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Based on two theorems of Chrystal concerning a systematic notational scheme for numbers, a one-to-one correspondence may be established between the position of a permutation in an orderly listing and the permutation itself. From the signature of a permutation in the list, information with respect to the number of inversions in the permutation is obtained in the methods of Netto, Lehmer, Johnson, and Hall. Lehmer's method is shown to be equivalent to that of Netto. A procedure for establishing the one-to-one correspondence is given explicitly for the Tompkins-Paige method.

ON THE ORDERLY LISTING OF PERMUTATIONS

by

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CHAPTER I

INTRODUCTION

D.H. Lehmer [5]* states that by an orderly listing of permutations is meant a generation for which it is possible to obtain the k th permutation directly from the number k, and conversely, given a permutation, it is possible to determine at once its rank, or serial number, in the list without generating any others. In the following discussion several methods of obtaining an orderly listing are considered, especially with respect to the recovery of information regarding the number of inversions in a given permutation.

*Numbers in brackets refer to numbered references in the bibliography.

CHAPTER II

PRELIMINARIES

In [1] Chrystal cites the following theorem as fundamental to a systematic notational scheme for numbers.

Theorem 1. Let $r_1, r_2, \ldots, r_n, r_{n+1}, \ldots$ denote an infinite series of integers restricted in no way except that each is to be greater than 1. Then any nonnegative integer N may be expressed in the finite form

$$(1)\quad N = p_0+p_1r_1+p_2r_1r_2+p_3r_1r_2r_3+\cdots+p_nr_1r_2\cdots r_n,$$

where $0\leq p_0<r_1,\ 0\leq p_1<r_2,\ 0\leq p_2<r_3,\ \ldots,\ 0\leq p_n<r_{n+1}$. When $r_1, r_2, r_3, \ldots$ are given, this is done uniquely.

For the proof, we use repeated application of the division algorithm. The case N=0 is trivial. Thus for $N>0$,

$$\begin{aligned} N &= p_0+N_1r_1, & 0\leq p_0<r_1,\\ N_1 &= p_1+N_2r_2, & 0\leq p_1<r_2,\\ N_2 &= p_2+N_3r_3, & 0\leq p_2<r_3,\\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\\ N_{n-1} &= p_{n-1}+p_nr_n, & 0\leq p_{n-1}<r_n. \end{aligned}$$

Combining these, we have (1).

That the representation is unique follows from the uniqueness of the division algorithm. An alternate proof is in [1]. That n is finite can be shown by the consideration of two cases: if infinitely many of the r_k are distinct, then for n sufficiently large, $r_{n+1} > N > N_i$ since each $N_i < N$; if only finitely many of the r_k are distinct and the process has not terminated when these are used, then, assuming without loss of generality that the r_k are all equal after a certain point, we have that the N_i form a strictly decreasing sequence of positive integers, which is of course finite.

Along with the application of the division algorithm it is easy to obtain, successively,

$$p_i = \left[\frac{N}{r_0 r_1 \cdots r_i}\right] - r_{i+1}\left[\frac{N}{r_0 r_1 \cdots r_{i+1}}\right],$$

for $i=0, 1, \ldots, n$, where $[x]$ is the greatest integer function and $r_0=1$.

As an immediate result of Theorem 1, we have

<u>Corollary 1</u>. Each integer N, $0 \leq N \leq n!-1$, has a unique factorial representation

$$\text{(2)} \qquad N = a_1 \cdot 1! + a_2 \cdot 2! + \ldots + a_{n-1} \cdot (n-1)!,$$

where $0 \leq a_k \leq k$.

For the proof, in Theorem 1, let $r_i = i+1$ and put $a_k = p_{k-1}$. The division process must terminate after the (n-1)st step.

Here,

$$a_i = \left[\frac{N}{i!}\right] - (i+1)\left[\frac{N}{(i+1)!}\right],$$

for $i=1, 2, \ldots, n-1$.

A second result of Theorem 1 is

Theorem 2. Let $r_1, r_2, \ldots, r_n, r_{n+1}, \ldots$ denote an infinite series of integers restricted in no way except that each is to be greater than 1. Then any proper fraction $\frac{A}{B}$ can be expressed in the form

$$(3) \qquad \frac{A}{B} = \frac{p_1}{r_1} + \frac{p_2}{r_1 r_2} + \frac{p_3}{r_1 r_2 r_3} + \ldots + \frac{p_n}{r_1 r_2 \ldots r_n} + F,$$

where $0 \leq p_1 < r_1$, $0 \leq p_2 < r_2$, $\ldots$, $0 \leq p_n < r_n$; and F is either zero or can be made as small as we please by taking a sufficient number of the integers $r_1, r_2, \ldots, r_n$. When $r_1, r_2, \ldots, r_n, \ldots$ are given, this resolution can be effected in one way only.

Although Chrystal's proof essentially is a variation of the division algorithm, he states that the proof might be deduced from Theorem 1. Following is a proof which does indeed use the result of Theorem 1.

Let the given infinite series of positive integers be B, r_1, r_2, ..., r_n, ... If A=0, take all the numerators to be 0. Then, for any A>0, by Theorem 1, we have

$$Ar_n r_{n-1}\cdots r_1 = q_0+q_1B+q_2Br_n+q_3Br_nr_{n-1}+\cdots+q_nBr_n\cdots r_2,$$

where $0\le q_0 < B$, $0\le q_1 < r_n$, ..., $0\le q_n < r_1$. Dividing by $Br_nr_{n-1}\cdots r_1$ and reversing the sum gives

$$\frac{A}{B} = \frac{q_n}{r_1} + \frac{q_{n-1}}{r_1r_2} + \cdots + \frac{q_1}{r_1r_2\cdots r_n} + \frac{q_0}{Br_1r_2\cdots r_n}.$$

Relabeling,

$$\frac{A}{B} = \frac{p_1}{r_1} + \frac{p_2}{r_1r_2} + \cdots + \frac{p_n}{r_1r_2\cdots r_n} + \frac{s_n}{Br_1r_2\cdots r_n},$$

where $0\le p_1 < r_1$, $0\le p_2 < r_2$, ..., $0\le p_n < r_n$, and $0\le s_n < B$, as required.

F=0 if and only if $B \mid r_1r_2\cdots r_n$ for (A,B)=1. Rewrite the fractional expansion as

$$s_n = Ar_1r_2\cdots r_n - Bp_1r_2\cdots r_n - Bp_2r_3\cdots r_n - \cdots - Bp_{n-1}r_n - Bp_n$$
$$= Ar_1r_2\cdots r_n - B(p_1r_2\cdots r_n+p_2r_3\cdots r_n+\cdots+p_{n-1}r_n+p_n).$$

Since B divides the right side, B also divides the left side. If $(A,B)=1$, then $B \mid r_1 r_2 \cdots r_n$. Conversely, suppose $B \mid r_1 r_2 \cdots r_n$. since B then divides both elements of the right side of the equation, $B \mid s_n$. Since $s_n < B$ and $B > 1$, $s_n = 0$. Thus $F=0$ also.

The uniqueness of the fractional expansion follows from that of Theorem 1.

Here, for $r_0 = 1$ and $i=1, 2, \ldots, n$,

$$p_i = \left[\frac{Ar_0 r_1 \cdots r_i}{B}\right] - r_i \left[\frac{Ar_0 r_1 \cdots r_{i-1}}{B}\right]$$

and

$$s_i = Ar_0 r_1 \cdots r_i - B\left[\frac{Ar_0 r_1 \cdots r_i}{B}\right] .$$

Note, especially, the case $i=n$; then

$$s_n = Ar_0 r_1 \cdots r_n - B\left[\frac{Ar_0 r_1 \cdots r_n}{B}\right] ,$$

so that if $(A,B)=1$, then $s_n = 0$ if and only if $B \mid r_0 r_1 \cdots r_n$. The same appears to be true if $B \mid Ar_0 r_1 \cdots r_n$. However, for the general problem one may as well take $(A,B)=1$. There is no gain otherwise.

In Theorem 2, suppose we let $A=N$, $r_i = i+1$ and $B=n!$. Then we obtain

<u>Corollary 2</u>. Any nonnegative integer N, $0 \leq N \leq n!-1$, can be expressed uniquely by the expansion

$$(4) \qquad \frac{N}{n!} = \frac{d_2}{2!} + \frac{d_3}{3!} + \ldots + \frac{d_n}{n!},$$

where $0 \leq d_i \leq i-1$. Note that $F=0$ since $n! \mid n!$.

Here,

$$d_i = \left[\frac{i!\,N}{n!}\right] - i\left[\frac{(i-1)!\,N}{n!}\right],$$

where $2 \leq i \leq n$.

CHAPTER III

LEXICOGRAPHICAL ORDERING

Perhaps the most natural orderly listing of the n! permutations of the n marks 1, 2, 3, ..., n is lexicographical. By lexicographical ordering is meant that the permutation $y_1y_2 \ldots y_n$ precedes $z_1z_2 \ldots z_n$ in the list if the first nonvanishing difference $z_i - y_i$, $i=1, \ldots, n$, is positive.

For example, with n=3 the list is

N	P_N
0	123
1	132
2	213
3	231
4	312
5	321

where N is the position number.

With slight modification we use a result given by Netto [6]. From this it is seen that each permutation in the list has a unique position number and a unique signature for its natural position in the list.

Let $y_1y_2 \ldots y_n$ be a given permutation of 1, 2, ..., n. Then y_1 is the j_1th element in the original order 1, 2, ..., n, and there are $(j_1-1)(n-1)!$ permutations preceding it with smaller first element. The first permutation with y_1 as its leading element has position $(j_1-1)(n-1)!$. Next, y_2 is the j_2th element in the natural order of the n-1 elements 1, 2, ..., y_1-1, y_1+1, ..., n, and there are $(j_1-1)(n-1)!+(j_2-1)(n-2)!$ preceding permutations whose first two elements are smaller than the respective elements y_1 and y_2. The first permutation with y_1y_2 initially has position $(j_1-1)(n-1)!+(j_2-1)(n-2)!$. Generally, $y_1y_2\ldots y_n$ has position

(5) $$N = (j_1-1)(n-1)!+(j_2-1)(n-2)!+\ldots+(j_{n-1}-1)1!,$$

where $0\leq j_i-1\leq n-i$ and $0\leq N\leq n!-1$. By Theorem 1, the representation is unique.

Thus to each N=0, 1, ..., n!-1, there corresponds a unique (n-1)tuple,

$$(a_{n-1}, a_{n-2}, \ldots, a_1), \quad 0\leq a_i\leq i,$$

called the signature of N. Note that in (5) the signature for N is obtained by $a_{n-i}=j_i-1$ for i=1, 2, ..., n-1. In a lexicographical ordering of permutations, the order of the signatures for any two values of N is the same as the order of the values of N.

Conversely, given the position number N and the number of elements n in a permutation, the permutation may be produced by rewriting N via division in the form of equation (5) and applying the definition of the j_i.

As an example of the method in Netto, to find the position number of the permutation 25413 in the lexicographical listing, we have

$j_1=2$ since 2 is second among 1, 2, 3, 4, 5;

$j_2=4$ since 5 is fourth among 1, 3, 4, 5;

$j_3=3$ since 4 is third among 1, 3, 4;

$j_4=1$ since 1 is first among 1, 3.

Since $(2-1)\cdot 4!+(4-1)\cdot 3!+(3-1)\cdot 2!+(1-1)\cdot 1! = 46$, 25413 has position number $N=46$ in the list.

To find the permutation that occupies the forty-sixth place in the lexicographical listing, write

$$46 = 1\cdot 4!+3\cdot 3!+2\cdot 2!+0\cdot 1!,$$

using division to obtain the factorial digits. Then, adding 1 to each of the factorial digits, $j_1=2$, $j_2=4$, $j_3=3$, and $j_4=1$. In the permutation then,

$y_1=2$ since 2 is second among 1, 2, 3, 4, 5;

$y_2=5$ since 5 is fourth among 1, 3, 4, 5;

$y_3=4$ since 4 is third among 1, 3, 4;

$y_4=1$ since 1 is first among 1, 3; and $y_5=3$.

Hence the permutation for $N=46$ is 25413.

As an immediate result of the derivation of equation (5), since j_i-1 gives the number of elements less than y_i which appear to its right in $y_1y_2\ldots y_n$, we have that

(6) $$I_N = (j_1-1)+(j_2-1)+\ldots+(j_{n-1}-1)$$

gives the total number of inversions of order in $y_1 y_2 \ldots y_n$.

In the permutation 25413 with signature (1, 3, 2, 0), for example, $a_4=1$ implies that the first element in the permutation has 1 element less than it to its right; so 2 and 1 are inverted in order. Similarly, $a_3=3$ implies the 3 inversions 54, 53, and 51; $a_2=2$ implies the 2 inversions 43 and 41; and $a_1=0$ implies that 1 and 3 are in their original order.

In this example one might also proceed as follows. $j_1=2$ since 2 is second among 1, 2, 3, 4, 5. Delete 2 and write the permutation as 4312, reducing each element greater than 2 by unity. Then $j_2=4$ since 4 is fourth among 1, 2, 3, 4. Delete 4 and write 312. Then $j_3=3$ since 3 is third among 1, 2, 3. Delete 3 and write 12. Then $j_4=1$. This process is that of D. N. Lehmer [5].

Lehmer systematizes this in the array

$$\begin{array}{rcccc}
 & & & & 2 \\
 & & & 4 & 5 \\
 & & 3 & 3 & 4 \\
 & 1 & 1 & 1 & 1 \\
(1) & 2 & 2 & 2 & 3
\end{array}$$

where the steps are clear. For a general description see [4] .

Since equation (6) gives the total number of inversions from the natural order 1, 2, ..., n in each permutation, given the signatures of any two permutations P_N and $P_{N'}$ in the list, we can get from P_N to $P_{N'}$ by $I_N+I_{N'}$ inversions. However, by adding the total inversions, we have needlessly twice reversed the order of each pair of inverted elements that P_N and $P_{N'}$ have in common: for each occurrence of a pair in common, $I_{N'}$ counts the inversion necessary to restore the elements of the pair to the natural order and then I_N counts the inversion required to obtain their inverted order in P_N. In effect then, $I_N+I_{N'}$ counts the inverted pairs common to P_N and $P_{N'}$ twice, whereas they do not need to be counted at all in the number of inversions required to obtain $P_{N'}$ from P_N.

From the example following equation (6), we see that, given the signature of a permutation, it is possible to recover not only information concerning the total number of inversions of order but the inversions themselves. Thus to find the minimum number of inversions required to obtain $P_{N'}$ from P_N, we recover the actual inversions for each permutation and then count only those pairs of inversions that are not common to P_N and $P_{N'}$.

As an example of this process, we consider the signatures (1, 2, 0) and (3, 1, 1) of the permutations $P_{10}=2413$ and $P_{21}=4231$. Since $I_{10}=3$ and $I_{21}=5$, $I_{10}+I_{21}=8$ gives the number of inversions for obtaining P_{21} from P_{10} when the pairs in common are counted twice.

For the minimum number of inversions, the procedure is the following: in P_{10} the 3 inversions are 21 since $a_3=1$, 43, 41 since $a_2=2$, and no inversion for the pair 13 since $a_1=0$; in P_{21} the 5 inversions are 43, 42, 41 since $a_3=3$, 21 since $a_2=1$, and 31 since $a_1=1$. The minimum number of inversions is 2 since 42 and 31 are the only pairs that must be inverted to obtain P_{21} from P_{10}.

CHAPTER IV
ADJACENT MARK METHOD

A method for listing permutations developed by Johnson [3] is based on the expansion of N according to equation (4):

$$N = n!\left\{\frac{d_2}{2!} + \frac{d_3}{3!} + \ldots + \frac{d_n}{n!}\right\},$$

where $0 \leq d_k \leq k-1$ and $d_1 = d_0 = 0$. Since the representation in (4) is unique, again for each N, $0 \leq N \leq n!-1$, there is a unique signature $(d_2, d_3, \ldots, d_n)$.

For each fixed permutation on the marks from 1 to k, the mark k+1 is moved in one direction through every possible position, beginning at the extreme right (or left), by means of interchanges with some smaller mark to the immediate left (or right), according as the permutation on the k marks is even (or odd). Thus if each permutation on k marks appears once and only once, then each permutation on k+1 marks is given once and only once. For each k, $3 \leq k \leq n$, if the fixed permutation on k-1 elements is even (or odd), then the number of inversions in the permutation on k elements clearly is d_k (or $k-1-d_k$) more than the number of inversions in the permutation on the k-1 elements.

As an example, to find the position in the list of the permutation 25413, we first find the signature by the above method: $d_2=1$ since 2 and 1 are interchanged once; $d_3=2$ since the 3 has been moved twice to the right through the odd permutation 21; $d_4=1$ since the 4 has been moved once to the right through the odd permutation 213; $d_5=1$ since the 5 has been moved once to the right through the odd permutation 2413. So we have N=106.

Conversely, given N=51 with signature (0, 2, 2, 1), the permutation is found as follows: $d_2=0$ implies that 1 and 2 are not interchanged; $d_3=2$ implies that 3 has been moved 2 to the left through the even permutation 12 to get 312; $d_4=2$ implies that 4 has been moved 2 to the left through the even permutation 312 to get 3412; $d_5=1$ implies that 5 has been moved 1 to the right through the odd permutation 3412 to get 35412. Thus $P_{51}=35412$.

CHAPTER V

OTHER METHODS

In the remaining discussion, the methods of obtaining an orderly listing do not seem to depend directly on the two theorems in Chrystal for their development. There is in each case, of course, a one-to-one correspondence between the position N in the list and the factorial digits.

Derangement Method

In this method developed by Hall [5] , for each k, $2 \leq k \leq n+1$, let a_{k-1} denote the number of the k-1 elements less than k that actually follow k in a given permutation on the elements 1, 2, ..., n; then for each k, we have the number of inversions for each element in a given permutation. As in the method in Netto then, the sum of the signature entries gives the total number of inversions of the natural order of the n elements.

Given two permutations P_N and $P_{N'}$ with Hall's signatures $(a_{n-1}, \ldots, a_1)$ and $(b_{n-1}, \ldots, b_1)$, respectively, the minimum number of inversions required to obtain $P_{N'}$ from P_N is

$$I = \sum_{k=1}^{n-1} |b_k - a_k| ,$$

since $b_{n-1}-a_{n-1}>0$ (or <0) implies that the n th element must be moved $|b_{n-1}-a_{n-1}|$ places to the left (or right) from its position in P_N to reach its position in $P_{N'}$; then $b_{n-2}-a_{n-2}>0$ (or <0) implies that the (n-1)st element must be moved $|b_{n-2}-a_{n-2}|$ places to the left (or right); finally, $b_1-a_1>0$ (or <0) implies that the second element must be moved $|b_1-a_1|$ places to the left (or right). For each case k, there are $|b_{n-k}-a_{n-k}|$ inversions.

For example, suppose we wish to obtain $P_{21}=4231$ with signature (3, 1, 1) from $P_{10}=2413$ with signature (2, 0, 1). Then 3-2=1 implies that the 4 in P_{10} must be moved 1 place to the left to get 4213; 1-0=1 implies that the 3 in the new permutation must be moved 1 place to the left to get 4231; 1-1=0 implies that the 2 in 4231 is not to be moved.

<u>Addition Method</u>

Howell [2] gives a method for generating permutations which is not an orderly listing. Beginning with the permutation 012...n-1 on the n elements 0, 1, ..., n-1, the next permutation in the list is obtained by adding n-1 modulo n; this gives the permutation 01...n-1 n-2. By successive addition of n-1 modulo n, all of the n! permutations are obtained in lexicographical order. Since this process also generates all numbers modulo n that are between 01...n-1 and n-1 n-2...1 without distinguishing between those numbers that are or are not permutations, it is not a very efficient process.

For n=3 the list is

012#	102#	122
021#	111	201#
100	120#	210#

where # denotes a permutation.

Transposition Method

In this method by Wells [5] , each permutation is obtained from its predecessor by interchanging two elements; the process differs from Johnson's since the two elements that are interchanged are not necessarily adjacent.

From the signature define h=h(N) to be the least subscript i such that $a_i \neq i$. To obtain P_{N+1} from P_N, interchange the elements in places h and h-1 if h is odd or if h is even and $a_{h+1}<2$; otherwise interchange the elements in places h and $h-a_{h+1}$. The places are numbered 0, 1, ..., n-1 and a negative place is defined to be zero.

For $n=3$ the list is

N	(a_2, a_1)	h	P_N
0	(0, 0)	1	123
1	(0, 1)	2	213
2	(1, 0)	1	231
3	(1, 1)	2	321
4	(2, 0)	1	312
5	(2, 1)	3	132

Tompkins-Paige Method

A permutation of 1, 2, ..., n has degree a_k and order k+1 provided the last k+1 elements of the permutation 12...n have been mounted on a wheel and the wheel rolled forward a_k spokes. Since each permutation on n marks is the result of n-1 transformations of order k+1 and degree a_k, $0 \leq a_k \leq k$ and $k=1, 2, \ldots, n-1$, the a_k become the entries in the signature of the permutation in this method [4].

For example, given N=40 with signature (1, 2, 2, 0), the permutation P_{40} is obtained as follows: $a_4=1$ implies that the last 5 elements of 12345 have been subjected to an end-around shift of 1 to the left to become 23451; $a_3=2$ implies that the last 4 elements of 23451 have been subjected to an end-around shift of 2 to the left to become 25134; $a_2=2$ implies that the last 3 elements of

25134 have been subjected to an end-around shift of 2 to the left to become 25413; and $a_1=0$ implies no shift in the last 2 elements of 25413. Hence $P_{40}=25413$.

Conversely, given a permutation in the list, its position in the list may be found by reversing the above procedure.

CHAPTER VI

SUMMARY

The two theorems in Chrystal, basic for any systematic method of denoting numbers, yield unique factorial and fractional representations for any non-negative integer. In particular, it is possible to establish a one-to-one correspondence between the position in an orderly listing of each of the permutations on n elements and the n-1 factorial or fractional digits. In the orderly listings of Netto, Lehmer, Hall, Wells, and Tompkins and Paige, the correspondence is given by the first theorem from Chrystal; in the orderly listing of Johnson, the correspondence is established by the second theorem.

In the method of Netto, Lehmer, and Howell, the listing is a lexicographical ordering of the permutations. Procedures for finding the number of inversions in each permutation and the minimum number of inversions required to obtain any permutation in the list from a given permutation are given for the method in Netto. The method of Lehmer is shown to be equivalent to that in Netto. The method of Johnson depends on the parity of a given permutation in order to generate the next permutation in the list by the interchange of two adjacent elements. Accordingly, the procedure given for finding the increase in the number of inversions between the two depends on whether the given permutation is even or odd. For the method of Hall, which uses the relative positions of the elements in a given permutation to obtain the position, a formula for finding the minimum number of inversions between any two in the list is derived. In the method of Wells, each permutation is obtained from its predecessor by the interchange of two elements. Finally, the listing of Tompkins and Paige is generated by a product of cyclic permutations.

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