SOME INVESTIGATIONS IN
GAME THEORY

by

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PART I

An Introduction to Game Theory

Game theory, or the theory of games of strategy is a relatively new field in mathematics. Very little was said about this theory prior to 1944 and the appearance of The Theory of Games and Economic Behavior by John von Neumann and Oskar Morgenstern. Since that time, both theoretical and practical aspects of the theory have been developed, and applications ranging from economic problems to warfare have been investigated.

The theory of linear programming, dealing with maximizing and minimizing problems, has been applied to the solution of games of strategy. Most often, however, the applications of the theory involve such extensive computation that automatic computers are required for solution.[1]

Pure games of chance have no place in game theory. The game theorist is concerned only with games of strategy or decision making on the part of rational players. This is not to say that chance may not play an important role in a game of strategy. Poker is a prime example of a game where chance determines how the cards are dealt, and decision making, how the game will go. Indeed, a player's own choices are often determined by some chance

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[1] S, Vajda, The Theory of Games and Linear Programming (New York, 1961), p.l.

device that he uses.

In the material written on the subject, the theory of games has been developed from two standpoints: 1) in a language as mathematically rigorous as possible, with a thorough theoretical approach, and with applications involving this rigorous system, and 2) in a more intuitive manner, also with applications to practical problems, but in a more informal presentation. In this paper, the purpose of which is to provide an introduction to the theory of games sufficiently complete to make understanding of the applications of linear programming to some original games possible, a combination of the two approaches will be used.

Intuitively speaking, a particular game is all of the rules of which it is made. It is a sequence of rational moves on the part of players, a move being the occasion of a choice between alternatives. Usually games are played for some end, and it is important in game theory to make all rewards in terms of money, that is, to arrange a payoff for winners and losers at the end of some specified sequence of moves.

There are several features common to any game. First, there must be at least two players since one player would have to be playing against nature, or chance, not a rational opponent. The first occurrence in a game is a move by one of the players and is performed by his deciding among alternatives. There are two kinds of moves a player may make: 1) a <u>personal move</u>, where he makes a choice by his own free decision, and 2) a <u>chance move</u>, whereby some chance device determines the choice for him. This move is followed by some prescribed situation which is set up in the rules and which somehow determines who is to make the next move and what alternatives he may choose from.

Each player may or may not know the choices his opponent has made; he may or may not possess full information. Herein lies another field called information theory. In games such as chess or tic-tac-toe, where each player is fully aware of all previous moves by his opponent, the players are said to have perfect information.

Another feature of a game is a terminating rule. Some situation must be defined as ending the game. Finally, there must be a situation describing the payoff or winnings at termination. Rapoport sums this up by saying, "A particular game is defined when the choices open to the players in each situation, the situation defining the end of a play, and the payoffs associated with each play-terminating situation have been specified."[2]

The game theorist is not concerned as much with how to play a particular game as he is with analyzing games in general. He is also very much interested in the classification of games into different categories. Games may be classified according to the number of players, how many moves the game has, whether the game

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[2] Anatol Rapoport, Two Person Game Theory (Ann Arbor, 1966), p. 21.

is finite or infinite, and the amount of information available to players. To say that a game is an n-person game does not necessarily mean that there are n players. Bridge, for example, is essentially a two-person game with partners acting as one player. Tic-tac-toe may be classified according to the number of moves since there are always exactly nine possible moves in this game. A finite game has a finite number of moves, each involving a finite number of alternatives from which to choose. All other games are infinite. Classification regarding the information available to players at any point in the game has been mentioned briefly above.

The most simple game and the one on which most research has been conducted is the zero-sum, two-person game. A game is said to be zero-sum if the sum of all payments received by all the players at the end of the game is zero. Expressed mathematically in the notation of Von Neumann, if a game has players $P_1, P_2, \ldots, P_n$ and if p_i $(i-1,2,\ldots,n)$ is the payment made to P_i at the end of the play and if $\sum_{i=1}^{n} p_i = 0$, then the play is called zero-sum. Furthermore, if every possible play is a zero-sum play, then the game itself is said to be zero-sum. Thus, a zero-sum, two-person game is a game with players P_1 and P_2 where p_1 is the payment made to P_1 and p_2 is the payment made to P_2 at the end of a play, and $p_1 + p_2 = 0$ for each play. This is saying nothing more than that one player's winnings is the other player's losses. Most often, zero-sum games are found in parlor games rather than in economic or warfare games where, in winning a particular play, one player may gain more than his opponent loses. A non-zero-sum game has an element similar to nature; by some outside doing, what one player wins is not the same as what his opponent loses, or vice versa.

The games dealt with in this paper will be zero-sum, two-person games. This means there will be no discussion of coalitions for, in a zero-sum, two-person game, the players are in such direct conflict as opponents that the "ethics" of the game prohibit co-operation or coalition on the part of the players. It should be mentioned that there is a complete theory of co-operative, as well as non-co-operative games.

Almost all games are found originally in what the game theorist calls <u>extensive</u> <u>form</u>. This is the sum total of the rules of an arbitrary game. It is the complete and extensive game itself with all possible courses of action and every possible sequence of moves. Any extensive game in which the two players make their choices from alternatives belonging to a finite set is identical to what is called the <u>normal</u> <u>form</u> of the game. To normalize a game, or to put it in normal form, is to find the rectangular game equivalent to the extensive form. One may think of the rectangular form of a game as a matrix describing the payoffs at each play.

To completely describe rectangular form of a game, a few other definitions are necessary. First, strategy needs to be defined. A strategy for a particular player is a complete plan of action throughout the particular game under consideration. It contains much more information than the player will ever need to use because it tells him how to act (that is, specifies his behavior) under all conceivable circumstances of play. A player adopts his strategy before the beginning of the game. In game theory, since it is assumed that each player is a rational being who is out to win, it is also assumed that each player will try to play his best strategy -- that he will not deliberately lose a game and that he is intelligent enough to be able to adopt such a strategy.

Another definition needed in describing a rectangular game is that of matrix, which is often defined as a rectangular array of numbers, but which is more meaningfully defined in game theory as a real valued function of two variables. Thus, the function $f(i,j)$, $i = 1,2,\ldots,m$; $j = 1,2,\ldots,n$, is defined by the equation $f(i,j) = a_{ij}$ and is represented as the m by n matrix

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mj} & \cdots & a_{mn} \end{bmatrix}$$

If one considers the function $f(i,j)$ as the payoff function for a particular game and if rows $1,2,\ldots m$ represent alternatives from which P_1 may choose and columns 1,2,...,n represent alternatives from which P_2 may choose (with a_{ij} being the payoff to P_1 if he chooses alternative i and P_2 chooses alternative j), then the m by n rectangular array of numbers is called the payoff matrix of the game. If a_{ij} is positive, then P_2 pays P_1 that positive amount since, as stated above, the numbers in the matrix represent payoffs <u>to</u> P_1 <u>from</u> P_2. This is the usual game-theoretic form. On the other hand, if an entry is negative, it represents a positive amount paid by P_1 to P_2. In other words, P_1 receives a negative payoff. A strategy for P_1 or P_2 includes which particular alternative he will choose under all situations. For example, if P_1's strategy were (1,0,0,...,0), then he would be choosing his first alternative each time there is a choice to make, and his resulting payoff would thus be found in the row of the matrix for alternative one: a_{11} a_{12} a_{13} ... a_{1j} ... a_{1n}. Which of these n payoffs he really receives then depends on whether P_2 chooses alternatives 1,2,...,j, ..., or n from his collection of alternatives. It should be apparent here why, in rectangular games, the number of alternatives open to a player must be finite if a finite optimal strategy is to be found.

Von Neumann himself defines strategy as a complete plan which specifies what choices a player will make in every possible situation, for every possible actual information which he might possess at that time, and in conformity with the pattern or information which the rules of the game provide for that particular situation.[3]

All of the possible strategies which a player may adopt are called his space of strategies; thus, in the n by m matrix, a player's space of strategies is the set of all n-tuples in which the sum of the n terms is one. A strategy such as the (1,0,0,...,0) mentioned above is called a <u>pure</u> <u>strategy</u> for the player is choosing the same alternative always. On the other hand, a strategy such as $(\frac{1}{4},0,\frac{1}{2},0,0,\ldots,0,\frac{1}{4})$ is called a <u>mixed</u> <u>strategy</u> and states that the player is choosing his first alternative one-fourth of the time, his third alternative one-half of the time, and his last alternative one fourth of the time. This is just another way of saying that each element of a player's particular n-tuple strategy is the frequency with which he plays that corresponding alternative. So, for P_1 to play pure strategy 1 is equivalent to his playing the mixed strategy $(x_1, x_2, \ldots, x_m)$ where $x_i = 1$ and $x_k = 0$ for each $k \neq i$. S_m designates the set of all m-tuple strategies open to P_1, and S_n, the set of all n-tuple strategies open to P_2. P_2's strategies are represented by $(y_1, y_2, \ldots, y_n)$. It is obvious that the sum of the elements of each strategy, or each n-tuple and each m-tuple, must be one. Thus, $\sum_{i=1}^{m} x_i = 1$ and $\sum_{j=1}^{n} y_j = 1$.

Any game characterized by the elements and features previously

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[3] John von Neumann and Oskar Morgenstern, <u>The Theory of Games and Economic Behavior</u> (Princeton, 1953), pp. 79-84.

described, and found in extensive form, may be normalized or changed into a rectangular game and given a payoff matrix. The most important question the game theorist seeks to answer, after the game is in rectangular form, is whether or not there is an optimal way of playing. When Von Neumann proved the Fundamental Theorem for Arbitrary Rectangular Games (which really marked the beginning of game theory), this question was answered with a definite yes. Before stating this theorem, it may be profitable to discuss briefly a player's expectation function.

In general, if P_1 chooses alternative i, he can be assured of obtaining at least the minimum payment in the ith row, that is, the $\min_j a_{ij}$. But since he can choose any row he wants, he can make the $\min_j a_{ij}$ as large as possible. It must be remembered that P_1 is seeking to make his winnings as great as he can. There is one choice he can make which will give him at least $\max_i \min_j a_{ij}$; this is the largest of the minimum payments considered by rows. In a similar way, P_2 hopes to minimize his losses. (It is well to think of P_1's seeking to maximize gain and P_2's seeking to minimize loss since the payoff matrix is written in terms of payoff to P_1 from P_2.) If P_2 chooses alternative j, he can be assured of losing not more than the maximum payment in the jth column, or $\max_i a_{ij}$, and by choosing the proper column, he can expect to minimize these maximums, thus finding $\min_j \max_i a_{ij}$. In summary, then, P_1 is assured of $\max_i \min_j a_{ij}$, and P_2 is assured of $\min_j \max_i a_{ij}$. In any m by n matrix A, since the matrix can be regarded as a real valued function $f(i,j) = a_{ij}$, $\max_i \min_j a_{ij} \leq \min_j \max_i a_{ij}$.

The expectation of a player is defined by McKinsey, who says that if P_1 uses mixed strategy $X = (x_1, x_2, \ldots, x_m)$ and if P_2 uses mixed strategy $Y = (y_1, y_2, \ldots, y_n)$, then the mathematical expectation of P_1 is given by $E(X,Y) = \sum_{j=1}^{n} \sum_{i=1}^{m} a_{ij} x_i y_j$. This can also be expressed as the product of the two strategies and the matrix, $E(X,Y) = X \cdot A_{mxn} \cdot Y$. If it happens that, for some X^* in P_1's space of strategies and for some Y^* in P_2's space of strategies, $E(X,Y^*) \leq E(X^*,Y^*) \leq E(X^*,Y)$ for each X and Y in the respective spaces of strategies, then X^* and Y^* are optimal strategies for P_1 and P_2, and $E(X^*,Y^*)$ is called the <u>value of the game</u>.[4]

If the value of a game is zero, the game is called fair. If v_1 and v_2 exist where $v_1 = \max_X \min_Y E(X,Y)$ and $v_2 = \min_Y \max_X E(X,Y)$ and if v_1 and v_2 are equal, then the condition above is satisfied, and the game has value $v = v_1 = v_2$, and optimal strategies exist. The fundamental theorem guarantees that this will always be true -- that $v_1 = v_2$ in each rectangular game.

Expressed simply, the fundamental theorem of game theory states that every rectangular game has a value and that a player of a rectangular game always has an optimal strategy. More formally expressed by J. C. C. McKinsey: Let the matrix A equal to

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[4] J. C. C. McKinsey, Introduction to the Theory of Games (New York, 1963), pp. 21-25.

$\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \cdot & & \cdot \\ \cdot & & \cdot \\ \cdot & & \cdot \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$ be any matrix, and let the expectation function $E(X,Y)$ for any $X = (x_1, x_2, \ldots, x_m)$ and any $Y = (y_1, y_2, \ldots, y_n)$ that are members of S_m and S_n respectively, be defined as follows: $E(X,Y) = \sum_{i=1}^{m} \sum_{j=1}^{n} a_{ij}\, x_i\, y_j$. Then the quantities $\max_X \min_Y E(X,Y)$ and $\min_Y \max_X E(X,Y)$ exist and are[5] equal. A proof of this theorem may be found in McKinsey's book.

Sometimes a game has what is called a <u>saddle</u> <u>point</u>. A saddle point is just a pair of integers (i, j) such that a_{ij} is simultaneously the row minimum and the column maximum or, expressed differently, both the maximum of the row minimums and the minimum of the column maximums. If a matrix has a saddle point at $a_{i_0 j_0}$, then the optimal strategies of P_1 and P_2 are pure strategies of playing alternatives i_0 and j_0. Also, $f(i_0, j_0)$ is called the value of the game. The chances that a matrix of random numbers will have a saddle point decreases rapidly as the deminsion of the matrix increases.

The solution of a rectangular game consists of X, Y, and the value of the game (X and Y here are optimal strategies for P_1 and P_2 respectively). Because of the convexity of strategy sets, a rectangular game has either just one solution or infinitely many solutions.

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[5] McKinsey, pp. 31-37.

There are many interesting and important properties of optimal strategies, some of which are investigated, discussed, and proved by McKinsey, who uses Von Neumann's language in many instances. The following theorem, somewhat restated, is dealt with by McKinsey[6]: Let E be the expectation function of an m by n rectangular game, and let X* and Y* be members of S_m and S_n respectively. Then the following conditions are equivalent:

i) X* is an optimal strategy for P_1 and Y* is an optimal strategy for P_2.

ii) If X is any member of S_m and Y is any member of S_n, then $E(X,Y*) \leq E(X*,Y*) \leq E(X*,Y)$.

iii) If i and j are any integers such that $1 \leq i \leq m$ and $1 \leq j \leq n$, then $E(x_i, Y*) \leq E(X*,Y*) \leq E(X*, y_j)$. ($x_i$ is the member of S_m whose ith component is 1, and y_j is the member of S_n whose jth component is 1.)

Before going into the application of linear programming theory to the solution of games of conflict or strategy, it will be well to discuss relations of <u>dominance</u>, the <u>skew-symmetric game</u>, and the <u>strictly determined</u> game.

A mixed strategy X is said to dominate a mixed strategy X' if, for each pure strategy y_j for P_2, $E(X, y_j) \geq E(X', y_j)$, and there exists at least one strategy y_j for P_2 such that $E(X, y_j) > E(X', y_j)$. In the payoff matrix itself, a pure strategy x_i dominates another

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[6] McKinsey, pp. 42,43.

pure strategy x_j if each element in the ith row is, term for term, greater than or equal to its corresponding term in the jth row, with at least one term being strictly greater than its corresponding term. A similar explanation goes for column dominance except that, since P_2's payoffs are expressed as losses, the dominance occurs where terms are less than their corresponding terms.

In solving a game which has dominated rows or columns, the dominated row, say row i, or column, say column j, may be deleted and a zero substituted in the strategy of P_1 in the ith position and in the strategy of P_2 in the jth position, for these dominated rows and columns would logically be played none of the time. Dominance of this type is called strict dominance. Non-strict dominance occurs when, although the elements in a particular row (or column) are not all smaller (greater) than the elements in another row (column), they are all smaller than certain convex linear combinations of the corresponding elements of the entries in that row (column).

A skew-symmetric matrix always has value zero and is thus always a fair game, since, by definition, a fair game is one with zero value. However, a game may be fair -- that is, it may have value zero -- without being skew-symmetric. The optimal strategies in a skew-symmetric game are the same for both players, and each player can avoid loss, no matter what his opponent does, if he plays his optimal strategy.

A strictly determined game is one in which both optimal strategies are pure strategies; thus, if a game has a saddle point, it is a strictly determined game.

PART II

The Linear Programming Method of Solution

A two by two matrix game can be solved with relative ease without the use of linear programming methods. For instance, in the game of matching pennies, where if both players match with two heads or two tails, P_1 wins, and if they fail to match with a head and a tail, P_2 wins, the matrix is $\begin{array}{c|cc} & H & T \\ \hline H & 1 & -1 \\ T & -1 & 1 \end{array}$. If x equals the amount of time P_1 plays heads, then 1 - x represents the amount of time he plays tails, and his expectation function E is then

$$\begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} x & 1-x \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

depending upon whether P_2 plays heads or tails. Performing the multiplication, $E = x - (1-x)$ or $E = -x + (1-x)$. Thus $E = 2x - 1$ or $E + -2x + 1$, and solving for x, $x = \frac{1}{2}$ and $(1-x) = \frac{1}{2}$. Therefore, the optimal strategy for P_1 is $(\frac{1}{2}, \frac{1}{2})$. In like manner, this is also the optimal strategy for P_2 because he has expectation function

$$\begin{bmatrix} y & 1-y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{or}$$

$$\begin{bmatrix} y & 1-y \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} ,$$

and the solution is the same. The matrix for P_2 is the transpose of the matrix for P_1. So each player will be doing best if he plays heads one half the time and tails one half the time. It is here that chance may come in because the best way to assure himself that his opponent will not figure out what he will do next is to let some sort of chance device determine for him whether to play heads or tails at each move. In fact, he could flip a coin to decide for himself.

After passing two by two matrix games, the number of unknowns soon becomes so great that this method is not practical. Linear programming theory here provides an excellent method of solution.[7] The method is perhaps best explained by an example. Suppose the game with matrix

$$\begin{bmatrix} 4 & 1 & 2 \\ 2 & 0 & 1 \\ 0 & 2 & 3 \end{bmatrix}$$

is to be solved.

Denoting any strategy for P_1 by (x, y, z) so that

$$\begin{cases} x + y + z = 1 \\ x \geq 0,\ y \geq 0,\ z \geq 0, \end{cases}$$

P_1's expectation against any of P_2's three pure strategies are $4x + 2y$, $x + 2z$, and $2x + y + 3z$. Letting g represent the smallest of these three payoffs, $4x + 2y \geq g$, $x + 2y \geq g$, and $2x + y + 3z \geq g$.

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[7] The following discussion of the simplex method of linear programming relies heavily upon Glicksman's <u>Linear Programming and the Theory of Games</u>.

In this example all the entries of the original matrix are positive. Many times this is not the case; if, however, any entry were negative, all entries could be made positive by adding a positive quantity equal to the absolute value of the smallest negative entry. This may be done first if there are negative values in the payoff matrix. Adding the same number to each entry in the matrix does not affect the optimal strategies of the game; it does, however, change the value of the game by the amount of the number added.

Dividing the statements $x + y + z = 1$, $x \geq 0$, $y \geq 0$, $z \geq 0$, $4x + 2y \geq g$, $x + 2y \geq g$, and $2x + y + 3z \geq g$, by g, one gets

$$\frac{x}{g} + \frac{y}{g} + \frac{z}{g} = \frac{1}{g}, \quad \frac{x}{g} \geq 0, \quad \frac{y}{g} \geq 0, \quad \frac{z}{g} \geq 0$$

$$\frac{4x}{g} + \frac{2y}{g} \geq 1, \quad \frac{x}{g} + \frac{2y}{g} \geq 1, \quad \frac{2x}{g} + \frac{y}{g} + \frac{3z}{g} \geq 1.$$

Player I can maximize g by minimizing $\frac{1}{g}$. If the notation $x' = \frac{x}{g}$, $y' = \frac{y}{g}$, $z' = \frac{z}{g}$, and $m = \frac{1}{g}$ is introduced, then P_1 wants to find $x' \geq 0$, $y' \geq 0$, and $z' \geq 0$ such that

$$\begin{cases} 4x' + 2y' \geq 1 \\ x' + 2z' \geq 1 \\ 2x' + y' + 3z' \geq 1 \end{cases}$$

and so that $x' + y' + z' = m$ is minimized. This is a full-fledged linear programming problem which can be solved by the simplex method.

Since P_2 wants to minimize his payoff, his problem is the dual problem of linear programming theory. Because of the minmax theorem, the solutions to both players' problems can be read from the solution matrix of either player's problem. And since the original matrix of the game does not have to be transposed to form P_2's problem, his, perhaps, is the better to solve.

If P_2's strategy is denoted by (p, q, r) so that $p \geq 0$, $q \geq 0$, and $r \geq 0$, the amount P_2 expects to pay for each of P_1's pure strategies is $4p + q + 2r$, $2p + r$, and $2q + 3r$. Furthermore, if, for each (p, q, r), h is the maximum of these three payoffs, then $4p + q + 2r \leq h$, $2p + r \leq h$, and $2q + 3r \leq h$. Dividing as before, this time by h,

$$\frac{p}{h} \geq 0, \quad \frac{q}{h} \geq 0, \quad \frac{r}{h} \geq 0, \text{ and } \frac{p}{h} + \frac{q}{h} + \frac{r}{h} = \frac{1}{h}.$$

$$\text{Also, } \frac{4p}{h} + \frac{q}{h} + \frac{2r}{h} \leq 1, \ \frac{2p}{h} + \frac{r}{h} \leq 1, \text{ and } \frac{2q}{h} + \frac{3r}{h} \leq 1.$$

Since P_2 desires to minimize h, he can do this by maximizing $\frac{1}{h}$. Letting $p' = \frac{p}{h}$, $q' = \frac{q}{h}$, $r' = \frac{r}{h}$, and $M = \frac{1}{h}$, P_2 wants to find $p' \geq 0$, $q' \geq 0$, and $r' \geq 0$ so that

$$\begin{cases} 4p' + q' + 2r' \leq 1 \\ 2p' + r' \leq 1 \\ 2q' + 3r' \leq 1 \end{cases}$$

and so that $p' + q' + r' = M$ is maximized.

Thus, from its form, it can be seen that P_2's problem is also a linear programming one and is the dual of P_1's problem.

Putting it into linear programming matrix form to be handled by the simplex method, three <u>slack</u> <u>variables</u> a, b, and c are first introduced to change the inequalities to equations. Then, $p' \geq 0$, $q' \geq 0$, $r' \geq 0$, $a \geq 0$, $b \geq 0$, $c \geq 0$,

$$\left\{\begin{aligned} &4p' + q' + 2r' + a = 1 \\ &2p' + r' + b = 1 \\ &2q' + 3r' + c = 1 \end{aligned}\right.$$

and $p' + q' + r' = M$ is to be maximized.

This yields the matrix

$$\begin{bmatrix} 4 & 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 3 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & M \end{bmatrix}.$$

The three by three submatrix in the upper left corner is the original game matrix.

The simplex method of solution starts with changing entries in the last row so that each element in that row is less than or equal to zero. One begins, then, by choosing one of the columns with last entry greater than zero, say the first in this example. Then, by forming the ratios $\frac{1}{4}$ and $\frac{1}{2}$, that is, by comparing by division the <u>positive</u> numbers in the last column with the corresponding <u>positive</u> numbers in the column chosen and selecting the smallest of these ratios, $\frac{1}{4}$, a <u>pivot</u> <u>point</u> is found at the entry 4.

$$\begin{bmatrix} \textcircled{4} & 1 & 2 & 1 & 0 & 0 & 1 \\ 2 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 3 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & M \end{bmatrix}$$

All other elements in this column must be made zero by dividing row one by 4 and then adding to row two -2 times row one, to row three, 0 times row one, and to row four -1 times row one.

$$\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & 2 & 3 & 0 & 0 & 1 & 1 \\ 0 & \frac{3}{4} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & M-\frac{1}{4} \end{bmatrix}$$

Three-fourths and one-half are both positive numbers in row four, so a pivot point is to be found in the column containing $\frac{3}{4}$, or in the one whose last entry is $\frac{1}{2}$. Taking the third column, $\frac{1}{3}$ $\frac{1}{2}$, so 3 is the second pivot point if column three is chosen next.

$$\begin{bmatrix} 1 & \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & 2 & \textcircled{3} & 0 & 0 & 1 & 1 \\ 0 & \frac{3}{4} & \frac{1}{2} & -\frac{1}{4} & 0 & 0 & M-\frac{1}{4} \end{bmatrix}$$

Using the same process as before to get all other elements in the third column to be zero and the pivot point entry to be one, the resulting matrix is

$$\begin{bmatrix} 1 & -\frac{1}{12} & 0 & \frac{1}{4} & 0 & -\frac{1}{6} & \frac{1}{12} \\ 0 & -\frac{1}{2} & 0 & -\frac{1}{2} & 1 & 0 & \frac{1}{2} \\ 0 & \frac{2}{3} & 1 & 0 & 0 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{5}{12} & 0 & -\frac{1}{4} & 0 & -\frac{1}{6} & M-\frac{5}{12} \end{bmatrix}$$

Since $\frac{5}{12}$ is the only positive entry remaining in row four, the column containing $\frac{5}{12}$, the second column, contains the third pivot point which can be $\frac{2}{3}$ only. Thus, applying the same method, the result is

$$\begin{bmatrix} 1 & 0 & \frac{1}{8} & \frac{1}{4} & 0 & -\frac{1}{8} & \frac{1}{8} \\ 0 & 0 & \frac{3}{4} & -\frac{1}{2} & 1 & \frac{1}{4} & \frac{3}{4} \\ 0 & 1 & \frac{3}{2} & 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & -\frac{5}{8} & -\frac{1}{4} & 0 & -\frac{3}{8} & M-\frac{5}{8} \end{bmatrix}$$

Since the first column originally contained the coefficients of p', if the entries in the first column of the resulting matrix are all zero except for one, the entry in the last column and the row of the one non-zero entry is the value of p'. So p' in this game is $\frac{1}{8}$. Likewise, q' = $\frac{1}{2}$. Since the r' column does not satisfy the conditions previously described, r' = 0.

For Player 1, x', y', and z' may be read off the solution matrix as the negative values of the last entries in the columns of the slack variables a, b, and c. Thus, x' = $\frac{1}{4}$, y' = 0, and z' = $\frac{3}{8}$. M equals $\frac{5}{8}$ in this game; since M = $\frac{1}{h}$, h, or the value of the game, is $\frac{8}{5}$; thus g is also $\frac{8}{5}$. To find optimal strategies for each player, the substitutions p = p'h, q = q'h, r = r'h, x = x'g, y = y'g, z = z'g yield the values p = $\frac{1}{5}$, q = $\frac{4}{5}$, r = 0, x = $\frac{2}{5}$, y = 0, and z = $\frac{3}{5}$. So Player 1's optimal strategy is $(\frac{2}{5}, 0, \frac{3}{5})$, and Player 2's optimal strategy is $(\frac{1}{5}, \frac{4}{5}, 0)$. Thus the game is solved by the simplex method of linear programming. This same method may be employed to solve any rectangular game.

PART III

Some Original Games

The following are original zero-sum, two-person games, which are worthless as far as providing interesting pastimes as games, but which may be of some value when considered from the point of view of game theory. The objective in devising these games was to develop a game whose matrix would create a pattern as the dimension of the matrix increased, to generalize the game to an m by n matrix, and then, after solving the first few games of small dimension, to predict the solution to the m by n matrix game. Some success was achieved in developing matrices which followed a pattern as their dimensions increased and in generalizing a few of the games to nth form, but no method of predicting a solution to the nth game in any one of the examples presented itself. The games and some solutions follow.[8]

GAME ONE

Rules

Player 1 chooses to hold a certain number of cents covered in his hand. (Player 2 knows the total amount of money which Player 1 possesses at the beginning of the game.) Player 2 then attempts to

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[8] Some of the games have more than one solution even though only one is given.

guess the amount held by Player 1. If he guesses correctly, he gets the amount held. If he guesses incorrectly, he receives the difference between the two amounts provided that both his guess and the amount played are as much as or more than one half the total amount possessed by Player 1 at the beginning. Otherwise, Player 1 gets the difference.

Since this game would be rather meaningless if played with only one cent, the first matrix is the two-cent game.

$$\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}$$

For any strategy $(x_1, x_2, \ldots, x_n)$ of Player 1 in this game, x_1 means playing one cent, x_2 means playing two cents, etc. Likewise, in any strategy $(y_1, y_2, \ldots, y_n)$ for Player 2, y_1 means guessing one cent, y_2 means guessing two cents, etc. The solution reached by methods described above is

$$v = -1$$
$$P_1\ (1, 0)$$
$$P_2\ (1, 0)$$

since there is a saddle point at the first element in the matrix.

The three-cent game has no saddle point.

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & -2 & -1 \\ 2 & -1 & -3 \end{bmatrix}$$

Solution: $v = \frac{2}{15}$, $P_1\ (\frac{9}{15}, \frac{1}{15}, \frac{5}{15})$, $P_2\ (\frac{9}{15}, \frac{1}{15}, \frac{5}{15})$.

This solution was reached by first setting the problem up as a linear programming problem with inequalities. Before this is done, however, it is necessary to add the absolute value of -3, the smallest entry in the matrix, to each entry in the matrix so all entries will be positive. This yields the matrix

$$\begin{bmatrix} 2 & 4 & 5 \\ 4 & 1 & 2 \\ 5 & 2 & 0 \end{bmatrix}$$

The set of inequalities then is

$$\begin{cases} 2p + 4q + 5r \leq h \\ 4p + q + 2r \leq h \\ 5p + 2q \leq h \\ p \geq 0,\ q \geq 0,\ r \geq 0 \end{cases}$$

$$\text{and } p + q + r = 1$$

Dividing each expression by h and letting $p' = \frac{p}{h}$, $q' = \frac{q}{h}$, and $r' = \frac{r}{h}$,

$$\begin{cases} 2p' + 4q' + 5r' \leq 1 \\ 4p' + q' + 2r' \leq 1 \\ 5p' + 2q' \leq 1 \\ p' \geq 0,\ q' \geq 0,\ r' \geq 0 \end{cases}$$

$$\text{and } p' + q' + r' = \frac{1}{h} \quad .$$

The next step is to introduce slack variables a, b, and c so that the inequalities will be equations. These variables must also be positive.

$$\begin{cases} 2p' + 4q' + 5r' + a = 1 \\ 4p' + q' + 2r' + b = 1 \\ 5p' + 2q' + c = 1 \\ p' \quad 0,\ q' \quad 0,\ r' \quad 0,\ a \quad 0,\ b \quad 0,\ c \quad 0 \end{cases}$$

$$p' + q' + r' = \frac{1}{h} = M$$

The matrix to be solved by the simplex method turns out to be

$$\begin{bmatrix} 2 & 4 & 5 & 1 & 0 & 0 & 1 \\ 4 & 1 & 2 & 0 & 1 & 0 & 1 \\ 5 & 2 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & M \end{bmatrix}$$

The three-cent game does not have strict dominance, either, but some rows and columns are dominated, as the solution reveals.

$$\begin{bmatrix} -1 & 1 & 2 & 3 \\ 1 & -2 & -1 & -2 \\ 2 & -1 & -3 & -1 \\ 3 & -2 & -1 & -4 \end{bmatrix}$$

Solution: $v = \frac{1}{5}$, $P_1\ (\frac{3}{5}, 0, \frac{2}{5}, 0)$, $P_2\ (\frac{2}{5}, \frac{3}{5}, 0, 0)$.

Generalizing the game now, the n by n matrix takes on one of two forms, depending upon whether n is an odd or even integer. If n is an even integer, the payoff matrix is

$$\begin{bmatrix}
-1 & 1 & 2 & 3 & \cdots & \frac{n}{2}-1 & \frac{n}{2} & \frac{n}{2}+1 & \cdots & n-1 \\
1 & -2 & 1 & 2 & \cdots & \frac{n}{2}-2 & \frac{n}{2}-1 & \frac{n}{2} & \cdots & n-2 \\
2 & 1 & -3 & 1 & \cdots & \frac{n}{2}-3 & \frac{n}{2}-2 & \frac{n}{2}-1 & \cdots & n-3 \\
3 & 2 & 1 & -4 & \cdots & \frac{n}{2}-4 & \frac{n}{2}-3 & \frac{n}{2}-2 & \cdots & n-4 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{n}{2}-1 & \frac{n}{2}-2 & \frac{n}{2}-3 & \frac{n}{2}-4 & \cdots & -\frac{n}{2} & -1 & -2 & \cdots & -\frac{n}{2} \\
\frac{n}{2} & \frac{n}{2}-1 & \frac{n}{2}-2 & \frac{n}{2}-3 & \cdots & -1 & -(\frac{n}{2}+1) & -1 & \cdots & -(\frac{n}{2}-1) \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
n-1 & n-2 & n-3 & n-4 & \cdots & -\frac{n}{2} & -(\frac{n}{2}-1) & -(\frac{n}{2}-2) & \cdots & -n
\end{bmatrix}$$

But if n is odd, then the matrix is as follows:

$$\begin{bmatrix}
-1 & 1 & 2 & 3 & \cdots & \frac{n}{2}-\frac{3}{2} & \frac{n}{2}-\frac{1}{2} & \frac{n}{2}+\frac{1}{2} & \cdots & n-1 \\
1 & -2 & 1 & 2 & \cdots & \frac{n}{2}-\frac{5}{2} & \frac{n}{2}-\frac{3}{2} & \frac{n}{2}-\frac{1}{2} & \cdots & n-2 \\
2 & 1 & -3 & 1 & \cdots & \frac{n}{2}-\frac{7}{2} & \frac{n}{2}-\frac{5}{2} & \frac{n}{2}-\frac{3}{2} & \cdots & n-3 \\
3 & 2 & 1 & -4 & \cdots & \frac{n}{2}-\frac{9}{2} & \frac{n}{2}-\frac{7}{2} & \frac{n}{2}-\frac{5}{2} & \cdots & n-4 \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
\frac{n}{2}-\frac{3}{2} & \frac{n}{2}-\frac{5}{2} & \frac{n}{2}-\frac{7}{2} & \frac{n}{2}-\frac{9}{2} & \cdots & -(\frac{n}{2}-\frac{1}{2}) & 1 & 2 & \cdots & \frac{n}{2}+\frac{1}{2} \\
\frac{n}{2}-\frac{1}{2} & \frac{n}{2}-\frac{3}{2} & \frac{n}{2}-\frac{5}{2} & \frac{n}{2}-\frac{7}{2} & \cdots & 1 & -(\frac{n}{2}+\frac{1}{2}) & -1 & \cdots & -(\frac{n}{2}-\frac{1}{2}) \\
\frac{n}{2}+\frac{1}{2} & \frac{n}{2}-\frac{1}{2} & \frac{n}{2}-\frac{3}{2} & \frac{n}{2}-\frac{5}{2} & \cdots & 2 & -1 & -(\frac{n}{2}+\frac{3}{2}) & \cdots & -(\frac{n}{2}-\frac{3}{2}) \\
\vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\
n-1 & n-2 & n-3 & n-4 & \cdots & (\frac{n}{2}+\frac{3}{2}) & -(\frac{n}{2}+\frac{1}{2}) & -(\frac{n}{2}-\frac{1}{2}) & \cdots & -n
\end{bmatrix}$$

GAME TWO

This game is almost identical to game one except for a slight change in the rules, which makes the matrices and solutions quite different from those above.

Rules

Player 1 chooses to hold in his hand a number of coins. (Player 2 knows the total number of coins Player 1 has at the beginning of the game.) Player 2 then tries to guess the number of coins held by Player 1. If both the number held and the number guessed are odd integers, or if both are even integers, Player 1 gets the difference between the two numbers. If one is odd and the other even, Player 2 gets the difference.

The two-coin matrix is

$$\begin{bmatrix} -1 & -1 \\ -1 & -2 \end{bmatrix}.$$

There is a saddle point at the first entry in the matrix. Thus, the solution is

$$v = -1$$

$$P_1\ (1,0)$$

$$P_2\ (1,\ 0)$$

The three-coin matrix takes the form

$$\begin{bmatrix} -1 & -1 & 2 \\ -1 & -2 & -1 \\ 2 & -1 & -3 \end{bmatrix}.$$

Again, there is a saddle point. This time it is in the first row and the second column at -1. The solution here is

$$v = -1$$

$$P_1\ (1, 0, 0)$$

$$P_2\ (0, 1, 0)$$

The four-coin game,

$$\begin{bmatrix} -1 & -1 & 2 & -3 \\ -1 & -2 & -1 & 2 \\ 2 & -1 & -3 & -1 \\ -3 & 2 & -1 & -4 \end{bmatrix}$$

has solution,

$$v = -\frac{92}{115}$$

$$P_1\ \left(\frac{23}{115}, \frac{46}{115}, \frac{23}{115}, \frac{23}{115}\right)$$

$$P_2\ \left(\frac{23}{115}, \frac{46}{115}, \frac{23}{115}, \frac{23}{115}\right)$$

Generalizing in this game, the n-coin matrix has two forms again, depending upon whether or not n is odd or even. If n is even, the first sign in each $\pm$ and $\mp$ combination in the following matrix applies; if n is odd, the second sign applies.

$$\begin{bmatrix} -1 & -1 & 2 & -3 & \cdots & \pm(n-2) & \mp(n-1) \\ -1 & -2 & -1 & 2 & \cdots & \mp(n-3) & \pm(n-2) \\ 2 & -1 & -3 & -1 & \cdots & \pm(n-4) & \mp(n-3) \\ -3 & 2 & -1 & -4 & \cdots & \mp(n-5) & \pm(n-4) \\ \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots \\ \pm(n-2) & \mp(n-3) & \pm(n-4) & \mp(n-5) & \cdots & -(n-1) & -[n-(n-1] \\ \mp(n-1) & \pm(n-2) & \mp(n-3) & \pm(n-4) & \cdots & -[n-(n-1)] & -n \end{bmatrix}$$

GAME THREE

Rules

Player 1 has a certain quantity of money at the beginning of this game. Player 2 always knows exactly how much his opponent has. Player 1 chooses to hold n cents in his hand. Player 2 then tries to guess the amount held by Player 1. He makes his guess. Hearing this guess and knowing whether it is right or wrong, Player 1 then has the option of either betting that Player 2 has made a wrong guess or not betting. (Logically, his only reason for betting in the case that he knows Player 2 has guessed incorrectly would be to bluff his opponent.) Player 2, having heard whether or not his opponent is betting that he is wrong, then decides whether or not he wants to bet that he is right. A summary of the payoffs is as follows:

a. Neither bets:

If Player 2 guesses correctly....Player 2 receives n¢

If Player 2 guesses $k > n$.........Player 2 pays (n-k)¢

If Player 2 guesses $k < n$.........Player 2 receives (k-n)¢

b. Both bet:

If Player 2 guesses correctly....Player 2 receives 3n¢

If Player 2 guesses $k \neq n$........Player 2 receives $(|n-k|+1)$¢

c. Only Player 1 bets:

If Player 2 guesses any k........Player 2 receives 1¢

d. Only Player 2 bets:

If Player 2 guesses correctly....Player 2 receives 2n¢

If Player 2 guesses $k \neq n$........Player 2 pays 1¢

Just by reading the rules and observing the payoff schedule, one gets the impression that the game is not advantageous to Player 1, and it is not.

In the matrices for this game, the first row strategy means playing one cent and not betting; the alternative represented by row two means playing one cent and betting, etc. Likewise, the alternative represented by column one means guessing one cent and not betting, while column two is guessing one cent and betting, etc.

The one-cent game would not be meaningful here, either, since there would be no chance for Player 2 to guess incorrectly. The two-cent game is as follows:

$$\begin{bmatrix} -1 & -2 & 1 & 1 \\ -1 & -3 & -1 & -2 \\ -1 & 1 & -2 & -4 \\ -1 & -2 & -1 & -6 \end{bmatrix}$$

Eliminating dominated rows and columns, the following matrix is the result:

$$\begin{bmatrix} -1 & -2 & 1 \\ -1 & -3 & -2 \\ -1 & 1 & -4 \end{bmatrix}$$

and adding the absolute value of -4 to each element of the matrix,

$$\begin{bmatrix} 3 & 2 & 5 \\ 3 & 1 & 2 \\ 3 & 5 & 0 \end{bmatrix}$$

The solution is $v = -1$, $P_1\ (\frac{3}{5}, 0, \frac{2}{5})$, $P_2\ (1, 0, 0)$. And checking,

$$\begin{bmatrix} \frac{3}{5} & 0 & \frac{2}{5} \end{bmatrix} \begin{bmatrix} -1 & -2 & 1 \\ -1 & -3 & -2 \\ -1 & 1 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = -1$$

The three-coin game is

$$\begin{bmatrix} -1 & -2 & 1 & 1 & 2 & 1 \\ -1 & -3 & -1 & -2 & -1 & -3 \\ -1 & 1 & -2 & -4 & 1 & 1 \\ -1 & -2 & -1 & -6 & -1 & -2 \\ -2 & 1 & -1 & 1 & -3 & -6 \\ -1 & -3 & -1 & -2 & -1 & -9 \end{bmatrix}$$

Notice that with each one-cent increase in n, the corresponding matrix increases in dimension by two rows and two columns. Thus, arriving at a solution of a game of this size soon becomes impossible without the use of a computer. However, in this matrix, since there is dominance, the simplex method is not too complicated. The matrix resulting after dominated rows and columns have been eleminated is

$$\begin{bmatrix} -1 & -2 & 1 & 1 & 1 \\ -1 & 1 & -2 & -4 & 1 \\ -2 & 1 & -1 & 1 & -6 \end{bmatrix}$$

A solution to this matrix by the simplex method of linear programming is $v = -\frac{5}{4}$, $P_1\ (\frac{3}{4}, 0, \frac{1}{4})$, $P_2\ (\frac{3}{4}, \frac{1}{4}, 0, 0, 0)$. For the original matrix game $v = -\frac{5}{4}$, $P_1\ (\frac{3}{4}, 0, 0, 0, \frac{1}{4}, 0)$, $P_2\ (\frac{3}{4}, \frac{1}{4}, 0, 0, 0, 0)$.

The matrix for this game played with n cents follows.

$$\begin{bmatrix}
-1 & -2 & 1 & 1 & 2 & 1 & \cdots & i-1 & 1 & \cdots & n-2 & 1 & n-1 & 1 \\
-1 & -3 & -1 & -2 & -1 & -3 & \cdots & -1 & -i & \cdots & -1 & -(n-1) & -1 & -n \\
-1 & 1 & -2 & -4 & 1 & 1 & \cdots & i-2 & 1 & \cdots & m-3 & 1 & m-2 & 1 \\
-1 & -2 & -1 & -6 & -1 & -2 & \cdots & -1 & -(i-1) & \cdots & -1 & -(n-2) & -1 & -(n-1) \\
-2 & 1 & -1 & 1 & -3 & -6 & \cdots & i-3 & 1 & \cdots & n-4 & 1 & n-3 & 1 \\
-1 & -3 & -1 & -2 & -1 & -9 & \cdots & -1 & -(i-2) & \cdots & -1 & -(n-3) & -1 & -(n-2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
-(i-1) & 1 & -(i-2) & 1 & -(i-3) & 1 & \cdots & -i & -2i & \cdots & n-1-i & 1 & n-i & 1 \\
-1 & -i & -1 & -(i-1) & -1 & -(i-2) & \cdots & -1 & -3i & \cdots & -1 & -(n-i) & -1 & -(n-i+1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
-(n-2) & 1 & -(n-3) & 1 & -(n-4) & 1 & \cdots & -(n-1-i) & 1 & \cdots & -n & -2(n-1) & 1 & 1 \\
-1 & -(n-1) & -1 & -(n-2) & -1 & -(n-3) & \cdots & 1 & -(n-i) & \cdots & -1 & -3(n-1) & -1 & -2 \\
-(n-1) & 1 & -(n-2) & 1 & -(n-3) & 1 & \cdots & -(n-i) & 1 & \cdots & -1 & 1 & -n & -2n \\
-1 & -n & -1 & -(n-1) & -1 & -(n-2) & \cdots & -1 & -(n-i+1) & \cdots & -1 & -2 & -1 & -3n
\end{bmatrix}$$

$$\begin{bmatrix}
-1 & -2 & 1 & 1 & 2 & 1 & \cdots & i-1 & 1 & \cdots & n-2 & 1 & n-1 & 1 \\
-1 & -3 & -1 & -2 & -1 & -3 & \cdots & -1 & -i & \cdots & -1 & -(n-1) & -1 & -n \\
-1 & 1 & -2 & -4 & 1 & 1 & \cdots & i-2 & 1 & \cdots & m-3 & 1 & m-2 & 1 \\
-1 & -2 & -1 & -6 & -1 & -2 & \cdots & -1 & -(i-1) & \cdots & -1 & -(n-2) & -1 & -(n-1) \\
-2 & 1 & -1 & 1 & -3 & -6 & \cdots & i-3 & 1 & \cdots & n-4 & 1 & n-3 & 1 \\
-1 & -3 & -1 & -2 & -1 & -9 & \cdots & -1 & -(i-2) & \cdots & -1 & -(n-3) & -1 & -(n-2) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
-(i-1) & 1 & -(i-2) & 1 & -(i-3) & 1 & \cdots & -i & -2i & \cdots & n-1-i & 1 & n-i & 1 \\
-1 & -i & -1 & -(i-1) & -1 & -(i-2) & \cdots & -1 & -3i & \cdots & -1 & -(n-i) & -1 & -(n-i+1) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & \vdots \\
-(n-2) & 1 & -(n-3) & 1 & -(n-4) & 1 & \cdots & -(n-1-i) & 1 & \cdots & -n & -2(n-1) & 1 & 1 \\
-1 & -(n-1) & -1 & -(n-2) & -1 & -(n-3) & \cdots & 1 & -(n-i) & \cdots & -1 & -3(n-1) & -1 & -2 \\
-(n-1) & 1 & -(n-2) & 1 & -(n-3) & 1 & \cdots & -(n-i) & 1 & \cdots & -1 & 1 & -n & -2n \\
-1 & -n & -1 & -(n-1) & -1 & -(n-2) & \cdots & -1 & -(n-i+1) & \cdots & -1 & -2 & -1 & -3n
\end{bmatrix}$$

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