THE LINE INTEGRAL AND ITS APPLICATIONS

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TABLE OF CONTENTS

1. Introduction. Mathematical theory has two important fields of application: (1) to physical situations and (2) in the development of further mathematical theory which may in turn have practical applications. It is logic or the deductive element in mathematical reasoning which enables us to produce a theory before we have observed the physical situation to which it is applicable. It is by induction that we are able to test our findings. It is because these two types of reasoning are so closely interwoven in the development of mathematical theory that it is necessary to consider both the major fields of application in order to establish the validity and to understand more fully the implications of our mathematical knowledge.

The line integral, with which this paper is concerned is a part of the subject matter of advanced calculus. As such, it is necessary to the development of the theory of functions, and hence is of prime importance to that whole field of mathematics known as mathematical analysis. In this paper we shall define the line integral, present the theory which establishes its existence and show how such an integral is useful in both the major fields of application.

2. Continuous and rectifiable curves. In defining the line integral, we will be dealing with functions which are continuous and curves which are both continuous and rectifiable. It is therefore necessary to recall what is meant by these concepts.

A function such as $x = x(t)$ is said to be continuous in an interval $[a, b]$ if it is continuous at every point in the interval. In order that $x(t)$ be continuous at $t = a$, for any preassigned positive number ϵ, there must be a positive number δ such that if $|t - a| < \delta$, then $|x(t) - x(a)| < \epsilon$. Geometrically this means that the graph of $x(t)$ is unbroken at the point. A curve given by $x = x(t)$, $y = y(t)$, $z = z(t)$ is continuous in an interval if each of these coordinate functions is single-valued and continuous on the parameter range.

A curve is <u>rectifiable</u> if it has finite length. The length of a line joining two points, $p_0:(x_0, y_0, z_0)$ and $p_1(x_1, y_1, z_1)$ is $\left[(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2\right]^{\frac{1}{2}}$. We may denote this distance by $d(p_0, p_1)$. Similarly the length of a polygonal curve formed by joining consecutively by straight line segments the points $p_0, p_1, \ldots, p_n$ is given by $d(p_0, p_1) + d(p_1, p_2) + \ldots + d(p_{n-1}, p_n)$ $= \sum_{i=1}^{n} d(p_{i-1}, p_i)$. The methods of the calculus afford a means of defining the notion of the length for general curves given parametrically by

$$x = x(t),\ y = y(t),\ z = z(t) \quad a \leq t \leq b.$$

Intuitively, one would expect the length of such a curve C to be so defined that its length is greater than or equal to the length of any polygonal curve inscribed in C, yet such that there do exist inscribed polygons whose lengths are arbitrarily close to the length of C.

Let us denote by $S(C:P)$ the sum $\sum_{i=1}^{n} d(p_{i-1}, p_i)$.

Geometrically $S(C:P)$ is the length of the polygon inscribed in C by joining consecutively the points $p_0, p_1, \ldots, p_n$. Now denote by $S(C)$ the least upper bound of $S(C:P)$. If $S(C)$ is finite, the curve C is said to be rectifiable, and $S(C)$ is defined to be the length of C.

3. Properties of rectifiable curves. Starting from this general definition of arc length, we can prove the following properties concerning rectifiable curves.[9][1]

(1) For a continuous curve C defined by $x = x(t)$, $y = y(t)$, $z = z(t)$, $a \leqq t \leqq b$, let C be an arbitrary value on $a < t < b$, and denote by C_1 and C_2 the curves defined by $x = x(t)$, $y = y(t)$, $z = z(t)$ for t on the sub-intervals $a \leqq t \leqq c$ and $c \leqq t \leqq b$, respectively. Then C is rectifiable if and only if both C_1 and C_2 are rectifiable. Moreover if C is rectifiable $S(C) = S(C_1) + S(C_2)$.

For a partition $P: a = t_0 < t_1 < \ldots < t_n = b$ we shall define the norm of P, written $N(P)$, as the greatest of the values $t_i - t_{i-1}$. The set of polygonal lengths $S(C:P)$ is said to tend to the finite limit S, or to converge to S, as the norm of P approaches zero, if to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that $|S - S(C:P)| < \epsilon$ for every partition P such that $N(P) < \delta$. We write $\lim_{N(P)\to 0} S(C:P) = S$. In particular, if $S(C:P)$ converges to S and $N(P)$ approaches zero, we have for every sequence of partitions (P_m) satisfying $\lim N(P_m) = 0$ that $\lim S(C:P_m) = S$.

With the aid of these definitions, we can state a second

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1. Numbers in square brackets will hereafter be used to refer to books listed in the bibliography to this paper.

property of a rectifiable curve.

(2) A continuous curve C defined by $x = x(t)$, $y = y(t)$, $z = z(t)$ is rectifiable if and only if the set of polygonal lengths $S(C:P)$ converges as $N(P)$ approaches zero. Moreover, for a rectifiable curve $\lim_{N(P)\to 0} S(C:P) = S(C)$.

The discussion thus far gives necessary and sufficient conditions for a curve to be rectifiable, but does not provide a formula for computing the length of a rectifiable curve. We shall now consider continuous curves $C : x = x(t)$, $y = y(t)$, $z = z(t)$ for which the coordinate functions have derivatives $x'(t)$, $y'(t)$ and $z'(t)$ that are piece-wise continuous on the interval $[a, b]$. That is, there is a finite set of division points $a = a_0 < a_1 < \dots < a_k = b$ such that on each of the sub-intervals these derivatives are continuous. We can now establish the integral representation for the length of such curves.[2]

<u>Theorem 3:1</u> If a continuous curve C is defined by $x = x(t)$, $y = y(t)$, $z = z(t)$ and the coordinate functions have derivatives which are piece-wise continuous on $[a, b]$, then C is rectifiable, and $S(C) = \int_a^b (x'^2 + y'^2 + z'^2)^{\frac{1}{2}}\, dt$.

4. <u>Definition of the line integral</u>. We are now in a position to define the line integral. We will consider the curve $C : x = x(t)$, $y = y(t)$, $z = z(t)$ which is continuous and rectifiable on the interval $[a, b]$ and suppose that $M(x, y, z)$ is a bounded,

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2. The proof for this theorem is given in full by most textbooks on advanced calculus. See for example [1], [9], and [13].

single-valued function of (x, y, z) in a region of space containing the curve C. In accordance with the usual method of the calculus, divide up the párameter range by a partition $P : a = t_0 < t_1 < \dots < t_n = b$ and let $p_i(x_i, y_i, z_i)$ denote the point on C determined by $t = t_i$. Let $\Delta_i x, \Delta_i y, \Delta_i z$ represent the increments of the respective coordinates in the i-th interval of the partition; $\Delta_i t = t_i - t_{i-1}$, $s_i = s(t_i)$ and $\Delta_i s = s_i - s_{i-1}$ where $s(t)$ is the arc length along C measured from the initial point. Let $\mu(P)$ denote the maximum of the values $\Delta_i s$.

Corresponding to the arbitrary point (x_i', y_i', z_i') in this i-th interval, we can form the sum

$$M(C:P) = \sum_{i=1}^{n} M(x_i', y_i', z_i') \Delta_i x.$$

The line integral of $M(x, y, z)$ with respect to x along C is said to exist and be equal to I_1 if $M(C:P)$ tends to the finite limit I_1 as $\mu(P)$ approaches zero; that is, if to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that $|I_1 - M(C:P)| < \epsilon$ for every partition P such that $\mu(P) < \delta$. This integral when it exists, is written

$$I_1 = \int_C M(x, y, z)\, dx.$$

Correspondingly the integral $\int_C N(x, y, z)\, dy$ is said to exist and be equal to I_2 if the associated sum $N(C:P) = \sum_{i=1}^{n} N(x_i', y_i', z_i') \Delta_i y$ converges to I_2 as $\mu(P)$ approaches zero.

Similarly $\int_C Q(x, y, z)\, dz$ exists and is equal to I_3

if the sum $Q(C:P) = \sum_{i=1}^{n} Q(x_i', y_i', z_i') \Delta_i z$ converges to I_3 as $\mu(P)$ approaches zero.

There are several sets of conditions under which the line integral can be shown to exist. These will be given in the form of theorems but the proofs will be omitted. Full proofs may be found in texts on advanced calculus or elementary function theory.[3]

Theorem 4:1 A necessary and sufficient condition for the line integral to exist is that to each $\epsilon > 0$ there corresponds a $\delta > 0$ such that $|M(C:P') - M(C:P'')| < \epsilon$ for all sums associated with arbitrary partitions P' and P" satisfying $\mu(P') < \delta$, $\mu(P'') < \delta$.

At first glance, this theorem does not seem to tell us much about the existence of the line integral for a specific function and a given curve. It is, however, the basis for the proof of the next theorem which is more readily applied.

Theorem 4:2 If $M(x, y, z)$ is continuous in a region containing a rectifiable arc C, then the integral along C exists. An extension of this theorem to apply to any curve made up of a finite number of rectifiable arcs is made possible by

Theorem 4:3 Suppose the curve C given by the usual parametric equations is rectifiable and for a given c on $a < c < b$, let C_1 and C_2 denote the curves determined for t on the intervals $[a, c]$ and $[c, b]$. Then the line integrals along C_1 and C_2 exist, and the integral along C is equal to the sum of the integrals along

3. See for example [5], [9].

C_1 and C_2 ; that is $\int_C Mdx = \int_{C_1} Mdx + \int_{C_2} Mdx$.

5. Evaluation of the line integral. So far we have no method for evaluating the line integral. The next theorem[4] gives us a relation between the line integral and the definite integral, by means of which the line integral may be evaluated.

Theorem 5:1 If $M(x, y, z)$ is continuous in a region containing continuous curve C whose coordinate functions have piecewise continuous derivatives on $[a, b]$ then

$$\int_C M(x, y, z)\, dx = \int_a^b M(x(t), y(t), z(t))\, x'(t)\, dt.$$

In other words if we substitute for x, y, and z their equals in terms of t as given by the equation of the curve C and for dx its equal $x'(t)\, dt$, we obtain an integral expressed in one variable t. The limits are from a to b since as the curve C is traced out t varies from a to b.

By similar reasoning the theorems of this section and the preceding one hold for the integrals I_2 and I_3 along C.

If C is a plane rectifiable curve $x = x(t)$, $y = y(t)$ on $[a, b]$, the definitions and properties of line integrals $\int_C M(x,y)\, dx$, $\int_C N(x, y)\, dy$ are obtained from the above by considering $z = 0$. The method for evaluating these integrals is the same as for the line integral in space.

By way of illustration, let us evaluate $\int_{(0,0)}^{(1,1)} \sqrt{y}\, dx +$

4. For proof of this theorem see [9], [10]

$(x - y)\,dy$ along several curves. First if we choose as the path of integration the straight line connecting the two points (0, 0) and (1, 1), given parametrically by $x = t$, $y = t$, the line integral becomes

$$\int_0^1 t^{1/2}\,dt = 2/3 \left[t^{3/2}\right]_0^1 = 2/3$$

If we evaluate this same integral along the parabola $x = t^2$, $y = t$, we obtain

$$\int_0^1 \left[t^{1/2}(2t) + (t^2 - t)\right] dt = \int_0^1 (2t^{3/2} + t^2 - t)\,dt =$$
$$\left[4/5\, t^{5/2} + t^3/3 - t^2/2\right]_0^1 = 4/5 + 1/3 - 1/2 = 19/30$$

Integration along a third path, the cubical parabola $x = t$, $y = t^3$, yields yet another result. The integral becomes

$$\int_0^1 \left[t^{3/2} + (t - t^3)(3t^2)\right] dt = \int_0^1 (t^{3/2} + 3t^3 - 3t^5)\,dt$$
$$= \left[2/5\, t^{5/2} + 3/4\, t^4 - 3/6\, t^6\right]_0^1 = 2/5 + 3/4 - 1/2 = 13/20$$

In general, as in this example, the value of the line integral depends upon the path of integration. There are, however, conditions under which its value is seen to depend only on the end points of the path. Under these conditions the line integral is said to be independent of the path. For instance, let us evaluate

$\int_{(0,0)}^{(1,1)} (x^2 + y^2)\,dx + 2xy\,dy$ along the same paths as those used in the preceding example. Integrating along the straight line $x = t$, $y = t$, we have

$$\int_0^1 (2t^2 + 2t^2)\,dt = 4\int_0^1 t^2\,dt = 4/3\left[t^3\right]_0^1 = 4/3.$$

Choosing $x = t^2$, $y = t$ as the path of integration, we get

$$\int_0^1 \left[(t^4 + t^2)(2t) + 2t^3\right] dt = \int_0^1 (2t^5 + 4t^3)\, dt$$
$$= \left[2t^6/6 + 4t^4/4\right]_0^1 = 1/3 + 1 = 4/3$$

If we integrate along the cubical parabola $x = t$, $y = t^3$, it is interesting to note that the result is still the same. Thus the line integral becomes

$$\int_0^1 \left[(t^2 + t^6) + 2t^4(3t^2)\right] dt = \int_0^1 (t^2 + 7t^6)\, dt$$
$$= \left[t^3/3 + 7t^7/7\right]_0^1 = 4/3$$

Integrals which are independent of the path will be more fully discussed and the conditions necessary for an integral to be of this type will be further investigated in section 7.

The discussion so far has dealt with curves given parametrically. There are many instances in which it is preferable to use parametric representation of a curve. One case in which this is true is in dealing with curves which cannot be expressed as the locus of an equation giving one of the coordinates as a single-valued function of the other coordinate. Such a case is illustrated by the circle $x^2 + y^2 = a^2$ which has the parametric representation

$$x = a \cos t, \quad y = a \sin t \quad 0 \leqq t \leqq 2\pi$$

where t is the radian measure of an associated central angle of the circle.

Parametric representation is also convenient when the sense, or direction in which the curve is traced, needs to be preserved. We will say that a curve is described in the positive direction if it is described so that a man walking along the curve in the direction of description has the enclosed area always on his left.

It is of prime importance when working with line integrals that the direction of integration be preserved. For it follows from the definition of the line integral that $\int_C M dx = -\int_{C'} M dx$ where C' is the same curve as C but traced in the opposite direction.

However, when a curve can be defined by an equation giving one of the coordinates as a single-valued function of the other and when there is no danger of confusing direction, then it may be simpler in specific examples to use the equation of the curve given in the form $y = f(x)$, $a \leqq x \leqq b$. With the equation in this form and with $f'(x)$ a continuous function, then $dy = f'(x)\, dx$, and substitution in the line integral gives the definite integral with respect to x; that is

$$\int_C M(x,y)\, dx + N(x,y)\, dy = \int_a^b \left[M(x, f(x)) + N(x, f(x))\, f'x\right] dx$$

For example if we wished to evaluate

$\int_{(0,\, 0)}^{(1,\, 3)} \left[x^2 y dx + (x^2 - y^2)\, dy\right]$ along the curve given by $y = 3x^2$,

we would have

$$\int_0^1 \left[x^2(3x^2) + (x^2 - 9x^4)(6x)\right] dx = \int_0^1 (3x^4 + 6x^3 - 54x^5) dx$$

$$= \left[3x^5/5 + 6x^4/4 - 54x^6/6\right]_0^1 = 3/5 + 3/2 - 9 = {}^-69/10.$$

6. Green's Theorem. We will now consider a theorem which gives a relation between the line integral in the plane and a double integral, thereby providing another method for evaluating the line

integral and giving a basis for some conclusions which may be drawn concerning independence of the path of integration of certain integrals. This is Green's Theorem. and

<u>Theorem 6:1</u>. If $M(x, y)$ and $N(x, y)$, $\frac{\partial M}{\partial y}$ and $\frac{\partial N}{\partial x}$ are continuous, single-valued functions over a closed region R, bounded by the curve C, then

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

the double integral being taken over the given region, and the curve C being described in the positive direction.

A continuous closed curve which does not cut itself is called simple. At first let us consider the region R bounded by the simple closed curve C which also has the property that no line parallel to either of the coordinate axes intersects C in more than two points. If the line is drawn parallel to OY then it intersects C in two points; where $y = y_1$ on the lower boundary and $y = y_2$ on the upper boundary. Let a and b be the extreme values of x for points in R. Now let $M(x, y)$ be any function which is continuous in the region R and

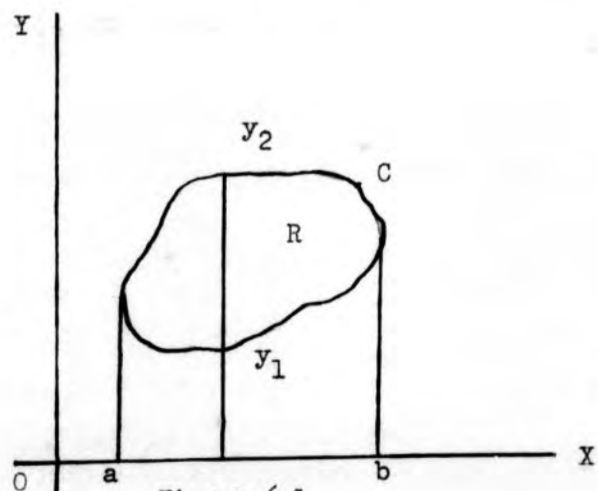

Figure 6:1

along its boundary C and for which $\frac{\partial M}{\partial y}$ is continuous. We shall consider the double integral of $\frac{\partial M}{\partial y}$ over the region R. The

$$\iint_R \frac{\partial M}{\partial y}\,dxdy = \int_a^b dx \int_{y_1}^{y_2} \frac{\partial M}{\partial y}\,dy$$

$$= \int_a^b \left[M(x, y_2) - M(x, y_1)\right] dx$$

$$= -\int_a^b M(x, y_1)dx - \int_b^a M(x, y_2)dx.$$

But by the definition of a line integral, the expression on the right is, except for sign, the line integral of Mdx around C in the positive direction. Hence we have

$$\iint_R \frac{\partial M}{\partial y}\,dxdy = -\int_C Mdx.$$

Similarly, if N is another function of x and y continuous in R and on C, and such that $\frac{\partial N}{\partial x}$ is continuous in R, we may show that

$$\iint_R \frac{\partial N}{\partial x}\,dxdy = \int_C Ndy$$

For if we let c and d be the extreme values of y in the region R and draw a line parallel to OX cutting C in points where $x = x_1$, $x = x_2$, we have

$$\iint_R \frac{\partial N}{\partial x}\,dxdy = \int_c^d dy \int_{x_1}^{x_2} \frac{\partial N}{\partial x}\,dx$$

$$= \int_c^d \left[N(x_2, y) - N(x_1, y)\right] dy$$

$$= \int_c^d N(x_2, y)dy + \int_d^c N(x_1, y)dy.$$

Again by definition of the line integral, the expression on the right is the line integral of Ndy around C in the positive direction. By adding equals to equals, we have finally

$$\int_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy.$$

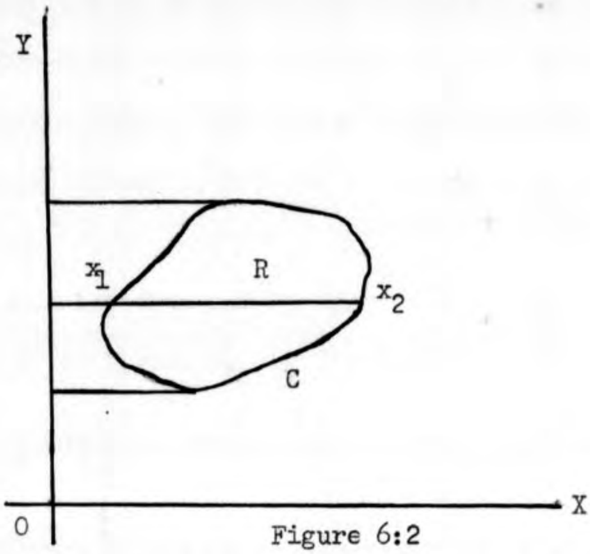

Figure 6:2

We have proved this result for a simple region R which also possessed the property that the boundary would be cut only twice by a line parallel to either of the coordinate axes. Our theorem may easily be extended to regions bounded by any continuous curve C such that it is possible to draw a finite number of lines which divide the region into subregions, each of the type considered in proving the theorem. Such a region is shown in Figure 6:3. If we draw the additional lines KL, GH, and PS, we have four subregions of the type used in the proof of the theorem. By adding the integrals obtained for each of the subregions, we obtain the integral along the whole curve plus the integrals along

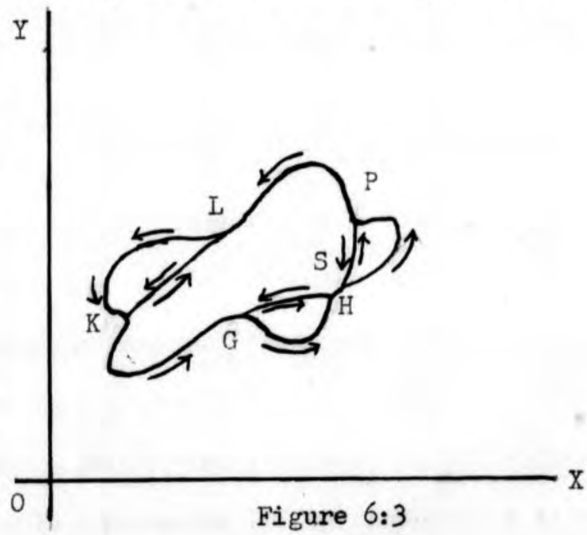

Figure 6:3

KL, GH, and PS. Examination of the arrows in the figure shows that we have integrated twice along each of these lines, but in opposite directions. The integrals along these lines therefore cancel, leaving only the integral around C, traversed continuously in the positive direction.

The theorem is also true for a region bounded by more than one curve. For example, in Figure 6:4 by drawing KL and GH the region is turned into one bounded by a single continuous curve. Again the two integrations along KL and GH cancel and we have left the integrals around the boundary curves each traversed in the positive direction.

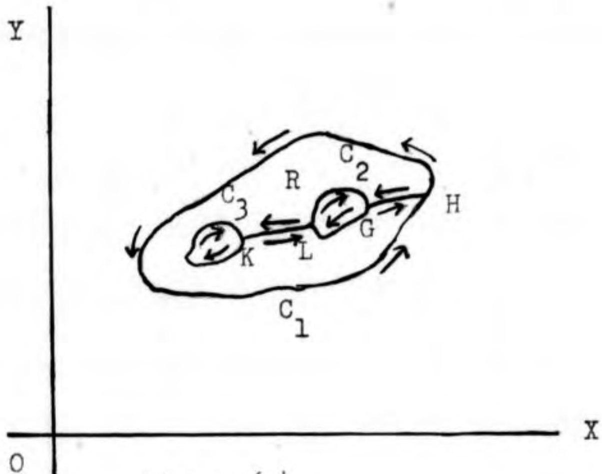

Figure 6:4

A region which has the property that any curve connecting two points in the region may be gradually deformed into any other curve connecting the same two points without passing out of the region is called a simply connected region. Thus, regions bounded by a circle, a rectangle, or an ellipse are simply connected. The region R in Figure 6:4 outside C_2 and C_3 and inside C_1 is not simply connected because an arc of a circle connecting points on opposite sides of C_2 cannot be deformed into a straight line without passing out of the region R. In other words, regions that have holes in them are not simply connected regions; such regions are called multiply connected.

7. Independence of the path of integration. Using Green's Theorem we are now able to derive conditions which are necessary and sufficient that the line integral connecting two points A and B of a region depend only on these points and not on the curve connecting them.

Theorem 7:1. Let M and N be two functions of x and y, such that M, N, $\frac{\partial M}{\partial y}$, and $\frac{\partial N}{\partial x}$ are continuous and single valued at every point of a simply connected region R. The necessary and sufficient condition that $\int_C Mdx + Ndy$ be independent of the curve C is that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ at all points of the region R. In this case the line integral is a function of the end points only.

In the first place, we can see from the figure that if the line integral along C_1 from A to B is equal to that along C_2 from A to B, then the integral along the closed curve formed by going from A to B along C_1 and from B to A along C_2 is zero. Hence, since the points A and B may be any two points and the curves C_1 and C_2 any two curves, the statement that the line integral between two points is independent of the path of integration is equivalent to the statement that the line integral

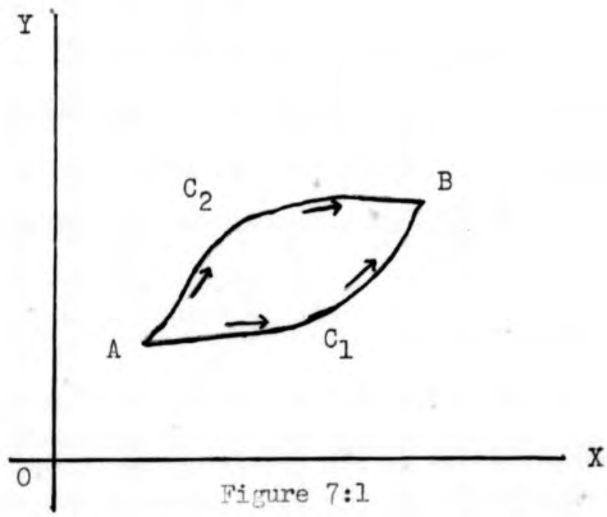

Figure 7:1

around any closed curve containing the two points is zero.

Green's Theorem gives us the relation

$$\int_C Mdx + Ndy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dxdy$$

From this formula, it is evident that if $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ at all points of R, then the integral on the left is equal to zero. This then is a sufficient condition that a line integral around any closed path is zero.

This condition is also necessary. For suppose $\frac{\partial N}{\partial x} \neq \frac{\partial M}{\partial y}$ at some interior point A of R and hence $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ must be either positive or negative throughout a suitably chosen neighborhood of A. Then the double integral could not vanish when taken over this region, and consequently the line integral which is equal to it could not vanish. Therefore, the condition is necessary for the line integral to vanish.

Thus we have proved that the condition $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ is a necessary and sufficient condition that $\int_C Mdx + Ndy$ vanish. For reasons given immediately preceding the proof of this theorem, this is equivalent to saying that the condition $\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$ is a necessary and sufficient condition that $\int_C Mdx + Ndy$ be independent of the path of integration.

When evaluating integrals in section 5 we saw that

$\int_{(0, 0)}^{(1, 1)} (x^2 + y^2)dx + 2xydy = 4/3$ when integrated along the straight line, the parabola, and the cubical parabola connecting the two points. Applying our test for independence which we have just developed, we see that

$$\frac{\partial N}{\partial x} = 2y \qquad \frac{\partial M}{\partial y} = 2y$$

so that

$$\frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

and we conclude that the value of the integral would be the same along any curve connecting (0, 0) and (1, 1).

A theorem in space which corresponds to Green's Theorem in the plane and which gives the basis for a test for independence of the path of integration in space is Stokes's Theorem.[5]

Theorem 7:2. Let $M(x, y, z)$, $N(x, y, z)$ and $Q(x, y, z)$ and their partial derivatives with respect to x, y and z be continuous and single valued functions in a region containing the surface S which is bounded by the closed curve C. Let dS be the element of area of S, and let $\cos\alpha$, $\cos\beta$ and $\cos\gamma$ be the direction cosines of the exterior normal to dS. Then

$$\int_C (Mdx + Ndy + Qdz) = \iint_S \left[\left(\frac{\partial Q}{\partial y} - \frac{\partial N}{\partial z}\right)\cos\alpha + \left(\frac{\partial M}{\partial z} - \frac{\partial Q}{\partial x}\right)\cos\beta + \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right)\cos\gamma\right] dS.$$

By means of this theorem it is possible to derive a test for independence of the path of integration in space similar to that derived for the plane by means of Green's Theorem. The derivation

5. This theorem will be given without proof since the proof involves surface integrals which will not be discussed in this paper. For proofs see tests on advanced calculus, especially [10] and [13].

is quite similar to that of the test for independence in the plane. This test[6] is given by

Theorem 7:3. Let the region of space considered be one in which $M(x, y, z)$, $N(x, y, z)$ and $Q(x, y, z)$ and their partial derivatives are continuous and single-valued functions of x, y, and z. Then the necessary and sufficient condition that

$$\int_C Mdx + Ndy + Qdz$$

be independent of the path of integration is that

$$\frac{\partial Q}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial Q}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}.$$

8. Area inside a closed curve. Having defined the line integral and discussed conditions for its existence, having found methods for evaluating it, and having observed some of its properties, we are now ready to see how such an integral can be useful.

The line integral may be used to find the area inside a closed curve. First consider the simple closed curve C such that a line parallel to either coordinate axis cuts it in only two points. Let C be bounded by the lines $x = a_1$, $x = a_2$, $y = b_1$ and $y = b_2$ which are tangent to C at A_1, A_2, B_1 and

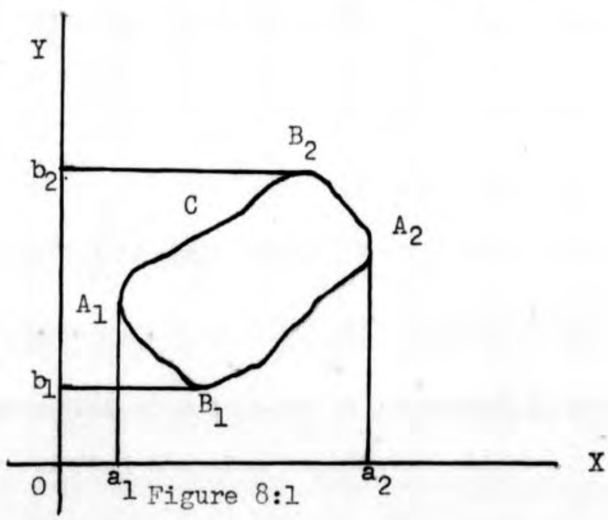

Figure 8:1

6. For derivation see [13].

B_2, respectively. Clearly C cannot be single valued. Therefore, let the equation of $A_1\ B_1\ A_2$ be given by $y_1 = f_1(x)$ and the equation of $A_1\ B_2\ A_2$ by $y_2 = f_2(x)$ where $f_1(x)$ and $f_2(x)$ are single valued functions. By use of definite integrals we can establish a formula for area which can be shown to be equivalent to a line integral expression. For by definite integrals

$$A = \int_{a_1}^{a_2} y_2 dx - \int_{a_1}^{a_2} y_1 dx$$

$$= -\int_{a_2}^{a_1} y_2 dx - \int_{a_1}^{a_2} y_1 dx = -\int_C ydx.$$

Similarly, if $x_1 = F_1(y)$ is the equation of $B_1\ A_1\ B_2$ and $x_2 = F_2(y)$ is the equation of $B_1\ A_2\ B_2$, then

$$A = \int_{b_1}^{b_2} x_2 dy - \int_{b_1}^{b_2} x_1 dy$$

$$= \int_{b_1}^{b_2} x_2 dy + \int_{b_2}^{b_1} x_1 dy = \int_C xdy.$$

In each of these formulas the integral around C is taken in the positive direction, thus keeping the area positive. By adding these integrals we get a formula which is a line integral expression in the form $\int_C Mdx + Ndy$, namely

$$A = 1/2 \int_C (xdy - ydx).$$

In many cases this formula has the advantage over the definite integral for computing area in that it is easier to evaluate. For instance, if we wish to find the area inside the ellipse given by the equations

$$x = a \cos \theta, \qquad y = b \sin \theta$$

by definite integrals, we have

$$A = 4\int_0^a ydx = -4ab\int_{\pi/2}^0 \sin^2 \theta d\theta = -4ab\int_{\pi/2}^0 \frac{1 - \cos 2\theta}{2} d\theta$$

$$= -2ab\left[\theta - \frac{\sin 2\theta}{2}\right]_{\pi/2}^0 = -2ab\left[-\pi/2\right]$$

$$= \pi ab.$$

The line integral formula in this case is more easily evaluated, for we have

$$A = 1/2\int_C xdy - ydx = 1/2\int_0^{2\pi} ab \cos^2 \theta d\theta + ab \sin^2 \theta d\theta$$

$$= ab/2\int_0^{2\pi} d\theta = ab/2\left[\theta\right]_0^{2\pi} = ab/2\left[2\pi\right]$$

$$= \pi ab$$

To illustrate further the use of this formula the areas of the triangle and the circle, which we already know from geometry will be found.

First we will find the area of the right triangle formed by the line $x/a + y/b = 1$ and the coordinate axes. From the line integral formula, we have

$$A = 1/2\int_C xdy - ydx.$$

Our curve C in this case is made up of the x axis from O to A, the line $x/a + y/b = 1$ from A to B and the y axis from B back to O. Along line AB $y = -b/a\,x + b$ and $dy = -b/a\,dx$. Along the y axis $x = 0$. $dx = 0$. The area of the triangle OAB can therefore be expressed by the integral

$$A = 1/2\int_a^0 \left[-b/a\,x - (-b/a\,x + b)\right] dx$$

$$= -\,b/2\int_a^0 dx = -b/2\Big[x\Big]_a^0 = -\,b/2\,\Big[-a\Big]$$

$$= ab/2$$

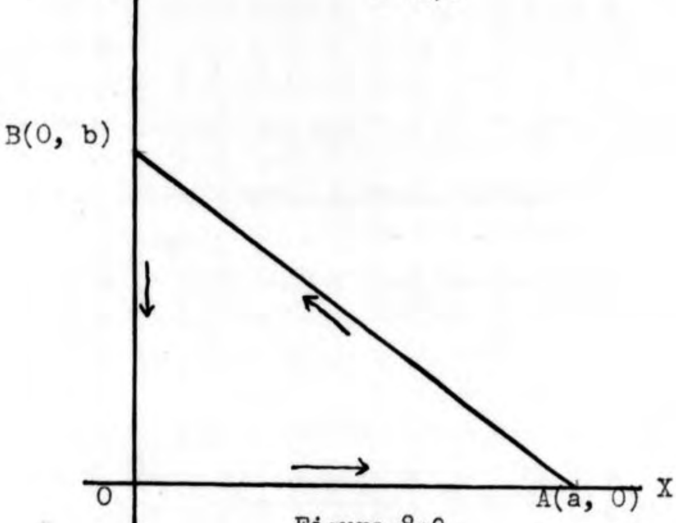

Figure 8:2

The process of finding the area of the circle is similar to that of finding the area of the ellipse which has already been shown. The circle may be given by the equations

$$x = a\cos\theta, \qquad y = a\sin\theta$$

From the line integral formula

$$A = 1/2\int_C xdy - ydx = 1/2\int_0^{2\pi}(a^2\cos^2\theta + a^2\sin^2\theta)d\theta$$

$$= a^2/2\Big[\theta\Big]_0^{2\pi} = a^2/2\Big[2\pi\Big] = \pi a^2.$$

Thus by applying the line integral formula to these curves we see that our results agree with those we have from geometry; that is, the area of the triangle is half the base times the altitude and the area of the circle is π times the radius squared. There are many curves, however, which lend themselves more readily to the methods of the calculus than to the methods of geometry. One such curve is the hypocycloid.

If we wish to find the area of the hypocycloid of four cusps given by the equations

$$x = a \cos^3 \theta, \qquad y = a \sin^3 \theta,$$

we have

$$dx = -\ 3a^2 \sin \theta d\theta, \ dy = 3a \sin^2 \theta \cos \theta d\theta$$

Applying the line integral formula, we have

$$A = 1/2 \int_0^{2\pi} (3a^2 \cos^4 \theta \sin^2 \theta + 3a^2 \sin^4 \theta \cos^2 \theta) d\theta$$

$$= 3a^2/2 \int_0^{2\pi} \cos^2 \theta \sin^2 \theta d\theta = 3a^2/2 \int_0^{2\pi} \left(\frac{1 + \cos 2\theta}{2}\right)\left(\frac{1 - \cos 2\theta}{2}\right) d\theta$$

$$= 3a^2/8 \int_0^{2\pi} \left[1 - \left(\frac{1 - \cos 4\theta}{2}\right)\right] d\theta = 3a^2/8 \left[\frac{\theta}{2} - \frac{\sin 4\theta}{8}\right]_0^{2\pi}$$

$$= 3a^2/8 [\pi] = 3/8 \pi a^2$$

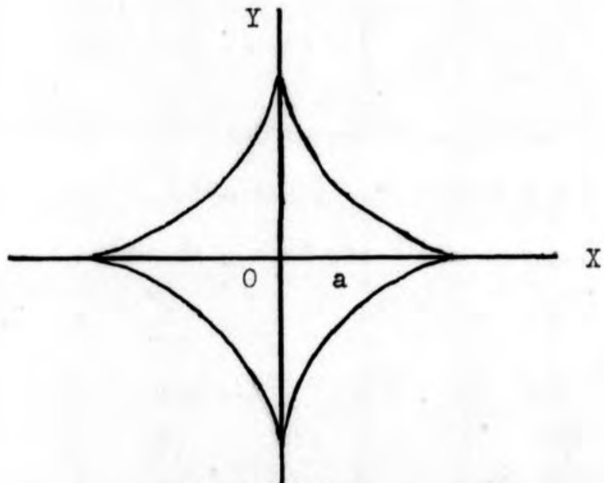

Figure 8:3

We can also find the area enclosed by the loop of the strophoid given by the equations

$$x = (1 - t^2)/(1 + t^2), \qquad y = t(1 - t^2)/(1 + t^2).$$

We have

$$dx = -4tdt/(1 + t^2)^2, \qquad dy = (1 - 4t^2 - t^4)/(1 + t^2)^2$$

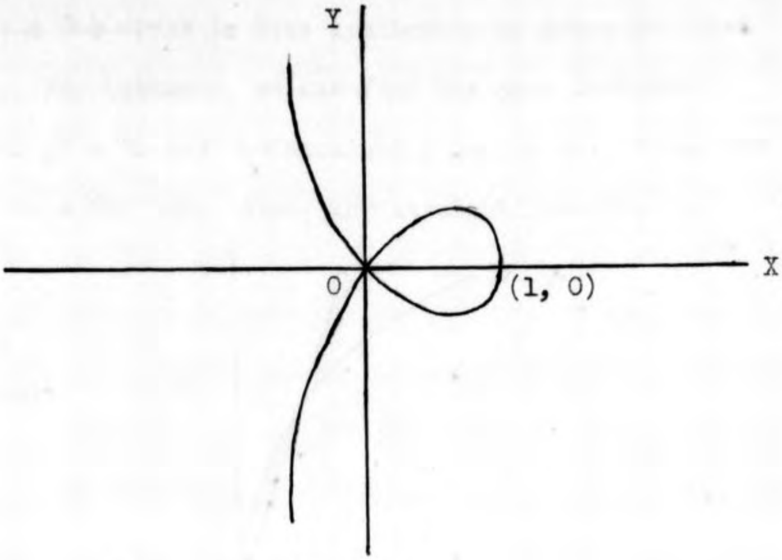

Figure 8:4

From the area formula

$$A = 1/2 \int_{-1}^{1} \left[\frac{(1 - t^2)(1 - 4t^2 - t^4) + t(1 - t^2)(+ 4t)}{(1 + t^2)^3} \right] dt$$

$$= 1/2 \int_{-1}^{1} \frac{1 - t^2 - t^4 + t^6}{(1 + t^2)^3} dt = 1/2 \int_{-1}^{1} \frac{(t^2 - 1)^2}{(t^2 + 1)^2} dt = 1/2 \int_{-1}^{1} \left[1 - \frac{4t^2}{(t^2 + 1)^2} \right] dt$$

In order to evaluate this last integral, make the substitution

$t = \tan z$ When $t = 1$

Then $dt = \sec^2 z dz$ $z = \pi/4$

$t = -1$

$z = -\pi/4$

Substituting

$$A = 1/2 \int_{-\pi/4}^{\pi/4} \left(1 - 4 \frac{\tan^2 z}{\sec^4 z}\right) \sec^2 z dz = 1/2 \int_{-\pi/4}^{\pi/4} (\sec^2 z - 4 \sin^2 z) dz$$

$$= 1/2 \int_{-\pi/4}^{\pi/4} \left[\sec^2 z - 2(1 - \cos 2z)\right] dz = 1/2 \Big[\tan z - 2z + \sin 2z \Big]_{-\pi/4}^{\pi/4}$$

$$= 1/2 \Big[(1 - \pi/2 + 1) - (-1 + \pi/2 - 1) \Big]$$

$$= 2 - \pi/2$$

The formula for areas is also applicable to areas included between two curves. For instance, we can find the area included between the parabola $y^2 = 9x$ and the straight line $y = 3x$. Along the parabola $x = y^2/9$, $dx = 2/9\, ydy$. Along the straight line $y = 3x$, $dy = 3dx$.

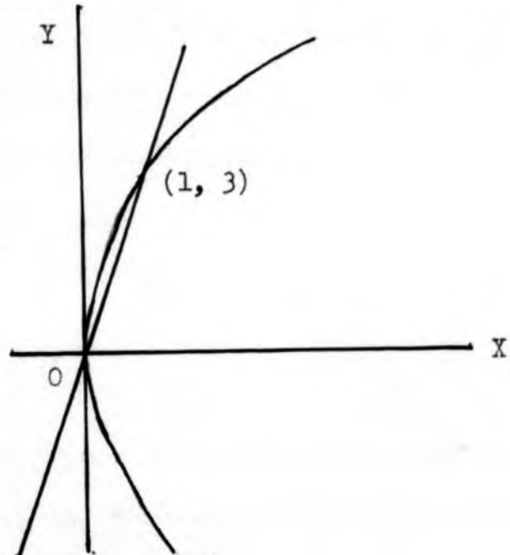

Figure 8:5

Solving these equations simultaneously we find that they intersect at (0, 0) and (1, 3). Integrating in the positive direction around the closed curve made up of the straight line from the origin to the point (1, 3) and the parabola from (1, 3) back to the origin, we have

$$A = 1/2\int_0^1 3xdx - 3xdx + 1/2\int_3^0 (y^2/9 - 2y^2/9)dy$$

$$= -1/2\left[y^3/27\right]_3^0 = -\,1/2\ \left[-\,27/27\right]$$

$$= 1/2.$$

9. Attraction of material curves. Another way in which the line integral is used is in finding the attraction of material curves. According to Newton's law of universal gravitation, each particle of matter in the universe attracts each other particle with a force whose direction is that of the line joining the two, and whose magnitude is directly proportional to the product of their masses

and inversely proportional to the square of their distance apart. That is

$$F = K\, m_1 m_2 / d^2$$

where m_1 and m_2 are the masses of the particles and d is their distance apart. The constant of proportionality K depends upon the units used.

In practice, one is usually concerned, not with material particles, but with continuously distributed matter. The natural procedure is then to think of dividing this body of matter into small parts, sum the vector forces corresponding to these parts, and consider the limiting value of this sum as the maximum dimensions of the parts approach zero.

A material curve is often referred to as a wire. More specifically, a material curve may be thought of as a wire of circular cross-section, with the centers of these cross-sections lying along a smooth curve C, a smooth curve being one which has at each point a tangent line whose direction changes continuously along the curve. The mass of the portion of the wire between any two planes perpendicular to C is thought of as being concentrated along C between these planes. Linear density D of a curve is the limit of the ratio of the mass of a segment to the length of this segment as the length of the segment tends toward zero. If D is a constant, C is said to be homogeneous.

Let us consider the attraction of the smooth material curve C, whose density $D = D(s)$ is a continuous function of arc length along C, on a particle $p_1(x_1, y_1, z_1)$ not on C. Corresponding to a partition of C, consider the mass $\Delta_i m$ of the piece of C corresponding to $\Delta_i s$ as as concentrated at a point $p_i'(x_i', y_i', z_i')$ of this piece. Then this mass particle will exert on a unit particle at (x_1, y_1, z_1) a force whose magnitude is $\frac{\Delta_i m}{d_i^2} = \frac{D(s_i')\Delta_i s}{d_i^2}$ where d_i is the distance between (x_i', y_i', z_i') and (x_1, y_1, z_1) and $\Delta_i m = D(s_i')\Delta_i s$ is derived from the fact that the mass of C along $\Delta_i s$ lies between the products of the least and greatest values of D(s) on this piece by $\Delta_i s$, the length of the piece. By continuity there is some value s_i' such that $D(s_i')\Delta_i s = \Delta_i m$. Now the direction cosines of the segment from p_1 to p_i' are $\frac{(x_i' - x_1)}{d_i}$, $\frac{y_i' - y_1}{d_i}$, and $\frac{(z_i' - z_1)}{d_i}$.

Hence the components of force due to this piece are

$$\Delta_i X = \frac{D(s_i')(x_i' - x_1)\Delta_i s}{d_i^3}, \quad \Delta_i Y = \frac{D(s_i')(y_i' - y_1)\Delta_i s}{d_i^3}$$

$$\Delta_i Z = \frac{D(s_i')(z_i' - z_1)\Delta_i s}{d_i^3}$$

Summing and taking the limits of these components as the maximum length of $\Delta_i s$ approaches zero, we are led to the following line integrals giving the components of the force exerted by the material curve on a unit particle at (x_1, y_1, z_1):

$$X = \int_C \frac{D(s)(x - x_1)}{d^3} ds, \quad Y = \int_C \frac{D(s)(y - y_1)}{d^3} ds, \quad Z = \int_C \frac{D(s)(z - z_1)}{d^3} ds$$

where (x, y, z) is any point on the curve C.

By way of illustration suppose that we wished to find the attraction of a straight homogeneous wire at any point p_1 of space, not on the wire. In this case D(s) is a constant D, and d_1 is the perpendicular distance of p_1, from the wire. If we let the wire coincide with the x axis and let the y axis contain p_1, we will have

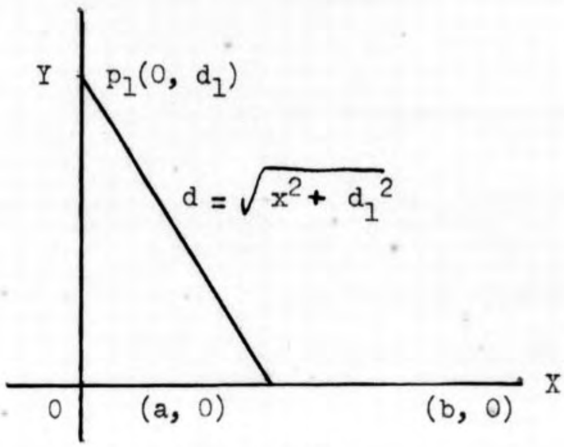

Figure 9:1

$$ds = dx$$

$$X = D\int_a^b \frac{xdx}{(x^2 + d_1^2)^{3/2}} = \frac{D}{2}\int_a^b \frac{2xdx}{(x^2 + d_1^2)^{3/2}}$$

$$= \left[\frac{-D}{\sqrt{x^2 + d_1^2}}\right]_a^b$$

$$Y = D\int_a^b \frac{-\,d_1 dx}{(x^2 + d_1^2)^{3/2}} = -\,d_1 D\int_{\arctan a/d_1}^{\arctan b/d_1} \frac{d_1 \sec^2 zdz}{\left[d_1^2(1 + \tan z\right]^{3/2}}$$

$$= -\frac{d_1^2 D}{d_1^3}\int_{\arctan a/d_1}^{\arctan b/d_1} \cos zdz = \frac{-D}{d_1}\Big[\sin z\Big]_{\arctan a/d_1}^{\arctan b/d_1}$$

$$= \frac{-D}{d_1}\left[\frac{b}{\sqrt{b^2 + d_1^2}} + \frac{a}{\sqrt{a^2 + d_1^2}}\right]$$

Since a plane may be passed containing a straight line and any point not on the line, there will be no Z component of this force.

$$Z = 0$$

If we lengthen the wire indefinitely in both directions, the resulting force is defined as the force due to the infinite wire. From the above results we have by substituting

$$a = -\infty \qquad b = +\infty,$$

$$X = 0$$

Therefore the resulting force is perpendicular to the wire.

$$Y = \frac{-D}{d_1}\Big[\sin z\Big]_{-\pi/2}^{\pi/2} = \frac{-D}{d_1}\Big[1 -(-1)\Big] = \frac{-2D}{d_1}.$$

Hence the attraction of the infinite wire on a unit particle outside the wire is equal to twice the linear density divided by the perpendicular distance of the particle from the wire.

10. Work. Suppose that a force $F = F(x, y)$ acts at every point in the x, y - plane; in general this force varies from point to point in magnitude and direction. We now propose to find the work done on a particle as it moves along some curve C in the region joining $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$.

First let C be the straight line connecting p_1 and p_2. If F is constant in magnitude and the direction of F is along p_1p_2, the work done by the force in moving the particle is defined to be force times displacement. But suppose that F makes an angle θ with p_1p_2. The components are $|F| \cos \theta$ and $|F| \sin \theta$, respectively. The work done by $|F| \cos \theta$ which acts along p_1p_2 is force times displacement or $|F|\, d \cos \theta$. Since the particle remains on the line p_1p_2 the component of force perpendicular to this line causes no motion; therefore we say that it does no work, and the work done by F in moving the particle from p_1 to p_2 along the straight line segment is

$$W = |F|\, d \cos \theta$$

Now let X and Y be the x - and y - components of F, and let α denote the angle that p_1p_2 makes with the positive x axis. Then the projection of F on p_1p_2 is equal to the sum of the projections of X and Y on p_1p_2, and $|F| \cos \theta = X \cos\alpha + Y \sin\alpha$. Since

$\dfrac{x_2 - x_1}{d} = \cos\alpha$, $\dfrac{y_2 - y_1}{d} = \sin\alpha$, we have $W = X(x_2 - x_1) + Y(y_2 - y_1)$.

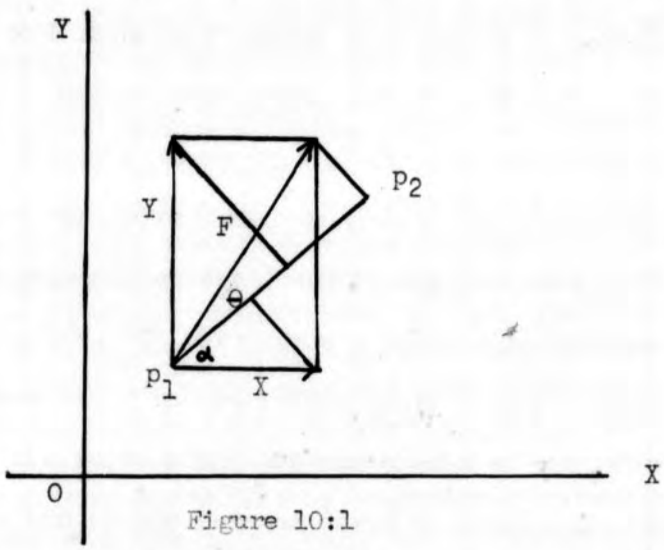

Figure 10:1

Now we consider the general case where F varies continuously in magnitude and direction throughout a region in the xy - plane, and consider the work done by this force in moving a particle along a curve $C: x = x(t)$, $y = y(t)$, lying in this region and joining the points A and B. Corresponding to an arbitrary partition of the parameter range for C, we replace the actual curve C by the polygonal path with successive vertices $p_0, p_1, p_2, \ldots p_n$, and instead of the actual force we consider a substitute force which along the segment $p_{i-1}p_i$ is constant and equal to the given force at an intermediate point (x_i', y_i'). Then the work done by this substitute force as the particle moves along the polygon from A to B is

$$\sum_{i=1}^{n} \left[X(x_i', y_i')\Delta_i x + Y(x_i', y_i')\Delta_i y \right].$$

The limit of this expression as the maximum length of any segment Δ_i s of the partition approaches zero is equal to the line integral

$$\int_C X(x, y)dx + Y(x, y)dy,$$

which is the work done by F on the particle as it moves along C from A to B.

To illustrate this type of work, suppose that there is a meteor, which may be regarded as a particle, which is attracted by the sun (considered at rest) and by all the rest of the matter in the solar system. It moves from a point A to a point B, which are at distances of r_1 and r_2 respectively from the sun. According to Newton's Law, the force which the sun exerts on the meteor is $|F| = \frac{Km}{d^2}$ since the meteor is considered to be of unit mass. Since d is variable in this problem, we will let r be the distance between the bodies in order to prevent confusion in the notation.

From similar figures we have

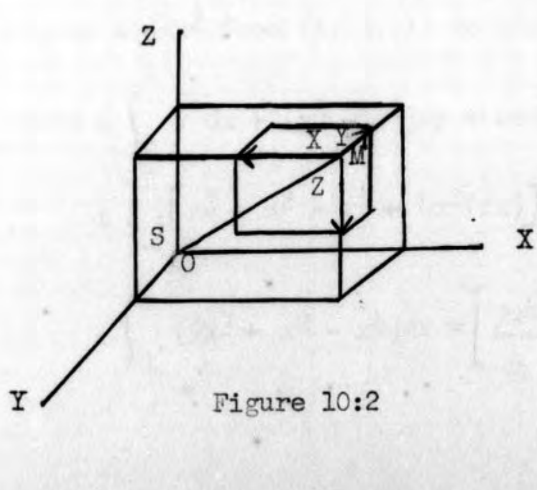

Figure 10:2

$$\frac{X}{|F|} = -\frac{x}{r} \qquad X = -\frac{Kmx}{r^3} \qquad \frac{Y}{|F|} = -\frac{y}{r} \qquad \frac{Z}{|F|} = -\frac{z}{r}$$

$$Y = -\frac{Kmy}{r^3} \qquad Z = -\frac{Kmz}{r^3}$$

Now applying the formula, we have

$$\text{Work} = -\int \frac{Km}{r^3}(xdx + ydy + zdz).$$

$$\text{But } r^2 = x^2 + y^2 + z^2$$

$$rdr = xdx + ydy + zdz$$

Therefore

$$\text{Work} = -Km\int \frac{dr}{r^2} = -Km\left[-\frac{1}{r}\right]_{r_1}^{r_2} = Km\left[\frac{1}{r_2} - \frac{1}{r_1}\right]$$

If we apply the independence test of section 7, we see that

$$\frac{\partial Z}{\partial y} = \frac{\partial Y}{\partial z} = 0 \qquad \frac{\partial X}{\partial z} = \frac{\partial Z}{\partial x} = 0 \qquad \frac{\partial Y}{\partial x} = \frac{\partial X}{\partial y} = 0$$

and so the work done by the sun is independent of the path of motion of the meteor. A field of force which possesses this property is called conservative or lamellar.

As a further illustration of work, let us suppose that the components of a force are

$$X = x^3, \qquad Y = y^2 - z^2, \qquad Z = 4z$$

and let us find the work done by the force as a particle moves along the curve $y = x$, $z = x^2$ from (1, 1, 1) to (2, 2, 4).

$$\text{Work} = \int_C x^3dx + (y^2 - z^2)dy + 4zdz$$

$$= {}_1^2\left[x^3 + x^2 - x^4 + 4x^2(2x)\right]dx$$

$$= \int_1^2 (9x^3 + x^2 - x^4)dx = \left[\frac{9x^4}{4} \quad \frac{x^3}{3} - \frac{x^5}{5}\right]_1^2$$

$$= (36 + \frac{8}{3} - \frac{32}{5}) - (\frac{9}{4} + \frac{1}{3} - \frac{1}{5})$$

$$= \frac{1793}{60}$$

Again testing for independence we have

$$\frac{\partial Z}{\partial y} = 0, \qquad \frac{\partial Y}{\partial z} = -2z$$

and $\frac{\partial Z}{\partial y} \neq \frac{\partial Y}{\partial z}$ and our test fails and the work done is seen to depend on the curve as well as the end points. For if we let C be the straight line connecting (1, 1, 1) and (2, 2, 4) given by equations $\frac{x-1}{1} = \frac{y-1}{1} = \frac{z-1}{3}$ from which $x = y$, $z = 3x - 2$

$$\text{Work} = \int_1^2 [x^3 + (x^2 - 9x^2 + 12x - 4) + 4(3x - 2)(3)]\,dx$$

$$= \int_1^2 (x^3 - 8x^2 + 48x - 28)dx$$

$$\text{Work} = \left[\frac{x^4}{4} - \frac{8x^3}{3} + \frac{48x^2}{2} - 28x\right]_1^2 = \left(4 - \frac{64}{3} + 96 - 56\right) - \left(\frac{1}{4} - \frac{8}{3} + 24 - 28\right)$$

$$= \frac{325}{12}$$

According to Newton's laws, the motion of a particle of mass is determined by the equations

$$X = mx'', \qquad Y = my'', \qquad Z = mz''$$

where x'', y'' and z'' denote derivatives with respect to time. If we multiply these equations by x', y' and z' respectively and integrate from t_0 to t_1 we have

$$\text{Work} = m \int_{t_0}^{t_1} \frac{dx}{dt}\frac{d^2x}{dt^2}\,dt + \frac{dy}{dt}\frac{d^2y}{dt^2}\,dt + \frac{dz}{dt}\frac{d^2z}{dt^2}\,dt$$

$$= \frac{m}{2}\left[\left(\frac{dx}{dt}\right)^2 \quad \left(\frac{dy}{dt}\right)^2 \quad \left(\frac{dz}{dt}\right)^2\right]_{t_0}^{t_1}$$

which is equal to the change in kinetic energy of the particle.

11. <u>Integrals of complex functions</u>. When real numbers are combined by addition, subtraction, multiplication or division with a non-vanishing divisor, the results are real numbers; such numbers form a closed system for these operations. But this is not always the case in root extraction, for no real number can be the square root of a negative real number. Therefore our number system has been extended to include complex numbers of the form $a + bi$ where a and b are real and $i = \sqrt{-1}$. The complex numbers include the real numbers and form a closed system with respect to addition, subtraction, multiplication, division, and root extraction. For the most part, the rules for manipulating complex numbers are the same as those for real numbers.

The definition of the derivative for a function of a complex variable is formally the same as for a function of a real variable. We say that $f(z)$ has a derivative $f'(z)$ at z if

$$\lim_{\Delta z \to 0} \frac{f(z + \Delta z) - f(z)}{\Delta z} = f'(z)$$

A function of a complex variable $z = x + iy$ is said to be analytic in the region R if a single value of $f(z)$ is defined for each z inside the region, and the function $f(z)$ has a finite derivative. A function is said to be analytic at z if it is analytic in some circular region including z as an interior point.

For a given function, a value of z that cannot be included in any circle C within which the function is analytic is called a singular point. For the elementary functions, the singular points include those values which make the function or its derivative infinite owing to the presence of a zero denominator. The elementary functions are analytic except for singular points.

Suppose $w = f(z)$ is a function of the complex variable $z = x + iy$ for all values of z in R, any region of the xy - plane, and $w = f(z)$ is an analytic function. We denote the real part of w by u and the imaginary part by v so that

$$w = u + iv.$$

The values of x and y determine z and hence determine $f(z)$. Thus u and v are functions of x and y so that

$$f(x + iy) = u(x, y) + iv(x, y).$$

Taking $f(z)$ as an analytic function, we can derive the Cauchy-Riemann differential equations[7] which are satisfied by u and v. On the other hand, we can prove that if the partial derivatives of $u(x, y)$ and $v(x, y)$ are continuous and satisfy the Cauchy-Riemann equations in the given region R, then the function $u + iv$ is an analytic function of $z = x + iy$ in R. These equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

It follows from the Cauchy-Riemann equations that

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial x \partial y} = \frac{\partial^2 v}{\partial y \partial x} = -\frac{\partial^2 u}{\partial y^2}$$

consequently

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

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7. For derivation see [2], [4], and [9]

This is known as Laplace's equation in two dimensions. It may be shown that any function $u(x, y)$ that satisfies Laplace's equation at all points of a region R necessarily has continuous partial derivatives of all orders at any interior point of R.[3], [4].

Let us recall the method for graphing complex numbers. Since the complex quantity $z = x + iy$ involves two variables x and y, we need the whole xy - plane for z. If we wish to take the integral of $f(z)$ from z_1 to z_2 it is evident that z must vary along some curve C. The integral of the complex function $f(z)$ along a path C is defined as the line integral along C of

$$f(z)dz = (u + iv)(dx + idy)$$

$$= (udx - vdy) + i(vdx + udy)$$

That is

$$\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy.$$

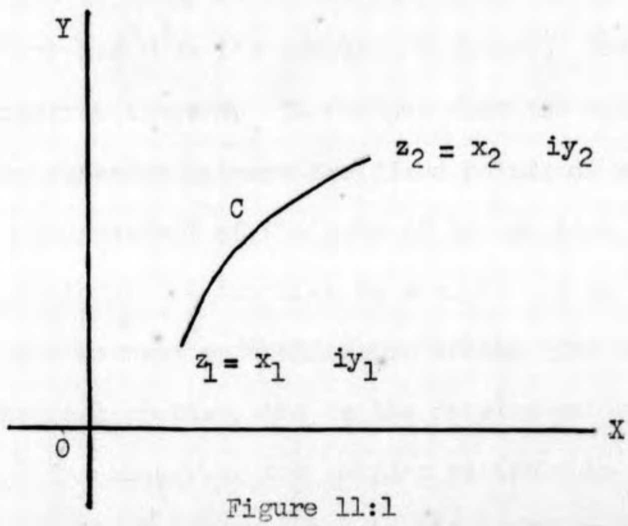

Figure 11:1

Each of the integrals on the right is a real line integral and could be evaluated by the methods already given.

Let C be the complete boundary of a region R of the type used in section 6. Suppose further that $u + iv = f(z)$ is an analytic function of z for all values of z in R and on C. Then we may use C as the path of integration and can transform the right member of the integral by Green's Theorem.

$$\int_C (wdx - vdy) + i\int_C (vdx + udy)$$

$$= \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy.$$

Since f(z) is analytic in R, the Cauchy-Riemann equations hold and the integrands on the right are zero. Hence the left number is also zero and

$$\int_C f(z)dz = 0$$

That is, the integral of f(z) is zero over every closed path C such that f(z) is analytic at all points on the path C and in the region R which has C as its complete boundary. This is known as Cauchy's integral theorem. It follows that the value of the integral of a complex function between two fixed points of a simply connected region R is independent of the path of integration, restricted to lie in R, providing the function is analytic in R.

Here we have an application of the line integral theory, not to a physical problem, but to the development of further mathematical theory. The theory of the complex variable in turn has practical applications. It also serves to simplify methods we already have for dealing with certain physical problems. Evaluation of certain real definite integrals is made simpler by a knowledge of poles and residues, which is a part of complex variable theory. Many mathematical results may be stated more simply, and obtained more readily, by the use of complex quantities in the intermediate stages, even if the final results involve real numbers only.

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