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The saturating topology on a product set is between the usual and box topologies and is different from each. On a product of compact spaces the saturating topology is the same as the usual topology, and hence, it is compact. On a product of connected spaces the saturating topology is connected. On a product of σ-regular spaces the saturating topology is regular. On a product of separable spaces the saturating topology is lattice isomorphic to a separable topology.

A COMPACTNESS PRESERVING TOPOLOGY

FOR PRODUCTS SETS

by

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INTRODUCTION

The purpose of this thesis is to investigate the properties of certain topologies for a countable product of lattice isomorphic factor spaces. The idea of using lattice isomorphisms between factors to define a topology on a product space is due to Goolsby in [4]. Both the usual and box topologies are well known. The saturating topology was originated by the author in an effort to describe a topology properly containing the usual topology, but still preserving compactness. The example showing that regualrity of the factor spaces does not guarantee regularity of the saturating topology is also due to Goolsby in a paper not published at the time of this writing. The result that connectivity of the factor spaces does not gaurantee connectivity of the box topology was first proved by Knight in [5], but the proof given here is due to Professor Jerry E. Vaughan of the University of North Carolina at Greensboro.

A working knowlege of set theory and elementary topology is assuned. The reader is referred to [1], [2], and [3] for definitions and results not covered in this thesis.

In Chapter I certain definitions and results regarding lattice isomorphisms are reviewed. In Chapter II definitions of the usual, saturating, and box topologies are given and

and certain basic results are proved about them. In Chapter III the semi-complete base and the basic similarity are defined, it is proved that a basic similarity may be extended to a lattice isomorphism, and this result is applied to the product topologies. In Chapter IV it is shown that for T_0, T_1, or T_2 factor spaces, each of the three topologies discussed is T_0, T_1, or T_2, respectively. Also in Chapter IV an example is given to show that regularity of factor spaces does not gaurantee regularity of the saturating topology. In Chapter V it is shown that the saturating topology on a product of connected factors is connected and an example is given to show that connectivity of factor spaces does not guarantee connectivity of the box topology. Also in Chapter V it is shown that the saturating topology on a product of separable spaces is lattice isomorphic to a separable topology. Finally in Chapter V it is shown that on a product of compact spaces the usual topology and the saturating topology are the same.

Emphasis is given to the saturating topology throughout this thesis. All Cartesian products referred to in this thesis are indexed by the set of positive integers and the simplified notation $\Pi_n U_n$ is used to denote the product of the sequence U of sets. The closure of a set A with respect to a topology T will be denoted by $cl_T(A)$. When context makes the meaning clear $cl(A)$ will be used.

CHAPTER I

PREPARATORY RESULTS

Definition 1. [1, Def. 1, p. 1] An ordered pair (X,R) is called a *partially ordered set* provided X is a set and R is a subset of $X \times X$ such that

(1) if $a \in X$, then $(a,a) \in R$,

(2) if $a,b \in X$, then if $(a,b) \in R$ and $(b,a) \in R$, then $a=b$, and

(3) if $a,b,c \in X$, then if $(a,b) \in R$ and $(b,c) \in R$, then $(a,c) \in R$.

If (X,R) is a partially ordered set, then $(a,b) \in R$ will be denoted by aRb.

Definition 2. [1, Def. 2,3; p. 1] Let $(X,<)$ be a partially ordered set. Let $b \in X$ and let $A \subset X$. Then b is called a *lower bound of* A provided if $a \in A$, then $b<a$; and b is called an *upper bound of* A provided if $a \in A$, then $a<b$. The element b is called a *greatest lower bound of* A provided b is a lower bound of A and b is an upper bound of the set of lower bounds of A. The element b is called a *least upper bound of* A provided b is an upper bound of A and b is a lower bound of the set of upper bounds of A.

Definition 3. [1, Def. 4, p. 2] A partially ordered set $(X,<)$ is called a *lattice* provided every finite, non-empty subset of X has both a least upper bound and a greatest lower bound.

Definition 4. [1, Def. 6, p. 2] Let $(K,<)$ and $(L,<<)$ be lattices and let f be a one-to-one function from K onto L. Then f is called a *lattice isomorphism from onto* $(K,<)$ $(L,<<)$ provided

(1) if $a,b \in K$ and $a<b$, then $f(a)<<f(b)$, and

(2) if $c,d \in L$ and $c<<d$, then $f^{-1}(c)<f^{-1}(d)$.

To say that $(K,<)$ *is lattice isomorphic to* $(L,<<)$ means that there is a lattice isomorphism from $(K,<)$ onto $(L,<<)$.

Theorem 1. Let $(X,<)$, $(Y,<<)$, and $(Z,<*)$ be lattices and suppose f is a lattice isomorphism from $(X,)$ onto $(Y,<<)$ and suppose g is a lattice isomorphism from $(Y,<<)$ onto $(Z,<*)$. Then $g \circ f$ is a lattice isomorphism from $(X,<)$ onto $(Z,<*)$.

Proof: This is clear from Definition 4.

The proofs of the remaining theorems in Chapter I may be found in [1] and [3].

Theorem 2. [1, Th. 2, p. 3] Let $(K,<)$ and $(L,<<)$ be lattices and suppose that f is a lattice isomorphism from $(K,<)$ onto $(L,<<)$. Then the inverse function f^{-1} is a lattice isomorphism from $(L,<<)$ onto $(K,<)$.

<u>Theorem 3</u>. [1, Th. 6, p. 7] Let (X,T) be a topological space. Then the usual subset relation defines a lattice on T.

The remainder of this paper is concerned with lattices defined on topologies. In all cases the order will be defined by the subset relation. When convient the topologies will be refered to as lattices.

<u>Theorem 4</u>. [1, Th. 5, p. 4] Let (X,S) and (Y,T) be topological spaces and suppose that F is a one-to-one function from S onto T. Then F is a lattice isomorphism from S onto T if and only if $\mathcal{U}\subset S$ implies

$$\cup\{F(U)|U\in\mathcal{U}\}=F(\cup\{U|U\in\mathcal{U}\}).$$

<u>Theorem 5</u>. [1, Ths. 9,10; pp. 8, 9] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Then

(1) $F(\phi)=\phi$ and $F(X)=Y$,

(2) if $\mathcal{U}$ is a open cover of X, then $\{F(U)|U\in\mathcal{U}\}$ is an open cover of Y, and

(3) if $\mathcal{U}$ is a finite subcollection of S, then

$$F(\cap\{U|U\in\mathcal{U}\})=\cap\{F(U)|U\in\mathcal{U}\}.$$

<u>Theorem 6</u>. [1, Th. 27, p. 32] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Then if (X,S) and (Y,T) are both T_1, then (X,S) is homeomorphic to (Y,T).

Definition 5. [1, Def. 14, p. 10] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. The collection of closed subsets of X will be denoted by $\mathcal{C}_S$ and likewise the closed subsets of Y will be denoted by $\mathcal{C}_T$. The function G with domain $\mathcal{C}_S$ defined by if $M \in \mathcal{C}_S$, then $G(M)=Y-F(X-M)$ will be called the dual of F and will be denoted by $\mathcal{D}_F$.

Theorem 7. [1, Th. 8, p. 8] Let (X,S) be a topological space. Then the superset relation defines a lattice on $\mathcal{C}_S$.

Theorem 8. [1, Ths. 11, 12; pp. 10, 11] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Then $\mathcal{D}_F$ is a lattice isomorphism from $\mathcal{C}_S$ onto $\mathcal{C}_T$ and $(\mathcal{D}_F)^{-1}=\mathcal{D}_{F^{-1}}$.

Theorem 9. [1, Ths. 5, 7; pp. 4,7] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Then if $\mathcal{M} \subset \mathcal{C}_S$, then

(1) $\mathcal{D}_F(\cap\{M \mid M \in \mathcal{M}\}) = \cap\{\mathcal{D}_F(M) \mid M \in \mathcal{M}\}$, and

(2) if $\mathcal{M}$ is finite, then $\mathcal{D}_F(\cup\{M \mid M \in \mathcal{M}\}) = \cup\{\mathcal{D}_F(M) \mid M \in \mathcal{M}\}$.

Theorem 10. [1, Ths. 9, 17; pp. 8, 15] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Then

(1) $\mathcal{D}_F(\phi)=\phi$ and $\mathcal{D}_F(X)=Y$, and

(2) if $\mathcal{O} \in S$, then $\mathrm{cl}(F(\mathcal{O}))=\mathcal{D}_F(\mathrm{cl}(\mathcal{O}))$.

Theorem 11. [3, Th. 6, p. 4] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Let $p \in X$, $\mathcal{O} \in S$, and $M \in \mathcal{C}_S$. Then

(1) if $M \subset \mathcal{O}$, then $\mathcal{D}_F(M) \subset F(\mathcal{O})$,

(2) $M \cap \mathcal{O} = \phi$ if and only if $\mathcal{D}_F(M) \cap F(\mathcal{O}) = \phi$, and

(3) if $V \in T$ such that $V \cap \mathcal{D}_F(cl(\{p\})) \neq \phi$, then p is a member of $F^{-1}(V)$.

Theorem 12. [1, Th. 13, p. 13] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Let $U \in S$ and suppose $p \in F(U)$. Then there exists $x \in U$ such that if $V \in S$ and $x \in V$, then $p \in F(V)$.

Theorem 13. [1, Th. 20, 21; pp. 19, 22] Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Then

(1) if (X,S) is connected, then (Y,T) is connected, and

(2) if (X,S) is compact, then (Y,T) is compact.

CHAPTER II

PRELIMINARY RESULTS

Definition 6. [4, Def. 1, p. 1] The ordered triple (X,T,F) is said to be a lattice limit sequence provided X is a sequence of sets, T is a sequence of topologies, and F is a sequence of functions such that if i is a positive integer, then

(1) T_i is a topology for X_i, and

(2) F_i is a lattice isomorphism from T_{i+1} onto T_i.

Suppose (X,T,F) is a lattice limit sequence. Then if i is a positive integer, then F_i^i will denote the identity map from T_i onto T_i, F_i^{i+1} will denote F_i, and if j is a positive integer, then F_i^{i+j+1} will denote $F_i^{i+j} \circ F_{i+j}$. Also if k and ℓ are positive integers and $k<\ell$, then F_ℓ^k will denote $(F_k^\ell)^{-1}$. Now, if m and n are positive integers, then F_n^m is a lattice isomorphism from T_m onto T_n.

Definition 7. Let $L=(X,T,F)$ be a lattice limit sequence. The limit collection of L, denoted by $C(L)$, is the usual product of sets $\Pi_n X_n$. For each positive integer i, the i^{th} projection of $C(L)$, denoted by π_i, is the usual projection from $C(L)$ into T_i, such that if $f \in C(L)$, then $\pi_i(f)=f(i)$.

Definition 8. Let $L=(X,T,F)$ be a lattice limit sequence. The usual topology on the limit collection of L, denoted by $U(L)$, is the usual topology generated by the collection B_U, to which a set O belongs provided there is a sequence U, such that

(1) if $i \in Z^+$, then $U_i \in T_i$,

(2) there exists a positive integer M, such that if $i \in Z^+$ and $i \geq M$, then $U_i = X_i$, and

(3) $O = \Pi_n U_n$.

The collection B_U will be called the usual base for $U(L)$.

Definition 9. Let $L=(X,T,F)$ be a lattice limit sequence. The saturating topology on the limit collection of L, denoted by $S(L)$, is the topology generated by the collection B_S, to which a set O belongs provided there is a sequence U, such that

(1) if $i \in Z^+$, then $U_i \in T_i$,

(2) there exists a positive integer M, such that if $i \in Z^+$ and $i \geq M$, then $U_i \subset F_i^{i+1}(U_{i+1})$,

(3) if $i \in Z^+$, then $X_i = \cup \{F_i^{i+k}(U_{i+k}) \mid k \in Z^+\}$, and

(4) $O = \Pi_n U_n$.

The collection B_S will be called the usual base for $S(L)$.

Definition 10. Let $L=(X,T,F)$ be a lattice limit sequence. The box topology on the limit collection of L, denoted by $B(L)$, is the usual box topology generated by the collection

$\mathcal{B}_{\mathcal{B}}$, to which a set $\mathcal{O}$ belongs provided there is a sequence U, such that

(1) if $i \in Z^+$, then $U_i \in T_i$, and

(2) $\mathcal{O} = \Pi_n U_n$.

The collection $\mathcal{B}_{\mathcal{B}}$ will be called the <u>usual base for</u> $\mathcal{B}(L)$.

<u>Lemma 14.1</u>. Let $L=(X,T,F)$ be a lattice limit sequence. Let $\mathcal{B}_S$ be the usual base for $S(L)$, the saturating topology on $C(L)$. Then if $A, B \in \mathcal{B}_S$, then $A \cap B \in \mathcal{B}_S$.

Proof: Let $A, B \in \mathcal{B}_S$. Now there exist sequences U and V, and there exist positive integers M and N, such that

(1) if $i \in Z^+$, then $U_i \in T_i$ and $V_i \in T_i$,

(2) if $i \in Z^+$, then if $i \geq M$, then $U_i \subset F_i^{i+1}(U_{i+1})$, and if $i \geq N$, then $V_i \subset F_i^{i+1}(V_{i+1})$,

(3) if $i \in Z^+$, then $X_i = \cup\{F_i^{i+k}(U_{i+k}) \mid k \in Z^+\}$ and $X_i = \cup\{F_i^{i+k}(V_{i+k}) \mid k \in Z^+\}$, and

(4) $A = \Pi_n U_n$ and $B = \Pi_n V_n$.

Let W be the sequence defined by if j is a positive integer, then $W_j = U_j \cap V_j$.

(1) If $i \in Z^+$, then $W_i = U_i \cap V_i$ and $U_i \cap V_i$ is in T_i.

(2) If $i \in Z^+$ and $i \geq M+N$, then $i \geq M$ and $i \geq N$, so

$$W_i = U_i \cap V_i \subset F^{i+1}(U_{i+1}) \cap F^{i+1}(V_{i+1}) = F^{i+1}(U_{i+1} \cap V_{i+1}),$$

but $W_{i+1} = U_{i+1} \cap V_{i+1}$, hence $W_i \subset F^{i+1}(W_{i+1})$.

(3) Let j be a positive integer. Clearly

$$X_j \supset \cup\{F_j^{j+k}(W_{j+k}) \mid k \in Z^+\}.$$

Let $x \in X_j$ and let $K=M+N$. Now, since

$$X_j = F_j^K(X_K) = F_j^K(\cup\{F_K^{K+k}(U_{K+k}) \mid k \in Z^+\}) = \cup\{F_j^K \circ F_K^{K+k}(U_{K+k}) \mid k \in Z^+\} = \cup\{F_j^{K+k}(U_{K+k}) \mid k \in Z^+\},$$

there is a positive integer L_u, such that $x \in F_j^{K+L_u}(U_{K+L_u})$. Likewise, there is a positive integer L_v, such that $x \in F_j^{K+L_v}(V_{K+L_v})$. Let $L=L_u+L_v$. Then $M \le K+L_u \le L$ and $N \le K+L_v \le L$, so $x \in F_j^{K+L}(U_{K+L})$ and $x \in F_j^{K+L}(V_{K+L})$. Thus

$$x \in F_j^{K+L}(U_{K+L}) \cap F_j^{K+L}(V_{K+L}) = F_j^{K+L}(U_{K+L} \cap V_{K+L}) = F_j^{K+L}(W_{K+L}).$$

Hence $x \in \cup\{F_j^{j+k}(W_{j+k}) \mid k \in Z^+\}$. It follows that $X_j \subset \cup\{F_j^{j+k}(W_{j+k}) \mid k \in Z^+\}$ and therefore $X_j = \cup\{F_j^{j+k}(W_{j+k}) \mid k \in Z^+\}$.

(4) The product $\Pi_n W_n$ is a member of $\mathcal{B}_S$.

Suppose $y \in \mathcal{A} \cap \mathcal{B}$, then $y \in \mathcal{A}$ and $y \in \mathcal{B}$. So if i is a positive integer, then $\pi_i(y) \in U_i$ and $\pi_i(y) \in V_i$ or $\pi_i(y) \in U_i \cap V_i = W_i$. Thus $y \in \Pi_n W_n$, and so $\mathcal{A} \cap \mathcal{B} \subset \Pi_n W_n$. Suppose $z \in \Pi_n W_n$, then if h is a positive integer, then $\pi_h(z) \in W_h$, or $\pi_h(z) \in U_h$ and $\pi_h(z) \in V_h$. Thus $z \in \Pi_n U_n = \mathcal{A}$ and $z \in \Pi_n V_n = \mathcal{B}$. Hence $z \in \mathcal{A} \cap \mathcal{B}$. It follows that $\Pi_n W_n \subset \mathcal{A} \cap \mathcal{B}$. Therefore $\Pi_n W_n = \mathcal{A} \cap \mathcal{B}$ and so $\mathcal{A} \cap \mathcal{B}$ is a member of $\mathcal{B}_S$.

<u>Theorem 14</u>. Let L be a lattice limit sequence and let $\mathcal{B}_S$ be the usual base for the saturating topology on $C(L)$. Then

$\mathcal{B}_S$ is a base for a topology.

Proof: This is clear from Lemma 14.1.

Theorem 15. Let L be a lattice limit sequence. Let $\mathcal{B}_U$ and $\mathcal{B}_B$ be the usual bases for $U(L)$ and $B(L)$, respectively. Then $\mathcal{B}_U$ and $\mathcal{B}_B$ are bases for topologies.

Proof: The proofs are similar to that of Theorem 14.

Theorem 16. Let L be a lattice limit sequence. Then $U(L) \subset S(L) \subset B(L)$.

Proof: Let $\mathcal{B}_U$, $\mathcal{B}_S$, and $\mathcal{B}_B$ be the usual bases for $U(L)$, $S(L)$, and $B(L)$, respectively. Clearly, $\mathcal{B}_U \subset \mathcal{B}_S \subset \mathcal{B}_B$, hence $U(L) \subset S(L) \subset B(L)$.

Example 1. Let X, T, and F be the sequences defined by if i is a positive integer, then X_i is the set of real numbers, T_i is the usual topology for the set of real numbers, and F_i is the identity lattice isomorphism from T_{i+1} onto T_i. Let $L=(X,T,F)$. Then L is a lattice limit sequence such that $U(L) \neq S(L)$ and $S(L) \neq B(L)$.

Proof: Clearly, L is a lattice limit sequence. Let U and V be the sequences defined by if i is a positive integer, then U_i is the open interval $(-i,i)$, and V_i is the open interval $(-1,1)$. Plainly, $\Pi_n U_n$ is a member of $S(L)$, but not a member of $U(L)$, hence $U(L) \neq S(L)$. Also, $\Pi_n V_n$ is a member of $B(L)$, but not a member of $S(L)$, hence $S(L) \neq B(L)$.

Theorem 17. Let $L=(X,T,F)$ be a lattice limit sequence. Let $\mathcal{T}$ be either $\mathcal{U}(L)$, $\mathcal{S}(L)$, or $\mathcal{B}(L)$. Then if m is a positive integer, then $\pi_m:(\mathcal{C}(L),\mathcal{T})\to(X_m,T_m)$ is both continuous and open.

Proof: Let m be a positive integer. From Theorem 16, $\mathcal{U}(L)\subset\mathcal{T}\subset\mathcal{B}(L)$.

Let $W\in T_m$, and let U be the sequence defined by if i is a positive integer, then if $i\neq m$, then $U_i=X_i$, and if $i=m$, then $U_i=W$. Now, $\Pi_n U_n$ is a member of $\mathcal{U}(L)$, and hence is a member of $\mathcal{T}$. Further, since $\pi_m^{-1}(W)=\Pi_n U_n$, then $\pi_m^{-1}(W)$ is a member of $\mathcal{T}$. It follows that π_m is continuous.

Let $\mathcal{O}\in\mathcal{T}$, then $\mathcal{O}\in\mathcal{B}(L)$. Let $\mathcal{B}_\mathcal{B}$ be the usual base for $\mathcal{B}(L)$. Plainly, if $B\in\mathcal{B}_\mathcal{B}$, then $\pi_m(B)$ is open. Since $\mathcal{O}=\cup\{B\,|\,B\in\mathcal{B}_\mathcal{B}\text{ and }B\subset\mathcal{O}\}$, then $\pi_m(\mathcal{O})=\cup\{\pi_m(B)\,|\,B\in\mathcal{B}_\mathcal{B}\text{ and }B\subset\mathcal{O}\}$ is a union of open sets, hence is open. It follows that π_m is open.

CHAPTER III

THE SATURATING BASE

Definition 11. [3, Def. 3, p. 9] Let (X,S) be a topological space. A base $\mathcal{B}$ for S is said to be semi-complete provided if B_1 and B_2 are members of $\mathcal{B}$, then $B_1 \cap B_2 \in \mathcal{B}$.

Definition 12. Let (X,S) and (Y,T) be topological spaces and let $\mathcal{B}$ be a base for S. A one-to-one function k with domain $\mathcal{B}$ and range a subset of T is called a simulacrum of $\mathcal{B}$ in T provided if $\mathcal{U}$ is a subcollection of $\mathcal{B}$, then

(1) if $\cup\{U|U\in\mathcal{U}\}\in\mathcal{B}$, then $k(\cup\{U|U\in\mathcal{U}\})=\cup\{k(U)|U\in\mathcal{U}\}$, and

(2) if $\mathcal{U}$ is finite and $\cap\{U|U\in\mathcal{U}\}\in\mathcal{B}$, then $k(\cap\{U|U\in\mathcal{U}\})=\cap\{k(U)|U\in\mathcal{U}\}$.

Definition 13. Let (X,S) and (Y,T) be topological spaces and let $\mathcal{B}$ be a base for S. Suppose that k is a simulacrum of $\mathcal{B}$ in T. The natural extension of k to S is the function with domain S defined by if $\mathcal{O}\in S$, then $K(\mathcal{O})=\cup\{k(U)|U\in\mathcal{B}$ and $U\subset\mathcal{O}\}$.

Definition 14. [3, Def. 4, p. 9] Let (X,S) and (Y,T) be topological spaces and let $\mathcal{B}_S$ and $\mathcal{B}_T$ semi-complete bases for S and T, respectively. A function k is said to be a basic similarity from $\mathcal{B}_S$ onto $\mathcal{B}_T$ provided k is a one-to-one function with domain $\mathcal{B}_S$ and range $\mathcal{B}_T$, such that (1) k is a simulacrum of $\mathcal{B}_S$ in T and (2) k^{-1} is a simulacrum of $\mathcal{B}_T$ in S. The bases $\mathcal{B}_S$ and $\mathcal{B}_T$ are said to be basically similar if there is a basic similarity from $\mathcal{B}_S$ onto $\mathcal{B}_T$.

Lemma 18.1. Let (X,S) and (Y,T) be topological spaces and let $\mathcal{B}$ be a base for S. Suppose that k is a simulacrum of $\mathcal{B}$ in T. Then if $A,B\in\mathcal{B}$ and $A\subset B$, then $k(A)\subset k(B)$.

Proof: Let $A,B\in\mathcal{B}$ with $A\subset B$. Then $B=A\cup B$, hence $k(A)\cup k(B)=k(A\cup B)=k(B)$ and therefore $k(A)\subset k(B)$.

Lemma 18.2. Let (X,S) and (Y,T) be topological spaces and let $\mathcal{B}_S$ and $\mathcal{B}_T$ be semi-complete bases for S and T, respectively. Suppose k is a basic similarity from $\mathcal{B}_S$ onto $\mathcal{B}_T$. Let K be the natural extension of k to S. Then if $\mathcal{O}\in T$, then $K(\cup\{k^{-1}(B)\,|\,B\in\mathcal{B}_T \text{ and } B\subset\mathcal{O}\})=\mathcal{O}$.

Proof: Let $\mathcal{O}\in T$ and let $\mathcal{U}=\{k^{-1}(B)\,|\,B\in\mathcal{B}_T \text{ and } B\subset\mathcal{O}\}$. Now, $\mathcal{U}\subset\mathcal{B}_S$ and therefore $\mathcal{O}\subset K(\cup\{U\,|\,U\in\mathcal{U}\})$.

Let $W\in\mathcal{B}_S$ such that $W\subset\cup\{U\,|\,U\in\mathcal{U}\}$. Let $\mathcal{C}=\{W\cap U\,|\,U\in\mathcal{U}\}$. Since $\mathcal{B}_S$ is semi-complete, $\mathcal{C}$ is a subcollection of $\mathcal{B}_S$. Plainly, $\cup\{C\,|\,C\in\mathcal{C}\}\subset W$. Let $w\in W$, then there exists $U_w\in\mathcal{U}$, such that $w\in U_w$. Now $w\in W\cap U_w$ and $W\cap U_w$ is a member of $\mathcal{C}$, hence $w\in\cup\{C\,|\,C\in\mathcal{C}\}$. Therefore $W\subset\cup\{C\,|\,C\in\mathcal{C}\}$. It follows that $W=\cup\{C\,|\,C\in\mathcal{C}\}$.

Let $D\in\mathcal{C}$, then there exists $U_D\in\mathcal{U}$, such that $D=U_D\cap W$. Also, there exists $V_D\in\mathcal{B}_T$, such that $V_D\subset\mathcal{O}$ and $U_D=k^{-1}(V_D)$. Now, $D=k^{-1}(V_D)\cap W$, hence $D\subset k^{-1}(V_D)$. From Lemma 18.1, $k(D)\subset k\circ k^{-1}(V_D)=V_D\subset\mathcal{O}$. It follows that if $C\in\mathcal{C}$, then $k(C)\subset\mathcal{O}$.

Now $k(W)=k(\cup\{C|C\in\mathcal{C}\})=\{k(C)|C\in\mathcal{C}\}\subset\mathcal{O}$. It follows that if $B\in\mathcal{B}_S$, such that $B\subset\cup\{U|U\in\mathcal{U}\}$, then $k(B)\subset\mathcal{O}$. Thus

$$K(\cup\{U|U\in\mathcal{U}\}=\cup\{k(B)|B\in\mathcal{B}_S \text{ and } B\subset\cup\{U|U\in\mathcal{U}\}\}\subset\mathcal{O}.$$

Therefore $K(\cup\{U|U\in\mathcal{U}\})=\mathcal{O}$, also $\{U|U\in\mathcal{U}\}=\{k^{-1}(B)|B\in\mathcal{B}_T$ and $B\subset\mathcal{O}\}$ by definition, so $K(\cup\{k^{-1}(B)|B\in\mathcal{B}_T$ and $B\subset\mathcal{O}\})=\mathcal{O}$.

<u>Lemma 18.3</u>. Let (X,S) and (Y,T) be topological spaces and let $\mathcal{B}_S$ and $\mathcal{B}_T$ be semi-complete bases for S and T, respectively. Suppose k is a basic similarity from $\mathcal{B}_S$ onto $\mathcal{B}_T$. Let K be the natural extension of k to S. Then K is one-to-one.

Proof: Let $U,V\in S$ such that $K(U)=K(V)$. Let $B\in\mathcal{B}_S$ such that $B\subset U$.

Let $\mathcal{C}=\{k(B)\cap k(W)|W\in\mathcal{B}_S$ and $W\subset V\}$. The range of k is $\mathcal{B}_T$, a semi-complete base, thus $\mathcal{C}$ is a subcollection of $\mathcal{B}_T$. Clearly, $\cup\{C|C\in\mathcal{C}\}\subset k(B)$. Furthermore,

$$k(B)\subset\cup\{k(W)|W\in\mathcal{B}_S \text{ and } W\subset U\}=K(U)=$$
$$K(V)=\cup\{k(Z)|Z\in\mathcal{B}_S \text{ and } Z\subset V\}.$$

Therefore if $b\in k(B)$, then there exists $W_b\in\mathcal{B}_S$, such that $W_b\in V$ and $b\in k(W_b)$, hence $b\in k(B)\cap k(W_b)$, which is a member of $\mathcal{C}$. Thus $k(B)\subset\cup\{C|C\in\mathcal{C}\}$, therefore $k(B)=\cup\{C|C\in\mathcal{C}\}$.

The inverse function k^{-1} is a simulacrum, so

$$B=k^{-1}(k(B))=k^{-1}(\cup\{C|C\in\mathcal{C}\})=\cup\{k^{-1}(C)|C\in\mathcal{C}\}=$$
$$\cup\{k^{-1}(k(B)\cap k(W)|W\in\mathcal{B}_S \text{ and } W\subset V\}=$$
$$\cup\{k^{-1}(k(B))\cap k^{-1}(k(W))|W\in\mathcal{B}_S \text{ and } W\subset V\}=$$
$$\cup\{B\cap W|W\in\mathcal{B}_S \text{ and } W\subset V\}\subset V,$$

or $B\subset V$. Since U is the union of the basic open sets contained in U, it follows that $U\subset V$. By a similar argument $V\subset U$, hence $U=V$. Therefore K is one-to-one.

<u>Theorem 18</u>. [3, Th. 9, p.10] Let (X,S) and (Y,T) be topological spaces and let $\mathcal{B}_S$ and $\mathcal{B}_T$ be semi-complete bases for S and T, respectively. Suppose k is a basic similarity from $\mathcal{B}_S$ onto $\mathcal{B}_T$. Let K be the natural extension of k to S. Then K is a lattice isomorphism from S onto T.

Proof: The function K has domain S, and from Lemma 18.3 K is one-to-one. Lemma 18.2 implies that K has range T.

Let $U,V\in S$ such that $U\subset V$. If $B\subset U$, then $B\subset V$; thus $K(U)=\cup\{k(B)|B\in\mathcal{B}_S \text{ and } B\subset U\}\subset\cup\{k(B)|B\in\mathcal{B}_S \text{ and } B\subset V\}=K(V)$. Let $Q,R\in T$ such that $Q\subset R$. It follows from Lemma 18.2 that $K^{-1}(Q)=\cup\{k^{-1}(B)|B\in\mathcal{B}_T \text{ and } B\subset Q\}\subset\cup\{k^{-1}(B)|B\in\mathcal{B}_T \text{ and } B\subset R\}=K^{-1}(R)$. Therefore K is a lattice isomorphism from S onto T.

<u>Theorem 19</u>. Let L be a lattice limit sequence and let B_S be the usual base for the saturating topology, $S(L)$. Then B_S is a semi-complete base.

Proof: This follows from Lemma 14.1.

<u>Theorem 20</u>. Let L be a lattice limit sequence and let B be the usual base for either $U(L)$ or $B(L)$. Then B is a semi-complete base.

Proof: The proof is similar to that of Theorem 19.

<u>Definition 15</u>. [4, Def. 4, p.15] Let $L=(X,T,F)$ be a lattice limit sequence. Let X*, T*, and I be the sequences defined by if j is a positive integer, then $X_j^*=X_1$, $T_j^*=T_1$, and I_j is the identity lattice isomorphism from T_{j+1}^* onto T_j^*. Then (X^*,T^*,I) is said to be the <u>corresponding identity lattice limit sequence for</u> L and will be denoted by L^*.

<u>Definition 16</u>. Let $L=(X,T,F)$ be a lattice limit sequence. Let G be the sequence defined by G_1 is the identity lattice isomorphism from T_1 onto T_1 and if j is a positive integer, then $G_{j+1}=G_j \circ F_j^{j+1}$. Let B be the usual base for either $U(L)$, $S(L)$, or $B(L)$. Let k be the function with domain B defined by if U is a sequence such that $\Pi_n U_n$ is a member of B, then $k(\Pi_n U_n)=\Pi_n(G_n(U_n))$. Then G is called the <u>identity matching map for</u> L and k is called the <u>transposing map of</u> B <u>into</u> $C(L^*)$.

Lemma 21.1. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence. Let G be the identity matching map for L. Then if j is a positive integer, then G_j is a lattice isomorphism from T_j onto T_j^*.

Proof: Let J be the set to which a positive integer j belongs provided G_j is a lattice isomorphism from T_j onto T_j^*. From Definition 16, G_1 is a lattice isomorphism from T_1 onto $T_1=T_1^*$, hence $1\in J$.

Suppose $m\in J$, then G_m is a lattice isomorphism from T_m onto T_m^*. From Definition 6, F_m^{m+1} is a lattice isomorphism from T_{m+1} onto T_m, thus by Theorem 13, $G_{m+1}=G_m\circ F_m^{m+1}$ is a lattice isomorphism from T_{m+1} onto $T_m^*=T_1=T_{m+1}^*$. Therefore $m+1$ is a member of J. It follows that if $j\in J$, then $j+1\in J$. Consequently, by induction, J is the set of positive integers.

If j is a positive integer, them $j\in J$, hence G_j is a lattice isomorphism from T_j onto T_j^*.

Lemma 21.2. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence. Let B_S be the usual base for $S(L)$ and let k be the transposing map of B_S into $C(L)$. Then k is one-to-one.

Proof: Let $A,B\in B_S$ such that $k(A)=k(B)$. Let G be the be the identity matching map for L. If j is a positive integer, then $G_j(\pi_j(A))=\pi_j(k(A))=\pi_j(k(B))=G_j(\pi_j(B))$ and therefore

$$A=\Pi_n(\pi_n(A))=\Pi_n(G_n^{-1}(G_n(\pi_n(A))))=$$
$$\Pi_n(G_n^{-1}(G_n(\pi_n(B))))=\Pi_n(\pi_n(B))=B.$$

Thus k is one-to-one.

Lemma 21.3. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Let B_S be the usual base for $S(L)$ and let B_S^* be the usual base for $S(L^*)$. Then the transposing map of B_S into $C(L^*)$ has range B_S^*.

Proof: Let k be the transposing map of B_S into $C(L^*)$. Let G be the identity matching map for L.

Let $O\in B_S^*$. Then there exists a sequence U^* and a positive integer M such that

(1) if $i\in Z^+$, then $U_i^*\in T_i^*$,

(2) if $i\in Z^+$ and $i\geq M$, then $U_i^*\subset I_i^{i+1}(U_{i+1}^*)$,

(3) if $i\in Z^+$, then $X_i^*=\cup\{I_i^{i+j}(U_{i+j}^*)|j\in Z^+\}$, and

(4) $O=\Pi_n U_n^*$.

Let U be the sequence defined by if i is a positive integer, then $U_i=G_i^{-1}(U_i^*)$. It follows that

(1) if $i\in Z^+$, then $U_i=G_i^{-1}(U_i^*)\in T_i$,

(2) if $i\in Z^+$ and $i\geq M$, then

$$U_i=G_i^{-1}(U_i^*)\subset G_i^{-1}(I_i^{i+1}(U_{i+1}^*))=G_i^{-1}(U_{i+1}^*)=$$
$$F_i^{i+1}\circ G_{i+1}^{-1}(U_{i+1}^*)=F_i^{i+1}(U_{i+1}),$$

(3) if $i\in Z^+$, then

$$X_i=G_i^{-1}(X_i^*)=G_i^{-1}(\cup\{I_i^{i+j}(U_{i+j}^*)\,|\,j\in Z^+\})=$$
$$G_i^{-1}(\cup\{U_{i+j}^*\,|\,j\in Z^+\})=\cup\{G_i^{-1}(U_{i+j}^*)\,|\,j\in Z^+\}=$$
$$\cup\{F_i^{i+j}(G_{i+j}^{-1}(U_{i+j}^*))\,|\,j\in Z^+\}=\cup\{F_i^{i+j}(U_{i+j})\,|\,j\in Z^+\},$$

and

(4) thus $\Pi_n U_n\in \mathcal{B}_S$.

Then, $k(\Pi_n U_n)=\Pi_n(G_n(U_n))=\Pi_n U_n^*=\mathcal{O}$. Now, it follows that k has range $\mathcal{B}_S^*$.

Theorem 21. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Let $\mathcal{B}_S$ be the usual base for $S(L)$ and let k be the transposing map of $\mathcal{B}_S$ into $C(L^*)$. Then k is a simulacrum of $\mathcal{B}_S$ in $S(L^*)$.

Proof: Lemma 21.2 shows that k is one-to-one and Lemma 21.3 shows that k has range a subset of $S(L^*)$. Let G be the identity matching map for L.

If V and W are members of $\mathcal{B}_S$ and $V\subset W$, then if i is a positive integer, then $\pi_i(V)\subset\pi_i(W)$, hence $G_i(\pi_i(V))\subset G_i(\pi_i(W))$. Thus if V and W are members of $\mathcal{B}_S$ and $V\subset W$, then $k(V)=\Pi_n(G_n(\pi_n(V)))\subset\Pi_n(G_n(\pi_n(W)))=k(W)$.

Let $\mathcal{U}$ be a subcollection of $\mathcal{B}_S$ and suppose that $\cup\{U\,|\,U\in\mathcal{U}\}$ is also a member of $\mathcal{B}_S$. Let $U_o=\cup\{U\,|\,U\in\mathcal{U}\}$. If $U\in\mathcal{U}$, then $U\subset U_o$, hence $k(U)\subset k(U_o)$. Therefore

$$\cup\{k(U)\,|\,U\in\mathcal{U}\}\subset k(U_o)=k(\cup\{U\,|\,U\in\mathcal{U}\}).$$

Let $p\in k(\cup\{U|U\in\mathcal{U}\})$. Let H be the sequence defined by if i is a positive integer, then H_i is the dual of the lattice isomorphism G_i. Let D be the sequence defined by if i is a positive integer, then $D_i=cl_{T_i^*}(\{\pi_i(p)\})$. Since $p\in k(U_o)$, if i is a positive integer, then $\pi_i(p)\in\pi_i(k(U_o))$. Also, if i is a positive integer, then $\pi_i(p)\in D_i$. Thus if i is a positive integer, then $\pi_i(p)\in\pi_i(k(U_o))\cap D_i$, hence $\pi_i(k(U_o))\cap D_i$ is not empty. From Theorem 12, if j is a positive integer, then $G_j^{-1}(\pi_j(k(U_o)))\cap H_j^{-1}(D_j)$ is not empty. Moreover, if j is a positive integer, then

$$G_j^{-1}(\pi_j(k(U_o)))=G_j^{-1}(\pi_j(\Pi_n(G_n(\pi_n(U_o)))))=$$
$$G_j^{-1}(G_j(\pi_j(U_o)))=\pi_j(U_o).$$

Therefore, if j is a positive integer, then $\pi_j(U_o)\cap H_j^{-1}(D_j)$ is not empty. Consequently, $\Pi_n(\pi_n(U_o)\cap H_n^{-1}(D_n))$ is not empty. Let $q\in\Pi_n(\pi_n(U_o)\cap H_n^{-1}(D_n))$. Now, if j is a positive integer, then $\pi_j(q)\in\pi_j(U_o)\cap H_j^{-1}(D_j)$, hence $\pi_j(q)\in\pi_j(U_o)$. By hypothesis, $U_o\in\mathcal{B}_S$, so $q\in\Pi_n(\pi_n(U_o))=U_o$. But $U_o=\cup\{U|U\in\mathcal{U}\}$, so there exists $\tilde{U}\in\mathcal{U}$ such that $q\in\tilde{U}$. If j is a positive integer, then $\pi_j(q)\in\pi_j(\tilde{U})$ and so $\pi_j(q)\in\pi_j(\tilde{U})\cap H_j^{-1}(D_j)$. Consequently, if j is a positive integer, then the set $\pi_j(\tilde{U})\cap H_j^{-1}(D_j)$ is not empty. Therefore, by Theorem 12, if i is a positive integer, then $\pi_i(p)\in(G_i^{-1})^{-1}(\pi_i(\tilde{U}))=G_i(\pi_i(\tilde{U}))$. Therefore $p\in\Pi_n(G_n(\pi_n(\tilde{U})))=k(\tilde{U})\subset\cup\{k(U)|U\in\mathcal{U}\}$. It follows that $k(\cup\{U|U\in\mathcal{U}\})\subset\cup\{k(U)|U\in\mathcal{U}\}$. Thus $k(\cup\{U|U\in\mathcal{U}\})=\cup\{k(U)|U\in\mathcal{U}\}$.

A parallel argument shows that if V is a subcollection of B_S such that $\cap\{V|V\in V\}$ is a member of B_S, then $k(\cap\{V|V\in V\})=\cap\{k(V)|V\in V\}$. It follows that k is a simulacrum of B_S in $S(L^*)$.

<u>Theorem 22</u>. Let L be a lattice limit sequence and let L^* be the corresponding identity lattice limit sequence for L. Let B_S and B_S^* be the usual bases for $S(L)$ and $S(L^*)$, respectively. Let k be the transposing map of B_S into $C(L^*)$. Then k^{-1} is a simulacrum of B_S^* in $S(L)$.

Proof: Lemma 21.2 shows that k is one-to-one, hence k^{-1} is a function. Plainly k^{-1} is one-to-one. Lemma 21.3 shows that k has range B_S^*, hence k^{-1} has domain B_S^*. Plainly k^{-1} has range B_S, a subset of $S(L)$. The proof that if U is a subcollection of B_S^*, then

(1) if $\cup\{U|U\in U\}\in B_S^*$, then $k^{-1}(\cup\{U|U\in U\})=\cup\{k^{-1}(U)|U\in U\}$,

and

(2) if $\cap\{U|U\in U\}\in B_S^*$, then $k^{-1}(\cap\{U|U\in U\})=\cap\{k^{-1}(U)|U\in U\}$,

is similar to the proof of Theorem 21. It follows that k^{-1} is a simulacrum of B_S^* in $S(L)$.

<u>Theorem 23</u>. Let L be a lattice limit sequence and let L^* be the corresponding identity lattice limit sequence for L. Let B_S be the usual base for $S(L)$ and let k be the transposing map of B_S into $C(L^*)$. Then the natural extension of k to $S(L)$ is a lattice isomorphism from $S(L)$ onto $S(L^*)$.

Proof: Let B_S^* be the usual base for $S(L^*)$. From Lemma 21.2 k is one-to-one. Lemma 21.3 shows that k has range B_S^*. Theorem 19 shows that both B_S and B_S^* are semi-complete bases. Theorem 21 shows that k is a simulacrum of B_S in $S(L^*)$ and Theorem 22 shows that k^{-1} is a simulacrum of B_S^* in $S(L)$. Therefore k is a basic similarity from B_S onto B_S^*. Thus by Theorem 18, the natural extension of k to $S(L)$ is a lattice isomorphism from $S(L)$ onto $S(L^*)$.

<u>Theorem 24</u>. Let L be a lattice limit sequence and let L^* be the corresponding identity lattice limit sequence for L. Let T be either $U(L)$ or $B(L)$ and let T^* be either $U(L^*)$ or $B(L^*)$. Let B be the usual base for T and let k be the transposing map of B into $C(L^*)$. Then the natural extension of k to T is a lattice isomorphism from T onto T^*.

Proof: The proof is similar to that of Theorem 23.

CHAPTER IV
SEPARATION AXIOMS FOR THE PRODUCT TOPOLOGIES

Theorem 25. Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is Hausdorff. Then $(C(L),S(L))$ is Hausdorff.

Proof: Let p and q be distinct points of $C(L)$. Then there exists a positive integer i such that $\pi_i(p)\neq\pi_i(q)$. But (X_i,T_i) is Hausdorff, so there exist disjoint members U_p and U_q of T_i such that $\pi_i(p)\in U_p$ and $\pi_i(q)\in U_q$.

Let A and B be the sequences defined by if j is a positive integer, then if $j\neq i$, then $A_j=B_j=X_j$, and if $j=i$, then $A_j=U_p$ and $B_j=U_q$. Let $A=\Pi_n A_n$ and let $B=\Pi_n B_n$. Now, A and B are in $S(L)$ and $p\in A$ and $q\in B$. If $r\in A\cap B$, then $\pi_i(r)\in\pi_i(A)\cap\pi_i(B)=U_p\cap U_q=\phi$, which is impossible. Thus $A\cap B=\phi$. It follows that $(C(L),S(L))$ is Hausdorff.

Theorem 26. Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is T_0 (T_1) (T_2). Let T be either $U(L)$, $S(L)$, or $B(L)$. Then $(C(L),T)$ is T_0 (T_1) (T_2).

Proof: The proofs are similar to that of Theorem 23.

<u>Theorem 27</u>. Let $L=(X,T,F)$ be a lattice limit sequence. Let M be a sequence such that if j is a positive integer, then M_j is closed in (X_j,T_j). Then $\Pi_n M_n$ is closed in $(C(L),S(L))$.

Proof: Let B_S be the usual base for $S(L)$. Let $p\in cl(\Pi_n M_n)$.

Let m be a positive integer. Let $A\in T_m$ such that $\pi_m(p)\in A$. Let V be the sequence defined by if i is a positive integer, then if $i=m$, then $V_i=A$ and if $i\neq m$, then $V_i=X_i$. Then $\Pi_n V_n$ is a member of B_S and $p\in \Pi_n V_n$. Therefore $\Pi_n V_n \cap \Pi_n M_n$ is not empty. Let $q\in \Pi_n V_n\cap\Pi_n M_n$. Then $\pi_m(q)\in V_m=A$ and $\pi_m(q)\in M_m$, hence $M_m\cap A$ is not empty. It follows that if $A^*\in T_m$ and $\pi_m(p)\in A^*$, then $M_m\in A^*$ is not empty. Therefore $\pi_m(p)\in cl(M_m)=M_m$. Now, it is concluded that if j is a positive integer, then $\pi_j(p)\in M_j$. Thus $p\in\Pi_n M_n$.

It follows that $cl(\Pi_n M_n)\subset\Pi_n M_n$. Plainly, $\Pi_n M_n\subset cl(\Pi_n M_n)$ and so $\Pi_n M_n=cl(\Pi_n M_n)$. Therefore $\Pi_n M_n$ is closed.

<u>Theorem 28</u>. Let $L=(X,T,F)$ be a lattice limit sequence. Let T be either $U(L)$ or $B(L)$. Let M be a sequence such that if j is a positive integer, then M_j is closed in (X_j,T_j). Then $\Pi_n M_n$ is closed in $(C(L),T)$.

Proof: The proof is similar to that of Theorem 27.

<u>Theorem 29</u>. Let $L=(X,T,F)$ be a lattice limit sequence.

Let W be a sequence such that if j is a positive integer, then $W_j \subset X_j$. Let $S(L)$ be the saturating topology on $C(L)$. Then $\Pi_n \mathrm{cl}(W_n) = \mathrm{cl}(\Pi_n W_n)$.

Proof: Let $p \in \Pi_n \mathrm{cl}(W_n)$. Let $\mathcal{B}_S$ be the usual base for $S(L)$ and let $B \in \mathcal{B}_S$ such that $p \in B$. If j is a positive integer, then $\pi_j(p) \in \mathrm{cl}(W_j)$ and $\pi_j(p) \in \pi_j(B)$, but $\pi_j(B) \in T_j$, so $\pi_j(B) \cap W_j$ is not empty. Therefore $\Pi_n(\pi_n(B) \cap W_n)$ is not empty. But $\Pi_n(\pi_n(B) \cap W_n) \subset \Pi_n(\pi_n(B)) \cap \Pi_n W_n$, and thus $\Pi_n(\pi_n(B)) \cap \Pi_n W_n = B \cap \Pi_n W_n$ is not empty. It follows that if $B^* \in \mathcal{B}_S$ such that $p \in B^*$, then $B^* \cap \Pi_n W_n \neq \phi$. Therefore $p \in \mathrm{cl}(\Pi_n W_n)$. Now, it follows that $\Pi_n \mathrm{cl}(W_n) \subset \mathrm{cl}(\Pi_n W_n)$.

Clearly, $\Pi_n W_n \subset \Pi_n \mathrm{cl}(W_n)$ and by Theorem 27, $\Pi_n \mathrm{cl}(W_n)$ is closed, hence $\mathrm{cl}(\Pi_n W_n) \subset \Pi_n \mathrm{cl}(W_n)$. Thus $\Pi_n \mathrm{cl}(W_n) = \mathrm{cl}(\Pi_n W_n)$.

<u>Theorem 30</u>. [4, Th. 4, p. 10] Let $L = (X,T,F)$ be a lattice limit sequence. Let $\mathcal{T}$ be either $U(L)$ or $B(L)$. Let W be a sequence such that if j is a positive integer, then $W_j \subset X_j$. Then $\Pi_n \mathrm{cl}(W_n) = \mathrm{cl}_{\mathcal{T}}(\Pi_n W_n)$.

Proof: The proof is similar to that of Theorem 29.

<u>Theorem 31</u>. [2, Th. 14.4, p. 93] Let $L = (X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j, T_j) is regular. Then $(C(L), U(L))$ is regular.

Proof: Let $\mathcal{O} \in U(L)$ and let $p \in \mathcal{O}$. Let $\mathcal{B}_U$ be the usual base for $U(L)$. Then there exists $B \in \mathcal{B}_U$ such that $p \in B$. Since $B \in \mathcal{B}_U$, there exists a sequence U and a positive integer M such that

(1) if $i \in Z^+$, then $U_i \in T_i$,

(2) if $i \in Z^+$ and $i \geq M$, then $U_i = X_i$, and

(3) $B = \Pi_n U_n$.

If j is a positive integer, then $\pi_j(p) \in U_j$. But if j is a positive integer, then (X_j, T_j) is regular. Therefore, there exists a sequence W such that if j is a positive integer, then $W_j \in T_j$ and $\pi_j(p) \in W_j \subset cl(W_j) \subset U_j$. Let V be the sequence defined by if i is a positive integer, then if $i < M$, then $V_i = W_i$ and if $i \geq M$, then $V_i = X_i$. Plainly, $\Pi_n V_n$ is a member of $\mathcal{B}_{\mathcal{U}}$. If j is a positive integer, then $\pi_j(p) \in W_j \subset V_j$. Therefore $p \in \Pi_n V_n$. If j is a positive integer, then

(1) if $j < M$, then $cl(V_j) = cl(W_j) \subset U_j$, and

(2) if $j \geq M$, then $cl(V_j) = cl(X_j) = X_j = U_j$.

Therefore, by Theorem 30, $cl(\Pi_n V_n) = \Pi_n cl(V_n) \subset \Pi_n U_n = B \subset \mathcal{O}$. Now, $p \in \Pi_n V_n \subset cl(\Pi_n V_n) \subset \mathcal{O}$. It follows that if $\mathcal{O}^* \in \mathcal{U}(\mathcal{L})$ and $p^* \in \mathcal{O}^*$, then there exists $A^* \in \mathcal{U}(\mathcal{L})$ such that $p^* \in A^* \subset cl(A^*) \subset \mathcal{O}^*$. Therefore $(\mathcal{C}(\mathcal{L}), \mathcal{U}(\mathcal{L}))$ is regular.

Theorem 32. Let $\mathcal{L} = (X, T, F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j, T_j) is regular. Then $(\mathcal{C}(\mathcal{L}), \mathcal{B}(\mathcal{L}))$ is regular.

Proof: The proof is similar to that of Theorem 31.

Example 2. [5, Ex. 1, p. 1] There exists a lattice limit sequence $\mathcal{L} = (X, T, F)$ such that if j is a positive integer, then (X_j, T_j) is regular, but $(\mathcal{C}(\mathcal{L}), \mathcal{S}(\mathcal{L}))$ is not regular.

Proof: Let L be the sequence of intervals in the plane defined by if i is a positive integer, then $L_i=\{(x,\frac{1}{i})|x\in[0,1)\}$. Also, let $L_0=\{(x,0)|x\in(0,1)\}$. Let N denote the set of non-negative integers. Let $X=\cup\{L_i|i\in N\}$.

Let $\mathcal{B}$ be the collection to which B belongs provided $B\subset X$ and either

(1) there exists $i\in Z^+$ and $x\in(0,1)$ such that $B=\{(x,\frac{1}{i})\}$,

(2) there exists $i\in Z^+$ such that $(0,\frac{1}{i})\in B$, $B\in L_i$, and L_i-B is finite,

(3) there exists $m\in Z^+$ and $x\in(0,1)$ such that $B=\{(x,0)\}\cup\{(x,\frac{1}{i})|i\in Z^+$ and $i>m\}$, or

(4) either $B=X$ or $B=\phi$.

Let T be the topology for X generated by $\mathcal{B}$.

Let X, T, and F be the sequences defined by if j is a positive integer, then $X_j=X$, $T_j=T$, and F_j is the identity lattice isomorphism from T_{j+1} onto T_j. Then, let $L=(X,T,F)$. Plainly, L is a lattice limit sequence.

The intersection of two elements of $\mathcal{B}$ is an element of $\mathcal{B}$, thus $\mathcal{B}$ is a basis for T. The topological space (X,T) is regular since each element of $\mathcal{B}$ is both open and closed. Consequently, if j is a positive integer, then $(X_j,T_j)=(X,T)$ is regular. Suppose $(C(L),S(L))$ is regular.

Let H be the sequence defined by if j is a positive integer, then $H_j:(0,1)\to T_j$ is the function defined by if $x\in(0,1)$, then $H_j(x)=\{(x,0)\}\cup\{(x,\frac{1}{i})|i\in Z^+$ and $i>j\}$.

Let $V:(0,1)\to T$ be the function defined by if $x\in(0,1)$, then $V(x)=\cup\{H_1(z)\mid z\in[x,1)\}$. If j is a positive integer, then L_j has a finite (empty) complement in L_j, hence $L_j\in B$. Let W be the sequence defined by if j is a positive integer, then if $j=1$, then $W_j=L_1$, but if $j\neq 1$, then $W_j=\cup\{L_k\mid k\in Z^+$ and $k\leq j\}\cup V(\frac{1}{j})$. If j is a positive integer, then W_j is a union of members of the topology T, thus $W_j\in T$.

Let ℓ be a positive integer. If $(x,y)\in X$, then either

(1) there exists $i\in Z^+$ such that $(x,y)\in L_i$, hence $(x,y)\in W_{\ell+i}$, or

(2) $(x,y)\in L_0$, thus $x\in(0,1)$ and $y=0$, consequently there exists $j\in Z^+$ such that $j\geq\frac{1}{x}$, and thus $x\geq\frac{1}{j}\geq\frac{1}{\ell+j}$, so $(x,y)\in V(\frac{1}{\ell+j})\subset W_{\ell+j}$.

It follows that if m is a positive integer, then

$$X_m=X=\cup\{W_{m+k}\mid k\in Z^+\}.$$

If j is a positive integer, then $W_j\subset W_{j+1}=F_j^{j+1}(W_{j+1})$. Therefore $\Pi_n W_n$ is a member of the usual base for $S(L)$. Let q be the sequence defined by if i is a positive integer, then if $i=1$, then $q_i=(\frac{1}{2},1)$, but if $i\neq 1$, then $q_i=(\frac{1}{i},0)$. Now, $q\in\Pi_n W_n$. Since $(C(L),S(L))$ is regular, there exists an open set O such that $q\in O\subset cl(O)\subset\Pi_n W_n$.

Let B_S be the usual base for $S(L)$. Then there exists $B\in B_S$ such that $q\in B\subset B_S$. Plainly, $cl(B)\subset cl(O)$.

Since $B \in \mathcal{B}_S$, there exists a sequence U and a positive integer M such that

(1) if $i \in Z^+$, then $U_i \in T_i$,

(2) if $i \in Z^+$ and $i \geq M$, then $U_i \subset F_i^{i+1}(U_{i+1}) = U_{i+1}$,

(3) if $i \in Z^+$, then $X_i = \cup\{F_i^{i+k}(U_{i+k}) \mid k \in Z^+\} = \cup\{U_{i+k} \mid k \in Z^+\}$, and

(4) $B = \Pi_n U_n$.

If j is a positive integer, then $\pi_j(q) \in U_j$.

Now $X = X_M = \cup\{U_{M+k} \mid k \in Z^+\}$, but L_0 is an uncountable subset of X, therefore there exists a positive integer m such that $m > M$ and $L_0 \cap U_m$ is uncountable. But U_m is a member of $T_m = T$, so if $(x,0) \in U_m$, then there exists a positive integer j such that $(x,0) \in H_j(x) \subset U_m$, because $\{H_i(x) \mid i \in Z^+\}$ is the collection of base elements that contain $(x,0)$. Let $N:(0,1) \to Z^+$ be the function defined by if $x \in (0,1)$, then if $(x,0) \notin U_m$, then $N(x)=1$, but if $(x,0) \in U_m$, then $N(x)$ is the least integer in the set $\{i \mid i \in Z^+ \text{ and } (x,0) \in H_i(x) \subset U_m\}$.

Let $K = \cup\{H_{N(x)}(x) \mid (x,0) \in U_m\}$. Then $K \subset U_m$, and thus if j is a positive integer, then $K \subset U_j$. Let $k: L_0 \cap U_m \to K - L_0$ be defined by if x is a real number such that $(x,0) \in L_0 \cap U_m$, then

$$k((x,0)) = (x, \frac{1}{N(x)+1}).$$

Plainly, k is one-to-one. Therefore the range of k is uncountable, but the range of k is a subset of $K - L_0$, thus $K - L_0$ is uncountable.

The set $K-L_0$ is a subset of $X-L_0=\cup\{L_i|i\in Z^+\}$. Thus there exists a positive integer τ such that $(K-L_0)\cap L_\tau$ is uncountable. Clearly, $(K-L_0)\cap L_\tau=K\cap L_\tau$. If x is a real number such that $(x,\frac{1}{\tau})\in K\cap L_\tau$, then $(x,\frac{1}{\tau})\in H_{N(x)}(x)$, hence $\tau>N(x)$. Let P be the sequence defined by if j is a positive integer, then if $j<\tau$, then $P_j=\phi$, but if $j\geq\tau$, then $P_j:K\cap L_\tau\to K\cap L_j$ is the function defined by if x is a real number such that $(x,\frac{1}{\tau})\in K\cap L_\tau$, then $P_j((x,\frac{1}{\tau}))=(x,\frac{1}{j})$. If j is a positive integer, then P_j is one-to-one. Thus, if j is a positive integer and $j\geq\tau$, then $K\cap L_j$ is uncountable. Recall that $K\subset U_m$, then if j is a positive integer and $j\geq\tau$, then $U_m\cap L_j$ is not countable, hence not finite.

Let $t=m+\tau$. Now, $t>\tau$ so $U_m\cap L_t$ is not finite. Let $D\in B$ such that $(0,\frac{1}{t})\in D$. Now, L_t-D is not finite. Therefore, $U_m\cap D$ is not empty. It follows that if $D^*\in B$ such that $(0,\frac{1}{t})\in D^*$, then $U_m\cap D^*$ is not empty. Therefore, $(0,\frac{1}{t})\in cl(U_m)$. But $t>m$, thus $(0,\frac{1}{t})$ is not a member of $\cup\{L_k|k\in Z^+ \text{ and } k\leq m\}\cup V(\frac{1}{m})=W_m$. Consequently, $cl(U_m)$ is not a subset of W_m. By Theorem 29 $\pi_m(cl(B))=cl(\pi_m(B))=cl(U_m)$. Thus $\pi_m(cl(B))$ is not a subset of W_m and so $cl(B)$ is not a subset of $\Pi_n W_n$. But recall that $cl(B)\subset cl(O)$ and $cl(O)\subset\Pi_n W_n$, thus $cl(B)\subset\Pi_n W_n$. This is a contradiction. It follows that $(C(L),S(L))$ is not regular.

Definition 17. Let (X,S) be a topological space and let $O\in S$. A subcollection U of S is said to be a σ-regular

<u>cover of</u> $\mathcal{O}$ provided $\mathcal{O}=\cup\{U|U\in\mathcal{U}\}$ and if $U\in\mathcal{U}$, then $cl(U)\subset\mathcal{O}$. A topological space is called <u>σ-regular</u> provided every open set has a countable σ-regular cover.

<u>Theorem 33</u>. Every metrizable space is σ-regular.

Proof: Let (X,T) be a metrizable space and let d be a metric on X which induces T. Let $[0,\to)$ denote the set of non-negative real numbers. Let $B:X\times[0,\to)\to T$ be the function defined by if $x\in X$ and $r\in[0,\to)$, then

$$B(x,r)=\{y|y\in X \text{ and } d(x,y)<r\}.$$

Let $\mathcal{B}=\{B(x,r)|x\in X \text{ and } r\in[0,\to)\}$. Then $\mathcal{B}$ is a base for T.

Let $\mathcal{O}\in T$. Let U be the sequence defined by if j is a positive integer, then

$$U_j=\cup\{B(x,\tfrac{1}{2j})|x\in\mathcal{O} \text{ and } B(x,\tfrac{1}{j})\subset\mathcal{O}\}.$$

If j is a positive integer, then $U_j\subset\mathcal{O}$, thus $\cup\{U_k|k\in Z^+\}\subset\mathcal{O}$. Let $q\in\mathcal{O}$. Then since $\mathcal{B}$ is a base for T, there is a positive real number r such that $q\in B(q,r)\subset\mathcal{O}$. Then there exists a positive integer m such that $m>\frac{1}{r}$. Now, $\frac{1}{m}<r$ and $q\in B(q,\frac{1}{2m})\subset B(q,\frac{1}{m})\subset B(q,r)\subset\mathcal{O}$. Thus $q\in U_m\subset\cup\{U_k|k\in Z^+\}$. It follows that $\mathcal{O}\subset\cup\{U_k|k\in Z^+\}$ and therefore $\mathcal{O}=\cup\{U_k|k\in Z^+\}$.

Let i be a positive integer and let $p\in cl(U_i)$. Then $B(p,\frac{1}{2i})\cap U_i$ is not empty. Let $x\in B(p,\frac{1}{2i})\cap U_i$. Since $x\in U_i$, there exists $y\in\mathcal{O}$ such that $x\in B(y,\frac{1}{2i})\subset B(y,\frac{1}{i})\subset\mathcal{O}$. Thus $d(p,x)<\frac{1}{2i}$ and $d(x,y)<\frac{1}{2i}$, and hence

$$d(p,y) \le d(p,x)+d(x,y) < \frac{1}{2i}+\frac{1}{2i}=\frac{1}{i}.$$

Therefore $p \in B(p,\frac{1}{i}) \subset \mathcal{O}$. It follows that $cl(U_i) \subset \mathcal{O}$. Now, it follows that if j is a positive integer, then $cl(U_j) \subset \mathcal{O}$. Therefore $\{U_k | k \in Z^+\}$ is a σ-regular cover of $\mathcal{O}$. Plainly, $\{U_k | k \in Z^+\}$ is countable. It follows that every open set has a countable σ-regular cover. Therefore (X,T) is σ-regular. It may be concluded that every metrizable space is σ-regular.

Theorem 34. Let (X,T) be a topological space. If (X,T) is σ-regular, then (X,T) is regular.

Proof: Suppose (X,T) is σ-regular. Let $\mathcal{O} \in T$ and let $p \in \mathcal{O}$. There exists a σ-regular cover $\mathcal{U}$ of $\mathcal{O}$. Now, $\mathcal{O}=\cup\{U | U \in \mathcal{U}\}$. Thus there exists $U \in \mathcal{U}$ such that $p \in U$. Also, since $\mathcal{U}$ is a σ-regular cover, $U \in T$ and $cl(U) \subset \mathcal{O}$. Thus, $p \in U \subset cl(U) \subset \mathcal{O}$. It follows that if $\mathcal{O}^* \in T$ and $p^* \in \mathcal{O}^*$, then there exists $U^* \in T$ such that $p^* \in U^* \subset cl(U^*) \subset \mathcal{O}^*$. Therefore, (X,T) is regular.

Theorem 35. Let (X,S) and (Y,T) be topological spaces and suppose F is a lattice isomorphism from S onto T. Let $\mathcal{O} \in S$. Then if $\mathcal{U}$ is a σ-regular cover of $\mathcal{O}$, then $\{F(U) | U \in \mathcal{U}\}$ is a σ-regular cover of $F(\mathcal{O})$.

Proof: Let $\mathcal{U}$ be a σ-regular cover of $\mathcal{O}$. Then by Theorem 2, $F(\mathcal{O})=F(\cup\{U | U \in \mathcal{U}\})=\cup\{F(U) | U \in \mathcal{U}\}$.

Let $\mathcal{D}_F$ be the dual of F. Then if $U \in \mathcal{U}$, then by Theorem 10, $cl(F(U))=\mathcal{D}_F(cl(U))$, but $cl(U) \subset \mathcal{O}$, so by

Theorem 11, $\mathcal{D}_F(\mathrm{cl}(U))\subset F(\mathcal{O})$. Thus if $U\in\mathcal{U}$, then $\mathrm{cl}(F(U))$ is a subset of $F(\mathcal{O})$. Therefore, $\{F(U)|U\in\mathcal{U}\}$ is a σ-regular cover of $\mathcal{O}$.

Corollary 35.1. Let (X,S) and (Y,S) be topological spaces and suppose F is a lattice isomorphism from S onto T. Then if (X,S) is σ-regular, then (Y,T) is σ-regular.

Proof: This is clear.

Theorem 36. Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is σ-regular. Then $(C(L),S(L))$ is regular.

Proof: Let $\mathcal{B}_S$ be the usual base for $S(L)$. Let $\mathcal{O}$ be a member of $\mathcal{B}_S$ and suppose $p\in\mathcal{O}$. Since $\mathcal{O}\in\mathcal{B}_S$, there exists a sequence U and a positive integer M such that

(1) if $i\in Z^+$, then $U_i\in T_i$,

(2) if $i\in Z^+$, and $i\geq M$, then $U_i\subset F_i^{i+1}(U_{i+1})$,

(3) if $i\in Z^+$, then $X_i=\cup\{F_i^{i+k}(U_{i+k})|k\in Z^+\}$, and

(4) $\mathcal{O}=\Pi_n U_n$.

Now, if j is a positive integer, then $\pi_j(p)\in U_j$. If j is a positive integer, then (X_j,T_j) is σ-regular, and therefore regular, by Theorem 34. Also, if j is a positive integer, then $U_j\in T_j$ and $\pi_j(p)\in U_j$. Therefore, there exists a sequence W such that if j is a positive integer, then $\pi_j(p)\in W_j\subset\mathrm{cl}(W_j)\subset U_j$. Let Z be the sequence defined by if j is a positive integer, then Z_j is a sequence such that $\{Z_j(k)|k\in Z^+\}$ is a σ-regular cover of U_j.

Let V be the sequence defined by if i is a positive integer, then (1) if $i \le M$, then $V_i = W_i$, and (2) if $i > M$, then V_i is the union of the sets W_i, $F_i^{i-1}(V_{i-1})$, and $\cup\{F_i^k(Z_k(i-k)) \mid k \in Z^+ \text{ and } M \le k < i\}$.

Let $J = \{k \mid k \in Z^+ \text{ and } cl(V_k) \not\subset U_k\}$. Suppose that J is not empty. Then there exists a least member t of J. But t is not less than or equal to M, else $cl(W_t) = cl(V_t) \not\subset U_t$, contradicting the definition of W. Therefore $t > M$. Then $t-1 \ge M$, so $U_{t-1} \subset F_{t-1}^t(U_t)$. Also, $t-1$ is not a member of J, hence $cl(V_{t-1}) \subset U_{t-1}$. Thus $cl(V_{t-1}) \subset F_{t-1}^t(U_t)$. Let G be the dual of F_t^{t-1}, then by Theorem 11,

$$G(cl(V_{t-1})) \subset F_t^{t-1}(F_{t-1}^t(U_t)) = U_t.$$

Also, by Theorem 10, $G(cl(V_{t-1})) = cl(F_t^{t-1}(V_{t-1}))$. Therefore $cl(F_t^{t-1}(V_{t-1})) \subset U_t$. A similar argument shows that if k is a positive integer and $M \le k < t$, then $cl(F_t^k(Z_k(t-k))) \subset U_t$. Thus $\cup\{cl(F_t^k(Z_k(t-k))) \mid k \in Z^+ \text{ and } M \le k < t\}$ is a subset of U_t. Since $\{F_t^k(Z_k(t-k)) \mid k \in Z^+ \text{ and } M \le k < t\}$ is finite, then

$$cl(\cup\{F_t^k(Z_k(t-k)) \mid k \in Z^+ \text{ and } M \le k < t\}) =$$
$$\cup\{cl(F_t^k(Z_k(t-k))) \mid k \in Z^+ \text{ and } M \le k < t\} \subset U_t.$$

Plainly, $cl(W_t) \subset U_t$. But V_t is the union of the sets W_t, $F_t^{t-1}(V_{t-1})$, and $\cup\{F_t^k(Z_k(t-k)) \mid k \in Z^+ \text{ and } M \le k < t\}$. Thus $cl(V_t)$ is the union of the sets $cl(W_t)$, $cl(F_t^{t-1}(V_{t-1}))$, and $cl(\cup\{F_t^k(Z_k(t-k)) \mid k \in Z^+ \text{ and } M \le k < t\})$, each of which is a subset of U_t. Therefore $cl(V_t) \subset U_t$ and hence, t is not a member of J. But t is the least member of J. This is a contradiction. It follows that J is empty. Thus if j is a positive integer, then $cl(V_j)\subset U_j$. Therefore, by Theorem 29, $cl(\Pi_n V_n)=\Pi_n cl(V_n)\subset\Pi_n U_n=\emptyset$.

Clearly, if j is a positive integer, then $V_j\in T_j$. If j is a positive integer and $j\geq M$, then $j+1>M$, hence $F^j_{j+1}(V_j)$ is a subset of V_{j+1}. Therefore, if j is a positive integer and $j\geq M$, then $V_j=F_j^{j+1}(F^j_{j+1}(V_j))\subset F_j^{j+1}(V_{j+1})$.

Let i be a positive integer. It is clear that $\cup\{F_i^{i+k}(V_{i+k})|k\in Z^+\}\subset X_i$. Let $x\in X_i$. Since $\Pi_n U_n\in\mathcal{B}_S$,

$$X_i=F_i^M(X_M)=F_i^M(\cup\{F_M^{M+k}(U_{M+k})|k\in Z^+\})=$$
$$\cup\{F_i^M(F_M^{M+k}(U_{M+k}))|k\in Z^+\}=\cup\{F_i^{M+k}(U_{M+k})|k\in Z^+\}.$$

Therefore, there exists a positive integer N such that $x\in F_i^{M+N}(U_{M+N})$. Now, $\{Z_{M+N}(k)|k\in Z^+\}$ is a σ-regular cover of U_{M+N}, thus by Theorem 35, $\{F_i^{M+N}(Z_{M+N}(k))|k\in Z^+\}$ is a σ-regular cover of $F_i^{M+N}(U_{M+N})$. Thus there exists a positive integer K such that $x\in F_i^{M+N}(Z_{M+N}(K))$. Let $m=M+N+K$. Then $M\leq M+N<M+N+K=m$, therefore

$$F_i^{M+N}(Z_{M+N}(K))=F_i^m(F_m^{M+N}(Z_{M+N}(m-(M+N))))\subset F_i^m(V_m).$$

Now, $M<m$, so $F_i^m(V_m)\subset F_i^{m+i}(V_{m+i})$. Thus $x\in F_i^{m+i}(V_{m+i})$ and therefore $x\in\cup\{F_i^{i+k}(V_{i+k})|k\in Z^+\}$. It follows that X_i is a subset of $\cup\{F_i^{i+k}(V_{i+k})|k\in Z^+\}$, thus $\{F_i^{i+k}(V_{i+k})|k\in Z^+\}$ is equal to X_i. It follows that if j is a positive integer, then $X_j=\cup\{F_j^{j+k}(V_{j+k})|k\in Z^+\}$.

It is concluded that $\Pi_n V_n$ is a member of $\mathcal{B}_S$. If j is a positive integer, then $W_j \subset V_j$. Therefore, $\Pi_n W_n$ is a subset of $\Pi_n V_n$. Recall that $p \in \Pi_n W_n$, thus $p \in \Pi_n V_n$. Now, $p \in \Pi_n V_n \subset cl(\Pi_n V_n) \subset \mathcal{O}$. It follows that if $\mathcal{O}^*$ is a member of $\mathcal{B}_S$ and $p^* \in \mathcal{O}^*$, then there exists $B \in \mathcal{B}_S$ such that $p^* \in B \subset cl(B) \subset \mathcal{O}^*$. Therefore $(\mathcal{C}(L), \mathcal{S}(L))$ is regular.

CHAPTER V

TOPOLOGICAL PROPERTIES OF THE PRODUCT TOPOLOGIES

Definition 18. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Let D be a subset of X_1. A sequence $x\in C(L^*)$ is called an echoing sequence in $C(L^*)$ if there exists a positive integer M such that if $j\in Z^+$ and $j\geq M$, then $\pi_j(x)=\pi_M(x)$. The echo of D in $C(L^*)$ is the set to which y belongs provided y is an echoing sequence in $C(L^*)$ and if j is a positive integer, then $\pi_j(y)\in D$. The echo of D in $C(L^*)$ will be denoted by $E(D,L^*)$.

Theorem 37. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Let D be a subset of X_1 such that D is dense in (X_1,T_1). Then $E(D,L^*)$ is dense in $(C(L^*),S(L^*))$.

Proof: Let B_S be the usual base for $S(L^*)$ and let O be a non-empty member of B_S. Then there exists a sequence U and a positive integer M such that

(1) if $i\in Z^+$, then $U_i\in T_i^*$,

(2) if $i\in Z^+$ and $i\geq M$, then $U_i\subset U_{i+1}$,

(3) if $i\in Z^+$, then $X_i=\cup\{U_{i+k}|k\in Z^+\}$, and

(4) $O=\Pi_n U_n$.

If i is a positive integer, then $U_i \in T_i^* = T_1$, hence $U_i \cap D$ is not empty. Therefore, there exists a sequence q such that if i is a positive integer, then $q_i \in U_i \cap D$. Then $q \in {}_n U_n = \mathcal{O}$. Let p be the sequence defined by if i is a positive integer, then

(1) if $i<M$, then $p_i = q_i$, and

(2) if $i \geq M$, then $p_i = q_M$.

Now, $p \in \mathcal{C}(\mathcal{L}^*)$ and if j is a positive integer, then there exists a positive integer k such that $\pi_j(p) = p_j = q_k \in U_k \cap D \subset D$, hence $p \in \mathcal{E}(D, \mathcal{L}^*)$. If j is a positive integer, then (1) if $j<M$, then $\pi_j(p) = p_j = q_j \in U_j \cap D \subset U_j$, and (2) if $j \geq M$, then $\pi_j(p) = p_j = q_M \in U_M \cap D \subset U_M \subset U_j$. Thus if j is a positive integer, then $\pi_j(p) \in U_j$. Therefore $p \in \Pi_n U_n = \mathcal{O}$. Now, $p \in \mathcal{E}(D, \mathcal{L}^*)$ and $p \in \mathcal{O}$, so $p \in \mathcal{O} \cap \mathcal{E}(D, \mathcal{L}^*)$. It follows that if $\hat{\mathcal{O}}$ is a non-empty member of $\mathcal{B}_S$, then there exists $\hat{p} \in \mathcal{C}(\mathcal{L}^*)$ such that $\hat{p}$ is a member of $\hat{\mathcal{O}} \cap \mathcal{E}(D, \mathcal{L}^*)$. Therefore, $\mathcal{E}(D, \mathcal{L}^*)$ is dense in $(\mathcal{C}(\mathcal{L}^*), \mathcal{S}(\mathcal{L}^*))$.

<u>Definition 19</u>. Let $\mathcal{L} = (X, T, F)$ be a lattice limit sequence and let $\mathcal{L}^*$ be the corresponding identity lattice limit sequence for $\mathcal{L}$. Let D be a subset of X_1. The <u>sequence of partial echos of D in</u> $\mathcal{C}(\mathcal{L}^*)$ is the sequence E defined by if j is a positive integer, then E_j is the set to which p belongs provided $p \in \mathcal{E}(D, \mathcal{L}^*)$ and if i is a positive integer and $i \geq j$, then $\pi_i(p) = \pi_j(p)$.

Definition 20. Let $L=(X,T,F)$ be a lattice limit sequence and let L^* be the corresponding identity lattice limit sequence for L. Let D be a subset of X_1 and let p be a member of $C(L^*)$. Let i be a positive integer. The i^{th} linear mimic of D in $C(L^*)$ through p, denoted by $L_i(p,D)$, is the set to which q belongs provided q is a member of $E(D,L^*)$ and if j is a positive integer and $i\neq j$, then $\pi_j(q)=\pi_j(p)$.

Theorem 38. Let $L=(X,T,F)$ be a lattice limit sequence and let L^* be the corresponding identity lattice limit sequence for L. Let D be a subset of X_1 and let E be the sequence of partial echos of D in $C(L^*)$. Then $E(D,L^*)$ is equal to $\cup\{E_k|k\in Z^+\}$.

Proof: From Definition 19, if i is a positive integer, then $E_i\subset E(D,L^*)$. Therefore, $\cup\{E_k|k\ Z^+\}\subset E(D,L^*)$. Let $p\in E(D,L^*)$. Then p is an echoing sequence, hence there exists a positive integer M such that if j is a positive integer and $j\geq M$, then $\pi_j(p)=\pi_M(p)$. Thus $p\in E_M$. It follows that $\cup\{E_k|k\in Z^+\}\supset E(D,L^*)$ and therefore $E(D,L^*)$ is equal to $\cup\{E_k|k\in Z^+\}$.

Theorem 39. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence. Let E be the sequence of partial echos of X_1 in $C(L^*)$ and let S be the relative topology for E_1 from $S(L^*)$. Let f be the restriction of the projection π_1 to E_1. Then f is a homeomorphism from (E_1,S) onto (X_1^*,T_1^*).

Proof: Clearly, f has domain E_1. Let $x\in X_1^*$. Let p be the sequence defined by if i is a positive integer, then $p_i=x$. Now $p\in E_1$, since $x\in X_1^*=X_1$. Then $f(p)=\pi_1(p)=x$. It follows that f has range X_1^*.

Suppose q and r are members of E_1 such that $f(q)=f(r)$. If $i\in Z^+$, then $\pi_i(q)=\pi_1(q)=f(q)=f(r)=\pi_1(r)=\pi_i(r)$. Thus $q=r$. It follows that f is one-to-one.

The continuity of f follows from that of π_1. Let $\mathcal{B}_S$ be the usual base for $S(L^*)$ and let $\mathcal{O}\in\mathcal{B}_S$. Then there exists a sequence U and a positive integer M such that

(1) if $i\in Z^+$, then $U_i\in T_i^*$,

(2) if $i\in Z^+$ and $i\geq M$, then $U_i\subset U_{i+1}$,

(3) if $i\in Z^+$, then $X_i^*=\cup\{U_{i+k}|k\in Z^+\}$, and

(4) $\mathcal{O}=\Pi_n U_n$.

Let $a\in\mathcal{O}\cap E_1$. Then $a\in E_1$, so if i is a positive integer, then $f(a)=\pi_1(a)=\pi_i(a)\in U_i$. Hence, $f(a)\in\cap\{U_k|k\in Z^+\}$. Thus $f(\mathcal{O}\cap E_1)\subset\cap\{U_k|k\in Z^+\}$. Let $y\in\cap\{U_k|k\in Z^+\}$. If i is a positive integer, then since $f^{-1}(y)\in E_1$,

$$\pi_i(f^{-1}(y))=\pi_1(f^{-1}(y))=f(f^{-1}(y))=y\in U_i.$$

Therefore, $f^{-1}(y)\in\Pi_n U_n=\mathcal{O}$. Hence, $f^{-1}(y)\in\mathcal{O}\cap E_1$. It follows that $\cap\{U_k|k\in Z^+\}\subset f(\mathcal{O}\cap E_1)$. Thus $\cap\{U_k|k\in Z^+\}=f(\mathcal{O}\cap E_1)$.

Clearly, $\cap\{U_k|k\in Z^+\}\subset\cap\{U_k|k\in Z^+ \text{ and } k\leq M\}$. Let z be a member of $\cap\{U_k|k\in Z^+ \text{ and } k\leq M\}$. If i is a positive integer, then (1) if $i \leq M$, then $z \in U_i$, and (2) if $i > M$, then $z \in U_M \subset U_i$. Thus $z \in \cap\{U_k | k \in Z^+\}$. It follows that $\cap\{U_k | k \in Z^+\}$ contains $\cap\{U_k | k \in Z^+$ and $k \leq M\}$ and therefore $\cap\{U_k | k \in Z^+\}$ is equal to $\cap\{U_k | k \in Z^+$ and $k \leq M\}$. The set $\{U_k | k \in Z^+$ and $k \leq M\}$ is a finite subcollection of T_1^*, thus $\cap\{U_k | k \in Z^+$ and $k \leq M\}$ is a member of T_1^*. Therefore $f(\mathcal{O} \cap E_1) \in T_1^*$. It follows that if $\hat{\mathcal{O}} \in \mathcal{B}_S$, then $f(\hat{\mathcal{O}} \cap E_1) \in T_1^*$. Since $\{B \cap E_1 | B \in \mathcal{B}_S\}$ is a base for S, then f is an open function. Therefore, f is a homeomorphism from (E_1, S) onto (X_1^*, T_1^*).

<u>Theorem 40</u>. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Let i be a positive integer and let $p \in C(L^*)$. Let L be the i^{th} linear mimic of X_1 in $C(L^*)$ through p and let S be the relative topology for L from $S(L^*)$. Let f be the restriction of the projection π_i to L. Then f is a homeomorphism from (L,S) onto (X_i^*, T_i^*).

Proof: Clearly, f has domain L. Let $x \in X_i^*$. Let q be the sequence defined by if j is a positive integer, then (1) if $j=i$, then $q_j = x$, and (2) if $j \neq i$, then $q_j = \pi_j(p)$. Now, $q \in L$ and $f(q) = \pi_i(q) = q_i = x$. It follows that f has range X_i^*.

Suppose u and v are members of L such that $f(u) = f(v)$. If j is a positive integer, then (1) if $j=i$, then $\pi_j(u) = \pi_i(u) = f(u) = f(v) = \pi_i(v) = \pi_j(v)$, and (2) if $j \neq i$, then $\pi_j(u)=\pi_j(p)=\pi_j(v)$. Thus $u=v$. It follows that f is one-to-one.

The continuity of f follows from that of π_i. The function f is open since if O is a member of the usual base for $S(L^*)$, then $f(O\cap L)=\pi_i(O\cap L)=\pi_i(O)$ which is in T_i^* by Theorem 17. It follows that f is a homeomorphism from (L,S) onto (X_i^*,T_i^*).

Corollary 40.1. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Let D be a subset of X_1. Let i be a positive integer and let $p\in E(D,L^*)$. Let $L_i(p,D)$ be the i^{th} linear mimic of D in $C(L^*)$ through p and let S be the relative topology for $L_i(p,D)$. Let f be the restriction of the projection π_i to $L_i(p,D)$. Then f is a homeomorphism from $(L_i(p,D),S)$ onto $(D,(T_i^*)_D)$.

Proof: This follows from Theorem 40.

Lemma 41.1. Let $L=(X,T,F)$ be a lattice limit sequence and let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Let D be a subset of X_1 and let E be the sequence of partial echos of D in $C(L^*)$. Let i be a positive integer. Then $E_{i+1}=\cup\{L_i(p,D)\mid p\in E_i\}$ and $E_i\subset E_{i+1}$.

Proof: Let $q\in E_{i+1}$ and let r be the sequence defined by if j is a positive integer, then (1) if $j<i$, then $r_j=\pi_j(q)$, and (2) if $j\geq i$, then $r_j=\pi_{i+1}(q)$.

Clearly, r is a member of E_i. If j is a positive integer and $j \neq i$, then (1) if $j<i$, then $\pi_j(q)=r_j=\pi_j(r)$, and (2) since $q \in E_{i+1}$, if $j>i$, then $\pi_j(q)=\pi_{i+1}(q)=r_j=\pi_j(r)$. Thus q is a member of $L_i(r,D)$, hence $q \in \cup\{L_i(p,D)|p \in E_i\}$. It follows that $E_{i+1} \subset \cup\{L_i(p,D)|p \in E_i\}$. Let s be a member of $\cup\{L_i(p,D)|p \in E_i\}$. Then there exists $p_s \in E_i$ such that $s \in L_i(p_s,D)$. If j is a positive integer and $j \geq i+1$, then $j \neq i$ and therefore $\pi_j(s)=\pi_j(p_s)=\pi_i(p_s)=\pi_{i+1}(p_s)=\pi_{i+1}(s)$. Hence, $s \in E_{i+1}$. It follows that $\cup\{L_i(p,D)|p \in E_i\} \subset E_{i+1}$, and therefore $E_{i+1}=\cup\{L_i(p,D)|p \in E_i\}$. Plainly, E_i is a subset of $\cup\{L_i(p,D)|p \in E_i\}=E_{i+1}$.

Theorem 41. Let $L=(X,T,F)$ be a lattice limit sequnce such that if j is a positive integer, then (X_j,T_j) is connected. Then $(C(L),S(L))$ is connected.

Proof: Let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. If j is a positive integer, then $(X_j^*,T_j^*)=(X_1,T_1)$ is connected.

The remainder of this proof is concerned with subspaces of $(C(L^*),S(L^*))$. When convienent a subset of $C(L^*)$ will be refered to as the corresponding subspace.

Let E be the sequence of partial echos of X_1 in $C(L^*)$. Let $J=\{j|j \in Z^+$ and E_j is connected$\}$. Then $1 \in J$, since by Theorem 39, E_1 is homeomorphic to (X_1,T_1), which is connected by hypothesis.

Suppose m is a member of J. Then E_m is connected. If $p \in E_m$, then $L_m(p,X_1)$ is connected, since by Theorem 40, $L_m(p,X_1)$ is homeomorphic to $(X_m^*,T_m^*)=(X_1,T_1)$. Also, if $p \in E_m$, then the intersection of E_m and $L_m(p,X_1)$ contains p, hence is non-empty. Therefore, if $p \in E_m$, then $E_m \cup L_m(p,X_1)$ is connected. Clearly, $\cap\{E_m \cup L_m(p,X_1) | p \in E_m\}$ is not empty. Therefore,

$$E_{m+1}=E_m \cup E_{m+1}=E_m \cup (\cup\{L_m(p,X_1)|p\in E_m\})=$$
$$\{E_m \cup L_m(p,X_1)|p\in E_m\}$$

is connected. Thus $m+1$ is a member of J. It follows that if $j \in J$, then $j+1 \in J$. Therefore, by induction, J is the set of positive integers. Hence, if i is a positive integer, then E_i is connected.

Clearly, if i is a positive integer, then $E_1 \subset E_i$. Thus $\cap\{E_k|k\in Z^+\}$ is not empty. Hence, $E(X_1,L^*)=\cup\{E_k|k\in Z^+\}$ is connected. By Theorem 37, $E(X_1,L^*)$ is dense in $(C(L^*),S(L^*))$, thus $(C(L^*),S(L^*))$ is connected. But by Theorem 23, $S(L^*)$ is lattice isomorphic to $S(L)$. Thus by Theorem , $(C(L),S(L))$ is connected.

<u>Theorem 42</u>. Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is connected. Then $(C(L),U(L))$ is connected.

Proof: This follows from the connectedness of $(C(L),S(L))$ by Theorem 41, and the fact that $U(L)\subset S(L)$ by Theorem 16.

Example 3. Let X, T, and I be the sequences defined by if j is a positive integer, then X_j is the set of real numbers, T_j is the usual topology for the set of real numbers, and I_j is the identity lattice isomorphism from T_{j+1} onto T_j. Let $\mathcal{L}=(X,T,I)$. Then $\mathcal{L}$ is a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is connected, but $(C(\mathcal{L}),\mathcal{B}(\mathcal{L}))$ is not connected.

Proof: Plainly, $\mathcal{L}$ is a lattice limit sequence. It is well known that the usual topology for the set of real numbers is connected [2, Ex. 26.9, p. 193]. Therefore, if j is a positive integer, then (X_j,T_j) is connected. Let $\mathcal{B}$ be the usual base for $\mathcal{B}(\mathcal{L})$.

Let C be the set of convergent sequences of real numbers. Then C is a subset of $C(\mathcal{L})$. Let $p\in C$ and let L be the limit of the sequence p. Let U be the sequence defined by if i is a positive integer, then U_i is the open interval $(\pi_i(p)-(1/i),\pi_i(p)+(1/i))$. Clearly, $\Pi_n U_n$ is a member of $\mathcal{B}$ and $p\in\Pi_n U_n$. Let $q\in\Pi_n U_n$ and let ε be a positive number. Since p is convergent to L, there exists a positive integer M such that if i is a positive integer and $i\geq M$, then $|\pi_i(p)-L|<\varepsilon/2$. Since $q\in\Pi_n U_n$, if i is a positive integer, then $|\pi_i(p)-\pi_i(q)|<1/i$. There exists a positive integer N such that $N>2/\varepsilon$, then $1/N<\varepsilon/2$. Now, if i is a positive integer and $i\geq M+N$, then $|\pi_i(q)-L|=|\pi_i(q)-\pi_i(p)+\pi_i(p)-L|\leq|\pi_i(q)-\pi_i(p)|+|\pi_i(p)-L|$ and $|\pi_i(q)-\pi_i(p)|<(1/i)<(1/N)<(\varepsilon/2)$ and $|\pi_i(p)-L|<\varepsilon/2$, hence

$|\pi_i(q)-L|<(\varepsilon/2)+(\varepsilon/2)=\varepsilon$. Therefore q converges to L. Thus q is a member of C. It follows that $\Pi_n U_n \subset C$. Now, it may be concluded that if p^* is a member of C, then there exists $B\epsilon\mathcal{B}$ such that $p^*\epsilon B\subset C$. Thus C is open.

Let $D=\mathcal{C}(L)-C$ and let $r\epsilon D$. Let V be the sequence defined by if i is a positive integer, then V_i is the open interval $(\pi_i(r)-(1/i),\pi_i(r)+(1/i))$. Let $s\epsilon\Pi_n V_n$ and suppose s is a convergent sequence. Then a similar argument to that above shows that r is a convergent sequence. But $r\epsilon D=\mathcal{C}(L)-C$, thus r is not a convergent sequence. This is a contradiction. Thus s is not a convergent sequence, hence $s\epsilon D$. It follows that $\Pi_n V_n\subset D$. Now, it may be concluded that if r^* is a member of D, then there exists $B\epsilon\mathcal{B}$ such that $r^*\epsilon B\subset D$. Thus $\mathcal{C}(L)-C=D$ is open and hence, C is closed. But C is also open and clearly neither $C=\phi$ nor $C=\mathcal{C}(L)$. Therefore $(\mathcal{C}(L),\mathcal{B}(L))$ is not connected.

<u>Theorem 43</u>. Let $L=(X,T,F)$ be a lattice limit sequence such that (X_1,T_1) is seperable. Let $L^*=(X^*,T^*,I)$ be the corresponding identity lattice limit sequence for L. Then $(\mathcal{C}(L^*),\mathcal{S}(L^*))$ is seperable.

Proof: Let D be a countable subset of X_1 such that D is dense in (X_1,T_1). Let E be the sequence of partial echos of D in $\mathcal{C}(L^*)$. Let J be the set to which j belongs provided j is a positive integer and E_j is countable. Now $1\epsilon J$, because Theorem 39 implies that the restriction of the projection π_1 to E_1 is a one-to-one function with range D.

Suppose m is a member of J. Then E_m is countable. If $p \in E_m$, then $L_m(p,D)$ is countable, since Corollary 40.1 implies that the restriction of the projection π_m to $L_m(p,D)$ is a one-to-one function with range D. Now, $E_{m+1} = \{L_m(p,D) \mid p \in E_m\}$ is a countable union of countable sets, hence is countable. Thus m+1 is a member of J. It follows that if $j \in J$, then $j+1 \in J$. Therefore, by induction, J is the set of positive integers. Thus, if i is a positive integer, then E_i is countable.

Now, $E(D,L^*) = \cup\{E_k \mid k \in Z^+\}$ is a countable union of countable sets, hence is countable. By Theorem 37 $E(D,L^*)$ is dense in $(C(L^*),S(L^*))$. Therefore $(C(L^*),S(L^*))$ is seperable.

Theorem 44. Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is both T_1 and seperable. Then $(C(L),S(L))$ is seperable.

Proof: Let L^* be the corresponding identity lattice limit sequence for L. Then by Theorem 23, $S(L)$ is lattice isomorphic to $S(L^*)$. By Theorem 26, both $(C(L),S(L))$ and $(C(L^*),S(L^*))$ are T_1. Therefore, by Theorem 6, $(C(L),S(L))$ is homeomorphic to $(C(L^*),S(L^*))$. Then $(C(L),S(L))$ is seperable, since by Theorem 43, $(C(L^*),S(L^*))$ is seperable.

Theorem 45. 2, Th. 17.8, p. 120 Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is compact. Then $(C(L),U(L))$ is compact.

Theorem 46. Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is compact. Then $U(L)=S(L)$.

Proof: Let B_U and B_S be the usual bases for $U(L)$ and $S(L)$, respectively. Let $O\in B_S$. Then there exists a sequence U and a positive integer M such that

(1) if $i\in Z^+$, then $U_i\in T_i$,

(2) if $i\in Z^+$ and $i\geq M$, then $U_i\subset F_i^{i+1}(U_{i+1})$,

(3) if $i\in Z^+$, then $X_i=\cup\{F_i^{i+k}(U_{i+k})|k\in Z^+\}$, and

(4) $O=\Pi_n U_n$.

In particular, $\{F_M^{M+k}(U_{M+k})|k\in Z^+\}$ is an open cover of X_M. But (X_M,T_M) is compact. Therefore, there exists a finite subset K of Z^+ such that $\{F_M^{M+k}(U_{M+k})|k\in K\}$ is a cover of X_M. Let K be the largest positive integer in K. If j is a positive integer and $M\leq j\leq K$, $U_{M+j}\subset F_{M+j}^{M+K}(U_{M+K})$, thus $F_M^{M+j}(U_{M+j})\subset F_M^{M+j}(F_{M+j}^{M+K}(U_{M+K}))=F_M^{M+K}(U_{M+K})$. It follows that $X_M=\cup\{F_M^{M+k}(U_{M+k})|k\in K\}=F_M^{M+K}(U_{M+K})$. Then by Theorem 4, U_{M+K} is equal to X_{M+K}. If j is a positive integer and $j\geq K$, then $X_{M+K}=U_{M+K}\subset F_{M+K}^{M+j}(U_{M+j})$, hence $X_{M+K}=F_{M+K}^{M+j}(U_{M+j})$. Now, by Theorem 4, if j is a positive integer and $j\geq K$, then $U_{M+j}=X_{M+j}$. Therefore $O=\Pi_n U_n\in B_U$. It follows that $B_S\subset B_U$.

Therefore $\mathcal{S}(L)\subset\mathcal{U}(L)$. By Theorem 16, $\mathcal{U}(L)\subset\mathcal{S}(L)$, thus $\mathcal{U}(L)=\mathcal{S}(L)$.

Example 4. Let X, T, and I be the sequences defined by if i is a positive integer, then $X_i=\{0,1\}$, T_i is the discrete topology on X_i, and I_i is the identity lattice isomorphism from T_{i+1} onto T_i. Let $L=(X,T,I)$. Then L is a lattice limit sequence such that if i is a positive integer, then (X_i,T_i) is compact, but $(C(L),\mathcal{S}(L))$ is not compact.

Proof: Clearly, L is a lattice limit sequence and for each positive integer i, (X_i,T_i) is compact. But $C(L)$ is infinite and $\mathcal{B}(L)$ is discrete, hence $(C(L),\mathcal{B}(L))$ is not compact.

Theorem 47. Let $L=(X,T,F)$ be a lattice limit sequence such that if j is a positive integer, then (X_j,T_j) is compact. Then $(C(L),\mathcal{S}(L))$ is compact.

Proof: By Theorem 46, $(C(L),\mathcal{S}(L))=(C(L),\mathcal{U}(L))$, which is compact by Theorem 45.

SUMMARY

In this thesis the saturating topology for a countable product of lattice isomorphic factors has been defined and investigated. It has been shown that among the topological properties inherited by the saturating topology from the factor spaces are T_0, T_1, T_2, connected, and compact. It was shown that the saturating topology does not inherit regularity from the factor spaces, but that if the factor spaces are σ-regular, then the saturating topology is regular. It is not known whether the saturating topology inherits σ-regularity from the factor spaces. The saturating topology on a product of separable spaces was shown to be lattice isomorphic to a separable topology, but it is not known whether such a saturating topology is itself separable.

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