, the last integral is independent of the path joining z and $z+\Delta z$ so long as the path in R. In particular, we can choose as path the straight line segment joining z and $z+\Delta z$ provided that we choose $|\Delta z|$ small enough so that this path lies in R.

Now by the continuity of f(z) we have for all points w on this straight line path $|f(w) - f(z)| < \varepsilon$ whenever $|w - z| < \delta$, which will be certainly true if $|\Delta z| < \delta$.

Furthermore, we have

$$\left|\int_z^{z+\Delta z} [f(w) - f(z)]dw\right| < \varepsilon|\Delta z| \tag{4}$$

so that from (3)

$$\left|\frac{F(z+\Delta z) - F(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\int_z^{z+\Delta z} [f(w) - f(z)]dw < \varepsilon$$

for $|\Delta z| < \delta$. This, however, amounts to saying that

$$\lim_{\Delta z \to 0} \frac{F(z+\Delta z - F(z)}{\Delta z} = f(z),$$

i.e., $F(z)$ is analytic and $F'(z) = f(z)$.

Theorem 3.4: If z_0 and z are any two points in R and $F'(z) = f(z)$, then

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0)$$

Proof: By the preceding theorem, the integral can differ at most from $F(z)$ by additive constant,

$$\int_{z_0}^{z_1} f(z)dz = F(z) + k.$$

Let $z = z_0$, this implies $k = -F(z_0)$. Then if $z = z_1$, the desired result is obtained

$$\int_{z_0}^{z_1} f(z)dz = F(z_1) - F(z_0).$$

Theorem 3.5: Let $f(z)$ be analytic in a region R bounded by two contours c_1 and c_2 and also on c_1 and c_2. Prove that

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz,$$

where c_1 and c_2 are traversed in the positive sense relative to their interiors [counterclockwise in Figure 4].

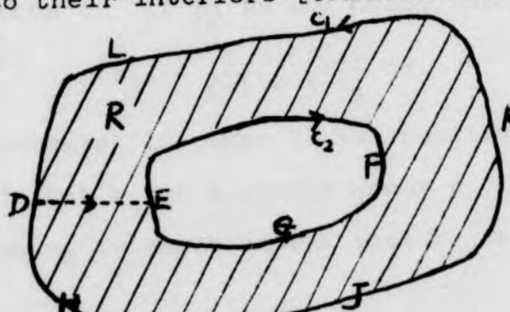

Figure 4. A Multiply-Connected Region

Proof: Construct cross-cut DE. Then since f(z) is analytic in the region R, we have Cauchy's theorem

$$\int_{DEFGEDHJKLD} f(z)dz = 0$$

or

$$\int_{DE} f(z)dz + \int_{EFGE} f(z)dz + \int_{ED} f(z)dz + \int_{DHJKLD} f(z)dz = 0.$$

Hence since

$$\int_{DE} f(z)dz = -\int_{ED} f(z)dz,$$

$$\int_{DHJKLD} f(z)dz = -\int_{EFGE} g(z)dz = \int_{EGFE} f(z)dz$$

or

$$\int_{c_1} f(z)dz = \int_{c_2} f(z)dz.$$

4. CAUCHY'S INTEGRAL FORMULA

Through a very simple application of Cauchy's theorem it becomes possible to represent an analytic function f(z) as a line integral in which the variable z enter as a parameter. This representation, known as Cauchy's integral formula, has numerous important applications. Above all, it enables us to study the local properties of an analytic function in great detail.

Theorem 4.1: Suppose f(z) is analytic within and on a closed contour c. If z_0 is any point interior to c, then

$$f(z_0) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-z_0}\,dz, \qquad (1)$$

where the integral is taken in the positive sense around c.

Proof: Let c_0 be a circle about z_0, $|z - z_0| = r_0$, whose radius r_0 is small enough that c_0 is interior to c.

The function $f(z)/(z-z_0)$ is analytic at all points within and on c except the point z_0. Hence the integral around the boundary of the ring-shaped region between c and c_0 is zero, according to the Theorem 3.5; that is

$$\int_c \frac{f(z)}{z-z_0}\,dz - \int_{c_0} \frac{f(z)}{z-z_0}\,dz = 0,$$

where both integrals are taken counterclockwise. Since the integrals around c and c_0 are equal, we can write

$$\int_c \frac{f(z)}{z-z_0}\,dz = f(z_0)\int_{c_0} \frac{dz}{z-z_0} + \int_{c_0} \frac{f(z)-f(z_0)}{z-z_0}\,dz. \tag{2}$$

But $z - z_0 = r_0 e^{i\Theta}$ on c_0 and $dz = ir_0 e^{i\Theta} d\Theta$, so that

$$\int_{c_0} \frac{dz}{z-z_0} = i\int_0^{2\pi} d\Theta = 2\pi i, \tag{3}$$

for every positive r_0. Also, f is continuous at point z_0. Hence, if we select any positive number ε, then a positive number δ exists such that

$$|f(z) - f(z_0)| < \varepsilon \quad \text{whenever } |z - z_0| < \delta.$$

We take r_0 equal that number δ. Then $|z - z_0| = \delta$, and

$$\left|\int_{c_0} \frac{f(z)-f(z_0)}{z-z_o}\,dz\right| \leq \int_{c_0} \frac{|f(z)-f(z_0)|}{|z-z_0|}\,|dz| < \frac{\varepsilon}{\delta}(2\pi\delta) = 2\pi\varepsilon.$$

The absolute value of the last integral in equation (2) can therefore be made arbitrarily small by taking r_0 sufficiently small. But since the other two integrals in the equation are independent of r_0, in view of equation (3), this one must be independent of r_0 also. Its value must therefore be zero. Equation (2) then reduces to the formula

$\int_c \frac{f(z)}{z-z_0}dz = 2\pi i f(z_0)$, and the theorem is proved.

The representation formula (1) gives us an ideal tool for the study of the local properties of analytic functions. In particular we can now show that analytic functions have derivatives of all orders, which are then also analytic. This is not necessarily true for functions of real variables.

Theorem 4.2: If f(z) is analytic within and on the closed contour c and that z_0 is within C then

$$f'(z_0) = \frac{1}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^2}\, dz \qquad (2)$$

Proof: We have, according to Cauchy's integral formula

$$\frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i \Delta z_0} \int_c \left(\frac{1}{z-z_0-\Delta z_0} - \frac{1}{z-z_0}\right) f(z)dz$$

$$= \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z-z_0-\Delta z_0)(z-z_0)} .$$

The last integral approaches the integral $\int_c \frac{f(z)}{(z-z_0)^2}\, dz$ as Δz_0 approaches zero; for the difference between that integral and this one reduces to

$$\Delta z_0 \int_c \frac{f(z)dz}{(z-z_0)^2(z-z_0-\Delta z_0)} .$$

Let M be the maximum value of $|f(z)|$ on c and let L be the length of c. Then, if d_0 is the shortest distance from z_0 to c and if $|\Delta z_0| < d_0$, we can write

$$\left| \Delta z_0 \int_c \frac{f(z)dz}{(z-z_0)^2(z-z_0-\Delta z_0)} \right| < \frac{ML|\Delta z_0|}{d_0^2(d_0-|\Delta z_0|)} ,$$

and the last fraction approaches zero when Δz_0 approaches zero. Consequently,

$$\lim_{\Delta z_0 \to 0} \frac{f(z_0+\Delta z_0)-f(z_0)}{\Delta z_0} = \frac{1}{2\pi i} \int_c \frac{f(z)dz}{(z-z_0)^2},$$

and formula (2) is established.

Theorem 4.3: Under the conditions of Theorem 4.2 we have

$$f^{(N)}(z_0) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-z_0)^{n+1}}\, dz \quad n = o,1,2,\ldots \quad (3)$$

Proof: The cases where $n = 0$ and 1 follow from Theorem 4.1 and Theorem 4.2 respectively provided we define $f^{(0)}(z_0) = f(z_0)$ and $0! = 1$.

To establish the case where $n = 2$, we use the same method that was used to establish formula (2). For it follows from formula (2) that

$$2\pi i \frac{f'(z_0+\Delta z_0)-f'(z_0)}{\Delta z_0} = \int_c \left[\frac{1}{(z-z_0-\Delta z_0)^2} - \frac{1}{(z-z_0)^2}\right] \frac{f(z)dz}{\Delta z_0}$$

$$= \int_c \frac{2(z-z_0)-\Delta z_0}{(z-z_0-\Delta z_0)^2(z-z_0)^2}\, f(z)dz.$$

Following the same procedure that was used before, we can show that the limit of the last integral, as Δz_0 approaches zero, is

$$2\int_c \frac{f(z)dz}{(z-z_0)^3},$$

and formula (3) for $n = 2$ follows at once.

In a similar manner we can establish the result for $n = 3, 4, \ldots$

We have now established the existence of the derivative of the function f' at each point z_0 interior to the region bounded by the curve c.

We recall our definition that a function f is analytic at a point z_1 if and only if there is a neighborhood about z_1 at each point of which f'(z) exists. Hence f is analytic in some neighborhood of the point. If the curve c used before is a circle $|z-z_1| = r_1$ in that neighborhood, then f"(z) exist at each point inside the circle, and therefore f' is analytic at z_1. We can apply the same argument to the function f' to conclude that its derivative f" is analytic at z_1, etc. Thus the following fundamental results are the consequences of formula (3).

<u>Theorem 4.4</u>: If a function f is analytic at a point, then its derivatives of all orders, f', f", ... are also analytic functions at that point.

<u>Theorem 4.5</u>: If f(z) has all derivatives f', f", f''', ..., $f^{(n)}$, . . . at z, then f has a first derivative f' at each point in some neighborhood of z.

<u>Theorem 4.6</u>: If f(z) has a first derivative at each point in some neighborhood of z, then it has all its derivatives (derivatives of all orders) at z.

The fact that the derivative of an analytic function is again an analytic function can be used to prove the following converse of Cauchy's theorem, known as Morera's theorem.

Theorem 4.7: If $f(z)$ is defined and continuous in a simply-connected region R, and if, for every closed contour c in R,

$$\int_c f(z)dz = 0$$

then $f(z)$ is analytic in R.

Proof: The hypothesis implies, as we have already remarked in Section 2.2, that $f(z)$ is the derivative of an analytic function $F(z)$. We know now that $f(z)$ is then itself analytic.

We end this section with Liouville's theorem concerning entire function.

Theorem 4.8: An entire function can not be bounded unless it reduces to a constant.

Proof: For the proof we make use of a simple estimate derived from (3). Let the radius r be ρ, and assume that $|f(z)| \leq M$ on r. If we apply (3) with $z_0 = a$, we obtain at once

$$|f^{(n)}(a)| \leq Mn!\rho^{-n}. \tag{4}$$

For Liouville's theorem we need only the case $n = 1$. The hypothesis means that $|f(z)| \leq M$ on all circles. Hence we can let ρ tend to ∞, and (4) leads to $f'(a) = 0$ for all

a. We conclude that the function is constant.

Liouville's theorem leads to an almost trivial proof of the fundamental theorem of algebra. Suppose that $p(z)$ is a polynomial of degree > 0. If $p(z)$ were never zero, the function $1/p(z)$ would be analytic in the whole plane. We know that $p(z) \to \infty$ as $z \to \infty$, and therefore $1/p(z)$ tends to zero. This implies boundedness, and by Liouville's theorem $1/p(z)$ would be constant. Since this is not so, the equation $p(z) = 0$ must have a root.

SUMMARY

In this thesis it was pointed out that analyticity at a point z_0 puts a very severe restriction on a function. It implies the existence of all higher derivatives in a neighborhood of z_0. This is in marked contrast to the behavior of real-valued functions, where it is possible to have existence and continuity of the first derivative without existence of the second derivative. With this important result, Morera's theorem was proved. Finally, Liouville's theorem was discussed.

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